

Solutions to Section 1.1

True-False Review:

1. **FALSE.** A derivative must involve *some* derivative of the function $y = f(x)$, not necessarily the first derivative.
2. **TRUE.** The initial conditions accompanying a differential equation consist of the values of $y, y', \dots$ at $t = 0$.
3. **TRUE.** If we define positive velocity to be oriented downward, then

$$\frac{dv}{dt} = g,$$

where g is the acceleration due to gravity.

4. **TRUE.** We can justify this mathematically by starting from $a(t) = g$, and integrating twice to get $v(t) = gt + c$, and then $s(t) = \frac{1}{2}gt^2 + ct + d$, which is a quadratic equation.
5. **FALSE.** The restoring force is directed in the direction *opposite* to the displacement from the equilibrium position.
6. **TRUE.** According to Newton's Law of Cooling, the rate of cooling is proportional to the *difference* between the object's temperature and the medium's temperature. Since that difference is greater for the object at $100^\circ F$ than the object at $90^\circ F$, the object whose temperature is $100^\circ F$ has a greater rate of cooling.
7. **FALSE.** The temperature of the object is given by $T(t) = T_m + ce^{-kt}$, where T_m is the temperature of the medium, and c and k are constants. Since $e^{-kt} \neq 0$, we see that $T(t) \neq T_m$ for all times t . The temperature of the object *approaches* the temperature of the surrounding medium, but never equals it.
8. **TRUE.** Since the temperature of the coffee is falling, the temperature *difference* between the coffee and the room is higher initially, during the first hour, than it is later, when the temperature of the coffee has already decreased.
9. **FALSE.** The slopes of the two curves are *negative* reciprocals of each other.
10. **TRUE.** If the original family of parallel lines have slopes k for $k \neq 0$, then the family of orthogonal trajectories are parallel lines with slope $-\frac{1}{k}$. If the original family of parallel lines are vertical (resp. horizontal), then the family of orthogonal trajectories are horizontal (resp. vertical) parallel lines.
11. **FALSE.** The family of orthogonal trajectories for a family of circles centered at the origin is the family of lines passing through the origin.

Problems:

1. Starting from the differential equation $\frac{d^2y}{dt^2} = g$, where g is the acceleration of gravity and y is the unknown position function, we integrate twice to obtain the general equations for the velocity and the position of the object:

$$\frac{dy}{dt} = gt + c_1 \quad \text{and} \quad y(t) = \frac{gt^2}{2} + c_1t + c_2,$$

where c_1, c_2 are constants of integration. Now we impose the initial conditions: $y(0) = 0$ implies that $c_2 = 0$,

and $\frac{dy}{dt}(0) = 0$ implies that $c_1 = 0$. Hence, the solution to the initial-value problem is

$$y(t) = \frac{gt^2}{2}.$$

The object hits the ground at the time t_0 for which $y(t_0) = 100$. Hence $100 = \frac{gt_0^2}{2}$, so that $t_0 = \sqrt{\frac{200}{g}} \approx 4.52$ s, where we have taken $g = 9.8 \text{ ms}^{-2}$.

2. Starting from the differential equation $\frac{d^2y}{dt^2} = g$, where g is the acceleration of gravity and y is the unknown position function, we integrate twice to obtain the general equations for the velocity and the position of the ball, respectively:

$$\frac{dy}{dt} = gt + c \quad \text{and} \quad y(t) = \frac{1}{2}gt^2 + ct + d,$$

where c, d are constants of integration. Setting $y = 0$ to be at the top of the boy's head (and positive direction downward), we know that $y(0) = 0$. Since the object hits the ground 8 seconds later, we have that $y(8) = 5$ (since the ground lies at the position $y = 5$). From the values of $y(0)$ and $y(8)$, we find that $d = 0$ and $5 = 32g + 8c$. Therefore, $c = \frac{5 - 32g}{8}$.

(a) The ball reaches its maximum height at the moment when $y'(t) = 0$. That is, $gt + c = 0$. Therefore,

$$t = -\frac{c}{g} = \frac{32g - 5}{8g} \approx 3.98 \text{ s}.$$

(b) To find the maximum height of the tennis ball, we compute

$$y(3.98) \approx -253.51 \text{ feet}.$$

So the ball is 253.51 feet *above* the top of the boy's head, which is 258.51 feet above the ground.

3. Starting from the differential equation $\frac{d^2y}{dt^2} = g$, where g is the acceleration of gravity and y is the unknown position function, we integrate twice to obtain the general equations for the velocity and the position of the rocket, respectively:

$$\frac{dy}{dt} = gt + c \quad \text{and} \quad y(t) = \frac{1}{2}gt^2 + ct + d,$$

where c, d are constants of integration. Setting $y = 0$ to be at ground level, we know that $y(0) = 0$. Thus, $d = 0$.

(a) The rocket reaches maximum height at the moment when $y'(t) = 0$. That is, $gt + c = 0$. Therefore, the time that the rocket achieves its maximum height is $t = -\frac{c}{g}$. At this time, $y(t) = -90$ (the negative sign accounts for the fact that the positive direction is chosen to be downward). Hence,

$$-90 = y\left(-\frac{c}{g}\right) = \frac{1}{2}g\left(-\frac{c}{g}\right)^2 + c\left(-\frac{c}{g}\right) = \frac{c^2}{2g} - \frac{c^2}{g} = -\frac{c^2}{2g}.$$

Solving this for c , we find that $c = \pm\sqrt{180g}$. However, since c represents the initial velocity of the rocket, and the initial velocity is negative (relative to the fact that the positive direction is downward), we choose $c = -\sqrt{180g} \approx -42.02 \text{ ms}^{-1}$, and thus the initial speed at which the rocket must be launched for optimal viewing is approximately 42.02 ms^{-1} .

(b) The time that the rocket reaches its maximum height is $t = -\frac{c}{g} \approx -\frac{-42.02}{9.81} = 4.28$ s.

4. Starting from the differential equation $\frac{d^2y}{dt^2} = g$, where g is the acceleration of gravity and y is the unknown position function, we integrate twice to obtain the general equations for the velocity and the position of the rocket, respectively:

$$\frac{dy}{dt} = gt + c \quad \text{and} \quad y(t) = \frac{1}{2}gt^2 + ct + d,$$

where c, d are constants of integration. Setting $y = 0$ to be at the level of the platform (with positive direction downward), we know that $y(0) = 0$. Thus, $d = 0$.

(a) The rocket reaches maximum height at the moment when $y'(t) = 0$. That is, $gt + c = 0$. Therefore, the time that the rocket achieves its maximum height is $t = -\frac{c}{g}$. At this time, $y(t) = -85$ (this is 85 m above the platform, or 90 m above the ground). Hence,

$$-85 = y\left(-\frac{c}{g}\right) = \frac{1}{2}g\left(-\frac{c}{g}\right)^2 + c\left(-\frac{c}{g}\right) = \frac{c^2}{2g} - \frac{c^2}{g} = -\frac{c^2}{2g}.$$

Solving this for c , we find that $c = \pm\sqrt{170g}$. However, since c represents the initial velocity of the rocket, and the initial velocity is negative (relative to the fact that the positive direction is downward), we choose $c = -\sqrt{170g} \approx -40.84 \text{ ms}^{-1}$, and thus the initial speed at which the rocket must be launched for optimal viewing is approximately 40.84 ms^{-1} .

(b) The time that the rocket reaches its maximum height is $t = -\frac{c}{g} \approx -\frac{-40.84}{9.81} = 4.16$ s.

5. If $y(t)$ denotes the displacement of the object from its initial position at time t , the motion of the object can be described by the initial-value problem

$$\frac{d^2y}{dt^2} = g, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = -2,$$

where g is the acceleration of gravity and y is the unknown position function. We integrate this differential equation twice to obtain the general equations for the velocity and the position of the object:

$$\frac{dy}{dt} = gt + c_1 \quad \text{and} \quad y(t) = \frac{gt^2}{2} + c_1t + c_2.$$

Now we impose the initial conditions: since $y(0) = 0$, we have $c_2 = 0$. Moreover, since $\frac{dy}{dt}(0) = -2$, we have $c_1 = -2$. Hence the solution to the initial-value problem is $y(t) = \frac{gt^2}{2} - 2t$. We are given that $y(10) = h$. Consequently, $h = \frac{g(10)^2}{2} - 2 \cdot 10 \implies h = 10(5g - 2) \approx 470$ m where we have taken $g = 9.8 \text{ ms}^{-2}$.

6. If $y(t)$ denotes the displacement of the object from its initial position at time t , the motion of the object can be described by the initial-value problem

$$\frac{d^2y}{dt^2} = g, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = v_0.$$

We integrate the differential equation twice to obtain the velocity and position functions, respectively:

$$\frac{dy}{dt} = gt + c_1 \quad \text{and} \quad y(t) = \frac{gt^2}{2} + c_1t + c_2.$$

Now we impose the initial conditions. Since $y(0) = 0$, we have $c_2 = 0$. Moreover, since $\frac{dy}{dt}(0) = v_0$, we have $c_1 = v_0$. Hence the solution to the initial-value problem is $y(t) = \frac{gt^2}{2} + v_0t$. We are given that $y(t_0) = h$. Consequently, $h = gt_0^2 + v_0t_0$. Solving for v_0 yields

$$v_0 = \frac{2h - gt_0^2}{2t_0}.$$

7. From $y(t) = A \cos(\omega t - \phi)$, we obtain

$$\frac{dy}{dt} = -A\omega \sin(\omega t - \phi) \quad \text{and} \quad \frac{d^2y}{dt^2} = -A\omega^2 \cos(\omega t - \phi).$$

Hence,

$$\frac{d^2y}{dt^2} + \omega^2 y = -A\omega^2 \cos(\omega t - \phi) + A\omega^2 \cos(\omega t - \phi) = 0.$$

Substituting $y(0) = a$, we obtain $a = A \cos(-\phi) = A \cos(\phi)$. Also, from $\frac{dy}{dt}(0) = 0$, we obtain $0 = -A\omega \sin(-\phi) = A\omega \sin(\phi)$. Since $A \neq 0$ and $\omega \neq 0$ and $|\phi| < \pi$, we have $\phi = 0$. It follows that $a = A$.

8. Taking derivatives of $y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$, we obtain

$$\frac{dy}{dt} = -c_1\omega \sin(\omega t) + c_2\omega \cos(\omega t)$$

and

$$\frac{d^2y}{dt^2} = -c_1\omega^2 \cos(\omega t) - c_2\omega^2 \sin(\omega t) = -\omega^2[c_1 \cos(\omega t) + c_2 \sin(\omega t)] = -\omega^2 y.$$

Consequently, $\frac{d^2y}{dt^2} + \omega^2 y = 0$. To determine the amplitude of the motion we write the solution to the differential equation in the equivalent form:

$$y(t) = \sqrt{c_1^2 + c_2^2} \left[\frac{c_1}{\sqrt{c_1^2 + c_2^2}} \cos(\omega t) + \frac{c_2}{\sqrt{c_1^2 + c_2^2}} \sin(\omega t) \right].$$

We can now define an angle ϕ by

$$\cos \phi = \frac{c_1}{\sqrt{c_1^2 + c_2^2}} \quad \text{and} \quad \sin \phi = \frac{c_2}{\sqrt{c_1^2 + c_2^2}}.$$

Then the expression for the solution to the differential equation is

$$y(t) = \sqrt{c_1^2 + c_2^2} [\cos(\omega t) \cos \phi + \sin(\omega t) \sin \phi] = \sqrt{c_1^2 + c_2^2} \cos(\omega t + \phi).$$

Consequently the motion corresponds to an oscillation with amplitude $A = \sqrt{c_1^2 + c_2^2}$.

9. We compute the first three derivatives of $y(t) = \ln t$:

$$\frac{dy}{dt} = \frac{1}{t}, \quad \frac{d^2y}{dt^2} = -\frac{1}{t^2}, \quad \frac{d^3y}{dt^3} = \frac{2}{t^3}.$$

Therefore,

$$2 \left(\frac{dy}{dt} \right)^3 = \frac{2}{t^3} = \frac{d^3y}{dt^3},$$

as required.

10. We compute the first two derivatives of $y(x) = x/(x+1)$:

$$\frac{dy}{dx} = \frac{1}{(x+1)^2} \quad \text{and} \quad \frac{d^2y}{dx^2} = -\frac{2}{(x+1)^3}.$$

Then

$$y + \frac{d^2y}{dx^2} = \frac{x}{x+1} - \frac{2}{(x+1)^3} = \frac{x^3 + 2x^2 + x - 2}{(x+1)^3} = \frac{(x+1) + (x^3 + 2x^2 - 3)}{(x+1)^3} = \frac{1}{(x+1)^2} + \frac{x^3 + 2x^2 - 3}{(1+x)^3},$$

as required.

11. We compute the first two derivatives of $y(x) = e^x \sin x$:

$$\frac{dy}{dx} = e^x(\sin x + \cos x) \quad \text{and} \quad \frac{d^2y}{dx^2} = 2e^x \cos x.$$

Then

$$2y \cot x - \frac{d^2y}{dx^2} = 2(e^x \sin x) \cot x - 2e^x \cos x = 0,$$

as required.

12. Starting from $(T - T_m)^{-1} \frac{dT}{dt} = -k$, we obtain $\frac{d}{dt}(\ln |T - T_m|) = -k$. The preceding equation can be integrated directly to yield $\ln |T - T_m| = -kt + c_1$. Exponentiating both sides of this equation gives $|T - T_m| = e^{-kt+c_1}$, which can be written as

$$T - T_m = ce^{-kt},$$

where $c = \pm e^{c_1}$. Rearranging this, we conclude that $T(t) = T_m + ce^{-kt}$.

13. After 4 p.m. In the first two hours after noon, the water temperature increased from 50° F to 55° F, an increase of five degrees. Because the temperature of the water has grown closer to the ambient air temperature, the temperature difference $|T - T_m|$ is smaller, and thus, the rate of change of the temperature of the water grows smaller, according to Newton's Law of Cooling. Thus, it will take longer for the water temperature to increase another five degrees. Therefore, the water temperature will reach 60° F more than two hours later than 2 p.m., or after 4 p.m.

14. The object temperature cools a total of 40° F during the 40 minutes, but according to Newton's Law of Cooling, it cools faster in the beginning (since $|T - T_m|$ is greater at first). Thus, the object cooled half-way from 70° F to 30° F in less than half the total cooling time. Therefore, it took less than 20 minutes for the object to reach 50° F.

15. Applying implicit differentiation to the given family of curves $x^2 + 4y^2 = c$ with respect to x gives

$$2x + 8y \frac{dy}{dx} = 0.$$

Therefore,

$$\frac{dy}{dx} = -\frac{x}{4y}.$$

Therefore, the collection of orthogonal trajectories satisfies:

$$\frac{dy}{dx} = \frac{4y}{x} \implies \frac{1}{y} \frac{dy}{dx} = \frac{4}{x} \implies \frac{d}{dx}(\ln |y|) = \frac{4}{x} \implies \ln |y| = 4 \ln |x| + c_1 \implies y = kx^4,$$

where $k = \pm e^{c_1}$.

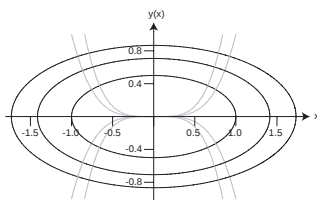


Figure 1: Figure for Problem 15

16. Differentiation of the given family of curves $y = \frac{c}{x}$ with respect to x gives

$$\frac{dy}{dx} = -\frac{c}{x^2} = -\frac{1}{x} \cdot \frac{c}{x} = -\frac{y}{x}.$$

Therefore, the collection of orthogonal trajectories satisfies:

$$\frac{dy}{dx} = \frac{x}{y} \implies y \frac{dy}{dx} = x \implies \frac{d}{dx} \left(\frac{1}{2} y^2 \right) = x \implies \frac{1}{2} y^2 = \frac{1}{2} x^2 + c_1 \implies y^2 - x^2 = c_2,$$

where $c_2 = 2c_1$.

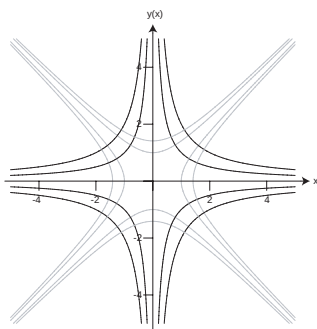


Figure 2: Figure for Problem 16

17. Solving the equation $y = cx^2$ for c gives $c = \frac{y}{x^2}$. Hence, differentiation leads to

$$\frac{dy}{dx} = 2cx = 2 \left(\frac{y}{x^2} \right) x = \frac{2y}{x}.$$

Therefore, the collection of orthogonal trajectories satisfies:

$$\frac{dy}{dx} = -\frac{x}{2y} \implies 2y \frac{dy}{dx} = -x \implies \frac{d}{dx}(y^2) = -x \implies y^2 = -\frac{1}{2}x^2 + c_1 \implies 2y^2 + x^2 = c_2,$$

where $c_2 = 2c_1$.

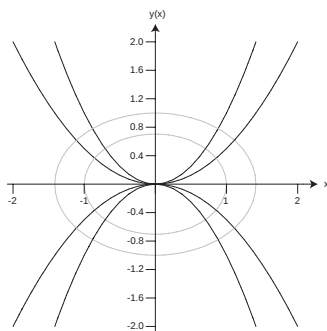


Figure 3: Figure for Problem 17

18. Solving the equation $y = cx^4$ for c gives $c = \frac{y}{x^4}$. Hence,

$$\frac{dy}{dx} = 4cx^3 = 4 \left(\frac{y}{x^4} \right) x^3 = \frac{4y}{x}.$$

Therefore, the collection of orthogonal trajectories satisfies:

$$\frac{dy}{dx} = -\frac{x}{4y} \implies 4y \frac{dy}{dx} = -x \implies \frac{d}{dx}(2y^2) = -x \implies 2y^2 = -\frac{1}{2}x^2 + c_1 \implies 4y^2 + x^2 = c_2,$$

where $c_2 = 2c_1$.

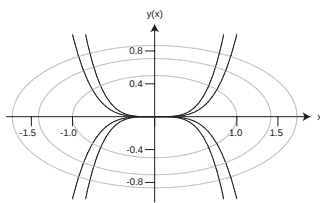


Figure 4: Figure for Problem 18

19. Implicit differentiation of the given family of curves $y^2 = 2x + c$ with respect to x gives $2y \frac{dy}{dx} = 2$. That is, $\frac{dy}{dx} = \frac{1}{y}$. Therefore, the collection of orthogonal trajectories satisfies:

$$\frac{dy}{dx} = -y \implies y^{-1} \frac{dy}{dx} = -1 \implies \frac{d}{dx}(\ln |y|) = -1 \implies \ln |y| = -x + c_1 \implies y = ke^{-x},$$

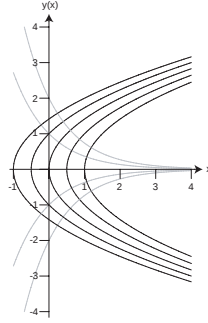


Figure 5: Figure for Problem 19

where $k = \pm e^{c_1}$.

20. Differentiating the given family of curves $y = ce^x$ with respect to x gives $\frac{dy}{dx} = ce^x = y$. Therefore, the collection of orthogonal trajectories satisfies:

$$\frac{dy}{dx} = -\frac{1}{y} \implies y \frac{dy}{dx} = -1 \implies \frac{d}{dx} \left(\frac{1}{2} y^2 \right) = -1 \implies \frac{1}{2} y^2 = -x + c_1 \implies y^2 = -2x + c_2,$$

where $c_2 = 2c_1$.

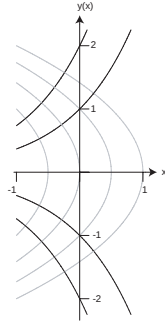


Figure 6: Figure for Problem 20

21. Differentiating the given family of curves $y = mx + c$ with respect to x gives $\frac{dy}{dx} = m$. Therefore, the collection of orthogonal trajectories satisfies:

$$\frac{dy}{dx} = -\frac{1}{m} \implies y = -\frac{1}{m}x + c_1.$$

22. Differentiating the given family of curves $y = cx^m$ with respect to x gives $\frac{dy}{dx} = cmx^{m-1}$. Since $c = \frac{y}{x^m}$, we have $\frac{dy}{dx} = \frac{my}{x}$. Therefore, the collection of orthogonal trajectories satisfies:

$$\frac{dy}{dx} = -\frac{x}{my} \implies y \frac{dy}{dx} = -\frac{x}{m} \implies \frac{d}{dx} \left(\frac{1}{2} y^2 \right) = -\frac{x}{m} \implies \frac{1}{2} y^2 = -\frac{1}{2m} x^2 + c_1 \implies y^2 = -\frac{1}{m} x^2 + c_2,$$

where $c_2 = 2c_1$.

23. Implicit differentiation of the given family of curves $y^2 + mx^2 = c$ with respect to x gives $2y \frac{dy}{dx} + 2mx = 0$.

That is, $\frac{dy}{dx} = -\frac{mx}{y}$. Therefore, the collection of orthogonal trajectories satisfies:

$$\frac{dy}{dx} = \frac{y}{mx} \implies y^{-1} \frac{dy}{dx} = \frac{1}{mx} \implies \frac{d}{dx}(\ln |y|) = \frac{1}{mx} \implies m \ln |y| = \ln |x| + c_1 \implies y^m = c_2 x,$$

where $c_2 = \pm e^{c_1}$.

24. Implicit differentiation of the given family of curves $y^2 = mx + c$ with respect to x gives $2y \frac{dy}{dx} = m$.

That is, $\frac{dy}{dx} = \frac{m}{2y}$. Therefore, the collection of orthogonal trajectories satisfies:

$$\frac{dy}{dx} = -\frac{2y}{m} \implies y^{-1} \frac{dy}{dx} = -\frac{2}{m} \implies \frac{d}{dx}(\ln |y|) = -\frac{2}{m} \implies \ln |y| = -\frac{2}{m}x + c_1 \implies y = c_2 e^{-\frac{2x}{m}},$$

where $c_2 = \pm e^{c_1}$.

25. Consider the coordinate curve $u = x^2 + 2y^2$ (i.e. u is constant). Differentiating implicitly with respect to x gives $0 = 2x + 4y \frac{dy}{dx}$. Therefore, $\frac{dy}{dx} = -\frac{x}{2y}$. Therefore, the collection of orthogonal trajectories satisfies:

$$\frac{dy}{dx} = \frac{2y}{x} \implies y^{-1} \frac{dy}{dx} = \frac{2}{x} \implies \frac{d}{dx}(\ln |y|) = \frac{2}{x} \implies \ln |y| = 2 \ln |x| + c_1 \implies y = c_2 x^2,$$

where $c_2 = \pm e^{c_1}$.

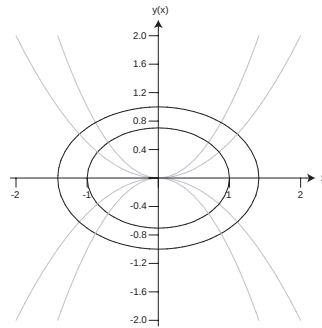


Figure 7: Figure for Problem 25

26. We have $m_1 = \tan(a_1) = \tan(a_2 - a) = \frac{\tan(a_2) - \tan(a)}{1 + \tan(a_2)\tan(a)} = \frac{m_2 - \tan(a)}{1 + m_2 \tan(a)}$.

Solutions to Section 1.2

True-False Review:

1. **FALSE.** The order of a differential equation is the order of the *highest* derivative appearing in the differential equation.
2. **TRUE.** This is condition 1 in Definition 1.2.11.
3. **TRUE.** This is the content of Theorem 1.2.15.
4. **FALSE.** There are solutions to $y'' + y = 0$ that do not have the form $c_1 \cos x + 5c_2 \cos x$, such as $y(x) = \sin x$. Therefore, $c_1 \cos x + 5c_2 \cos x$ does not meet the second requirement set forth in Definition 1.2.11 for the general solution.
5. **FALSE.** There are solutions to $y'' + y = 0$ that do not have the form $c_1 \cos x + 5c_1 \sin x$, such as $y(x) = \cos x + \sin x$. Therefore, $c_1 \cos x + 5c_1 \sin x$ does not meet the second requirement set forth in Definition 1.2.11 for the general solution.
6. **TRUE.** Since the right-hand side of the differential equation is a function of x only, we can integrate both sides n times to obtain the formula for the solution $y(x)$.

Problems:

1. 2, nonlinear.
2. 3, linear.
3. 2, nonlinear.
4. 2, nonlinear.
5. 4, linear.
6. 3, nonlinear.
7. We can quickly compute the first two derivatives of $y(x)$:

$$y'(x) = (c_1 + 2c_2)e^x \cos 2x + (-2c_1 + c_2)e^x \sin 2x \quad \text{and} \quad y''(x) = (-3c_1 + 4c_2)e^x \cos 2x + (-4c_1 - 3c_2)e^x \sin 2x.$$

Then we have

$$\begin{aligned} & y'' - 2y' + 5y \\ &= [(-3c_1 + 4c_2)e^x \cos 2x + (-4c_1 - 3c_2)e^x \sin 2x] - 2[(c_1 + 2c_2)e^x \cos 2x + (-2c_1 + c_2)e^x \sin 2x] + 5(c_1 e^x \cos 2x + c_2 e^x \sin 2x), \end{aligned}$$

which cancels to 0, as required. This solution is valid for all $x \in \mathbb{R}$.

8. We can quickly compute the first two derivatives of $y(x)$:

$$y'(x) = c_1 e^x - 2c_2 e^{-2x} \quad \text{and} \quad y''(x) = c_1 e^x + 4c_2 e^{-2x}.$$

Then we have

$$y'' + y' - 2y = (c_1 e^x + 4c_2 e^{-2x}) + (c_1 e^x - 2c_2 e^{-2x}) - 2(c_1 e^x + c_2 e^{-2x}) = 0.$$

Thus $y(x) = c_1 e^x + c_2 e^{-2x}$ is a solution of the given differential equation for all $x \in \mathbb{R}$.

9. The derivative of $y(x) = \frac{1}{x+4}$ is $y'(x) = -\frac{1}{(x+4)^2} = -y^2$. Thus $y(x) = \frac{1}{x+4}$ is a solution of the given differential equation for $x \in (-\infty, -4)$ or $x \in (-4, \infty)$.

10. The derivative of $y(x) = c_1\sqrt{x}$ is $y'(x) = \frac{c_1}{2\sqrt{x}} = \frac{y}{2x}$. Thus $y(x) = c_1\sqrt{x}$ is a solution of the given differential equation for all $x > 0$.

11. We can quickly compute the first two derivatives of $y(x) = c_1e^{-x}\sin(2x)$:

$$y'(x) = 2c_1e^{-x}\cos(2x) - c_1e^{-x}\sin(2x) \quad \text{and} \quad y''(x) = -3c_1e^{-x}\sin(2x) - 4c_1e^{-x}\cos(2x).$$

Therefore, we have

$$y'' + 2y' + 5y = -3c_1e^{-x}\sin(2x) - 4c_1e^{-x}\cos(2x) + 2[2c_1e^{-x}\cos(2x) - c_1e^{-x}\sin(2x)] + 5[c_1e^{-x}\sin(2x)] = 0,$$

which shows that $y(x) = c_1e^{-x}\sin(2x)$ is a solution to the given differential equation for all $x \in \mathbb{R}$.

12. We can quickly compute the first two derivatives of $y(x) = c_1\cosh(3x) + c_2\sinh(3x)$:

$$y'(x) = 3c_1\sinh(3x) + 3c_2\cosh(3x) \quad \text{and} \quad y''(x) = 9c_1\cosh(3x) + 9c_2\sinh(3x).$$

Therefore, we have

$$y'' - 9y = [9c_1\cosh(3x) + 9c_2\sinh(3x)] - 9[c_1\cosh(3x) + c_2\sinh(3x)] = 0,$$

which shows that $y(x) = c_1\cosh(3x) + c_2\sinh(3x)$ is a solution to the given differential equation for all $x \in \mathbb{R}$.

13. We can quickly compute the first two derivatives of $y(x) = \frac{c_1}{x^3} + \frac{c_2}{x}$:

$$y'(x) = -\frac{3c_1}{x^4} - \frac{c_2}{x^2} \quad \text{and} \quad y''(x) = \frac{12c_1}{x^5} + \frac{2c_2}{x^3}.$$

Therefore, we have

$$x^2y'' + 5xy' + 3y = x^2\left(\frac{12c_1}{x^5} + \frac{2c_2}{x^3}\right) + 5x\left(-\frac{3c_1}{x^4} - \frac{c_2}{x^2}\right) + 3\left(\frac{c_1}{x^3} + \frac{c_2}{x}\right) = 0,$$

which shows that $y(x) = \frac{c_1}{x^3} + \frac{c_2}{x}$ is a solution to the given differential equation for all $x \in (-\infty, 0)$ or $x \in (0, \infty)$.

14. We can quickly compute the first two derivatives of $y(x) = c_1\sqrt{x} + 3x^2$:

$$y'(x) = \frac{c_1}{2\sqrt{x}} + 6x \quad \text{and} \quad y''(x) = -\frac{c_1}{4\sqrt{x^3}} + 6.$$

Therefore, we have

$$2x^2y'' - xy' + y = 2x^2\left(-\frac{c_1}{4\sqrt{x^3}} + 6\right) - x\left(\frac{c_1}{2\sqrt{x}} + 6x\right) + (c_1\sqrt{x} + 3x^2) = 9x^2,$$

which shows that $y(x) = c_1\sqrt{x} + 3x^2$ is a solution to the given differential equation for all $x > 0$.

15. We can quickly compute the first two derivatives of $y(x) = c_1x^2 + c_2x^3 - x^2 \sin x$:

$$y'(x) = 2c_1x + 3c_2x^2 - x^2 \cos x - 2x \sin x \quad \text{and} \quad y''(x) = 2c_1 + 6c_2x + x^2 \sin x - 2x \cos x - 2x \cos - 2 \sin x.$$

Substituting these results into the given differential equation yields

$$\begin{aligned} x^2y'' - 4xy' + 6y &= x^2(2c_1 + 6c_2x + x^2 \sin x - 4x \cos x - 2 \sin x) \\ &\quad - 4x(2c_1x + 3c_2x^2 - x^2 \cos x - 2x \sin x) + 6(c_1x^2 + c_2x^3 - x^2 \sin x) \\ &= 2c_1x^2 + 6c_2x^3 + x^4 \sin x - 4x^3 \cos x - 2x^2 \sin x \\ &\quad - 8c_1x^2 - 12c_2x^3 + 4x^3 \cos x + 8x^2 \sin x + 6c_1x^2 + 6c_2x^3 - 6x^2 \sin x \\ &= x^4 \sin x. \end{aligned}$$

Hence, $y(x) = c_1x^2 + c_2x^3 - x^2 \sin x$ is a solution to the differential equation for all $x \in \mathbb{R}$.

16. We can quickly compute the first two derivatives of $y(x) = c_1e^{ax} + c_2e^{bx}$:

$$y'(x) = ac_1e^{ax} + bc_2e^{bx} \quad \text{and} \quad y''(x) = a^2c_1e^{ax} + b^2c_2e^{bx}.$$

Substituting these results into the differential equation yields

$$\begin{aligned} y'' - (a+b)y' + aby &= a^2c_1e^{ax} + b^2c_2e^{bx} - (a+b)(ac_1e^{ax} + bc_2e^{bx}) + ab(c_1e^{ax} + c_2e^{bx}) \\ &= (a^2c_1 - a^2c_1 - abc_1 + abc_1)e^{ax} + (b^2c_2 - abc_2 - b^2c_2 + abc_2)e^{bx} \\ &= 0. \end{aligned}$$

Hence, $y(x) = c_1e^{ax} + c_2e^{bx}$ is a solution to the given differential equation for all $x \in \mathbb{R}$.

17. We can quickly compute the first two derivatives of $y(x) = e^{ax}(c_1 + c_2x)$:

$$y'(x) = e^{ax}(c_2) + ae^{ax}(c_1 + c_2x) = e^{ax}(c_2 + ac_1 + ac_2x)$$

and

$$y''(x) = ea^{ax}(ac_2) + ae^{ax}(c_2 + ac_1 + ac_2x) = ae^{ax}(2c_2 + ac_1 + ac_2x).$$

Substituting these into the differential equation yields

$$\begin{aligned} y'' - 2ay' + a^2y &= ae^{ax}(2c_2 + ac_1 + ac_2x) - 2ae^{ax}(c_2 + ac_1 + ac_2x) + a^2e^{ax}(c_1 + c_2x) \\ &= ae^{ax}(2c_2 + ac_1 + ac_2x - 2c_2 - 2ac_1 - 2ac_2x + ac_1 + ac_2x) \\ &= 0. \end{aligned}$$

Thus, $y(x) = e^{ax}(c_1 + c_2x)$ is a solution to the given differential equation for all $x \in \mathbb{R}$.

18. We can quickly compute the first two derivatives of $y(x) = e^{ax}(c_1 \cos bx + c_2 \sin bx)$:

$$\begin{aligned} y'(x) &= e^{ax}(-bc_1 \sin bx + bc_2 \cos bx) + ae^{ax}(c_1 \cos bx + c_2 \sin bx) \\ &= e^{ax}[(bc_2 + ac_1) \cos bx + (ac_2 - bc_1) \sin bx], \\ y''(x) &= e^{ax}[-b(bc_2 + ac_1) \sin bx + b(ac_2 + bc_1) \cos bx] + ae^{ax}[(bc_2 + ac_1) \cos bx + (ac_2 + bc_1) \sin bx] \\ &= e^{ax}[(a^2c_1 - b^2c_1 + 2abc_2) \cos bx + (a^2c_2 - b^2c_2 - abc_1) \sin bx]. \end{aligned}$$

Substituting these results into the differential equation yields

$$\begin{aligned}
 y'' - 2ay' + (a^2 + b^2)y &= (e^{ax}[(a^2c_1 - b^2c_1 + 2abc_2)\cos bx + (a^2c_2 - b^2c_2 - abc_1)\sin bx]) \\
 &\quad - 2a(e^{ax}[(bc_2 + ac_1)\cos bx + (ac_2 - bc_1)\sin bx]) + (a^2 + b^2)(e^{ax}(c_1\cos bx + c_2\sin bx)) \\
 &= e^{ax}[(a^2c_1 - b^2c_1 + 2abc_2 - 2abc_2 - 2a^2c_1 + a^2c_1 + b^2c_1)\cos bx \\
 &\quad + (a^2c_2 - b^2c_2 - 2abc_1 + 2abc_1 - 2a^2c_2 + a^2c_2 + b^2c_2)\sin bx] \\
 &= 0.
 \end{aligned}$$

Thus, $y(x) = e^{ax}(c_1\cos bx + c_2\sin bx)$ is a solution to the given differential equation for all $x \in \mathbb{R}$.

19. From $y(x) = e^{rx}$, we obtain $y'(x) = re^{rx}$ and $y''(x) = r^2e^{rx}$. Substituting these results into the given differential equation yields $e^{rx}(r^2 + 2r - 3) = 0$, so that r must satisfy $r^2 + 2r - 3 = 0$, or $(r + 3)(r - 1) = 0$. Consequently $r = -3$ and $r = 1$ are the only values of r for which $y(x) = e^{rx}$ is a solution to the given differential equation. The corresponding solutions are $y(x) = e^{-3x}$ and $y(x) = e^x$.

20. From $y(x) = e^{rx}$, we obtain $y'(x) = re^{rx}$ and $y''(x) = r^2e^{rx}$. Substituting these results into the given differential equation yields $e^{rx}(r^2 - 8r + 16) = 0$, so that r must satisfy $r^2 - 8r + 16 = 0$, or $(r - 4)^2 = 0$. Consequently the only value of r for which $y(x) = e^{rx}$ is a solution to the differential equation is $r = 4$. The corresponding solution is $y(x) = e^{4x}$.

21. From $y(x) = x^r$, we obtain $y'(x) = rx^{r-1}$ and $y''(x) = r(r-1)x^{r-2}$. Substituting these results into the given differential equation yields $x^r[r(r-1) + r - 1] = 0$, so that r must satisfy $r^2 - 1 = 0$. Consequently $r = -1$ and $r = 1$ are the only values of r for which $y(x) = x^r$ is a solution to the given differential equation. The corresponding solutions are $y(x) = x^{-1}$ and $y(x) = x$.

22. From $y(x) = x^r$, we obtain $y'(x) = rx^{r-1}$ and $y''(x) = r(r-1)x^{r-2}$. Substituting these results into the given differential equation yields $x^r[r(r-1) + 5r + 4] = 0$, so that r must satisfy $r^2 + 4r + 4 = 0$, or equivalently $(r + 2)^2 = 0$. Consequently $r = -2$ is the only value of r for which $y(x) = x^r$ is a solution to the given differential equation. The corresponding solution is $y(x) = x^{-2}$.

23. From $y(x) = \frac{1}{2}x(5x^2 - 3) = \frac{1}{2}(5x^3 - 3x)$, we obtain $y'(x) = \frac{1}{2}(15x^2 - 3)$ and $y''(x) = 15x$. Substituting these results into the Legendre equation with $N = 3$ yields

$$(1 - x^2)y'' - 2xy' + 12y = (1 - x^2)(15x) + x(15x^2 - 3) + 6x(5x^2 - 3) = 0.$$

Consequently, the given function is a solution to the Legendre equation with $N = 3$.

24. From $y(x) = a_0 + a_1x + a_2x^2$, we obtain $y'(x) = a_1 + 2a_2x$ and $y''(x) = 4a_2$. Substituting these results into the given differential equation yields $(1 - x^2)(2a_2) - x(a_1 + 2a_2x) + 4(a_0 + a_1x + a_2x^2) = 0$. That is, $3a_1x + 2a_2 + 4a_0 = 0$. For this equation to hold for all x we require $3a_1 = 0$, and $2a_2 + 4a_0 = 0$. Consequently, $a_1 = 0$ and $a_2 = -2a_0$. The corresponding solution to the differential equation is $y(x) = a_0(1 - 2x^2)$. Imposing the normalization condition $y(1) = 1$ requires that $a_0 = -1$. Hence, the required solution to the differential equation is $y(x) = 2x^2 - 1$.

25. Differentiating $x \sin y - e^x = c$ implicitly with respect to x yields $x \cos y \frac{dy}{dx} + \sin y - e^x = 0$. Thus,

$$\frac{dy}{dx} = \frac{e^x - \sin y}{x \cos y}.$$

26. Differentiating $xy^2 + 2y - x = c$ implicitly with respect to x yields $2xy\frac{dy}{dx} + y^2 + 2\frac{dy}{dx} - 1 = 0$. Thus, $\frac{dy}{dx} = \frac{1 - y^2}{2(xy + 1)}$.

27. Differentiating $e^{xy} - x = c$ implicitly with respect to x yields $e^{xy}[x\frac{dy}{dx} + y] - 1 = 0$. Therefore, we have $xe^{xy}\frac{dy}{dx} + ye^{xy} = 1$. Hence, $\frac{dy}{dx} = \frac{1 - ye^{xy}}{xe^{xy}}$. Given that $y(1) = 0$, we have $e^{0(1)} - 1 = c$, so that $c = 0$. Therefore, $e^{xy} - x = 0$. Rearranging this equation and taking logarithms, we conclude that $y = \frac{\ln x}{x}$.

28. Differentiating $e^{y/x} + xy^2 - x = c$ implicitly with respect to x yields

$$e^{y/x} \left[\frac{x\frac{dy}{dx} - y}{x^2} \right] + 2xy\frac{dy}{dx} + y^2 - 1 = 0.$$

A short algebraic simplification yields

$$\frac{dy}{dx} = \frac{x^2(1 - y^2) + ye^{y/x}}{x(e^{y/x} + 2x^2y)}.$$

29. Differentiating $x^2y^2 - \sin x = c$ implicitly with respect to x yields $2x^2y\frac{dy}{dx} + 2xy^2 - \cos x = 0$. Rearranging, we obtain $\frac{dy}{dx} = \frac{\cos x - 2xy^2}{2x^2y}$. Since $y(\pi) = \frac{1}{\pi}$, we have $\pi^2 \left(\frac{1}{\pi}\right)^2 - \sin \pi = c$. Therefore, $c = 1$. Hence, $x^2y^2 - \sin x = 1$ so that $y^2 = \frac{1 + \sin x}{x^2}$. Since $y(\pi) = \frac{1}{\pi}$, take the branch of y where $x > 0$, so that $y(x) = \frac{\sqrt{1 + \sin x}}{x}$.

30. By integrating $\frac{dy}{dx} = \sin x$ with respect to x , we obtain $y(x) = -\cos x + c$ for all $x \in \mathbb{R}$.

31. By integrating $\frac{dy}{dx} = x^{-1/2}$ with respect to x , we obtain $y(x) = 2x^{1/2} + c$ for all $x > 0$.

32. By integrating $\frac{d^2y}{dx^2} = xe^x$ twice with respect to x , we obtain $\frac{dy}{dx} = xe^x - e^x + c_1$ and $y(x) = xe^x - 2e^x + c_1x + c_2$ for all $x \in \mathbb{R}$.

33. We consider three cases:

Case 1: $n = -1$: In this case, two integrations with respect to x yields $\frac{dy}{dx} = \ln|x| + c_1$ and $y(x) = x \ln|x| + c_1x + c_2$ for all $x \in (-\infty, 0)$ or $x \in (0, \infty)$.

Case 2: $n = -2$: In this case, two integrations with respect to x yields $\frac{dy}{dx} = -x^{-1} + c_1$ and $y(x) = c_1x + c_2 - \ln|x|$ for all $x \in (-\infty, 0)$ or $x \in (0, \infty)$.

Case 3: $n \neq -1, -2$: In this case, two integrations with respect to x yields $\frac{dy}{dx} = \frac{x^{n+1}}{n+1} + c_1$ and $y(x) = \frac{x^{n+2}}{(n+1)(n+2)} + c_1x + c_2$ for all $x \in \mathbb{R}$.

34. Integrating $\frac{dy}{dx} = \ln x$ with respect to x yields $y(x) = x \ln x - x + c$. Since $y(1) = 2$, we have $2 = 1(0) - 1 + c$, so that $c = 3$. Thus, $y(x) = x \ln x - x + 3$.

35. Integrating $\frac{d^2y}{dx^2} = \cos x$ twice with respect to x yields

$$\frac{dy}{dx} = \sin x + c_1 \quad \text{and} \quad y(x) = -\cos x + c_1x + c_2.$$

From $y'(0) = 1$, we obtain $c_1 = 1$, and from $y(0) = 2$, we obtain $c_2 = 3$. Thus, $y(x) = 3 + x - \cos x$.

36. Integrating $\frac{d^3y}{dx^3} = 6x$ three times with respect to x , we obtain

$$\frac{d^2y}{dx^2} = 3x^2 + c_1, \quad \frac{dy}{dx} = x^3 + c_1x + c_2, \quad y(x) = \frac{1}{4}x^4 + \frac{1}{2}c_1x^2 + c_2x + c_3.$$

From $y''(0) = 4$, we obtain $c_1 = 4$, from $y'(0) = -1$, we obtain $c_2 = -1$, and from $y(0) = 1$, we obtain $c_3 = 1$. Thus, $y(x) = \frac{1}{4}x^4 + 2x^2 - x + 1$.

37. Integrating $y''(x) = xe^x$ twice with respect to x , we obtain

$$y'(x) = xe^x - e^x + c_1 \quad \text{and} \quad y(x) = xe^x - 2e^x + c_1x + c_2.$$

From $y'(0) = 4$, we obtain $c_1 = 5$, and from $y(0) = 3$, we obtain $c_2 = 5$. Thus, $y(x) = xe^x - 2e^x + 5x + 5$.

38. Starting with $y(x) = c_1e^x + c_2e^{-x}$, we find that $y'(x) = c_1e^x - c_2e^{-x}$ and $y''(x) = c_1e^x + c_2e^{-x}$. Thus, $y'' - y = 0$, so $y(x) = c_1e^x + c_2e^{-x}$ is a solution to the differential equation on $(-\infty, \infty)$. Next we establish that every solution to the differential equation has the form $c_1e^x + c_2e^{-x}$. Suppose that $y = f(x)$ is any solution to the differential equation. Then according to Theorem 1.2.15, $y = f(x)$ is the unique solution to the initial-value problem

$$y'' - y = 0, \quad y(0) = f(0), \quad y'(0) = f'(0).$$

However, consider the function

$$y(x) = \frac{f(0) + f'(0)}{2}e^x + \frac{f(0) - f'(0)}{2}e^{-x}.$$

This is of the form $y(x) = c_1e^x + c_2e^{-x}$, where $c_1 = \frac{f(0) + f'(0)}{2}$ and $c_2 = \frac{f(0) - f'(0)}{2}$, and therefore solves the differential equation $y'' - y = 0$. Furthermore, evaluation this function at $x = 0$ yields

$$y(0) = f(0) \quad \text{and} \quad y'(0) = f'(0).$$

Consequently, this function solves the initial-value problem above. However, by assumption, $y(x) = f(x)$ solves the same initial-value problem. Owing to the uniqueness of the solution to this initial-value problem, it follows that these two solutions are the same:

$$f(x) = c_1e^x + c_2e^{-x}.$$

Consequently, every solution to the differential equation has the form $y(x) = c_1 e^x + c_2 e^{-x}$, and therefore this is the general solution on any interval I .

39. Integrating $\frac{d^2 y}{dx^2} = e^{-x}$ twice with respect to x , we obtain

$$\frac{dy}{dx} = -e^{-x} + c_1 \quad \text{and} \quad y(x) = e^{-x} + c_1 x + c_2.$$

From $y(0) = 1$, we obtain $c_2 = 0$, and from $y(1) = 0$, we obtain $c_1 = -\frac{1}{e}$. Hence, $y(x) = e^{-x} - \frac{1}{e}x$.

40. Integrating $\frac{d^2 y}{dx^2} = -6 - 4 \ln x$ twice with respect to x , we obtain

$$\frac{dy}{dx} = -2x - 4x \ln x + c_1 \quad \text{and} \quad y(x) = -2x^2 \ln x + c_1 x + c_2.$$

From $y(1) = 0$, we obtain $c_1 + c_2 = 0$, and from $y(e) = 0$, we obtain $ec_1 + c_2 = 2e^2$. Solving this system yields $c_1 = \frac{2e^2}{e-1}$ and $c_2 = -\frac{2e^2}{e-1}$. Thus, $y(x) = \frac{2e^2}{e-1}(x-1) - 2x^2 \ln x$.

41. We use the general solution $y(x) = c_1 \cos x + c_2 \sin x$ and seek values of c_1 and c_2 .

(a) From $y(0) = 0$, we find that $c_1 = 0$, and from $y(\pi) = 1$, we find that $1 = c_2(0)$, which is impossible. Hence, we have no solutions.

(b) From $y(0) = 0$, we find that $c_1 = 0$, and from $y(\pi) = 0$, we find that $0 = c_2(0)$, so c_2 can be any real number. Hence, we have infinitely many solutions.

42-47. Use some kind of technology to define each of the given functions. Then use the technology to simplify the expression given on the left-hand side of each differential equation and verify that the result corresponds to the expression on the right-hand side.

48. (a) Use some form of technology to substitute $y(x) = a + bx + cx^2 + dx^3 + ex^4 + fx^5$ where a, b, c, d, e, f are constants, into the given Legendre equation and set the coefficients of each power of x in the resulting equation to zero. The result is:

$$e = 0, \quad 20f + 18d = 0, \quad e + 2c = 0, \quad 3d + 14b = 0, \quad c + 15a = 0.$$

Now solve for the constants to find: $a = c = e = 0$, $d = -\frac{14}{3}b$, $f = -\frac{9}{10}d = \frac{21}{5}b$. Consequently, the corresponding solution to the Legendre equation is:

$$y(x) = bx \left(1 - \frac{14}{3}x^2 + \frac{21}{5}x^4 \right).$$

Imposing the normalization condition $y(1) = 1$ requires $1 = b(1 - \frac{14}{3} + \frac{21}{5}) \implies b = \frac{15}{8}$. Consequently, the required solution is $y(x) = \frac{15}{8}x(15 - 70x^2 + 63x^4)$.

49. (a) $J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k} = 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 + \dots$

(b) A Maple plot of $J(0, x, 4)$ is given in the accompanying figure.

From this graph, an approximation to the first positive zero of $J_0(x)$ is 2.4. Using the Maple internal function `BesselJZeros` gives the approximation 2.404825558.

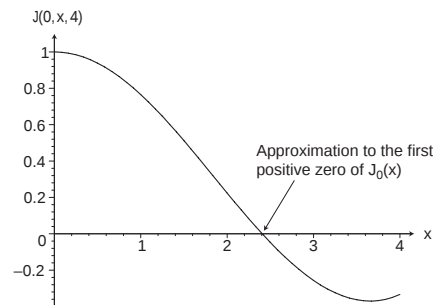


Figure 8: Figure for Problem 49(b)

(c) A Maple plot of the functions $J_0(x)$ and $J(0, x, 4)$ on the interval $[0, 2]$ is given in the accompanying figure. We see that to the printer resolution, these graphs are indistinguishable. On a larger interval, for example, $[0, 3]$, the two graphs would begin to differ dramatically from one another.

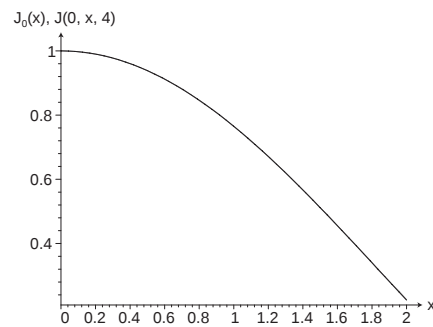


Figure 9: Figure for Problem 49(c)

(d) By trial and error, we find the smallest value of m to be $m = 11$. A plot of the functions $J(0, x)$ and $J(0, x, 11)$ is given in the accompanying figure.

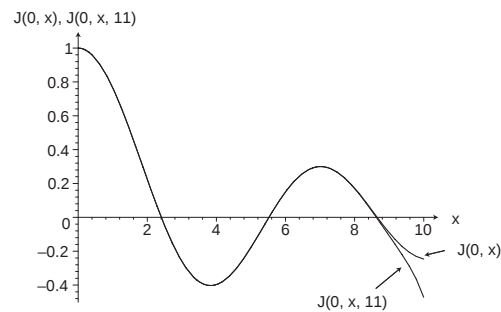


Figure 10: Figure for Problem 49(d)

Solutions to Section 1.3

True-False Review:

1. **TRUE.** This is precisely the remark after Theorem 1.3.2.
2. **FALSE.** For instance, the differential equation in Example 1.3.6 has no equilibrium solutions.
3. **FALSE.** This differential equation has equilibrium solutions $y(x) = 2$ and $y(x) = -2$.
4. **TRUE.** For this differential equation, we have $f(x, y) = x^2 + y^2$. Therefore, any equation of the form $x^2 + y^2 = k$ is an isocline, by definition.
5. **TRUE.** Equilibrium solutions are always horizontal lines. These are always parallel to each other.
6. **TRUE.** The isoclines have the form $\frac{x^2+y^2}{2y} = k$, or $x^2 + y^2 = 2ky$, or $x^2 + (y - k)^2 = k^2$, so the statement is valid.
7. **TRUE.** An equilibrium solution *is* a solution, and two solution curves to the differential equation $\frac{dy}{dx} = f(x, y)$ do not intersect.

Problems:

1. Taking the derivative of $y = cx^{-1}$, we obtain $\frac{dy}{dx} = -cx^{-2}$. Since $c = xy$, this becomes $\frac{dy}{dx} = -\frac{y}{x}$.
2. Taking the derivative of $y = cx^2$, we obtain $\frac{dy}{dx} = 2cx$. Since $c = \frac{y}{x^2}$, this becomes $\frac{dy}{dx} = \frac{2y}{x}$.
3. Implicit differentiation of $x^2 + y^2 = 2cx$ with respect to x yields $2x + 2y\frac{dy}{dx} = 2c$. Now, $c = \frac{x^2 + y^2}{2x}$, so $2x + 2y\frac{dy}{dx} = \frac{x^2 + y^2}{x}$. Hence, $y\frac{dy}{dx} = \frac{x^2 + y^2}{2x} - x$. Consequently, $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$.
4. Implicit differentiation of $y^2 = cx$ with respect to x yields $2y\frac{dy}{dx} = c$. Since $c = \frac{y^2}{x}$, we have $2y\frac{dy}{dx} = \frac{y^2}{x}$. Rearranging this equation, we obtain $\frac{dy}{dx} = \frac{y}{2x}$.
5. Implicit differentiation of $2cy = x^2 - c^2$ with respect to x yields $2c\frac{dy}{dx} = 2x$, so that $\frac{dy}{dx} = \frac{x}{c}$. Now from $c^2 + 2cy - x^2 = 0$, we use the quadratic equation to obtain

$$c = \frac{-2y \pm \sqrt{4y^2 + 4x^2}}{2} = -y \pm \sqrt{x^2 + y^2}.$$

Hence,

$$\frac{dy}{dx} = \frac{x}{c} = \frac{x}{-y \pm \sqrt{x^2 + y^2}}.$$

6. Implicit differentiation of $y^2 - x^2 = c$ with respect to x yields $2y\frac{dy}{dx} - 2x = 0$. Hence, $\frac{dy}{dx} = \frac{x}{y}$.

7. Algebraic simplification of $(x-c)^2 + (y-c)^2 = 2c^2$ yields $x^2 - 2cx + y^2 - 2cy = 0$. Implicit differentiation of this equation gives

$$2x - 2c + 2y \frac{dy}{dx} - 2c \frac{dy}{dx} = 0.$$

Therefore,

$$\frac{dy}{dx} = \frac{2c - 2x}{2y - 2c} = \frac{c - x}{y - c}.$$

We have $c = \frac{x^2 + y^2}{2(x + y)}$. Substituting this value for c into the previous formula for $\frac{dy}{dx}$ and simplifying, we obtain $\frac{dy}{dx} = -\frac{x^2 + 2xy - y^2}{y^2 + 2xy - x^2}$.

8. Implicit differentiation of $x^2 + y^2 = c$ with respect to x yields $2x + 2y \frac{dy}{dx} = 0$. Thus, $\frac{dy}{dx} = -\frac{x}{y}$.

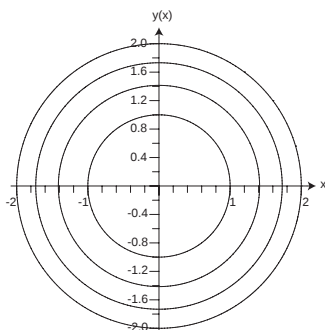


Figure 11: Figure for Problem 8

9. Differentiation of $y = cx^3$ with respect to x yields $\frac{dy}{dx} = 3cx^2 = 3\frac{y}{x^3}x^2 = \frac{3y}{x}$. The initial condition $y(2) = 8$ implies that $8 = c(2)^3$. Hence, $c = 1$. Thus the unique solution to the initial-value problem is $y = x^3$.

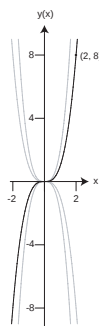


Figure 12: Figure for Problem 9

10. Implicit differentiation of $y^2 = cx$ with respect to x yields $2y \frac{dy}{dx} = c$. Thus, $2y \frac{dy}{dx} = \frac{y^2}{x}$, which can be rearranged to read $\frac{dy}{dx} = \frac{y}{2x}$. That is, $2x \cdot dy - y \cdot dx = 0$. The initial condition $y(1) = 2$ implies that $c = 4$, so that the unique solution to the initial-value problem is $y^2 = 4x$.

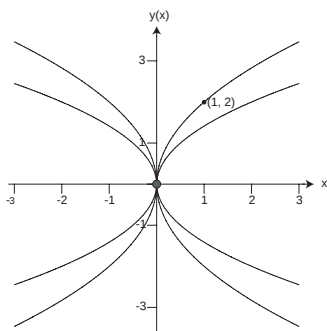


Figure 13: Figure for Problem 10

11. Expanding $(x - c)^2 + y^2 = c^2$ yields $x^2 - 2cx + c^2 + y^2 = c^2$, or $x^2 - 2cx + y^2 = 0$. Differentiating with respect to x gives $2x - 2c + 2y \frac{dy}{dx} = 0$. Solving for $\frac{dy}{dx}$, we have $\frac{dy}{dx} = \frac{c - x}{y}$. Now, $c = \frac{x^2 + y^2}{2x}$, and substituting this into the differential equation just obtained and simplifying yields $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$. Imposing the initial condition $y(2) = 2$ in the equation $x^2 - 2cx + y^2 = 0$, we find that $c = 2$. Therefore, the unique solution to the initial-value problem is $y = +\sqrt{x(4 - x)}$.

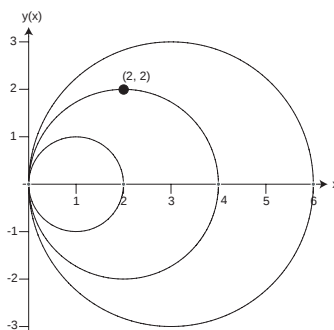


Figure 14: Figure for Problem 11

12. Let $f(x, y) = x \sin(x + y)$, which is continuous for all $x, y \in \mathbb{R}$. Then $\frac{\partial f}{\partial y} = x \cos(x + y)$, which is continuous for all $x, y \in \mathbb{R}$. By Theorem 1.3.2, the initial-value problem

$$\frac{dy}{dx} = x \sin(x + y), \quad y(x_0) = y_0$$

has a unique solution for some interval $I \in \mathbb{R}$.

13. Let

$$f(x, y) = \frac{x}{x^2 + 1}(y^2 - 9),$$

which is continuous for all $x, y \in \mathbb{R}$. Then

$$\frac{\partial f}{\partial y} = \frac{2xy}{x^2 + 1},$$

which is continuous for all $x, y \in \mathbb{R}$. Therefore, by the existence and uniqueness theorem, the initial-value problem stated above has a unique solution on any interval containing $(0, 3)$. By inspection we see that $y(x) = 3$ is the unique solution.

14. The initial-value problem does not necessarily have a unique solution since the hypothesis of the existence and uniqueness theorem are not satisfied at $(0, 0)$. This follows since $f(x, y) = xy^{1/2}$, so that $\frac{\partial f}{\partial y} = \frac{1}{2}xy^{-1/2}$, which is not continuous at $(0, 0)$.

15. (a) We have $f(x, y) = -2xy^2$, so $\frac{\partial f}{\partial y} = -4xy$. Both of these functions are continuous for all (x, y) , and therefore the hypothesis of the uniqueness and existence theorem are satisfied for any (x_0, y_0) in the xy -plane.

(b) For $y(x) = \frac{1}{x^2 + c}$, we have $y'(x) = -\frac{2x}{(x^2 + c)^2} = -2xy^2$.

(c) (i) From $y(0) = 1$, we find that $c = 1$. Hence, $y(x) = \frac{1}{x^2 + 1}$. This is valid on the interval $(-\infty, \infty)$.

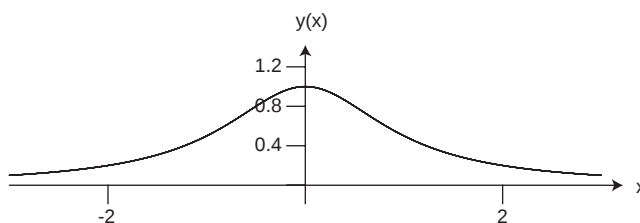


Figure 15: Figure for Problem 15c(i)

(ii) From $y(1) = 1$, we find that $c = 0$. Hence, $y(x) = \frac{1}{x^2}$. This is valid on the interval $(0, \infty)$.

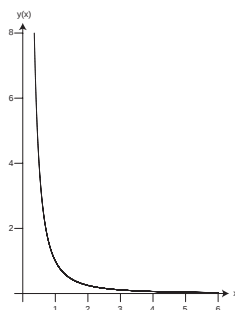


Figure 16: Figure for Problem 15c(ii)

- (iii) From $y(0) = -1$, we find that $c = -1$. Hence, $y(x) = \frac{1}{x^2 - 1}$. This is valid on the interval $(-1, 1)$.

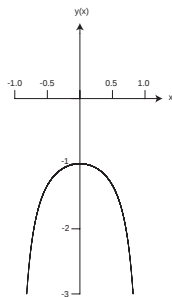


Figure 17: Figure for Problem 15c(iii)

- (d) Since, by inspection, $y(x) = 0$ satisfies the given initial-value problem, it must be the unique solution.

16. (a) Both $f(x, y) = y(y - 1)$ and $\frac{\partial f}{\partial y} = 2y - 1$ are continuous at all points (x, y) . Consequently, the hypothesis of the existence and uniqueness theorem are satisfied by the given initial-value problem for any (x_0, y_0) in the xy -plane.

(b) Equilibrium solutions: $y(x) = 0, y(x) = 1$.

(c) Differentiating the given differential equation yields $\frac{d^2 y}{dx^2} = (2y - 1) \frac{dy}{dx} = (2y - 1)y(y - 1)$. Hence the solution curves are concave up for $0 < y < \frac{1}{2}$ and $y > 1$, and concave down for $y < 0$ and $\frac{1}{2} < y < 1$.

(d) The solutions will be bounded provided $0 \leq y_0 \leq 1$.

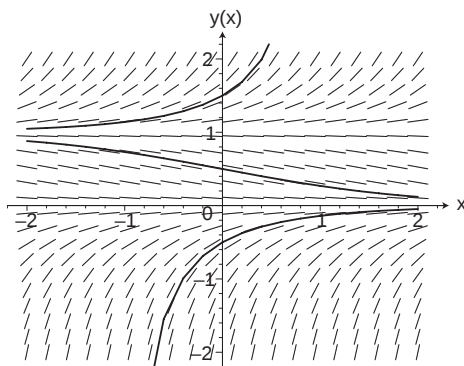


Figure 18: Figure for Problem 16(d)

17. There are no equilibrium solutions. The slope of the solution curves is positive for $x > 0$ and is negative for $x < 0$. The isoclines are the lines $x = \frac{k}{4}$.

Slope of Solution Curve	Equation of Isocline
-4	$x = -1$
-2	$x = -1/2$
0	$x = 0$
2	$x = 1/2$
4	$x = 1$

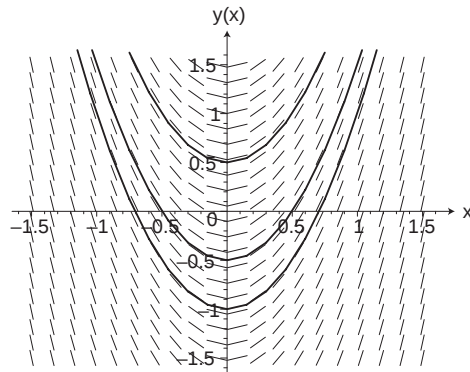


Figure 19: Figure for Problem 17

18. There are no equilibrium solutions. The slope of the solution curves is positive for $x > 0$ and increases without bound as $x \rightarrow 0^+$. The slope of the curve is negative for $x < 0$ and decreases without bound as $x \rightarrow 0^-$. The isoclines are the lines $\frac{1}{x} = k$.

Slope of Solution Curve	Equation of Isocline
± 4	$x = \pm 1/4$
± 2	$x = \pm 1/2$
$\pm 1/2$	$x = \pm 2$
$\pm 1/4$	$x = \pm 4$
$\pm 1/10$	$x = \pm 10$

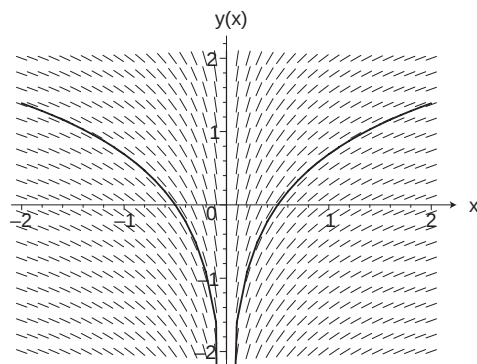


Figure 20: Figure for Problem 18

19. There are no equilibrium solutions. The slope of the solution curves is positive for $y > -x$, and negative for $y < -x$. The isoclines are the lines $y + x = k$.

Slope of Solution Curve	Equation of Isocline
-2	$y = -x - 2$
-1	$y = -x - 1$
0	$y = -x$
1	$y = -x + 1$
2	$y = -x + 2$

Since the slope of the solution curve along the isocline $y = -x - 1$ coincides with the slope of the isocline, it follows that $y = -x - 1$ is a solution to the differential equation. Differentiating the given differential equation yields: $y'' = 1 + y' = 1 + x + y$. Hence the solution curves are concave up for $y > -x - 1$, and concave down for $y < -x - 1$. Putting this information together leads to the slope field in the accompanying figure.

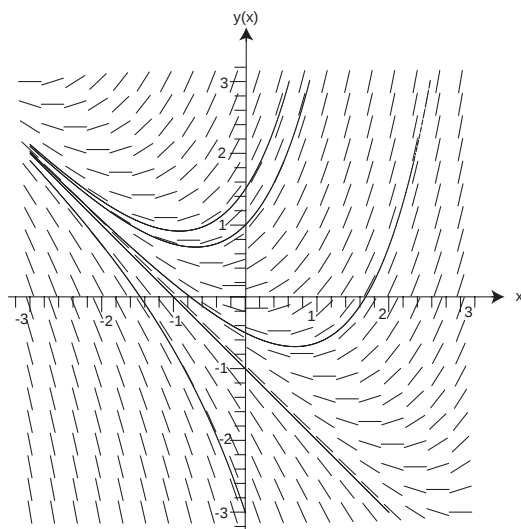


Figure 21: Figure for Problem 19

20. There are no equilibrium solutions. The slope of the solution curves is zero when $x = 0$. The solution has a vertical tangent line at all points along the x -axis (except the origin). Differentiating the differential equation yields: $y' = \frac{1}{y} - \frac{x}{y^2}y' = \frac{1}{y} - \frac{x^2}{y^3} = \frac{1}{y^3}(y^2 - x^2)$. Hence the solution curves are concave up for $y > 0$ and $y^2 > x^2$; $y < 0$ and $y^2 < x^2$ and concave down for $y > 0$ and $y^2 < x^2$; $y < 0$ and $y^2 > x^2$. The isoclines are the lines $\frac{x}{y} = k$.

Slope of Solution Curve	Equation of Isocline
± 2	$y = \pm x/2$
± 1	$y = \pm x$
$\pm 1/2$	$y = \pm 2x$
$\pm 1/4$	$y = \pm 4x$
$\pm 1/10$	$y = \pm 10x$

Note that $y = \pm x$ are solutions to the differential equation.

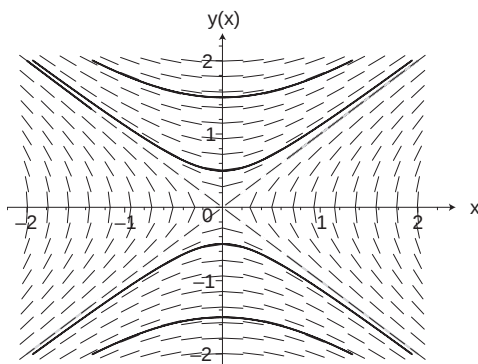


Figure 22: Figure for Problem 20

21. The slope is zero when $x = 0$ and $y \neq 0$. The solutions have a vertical tangent line at all points along the x -axis (except the origin). The isoclines are the lines $-\frac{4x}{y} = k$. Some values are given in the table below.

Slope of Solution Curve	Equation of Isocline
± 1	$y = \pm 4x$
± 2	$y = \pm 2x$
± 3	$y = \pm 4x/3$

Differentiating the given differential equation yields: $y' = -\frac{4}{y} + \frac{4xy'}{y^2} = -\frac{4}{y} - \frac{16x^2}{y^3} = -\frac{4(y^2 + 4x^2)}{y}$. Consequently the solution curves are concave up for $y < 0$, and concave down for $y > 0$. Putting this information together leads to the slope field in the accompanying figure.

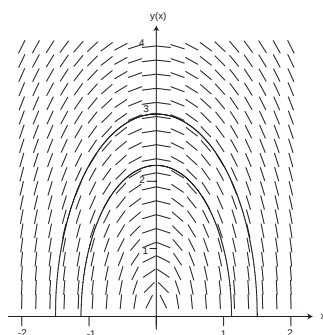


Figure 23: Figure for Problem 21

22. Equilibrium solution: $y(x) = 0$. Therefore, no solution curve can cross the x -axis. Slope: zero when $x = 0$ or $y = 0$. Positive when $y > 0$ (and $x \neq 0$); Negative when $y < 0$ (and $x \neq 0$). Differentiating the given differential equation yields: $\frac{d^2y}{dx^2} = 2xy + x^2 \frac{dy}{dx} = 2xy + x^4 y = xy(2 + x^3)$. So, when $y > 0$, the solution curves are concave up for $x \in (-\infty, (-2)^{1/3})$ and for $x > 0$, and the solution curves are concave down for $x \in ((-2)^{1/3}, 0)$. When $y < 0$, the solution curves are concave up for $x \in ((-2)^{1/3}, 0)$, and concave down for $x \in (-\infty, (-2)^{1/3})$ and for $x > 0$. The isoclines are the hyperbolas $x^2 y = k$.

Slope of Solution Curve	Equation of Isocline
± 2	$y = \pm 2/x^2$
± 1	$y = \pm 1/x^2$
$\pm 1/2$	$y = \pm 1/(2x)^2$
$\pm 1/4$	$y = \pm 1/(4x)^2$
$\pm 1/10$	$y = \pm 1/(10x)^2$
0	$y = 0$

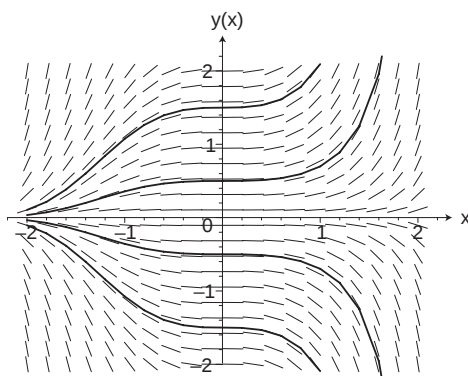


Figure 24: Figure for Problem 22

23. The slope is zero when $x = 0$. There are equilibrium solutions when $y = (2k + 1)\frac{\pi}{2}$. The slope field is best sketched using technology. The accompanying figure gives the slope field for $-\frac{\pi}{2} < y < \frac{3\pi}{2}$.

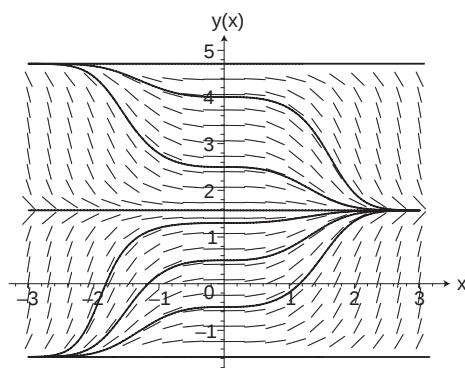


Figure 25: Figure for Problem 23

24. The slope of the solution curves is zero at the origin, and positive at all the other points. There are no equilibrium solutions. The isoclines are the circles $x^2 + y^2 = k$.

Slope of Solution Curve	Equation of Isocline
1	$x = \pm 1/4$
2	$x = \pm 1/2$
3	$x = \pm 2$
4	$x = \pm 4$
5	$x = \pm 10$

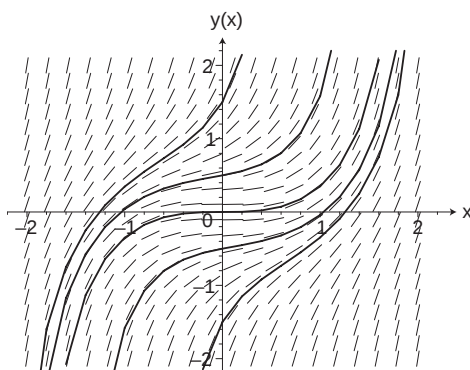


Figure 26: Figure for Problem 24

25. Substituting the constants given in the problem, the differential equation reads $\frac{dT}{dt} = -\frac{1}{80}(T - 70)$. Equilibrium solution: $T(t) = 70$. The slope of the solution curves is positive for $T > 70$, and negative for $T < 70$. Taking the derivative of both sides of the differential equation, we arrive at $\frac{d^2T}{dt^2} = -\frac{1}{80} \frac{dT}{dt} = -\frac{1}{6400}(T - 70)$. Hence the solution curves are concave up for $T > 70$, and concave down for $T < 70$. The isoclines are the horizontal lines $-\frac{1}{80}(T - 70) = k$.

Slope of Solution Curve	Equation of Isocline
$-1/4$	$T = 90$
$1/5$	$T = 86$
0	$T = 70$
$1/5$	$T = 54$
$1/4$	$T = 50$

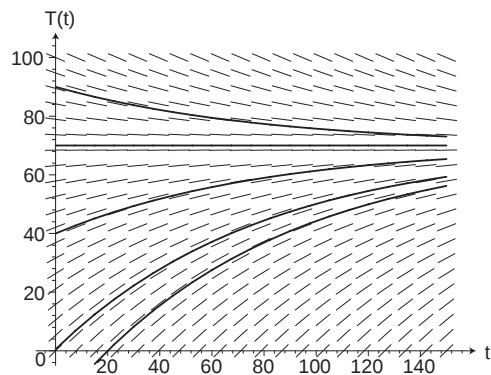


Figure 27: Figure for Problem 25

26.

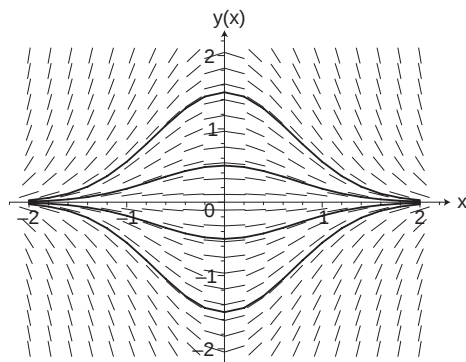


Figure 28: Figure for Problem 26

27.

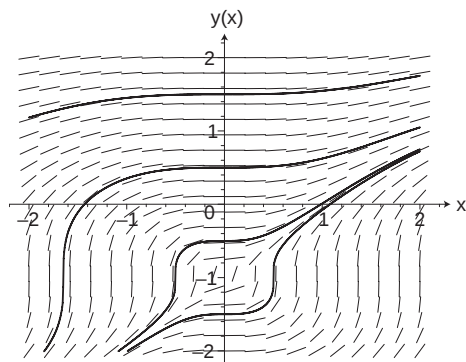


Figure 29: Figure for Problem 27

28.

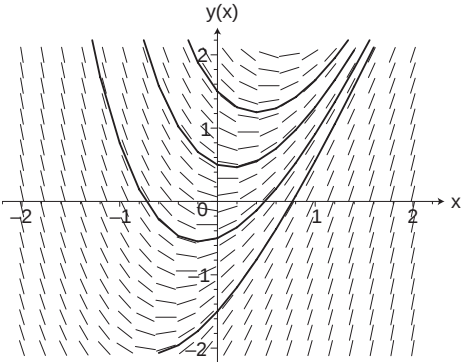


Figure 30: Figure for Problem 28

29.

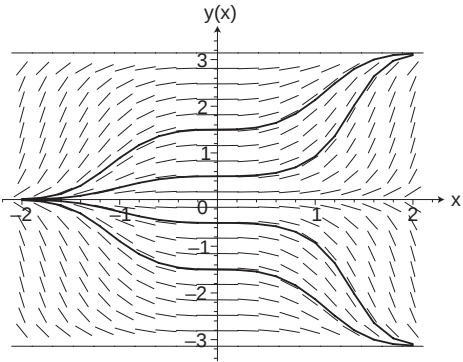


Figure 31: Figure for Problem 29

30.

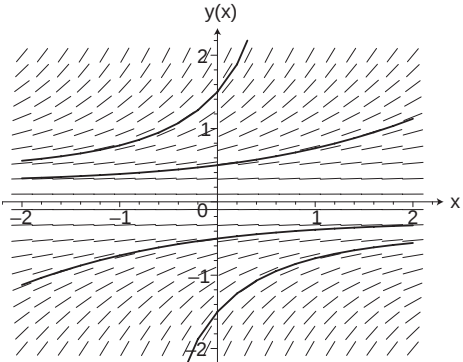


Figure 32: Figure for Problem 30

30

31.

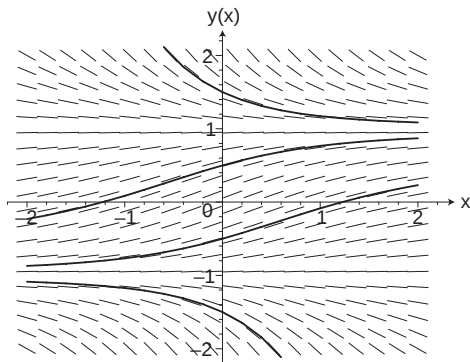


Figure 33: Figure for Problem 31

32.(a)

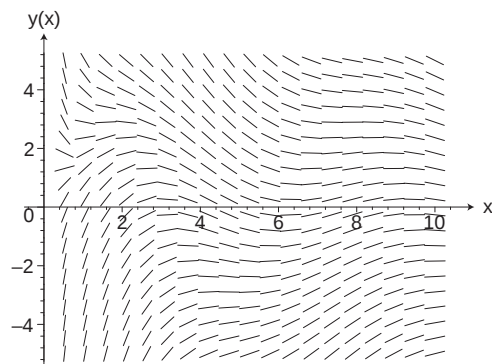


Figure 34: Figure for Problem 32(a)

(b) The figure suggests that the solutions to the differential equation are unbounded as $x \rightarrow 0^+$.

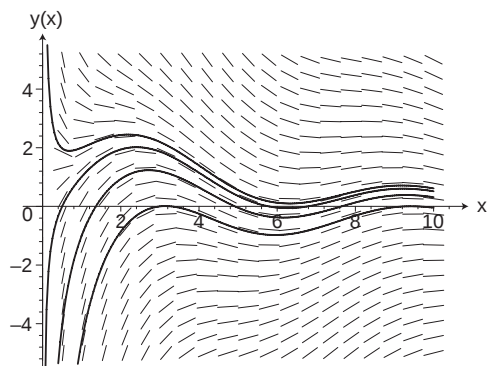


Figure 35: Figure for Problem 32(b)

(c)

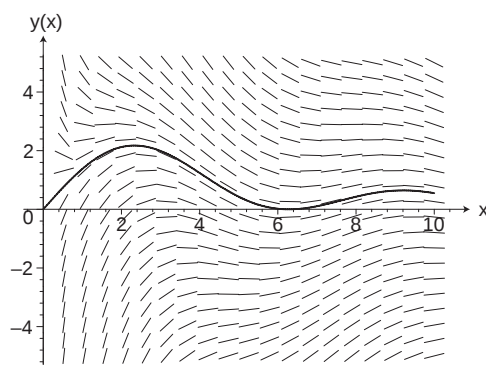


Figure 36: Figure for Problem 32(c)

This solution curve is bounded as $x \rightarrow 0^+$.

(d) In the accompanying figure we have sketched several solution curves on the interval $(0,15]$. The figure suggests that the solution curves approach the x -axis as $x \rightarrow \infty$.

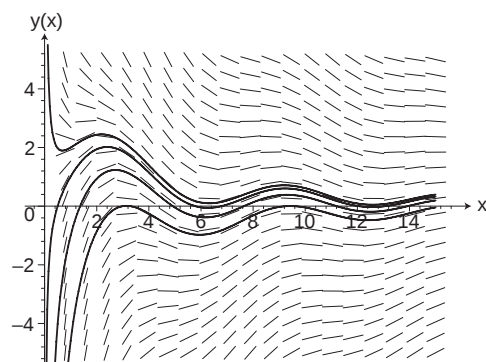


Figure 37: Figure for Problem 32(d)

33.(a) Differentiating the given equation gives $\frac{dy}{dx} = 2kx = 2\frac{y}{x}$. Hence the differential equation satisfied by the orthogonal trajectories is $\frac{dy}{dx} = -\frac{x}{2y}$.

(b) The orthogonal trajectories appear to be ellipses. This can be verified by integrating the differential equation derived in (a).

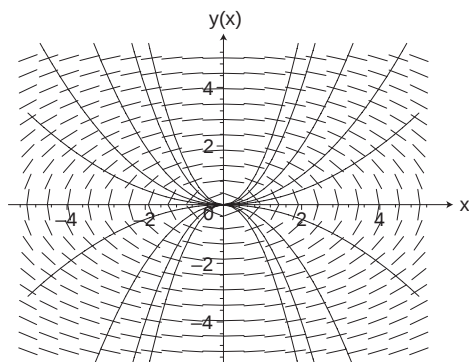
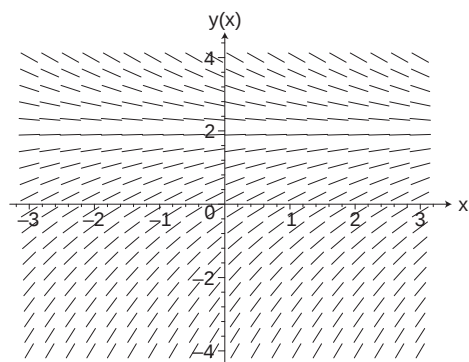
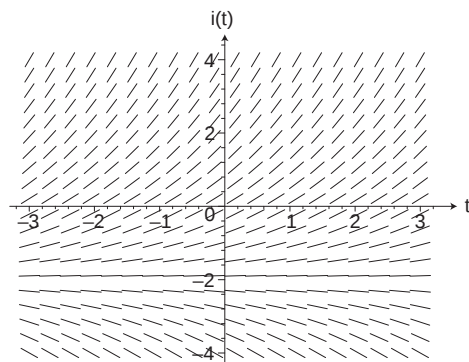


Figure 38: Figure for Problem 33(b)

34. If $a > 0$, then as illustrated in the following slope field ($a = 0.5, b = 1$), it appears that $\lim_{t \rightarrow \infty} i(t) = \frac{b}{a}$.

Figure 39: Figure for Problem 34 when $a > 0$

If $a < 0$, then as illustrated in the following slope field ($a = -0.5, b = 1$), it appears that $i(t)$ diverges as $t \rightarrow \infty$.

Figure 40: Figure for Problem 34 when $a < 0$

Finally, if $a = 0$ and $b \neq 0$, then once more $i(t)$ diverges as $t \rightarrow \infty$. The accompanying figure shows a representative case when $b > 0$. Here we see that $\lim_{t \rightarrow \infty} i(t) = +\infty$.

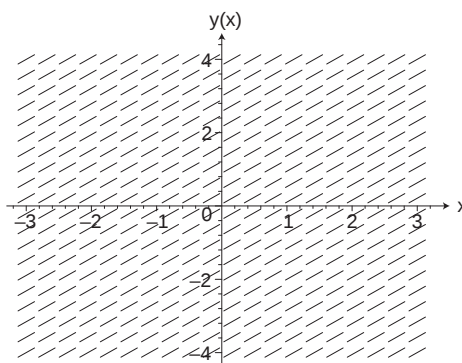


Figure 41: Figure for Problem 34 when $a = 0$

If $b < 0$, then $\lim_{t \rightarrow \infty} i(t) = -\infty$. If $a = b = 0$, then the general solution to the differential equation is $i(t) = i_0$ where i_0 is a constant.

Solutions to Section 1.4

True-False Review:

1. **TRUE.** The differential equation $\frac{dy}{dx} = f(x)g(y)$ can be written $\frac{1}{g(y)} \frac{dy}{dx} = f(x)$, which is the proper form, according to Definition 1.4.1, for a separable differential equation.
2. **TRUE.** A separable differential equation is a first-order differential equation, so the general solution contains one constant. The value of that constant can be determined from an initial condition, as usual.
3. **TRUE.** Newton's Law of Cooling is usually expressed as $\frac{dT}{dt} = -k(T - T_m)$, and this can be rewritten as

$$\frac{1}{T - T_m} \frac{dT}{dt} = -k,$$

and this form shows that the equation is separable.

4. **FALSE.** The expression $x^2 + y^2$ cannot be separated in the form $f(x)g(y)$, so the equation is not separable.
5. **FALSE.** The expression $x \sin(xy)$ cannot be separated in the form $f(x)g(y)$, so the equation is not separable.
6. **TRUE.** We can write the given equation as $e^{-y} \frac{dy}{dx} = e^x$, which is the proper form for a separable equation.
7. **TRUE.** We can write the given equation as $(1 + y^2) \frac{dy}{dx} = \frac{1}{x^2}$, which is the proper form for a separable equation.
8. **FALSE.** The expression $\frac{x+4y}{4x+y}$ cannot be separated in the form $f(x)g(y)$, so the equation is not separable.
9. **TRUE.** We can write $\frac{x^3y+x^2y^2}{x^2+xy} = xy$, so we can write the given differential equation as $\frac{1}{y} \frac{dy}{dx} = x$, which is the proper form for a separable equation.

Problems:

1. Separating the variables and integrating yields

$$\int \frac{dy}{y} = 2 \int x dx \implies \ln |y| = x^2 + c_1 \implies y(x) = ce^{x^2}.$$

2. Separating the variables and integrating yields

$$\int y^{-2} dy = \int \frac{dx}{x^2 + 1} \implies -\frac{1}{y} = \tan^{-1} x + c \implies y(x) = -\frac{1}{\tan^{-1} x + c}.$$

3. Separating the variables and integrating yields

$$\int e^y dy = \int e^{-x} dx \implies e^y = -e^{-x} + c \implies y(x) = \ln(c - e^{-x}).$$

4. Separating the variables and integrating yields

$$\int \frac{dy}{y} = \int \frac{(\ln x)^{-1}}{x} dx \implies \ln |y| = \ln |\ln x| + c \implies y(x) = c \ln x.$$

5. Separating the variables and integrating yields

$$\int \frac{dx}{x-2} = \int \frac{dy}{y} \implies \ln |y| = \ln |x-2| + c \implies \ln \left| \frac{y}{x-2} \right| = c \implies \frac{y}{x-2} = c_1 \implies y = c_1(x-2).$$

6. Separating the variables and integrating yields

$$\begin{aligned} \int \frac{dy}{y-1} &= \int \frac{2x}{x^2+3} dx \implies \ln |y-1| = \ln |x^2+3| + c_1 \\ &\implies \ln \left| \frac{y-1}{x^2+3} \right| = c \\ &\implies \frac{y-1}{x^2+3} = c_1 \\ &\implies y-1 = c_1(x^2+3) \implies y(x) = c_1(x^2+3) + 1. \end{aligned}$$

7. Regrouping the terms of the given equation, we have $x(2x-1)\frac{dy}{dx} = (3-y)$. Separating the variables and integrating yields

$$\begin{aligned} -\int \frac{dy}{y-3} &= \int \frac{dx}{x(2x-1)} \implies -\ln |y-3| = -\int \frac{dx}{x} + \int \frac{2}{2x-1} dx \\ &\implies -\ln |y-3| = -\ln |x| + \ln |2x-1| + c_1 \\ &\implies \ln \left| \frac{x}{(y-3)(2x-1)} \right| = c_1 \\ &\implies \frac{x}{(y-3)(2x-1)} = c_2 \\ &\implies c_2(y-3) = \frac{x}{2x-1} \\ &\implies y(x) = \frac{c_3 x}{2x-1} + 3 \implies y(x) = \frac{c_4 x - 3}{2x-1}. \end{aligned}$$

8. Using the trigonometric identity $\cos(x - y) = \cos x \cos y + \sin x \sin y$, we can rewrite the differential equation as $\frac{dy}{dx} = \frac{\cos(x - y)}{\sin x \sin y} - 1 = \frac{\cos x \cos y}{\sin x \sin y}$. Separating the variables, we have $\frac{\sin y}{\cos y} dy = \frac{\cos x}{\sin x} dx$. Therefore,

$$\begin{aligned} \int \frac{\sin y}{\cos y} dy &= \int \frac{\cos x}{\sin x} dx \implies -\ln |\cos y| = \ln |\sin x| + c_1 \\ &\implies \ln |\cos y| = -\ln |\sin x| + c_2 \\ &\implies \cos y = c_3 \csc x. \end{aligned}$$

9. Separating the variables, we have

$$\frac{dy}{(y+1)(y-1)} = \frac{1}{2} \frac{xdx}{(x-2)(x-1)},$$

where we must assume that $y \neq \pm 1$. We have partial fractions decompositions

$$\frac{1}{(y+1)(y-1)} = -\frac{1}{2} \cdot \frac{1}{y+1} + \frac{1}{2} \cdot \frac{1}{y-1} \quad \text{and} \quad \frac{1}{2} \cdot \frac{x}{(x-2)(x-1)} = \frac{1}{2} \left(\frac{2}{x-2} - \frac{1}{x-1} \right).$$

Integrating, we obtain

$$\begin{aligned} -\frac{1}{2} \int \frac{dy}{y+1} + \frac{1}{2} \int \frac{dy}{y-1} &= \frac{1}{2} \left(2 \int \frac{dx}{x-2} - \int \frac{dx}{x-1} \right) \implies -\ln |y+1| + \ln |y-1| = 2 \ln |x-2| - \ln |x-1| + c_1 \\ &\implies \ln \left| \frac{y-1}{y+1} \right| = \ln \left| \frac{(x-2)^2}{x-1} \right| + c_1 \\ &\implies \frac{y-1}{y+1} = c \frac{(x-2)^2}{x-1}. \end{aligned}$$

A short algebraic manipulation solves this expression for y :

$$y(x) = \frac{(x-1) + c(x-2)^2}{(x-1) - c(x-2)^2}.$$

We explicitly check the values $y = \pm 1$, which we had to exclude above. By inspection, we see that $y(x) = 1$ and $y(x) = -1$ are indeed also solutions of the given differential equation. The former is included in the above solution when $c = 0$.

10. We have $\frac{dy}{dx} = \frac{x^2 y - 32}{16 - x^2} + 2 = \frac{x^2 y - 2x^2}{16 - x^2}$, which can be separated as

$$\frac{1}{y-2} = \frac{x^2}{16-x^2} = -\left(1 + \frac{16}{x^2-16}\right) = -1 - 16 \left(-\frac{1}{8} \left(\frac{1}{x+4} \right) + \frac{1}{8} \left(\frac{1}{x-4} \right) \right) = -1 + \frac{2}{x+4} - \frac{2}{x-4}.$$

Integrating, we obtain

$$\ln |y-2| = -x + 2 \ln |x+4| - 2 \ln |x-4| + c_1 \implies y(x) = 2 + c \left(\frac{x+4}{x-4} \right)^2 e^{-x}.$$

11. Separating the variables, we have

$$\frac{dy}{y-c} = \frac{dx}{(x-a)(x-b)} = \frac{1}{a-b} \left(\frac{1}{x-a} - \frac{1}{x-b} \right) dx.$$

Integrating, we obtain

$$\ln |y - c| = \frac{1}{a - b} (\ln |x - a| - \ln |x - b|) + c = \frac{1}{a - b} \ln \left| \frac{x - a}{x - b} \right| + c.$$

Exponentiation of both sides and removal of absolute value gives

$$y - c = c_2 \left(\frac{x - a}{x - b} \right)^{1/(a-b)}.$$

Therefore,

$$y(x) = c + c_2 \left(\frac{x - a}{x - b} \right)^{1/(a-b)}.$$

12. Separating the variables, we have $\frac{dy}{1+y^2} = -\frac{dx}{1+x^2}$. Integrating both sides, we have $\tan^{-1} y = \tan^{-1} x + c$. Since $y(0) = 1$, we find that $c = \frac{\pi}{4}$. Thus, $\tan^{-1} y = \tan^{-1} x + \frac{\pi}{4}$. A short manipulation can be used to rewrite this as $y(x) = \frac{1-x}{1+x}$.

13. Separating the variables, we have $\frac{1}{a-y} = \frac{x}{1-x^2}$. Integration on each side gives

$$-\ln |a - y| = -\frac{1}{2} \ln |1 - x^2| + c_1 \implies y(x) = a + c\sqrt{1 - x^2}.$$

The initial condition $y(0) = 2a$ requires that $c = a$. Therefore, $y(x) = a(1 + \sqrt{1 - x^2})$.

14. Using the trigonometric identity $\sin(x + y) = \sin x \cos y + \cos x \sin y$, we can rewrite the differential equation as

$$\frac{dy}{dx} = 1 - \frac{\sin(x + y)}{\sin y \cos x} = -\tan x \cot y.$$

Separating the variables, we have

$$-\frac{\sin y}{\cos y} dy = \frac{\sin x}{\cos x} dx.$$

Integrating on each side gives

$$\ln |\cos y| = -\ln |\cos x| + c_1.$$

That is,

$$\ln |\cos x \cos y| = c_1.$$

Since $y(\frac{\pi}{4}) = \frac{\pi}{4}$, we find that $c = \ln \frac{1}{2}$. Hence, $\ln |\cos x \cos y| = \ln \frac{1}{2}$, so that

$$y(x) = \cos^{-1} \left(\frac{1}{2} \sec x \right).$$

15. Separation of variables gives $\frac{dy}{y^3} = \sin x dx$, where we must require $y \neq 0$ here. Integrating on each side gives $-\frac{1}{2y^2} = -\cos x + c$. However, we cannot impose the initial condition $y(0) = 0$ on the last equation

since it is not defined at $y = 0$. But, by inspection, $y(x) = 0$ is a solution to the given differential equation, and further, $y(0) = 0$; thus, the unique solution to the initial value problem is $y(x) = 0$.

16. For $y \neq 1$, we can separate variables to obtain $\frac{dy}{\sqrt{y-1}} = \frac{2}{3}dx$. Integration on each sides yields $2\sqrt{y-1} = \frac{2}{3}x + c$. The initial condition $y(1) = 1$ implies that $c = -\frac{2}{3}$. Therefore,

$$\sqrt{y-1} = \frac{1}{3}(x-1) \implies y(x) = 1 + \frac{1}{9}(x-1)^2.$$

This does not contradict the Existence-Uniqueness theorem because the hypothesis of the theorem is not satisfied when $x = 1$.

17.(a) Separating the variables v and t , we have $\frac{m}{k[(mg/k) - v^2]}dv = dt$. If we let $a = \sqrt{\frac{mg}{k}}$ then the preceding equation can be written as $\frac{m}{k} \frac{1}{a^2 - v^2} dv = dt$ which can be integrated directly to obtain

$$\frac{m}{2ak} \ln \left(\frac{a+v}{a-v} \right) = t + c.$$

Upon exponentiating both sides,

$$\frac{a+v}{a-v} = c_1 e^{\frac{2ak}{m}t}.$$

Imposing the initial condition $v(0) = 0$, yields $c_1 = 1$ so that

$$\frac{a+v}{a-v} = e^{\frac{2ak}{m}t}.$$

Now a short algebraic manipulation yields

$$v(t) = a \left(\frac{e^{\frac{2akt}{m}} - 1}{e^{\frac{2akt}{m}} + 1} \right),$$

which can be written in the equivalent form

$$v(t) = a \tanh \left(\frac{gt}{a} \right).$$

(b) No. As $t \rightarrow \infty, v \rightarrow a$ and as $t \rightarrow 0^+, v \rightarrow 0$.

(c) We integrate the velocity function $v(t) = \frac{dy}{dt}$ to obtain

$$y(t) = a \int \tanh \left(\frac{gt}{a} \right) dt \implies y(t) = \frac{a^2}{g} \ln(\cosh(\frac{gt}{a})) + c_1.$$

If $y(0) = 0$ then

$$y(t) = \frac{a^2}{g} \ln \left[\cosh \left(\frac{gt}{a} \right) \right].$$

18. The required curve is the solution curve to the initial-value problem $\frac{dy}{dx} = -\frac{x}{4y}$, $y(0) = \frac{1}{2}$. Separating the variables in the differential equation yields $4y^{-1}dy = -1dx$, which can be integrated directly to obtain

$2y^2 = -\frac{x^2}{2} + c$. Imposing the initial condition we obtain $c = \frac{1}{2}$, so that the solution curve has the equation $2y^2 = -x^2 + \frac{1}{2}$, or equivalently, $4y^2 + 2x^2 = 1$.

19. The required curve is the solution curve to the initial-value problem $\frac{dy}{dx} = e^{x-y}$, $y(3) = 1$. Separating the variables in the differential equation yields $e^y dy = e^x dx$, which can be integrated directly to obtain $e^y = e^x + c$. Imposing the initial condition we obtain $c = e - e^3$, so that the solution curve has the equation $e^y = e^x + e - e^3$, or equivalently, $y = \ln(e^x + e - e^3)$.

20. The required curve is the solution curve to the initial-value problem $\frac{dy}{dx} = x^2 y^2$, $y(-1) = 1$. Separating the variables in the differential equation yields $\frac{1}{y^2} dy = x^2 dx$, which can be integrated directly to obtain $-\frac{1}{y} = \frac{1}{3}x^3 + c$. Imposing the initial condition we obtain $c = -\frac{2}{3}$, so that the solution curve has the equation $y = -\frac{1}{\frac{1}{3}x^3 - \frac{2}{3}}$, or equivalently, $y = \frac{3}{2 - x^3}$.

21.(a) Separating the variables in the given differential equation yields $\frac{1}{1+v^2} dv = -dt$. Integrating we obtain $\tan^{-1}(v) = -t + c$. The initial condition $v(0) = v_0$ implies that $c = \tan^{-1}(v_0)$, so that $\tan^{-1}(v) = -t + \tan^{-1}(v_0)$. The object will come to rest if there is time, t_r , at which the velocity is zero. To determine t_r , we set $v = 0$ in the previous equation which yields $\tan^{-1}(0) = -t_r + \tan^{-1}(v_0)$. Consequently, $t_r = \tan^{-1}(v_0)$. The object does not remain at rest since we see from the given differential equation that $\frac{dv}{dt} < 0$ at $t = t_r$, and so v is decreasing with time. Consequently v passes through zero and becomes negative for $t > t_r$.

(b) From the chain rule we have $\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$. Then $\frac{dv}{dt} = v \frac{dv}{dx}$. Substituting this result into the differential equation (1.4.17) yields $v \frac{dv}{dx} = -(1+v^2)$. We now separate the variables: $\frac{v}{1+v^2} dv = -dx$. Integrating, we obtain $\ln(1+v^2) = -2x + c$. Imposing the initial condition $v(0) = v_0$ and $x(0) = 0$, we find that $c = \ln(1+v_0^2)$, so that $\ln(1+v^2) = -2x + \ln(1+v_0^2)$. When the object comes to rest the distance travelled by the object is $x = \frac{1}{2} \ln(1+v_0^2)$.

22.(a) Separating the variables, we have $v^{-n} dv = -kdt$. Let us consider two cases:

Case 1: $n \neq 1$. Integrating both sides of $v^{-n} dv = -kdt$ yields $\frac{1}{1-n} v^{1-n} = -kt + c$. Imposing the initial condition $v(0) = v_0$ yields $c = \frac{1}{1-n} v_0^{1-n}$, so that

$$v = [v_0^{1-n} + (n-1)kt]^{1/(1-n)}.$$

Observe that the object comes to rest in a finite time if there is a positive value of t for which $v = 0$. This requires $v_0^{1-n} + (n-1)kt = 0$. That is, $t = -\frac{v_0^{1-n}}{(n-1)k}$. If we assume $v_0 > 0$ and $k > 0$, t will be positive if and only if $n < 1$.

Case 2: $n = 1$. Integrating both sides of $v^{-n} dv = -kdt$ and imposing the initial conditions yields $v(t) = v_0 e^{-kt}$, and the object does not come to rest in a finite amount of time.

(b) If $n \neq 1, 2$, then $\frac{dx}{dt} = [v_0^{1-n} + (n-1)kt]^{1/(1-n)}$, where $x(t)$ denotes the distanced travelled by the object. Consequently,

$$x(t) = -\frac{1}{k(2-n)} [v_0^{1-n} + (n-1)kt]^{(2-n)/(1-n)} + c.$$

Imposing the initial condition $x(0) = 0$ yields $c = \frac{1}{k(2-n)} v_0^{2-n}$, so that

$$x(t) = -\frac{1}{k(2-n)} [v_0^{1-n} + n(n-1)kt]^{(2-n)/(1-n)} + \frac{1}{k(2-n)} v_0^{2-n}.$$

For $1 < n < 2$, we have $\frac{2-n}{1-n} < 0$, so that $\lim_{t \rightarrow \infty} x(t) = \frac{1}{k(2-n)}$. Hence the maximum distance that the object can travel in a finite time is less than $\frac{1}{k(2-n)}$.

If $n = 1$, then we can integrate to obtain $x(t) = \frac{v_0}{k}(1 - e^{-kt})$, where we have imposed the initial condition $x(0) = 0$. Consequently, $\lim_{t \rightarrow \infty} x(t) = \frac{v_0}{k}$. Thus in this case the maximum distance that the object can travel in a finite time is less than $\frac{v_0}{k}$.

(c) If $n > 2$, then $x(t) = -\frac{1}{k(2-n)} [v_0^{1-n} + n(n-1)kt]^{(2-n)/(1-n)} + \frac{1}{k(2-n)} v_0^{2-n}$ is still valid. However, in this case $\frac{2-n}{1-n} > 0$, and so $\lim_{t \rightarrow \infty} x(t) = +\infty$. Consequently, there is no limit to the distance that the object can travel.

If $n = 2$, then we return to $v = [v_0^{1-n} + (n-1)kt]^{1/(1-n)}$. In this case $\frac{dx}{dt} = (v_0^{-1} + kt)^{-1}$, which can be integrated directly to obtain $x(t) = \frac{1}{k} \ln(1 + v_0 kt)$, where we have imposed the initial condition that $x(0) = 0$. Once more we see that $\lim_{t \rightarrow \infty} x(t) = +\infty$, so that there is no limit to the distance that the object can travel.

23. Substituting the relation $p = p_0 \left(\frac{\rho}{\rho_0}\right)^{1/\gamma}$ into $dp = -g\rho dy$, we obtain

$$dp = -g\rho_0 \left(\frac{p}{p_0}\right)^{1/\gamma} dy,$$

or equivalently,

$$p^{-1/\gamma} dp = -\frac{g\rho_0}{p_0^{1/\gamma}} dy.$$

This can be integrated directly to obtain

$$\frac{\gamma p^{(\gamma-1)/\gamma}}{\gamma-1} = -\frac{g\rho_0 y}{p_0^{1/\gamma}} + c.$$

At the center of the Earth we have $p = p_0$. Imposing this initial condition on the preceding solution gives $c = \frac{\gamma p_0^{(\gamma-1)/\gamma}}{\gamma-1}$. Substituting this value of c into the general solution to the differential equation we find,

after some simplification,

$$p^{(\gamma-1)/\gamma} = p_0^{(\gamma-1)/\gamma} \left[1 - \frac{(\gamma-1)\rho_0 g y}{\gamma p_0} \right],$$

so that

$$p = p_0 \left[1 - \frac{(\gamma-1)\rho_0 g y}{\gamma p_0} \right]^{\gamma/(\gamma-1)}.$$

24. Substituting the given values into the differential equation for Newton's Law of Cooling, we have

$$\frac{dT}{dt} = -k(T - 75) \implies \frac{dT}{T - 75} = -k dt \implies \ln |T - 75| = -kt + c_1 \implies T(t) = 75 + ce^{-kt}.$$

Now, $T(0) = 135$ implies that $c = 60$. Therefore, $T(t) = 75 + 60e^{-kt}$. Next, $T(1) = 95$ implies that $95 = 75 + 60e^{-k}$, from which we quickly find that $k = \ln 3$. Thus, $T(t) = 75 + 60e^{-t \ln 3}$. Now if $T(t) = 615$, then $615 = 75 + 60e^{-t \ln 3}$. Therefore, $t = -2$ hours. Thus the object was placed in the room at 2 p.m.

25. Substituting the given values into the differential equation for Newton's Law of Cooling, we have $\frac{dT}{dt} = -k(T - 450)$. Solving by the usual process, we arrive at $T(t) = 450 + ce^{-kt}$. Since $T(0) = 50$, we compute that $c = -400$. Therefore, $T(t) = 450 - 400e^{-kt}$. Next, $T(20) = 150$ implies that $k = \frac{1}{20} \ln \frac{4}{3}$. Hence, $T(t) = 450 - 400(\frac{3}{4})^{t/20}$.

(i) After 40 minutes, the temperature is $T(40) = 450 - 400(\frac{3}{4})^2 = 225^\circ\text{F}$.

(ii) We must solve for t in the formula $T(t) = 350 = 450 - 400(\frac{3}{4})^{t/20}$. We have

$$(\frac{3}{4})^{t/20} = \frac{1}{4} \implies t = \frac{20 \ln 4}{\ln(4/3)} \approx 96.4 \text{ minutes.}$$

26. Substituting the given values into the differential equation for Newton's Law of Cooling, we have $\frac{dT}{dt} = -k(T - 34)$, which separates to $\frac{dT}{T - 34} = -k dt$. Solving by the usual process, we arrive at $T(t) = 34 + ce^{-kt}$. Since $T(0) = 38$ (by setting $t = 0$ at 2 p.m.), we compute that $c = 4$. Therefore, $T(t) = 34 + 4e^{-kt}$. Next, $T(1) = 36$ implies that $k = \ln 2$. Hence, $T(t) = 34 + 4e^{-t \ln 2}$. Now the temperature at the time of death is $T(t) = 98$, and we wish to solve for t :

$$T(t) = 34 + 4e^{-kt} = 98 \implies 2^{-t} = 16 \implies t = -4 \text{ hours.}$$

Thus $T(-4) = 98$ and Holmes was right, the time of death was 10 a.m.

27. We derive the solution to Newton's Law of Cooling as usual: $T(t) = 75 + ce^{-kt}$. Since $T(10) = 415$, we have $75 + ce^{-10k} = 415 \implies 340 = ce^{-10k}$. Moreover, since $T(20) = 347$, we have $75 + ce^{-20k} = 347 \implies 272 = ce^{-20k}$. Solving these two equations yields

$$k = \frac{1}{10} \ln \frac{5}{4} \quad \text{and} \quad c = 425.$$

Hence,

$$T(t) = 75 + 425 \left(\frac{4}{5} \right)^{t/10}.$$

(a) The furnace temperature is $T(0) = 500^\circ\text{F}$.

(b) If $T(t) = 100$, then

$$100 = 75 + 425 \left(\frac{4}{5} \right)^{t/10} \implies t = \frac{10 \ln 17}{\ln \frac{5}{4}} \approx 126.96 \text{ minutes.}$$

Thus the temperature of the coal was 100°F at 6:07 p.m.

28. Substituting the given values into the differential equation yields $\frac{dT}{dt} = -k(T - 72)$. With separation of variables, this becomes $\frac{dT}{T - 72} = -k dt$. Solving this by the usual process, we get the general solution $T(t) = 72 + ce^{-kt}$. Since $\frac{dT}{dt} = -20$, we have $-k(T - 72) = -20$. Therefore, $k = \frac{10}{39}$. Since $T(1) = 150$, we have $150 = 72 + ce^{-10/39} \implies c = 78e^{10/39}$. Consequently,

$$T(t) = 72 + 78e^{10(1-t)/39}.$$

(i) The initial temperature of the object is $T(0) = 72 + 78e^{10/39} \approx 173^\circ\text{F}$

(ii) The rate of change of the temperature after 10 minutes is $\frac{dT}{dt}$ when $t = 10$. Note that $T(10) = 72 + 78e^{-30/13}$ so after 10 minutes,

$$\frac{dT}{dt} = -k(T - 72) = -\frac{10}{39}(72 + 78e^{-30/13} - 72) \implies \frac{dT}{dt} = -20e^{-30/13} \approx -2^\circ\text{F per minute.}$$

Solutions to Section 1.5

True-False Review:

1. TRUE. The differential equation for such a population growth is $\frac{dP}{dt} = kP$, where $P(t)$ is the population as a function of time, and this is the Malthusian growth model described at the beginning of this section.

2. FALSE. The initial population could be greater than the carrying capacity, although in this case the population will asymptotically decrease towards the value of the carrying capacity.

3. TRUE. The differential equation governing the logistic model is (1.5.2), which is certainly separable as

$$\frac{D}{P(C - P)} \frac{dP}{dt} = r.$$

Likewise, the differential equation governing the Malthusian growth model is $\frac{dP}{dt} = kP$, and this is separable as $\frac{1}{P} \frac{dP}{dt} = k$.

4. TRUE. As (1.5.3) shows, as $t \rightarrow \infty$, the population does indeed tend to the carrying capacity C independently of the initial population P_0 . As it does so, its rate of change $\frac{dP}{dt}$ slows to zero (this is best seen from (1.5.2) with $P \approx C$).

5. TRUE. Every five minutes, the population doubles (increase 2-fold). Over 30 minutes, this population will double a total of 6 times, for an overall $2^6 = 64$ -fold increase.

6. TRUE. An 8-fold increase would take 30 years, and a 16-fold increase would take 40 years. Therefore, a 10-fold increase would take between 30 and 40 years.

7. FALSE. The growth rate is $\frac{dP}{dt} = kP$, and so as P changes, $\frac{dP}{dt}$ changes. Therefore, it is not always constant.

8. TRUE. From (1.5.2), the equilibrium solutions are $P(t) = 0$ and $P(t) = C$, where C is the carrying capacity of the population.

9. FALSE. If the initial population is in the interval $(\frac{C}{2}, C)$, then although it is less than the carrying capacity, its concavity does not change. To get a true statement, it should be stated instead that the initial population is less than *half* of the carrying capacity.

10. TRUE. Since $P'(t) = kP$, then $P''(t) = kP'(t) = k^2P > 0$ for all t . Therefore, the concavity is always positive, and does not change, regardless of the initial population.

Problems:

1. We solve $\frac{dP}{dt} = kP$ via separation of variables to obtain $P(t) = P_0e^{kt}$. Since $P(0) = 10$, then $P(t) = 10e^{kt}$. Since the doubling time is 3 hours, $P(3) = 20$, so that $2 = e^{3k} \implies k = \frac{\ln 2}{3}$. Thus $P(t) = 10e^{(t/3)\ln 2}$. Therefore, $P(24) = 10e^{(24/3)\ln 2} = 10 \cdot 2^8 = 2560$ bacteria.

2. We solve $\frac{dP}{dt} = kP$ via separation of variables to obtain $P(t) = P_0e^{kt}$. Since we are given $P(10) = 5000$ and $P(12) = 6000$, then $5000 = P_0e^{10k}$ and $6000 = P_0e^{12k}$. Dividing the second equation by the first, we obtain $e^{2k} = \frac{6}{5}$. Hence, $k = \frac{1}{2} \ln \frac{6}{5}$. Hence, the initial population is

$$P(0) = P_0 = 5000e^{-10k} = 5000e^{-5 \ln(6/5)} 5000 \left(\frac{5}{6}\right)^5 \approx 2009.4.$$

Also, the population doubles at time t such that $P(t) = 2P_0$. This occurs when $t = \frac{\ln 2}{k} = \frac{2 \ln 2}{\ln \frac{6}{5}} \approx 7.6$ hours.

3. From $P(t) = P_0e^{kt}$ and $P(0) = 2000$, it follows that $P(t) = 2000e^{kt}$. Since the doubling time is 4 hours, it follows that $k = \frac{\ln 2}{4}$, and so $P(t) = 2000e^{t \ln 2/4}$. Therefore, the time t at which the culture contains 10^6 cells satisfies $P(t) = 10^6 \implies 10^6 = 2000e^{t \ln 2/4} \implies t \approx 35.86$ hours.

4. We solve $\frac{dP}{dt} = kP$ via separation of variables to obtain $P(t) = P_0e^{kt}$, where we measure time in years. Since, $P(0) = 10000$ then $P(t) = 10000e^{kt}$. Since $P(5) = 20000$ then $20000 = 10000e^{5k} \implies k = \frac{\ln 2}{5}$. Hence $P(t) = 10000e^{(t \ln 2)/5}$.

(a) We have $P(20) = 10000e^{4 \ln 2} = 160000$.

(b) We set $1000000 = 10000e^{(t \ln 2)/5}$. Then $100 = e^{(t \ln 2)/5} \implies t = \frac{5 \ln 100}{\ln 2} \approx 33.22$ years.

5. In formulas (1.5.5) and (1.5.6) we have $P_0 = 500$, $P_1 = 800$, $P_2 = 1000$, $t_1 = 5$, and $t_2 = 10$. Hence,

$$r = \frac{1}{5} \ln \left[\frac{(1000)(300)}{(500)(200)} \right] = \frac{1}{5} \ln 3 \quad \text{and} \quad C = \frac{800[(800)(1500) - 2(500)(1000)]}{800^2 - (500)(1000)} \approx 1142.86,$$

so that

$$P(t) = \frac{(1142.86)(500)}{500 + 642.86e^{-0.2t \ln 3}} \approx \frac{571430}{500 + 642.86e^{-0.2t \ln 3}}.$$

Inserting $t = 15$ into the preceding formula yields $P(15) \approx 1091$.

6. In formulas (1.5.5) and (1.5.6) we have $P_0 = 50, P_1 = 62, P_2 = 76, t_1 = 2$, and $t_2 = 4$. Hence,

$$r = \frac{1}{2} \ln \left[\frac{(76)(12)}{(50)(14)} \right] \approx 0.132 \quad \text{and} \quad C = \frac{62[(62)(126) - 2(50)(76)]}{62^2 - (50)(76)} \approx 298.727,$$

so that

$$P(t) = \frac{14936.35}{50 + 248.727e^{-0.132t}}.$$

Inserting $t = 20$ into the preceding formula yields $P(20) \approx 221$.

7. (a) From (1.5.5), $r > 0$ requires $\frac{P_2(P_1 - P_0)}{P_0(P_2 - P_1)} > 1$. Rearranging the terms in this inequality and using the fact that $P_2 > P_1$ yields $P_1 > \frac{2P_0P_2}{P_0 + P_2}$. Further, $C > 0$ requires that $\frac{P_1(P_0 + P_2) - 2P_0P_2}{P_1^2 - P_0P_2} > 0$. From $P_1 > \frac{2P_0P_2}{P_0 + P_2}$ we see that the numerator in the preceding inequality is positive, and therefore the denominator must also be positive. Hence, in addition to $P_1 > \frac{2P_0P_2}{P_0 + P_2}$, we must also have $P_1^2 > P_0P_2$.

(b) We have $P_0 = 10000, P_1 = 12000$ and $P_2 = 18000$. We immediately see that $P_1^2 > P_0P_2$ fails for these values, so by part (a), we conclude that there is no solution to the logistic equation that fits this data.

8. Let $y(t)$ denote the number of passengers who have the flu at time t . Then we must solve

$$\frac{dy}{dt} = ky(1500 - y), \quad y(0) = 5, \quad y(1) = 10,$$

where k is a positive constant. Separating the differential equation and integrating yields $\int \frac{1}{y(1500 - y)} dy = k \int dt$. Using a partial fraction decomposition on the left-hand side gives $\int \left[\frac{1}{1500y} + \frac{1}{1500(1500 - y)} \right] dy = k \int dt$, so that

$$\frac{1}{1500} \ln \left(\frac{y}{1500 - y} \right) = kt + c,$$

which upon exponentiation yields

$$\frac{y}{1500 - y} = c_1 e^{1500kt}.$$

Imposing the initial condition $y(0) = 5$, we find that $c_1 = \frac{1}{299}$. Hence, $\frac{y}{1500 - y} = \frac{1}{299} e^{1500kt}$. The further condition $y(1) = 10$ requires $\frac{10}{1490} = \frac{1}{299} e^{1500k}$. Solving for k gives $k = \frac{1}{1500} \ln \frac{299}{149}$. Therefore, $\frac{y}{1500 - y} = \frac{1}{299} e^{t \ln (299/149)}$. Solving algebraically for y we find $y(t) = \frac{1500e^{t \ln (299/149)}}{299 + e^{t \ln (299/149)}} = \frac{1500}{1 + 299e^{-t \ln (299/149)}}$. Hence, $y(14) = \frac{1500}{1 + 299e^{-14 \ln (299/149)}} \approx 1474$.

9. (a) Equilibrium solutions: $P(t) = 0, P(t) = T$.

Slope: $P > T \Rightarrow \frac{dP}{dt} > 0, 0 < P < T \Rightarrow \frac{dP}{dt} < 0$.

Isoclines: $r(P - T) = k \Rightarrow P^2 - TP - \frac{k}{r} = 0 \Rightarrow P = \frac{1}{2} \left(T \pm \sqrt{\frac{rT^2 + 4k}{r}} \right)$.

We see that slope of the solution curves satisfies $k \geq \frac{-rT^2}{4}$.

Concavity: We have $\frac{d^2P}{dt^2} = r(2P - T)\frac{dP}{dt} = r^2(2P - T)(P - T)P$. Hence, the solution curves are concave up for $P > \frac{T}{2}$, and are concave down for $0 < P < \frac{T}{2}$.

(b)

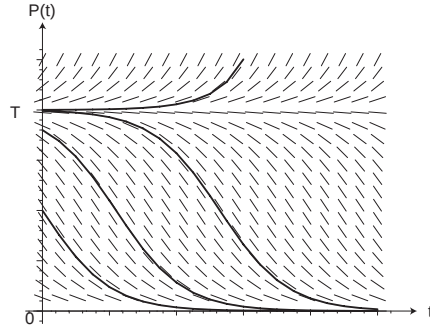


Figure 42: Figure for Problem 9(b)

(c) For $0 < P_0 < T$, the population dies out with time. For $P_0 > T$, there is a population growth. The term threshold level is appropriate since T gives the minimum value of P_0 above which there is a population growth.

10. (a) Separating the variables in differential equation (1.5.7) gives $\frac{1}{P(P-T)} \frac{dP}{dt} = r$, which can be written in the equivalent form $\left[\frac{1}{T(P-T)} - \frac{1}{TP} \right] \frac{dP}{dt} = r$. Integrating yields $\frac{1}{T} \ln \left(\frac{P-T}{P} \right) = rt + c$, so that $\frac{P-T}{P} = c_1 e^{Trt}$. The initial condition $P(0) = P_0$ requires $\frac{P_0 - T}{P_0} = c_1$, so that $\frac{P-T}{P} = \left(\frac{P_0 - T}{P_0} \right) e^{rTt}$. Solving algebraically for P yields

$$P(t) = \frac{TP_0}{P_0 - (P_0 - T)e^{rTt}}.$$

(b) If $P_0 < T$, then the denominator in $\frac{TP_0}{P_0 - (P_0 - T)e^{rTt}}$ is positive, and increases without bound as $t \rightarrow \infty$. Consequently $\lim_{t \rightarrow \infty} P(t) = 0$. In this case the population dies out as t increases.

(c) If $P_0 > T$, then the denominator of $\frac{TP_0}{P_0 - (P_0 - T)e^{rTt}}$ vanishes when $(P_0 - T)e^{rTt} = P_0$ (i.e., when

$t = \frac{1}{rT} \ln \left(\frac{P_0}{P_0 - T} \right)$. This means that within a finite time the population grows without bound. We can interpret this as a mathematical model of a population explosion.

11.

Equilibrium solutions: $P(t) = 0, P(t) = T, P(t) = C$. The slope of the solution curves is negative for $0 < P < T$, and for $P > C$. It is positive for $T < P < C$.

Concavity: We have

$$\frac{d^2P}{dt^2} = r^2[(C - P)(P - T) - (P - T)P + (C - P)P](C - P)(P - T)P,$$

which simplifies to

$$\frac{d^2P}{dt^2} = r^2(-3P^2 + 2PT + 2CP - CT)(C - P)(P - T).$$

Hence, changes in concavity occur when $P = \frac{1}{3}(C + T \pm \sqrt{C^2 - CT + T^2})$. A representative slope field with some solution curves is shown in the accompanying figure. We see that for $0 < P_0 < T$ the population dies out, whereas for $T < P_0 < C$ the population grows asymptotically to the equilibrium solution $P(t) = C$. If $P_0 > C$, then the solution decays towards the equilibrium solution $P(t) = C$.

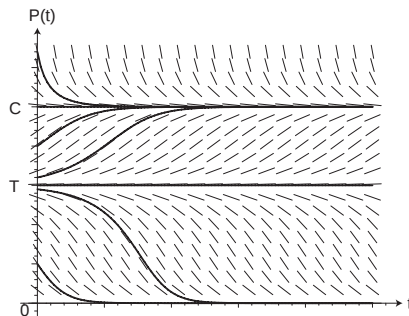


Figure 43: Figure for Problem 11

12.

Equilibrium solutions: $P(t) = C$. The slope of the solution curves is positive for $0 < P < C$, and negative for $P > C$.

Concavity: We have

$$\frac{d^2P}{dt^2} = r \left[\ln \left(\frac{C}{P} \right) - 1 \right] \frac{dP}{dt} = r^2 \left[\ln \left(\frac{C}{P} \right) - 1 \right] P \ln \frac{C}{P}.$$

Hence, the solution curves are concave up for $0 < P < \frac{C}{e}$ and $P > C$. They are concave down for $\frac{C}{e} < P < C$. A representative slope field with some solution curves are shown in the accompanying figure.

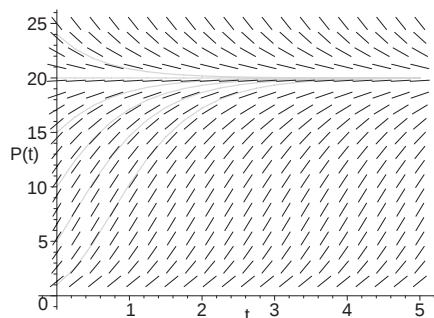


Figure 44: Figure for Problem 12

13. Separating the variables in (1.5.8) yields $\frac{dP}{P(\ln C - \ln P)} = r dt$, which can be integrated directly to obtain $-\ln(\ln C - \ln P) = rt + c$, where c is a constant of integration. Therefore, $\ln(\frac{C}{P}) = c_1 e^{-rt}$. The initial condition $P(0) = P_0$ requires that $\ln(\frac{C}{P_0}) = c_1$. Hence, $\ln(\frac{C}{P}) = e^{-rt} \ln(\frac{C}{P_0})$. Exponentiation of each side gives

$$\frac{C}{P} = \left(\frac{C}{P_0}\right)^{e^{-rt}} \implies P(t) = C \left(\frac{P_0}{C}\right)^{e^{-rt}}.$$

Since $\lim_{t \rightarrow \infty} e^{-rt} = 0$, it follows that $\lim_{t \rightarrow \infty} P(t) = C$.

14. We solve $\frac{dP}{dt} = kP$ via separation of variables to obtain $P(t) = P_0 e^{kt}$. The initial condition $P(0) = 400$ requires that $P_0 = 400$, so that $P(t) = 400e^{kt}$. We also know that $P(30) = 340$. This requires that $340 = 400e^{30k}$ so that $k = \frac{1}{30} \ln\left(\frac{17}{20}\right)$. Consequently,

$$P(t) = 400e^{\frac{t}{30} \ln\left(\frac{17}{20}\right)}.$$

(a) We have $P(60) = 400e^{2 \ln\left(\frac{17}{20}\right)} \approx 289$. Also, we have $P(100) = 400e^{\frac{10}{3} \ln\left(\frac{17}{20}\right)} \approx 233$.

(b) The half-life, t_H , is determine from

$$200 = 400e^{\frac{t_H}{30} \ln\left(\frac{17}{20}\right)} \implies t_H = 30 \frac{\ln 2}{\ln\left(\frac{20}{17}\right)} \approx 128 \text{ days}.$$

15. (a) More. From the given information, 20000 fans left the stadium in the first ten minutes. Because the rate of decay lessens with time under the exponential decay model, less than 20000 fans will leave in each of the next two ten minute intervals. Hence, less than 60000 fans will have departed after a total of 30 minutes. That is, more than 40000 fans will remain in the stadium after 30 minutes.

(b) We solve $\frac{dP}{dt} = kP$ via separation of variables to obtain $P(t) = P_0 e^{kt}$. The initial condition $P(0) = 100,000$ requires that $P_0 = 100,000$, so that $P(t) = 100,000e^{kt}$. We also know that $P(10) = 80,000$. This requires that $100,000 = 80,000e^{10k}$ so that $k = \frac{1}{10} \ln\left(\frac{4}{5}\right)$. Consequently,

$$P(t) = 100,000e^{\frac{t}{10} \ln\left(\frac{4}{5}\right)}.$$

The half-life is determined from $50,000 = 100,000e^{\frac{t_H}{10} \ln(\frac{4}{5})} \implies t_H = 10 \frac{\ln 2}{\ln(\frac{5}{4})} \approx 31.06$ min.

(c) There will be 15,000 fans left in the stadium at time t_0 , where

$$15,000 = 100,000e^{\frac{t_0}{10} \ln(\frac{4}{5})} \implies t_0 = 10 \frac{\ln(\frac{3}{20})}{\ln(\frac{4}{5})} \approx 85.02 \text{ min.}$$

16. We solve $\frac{dP}{dt} = kP$ via separation of variables to obtain $P(t) = P_0 e^{kt}$. Since the half-life is 5.2 years,

$$\frac{1}{2}P_0 = P_0 e^{5.2k} \implies k = -\frac{\ln 2}{5.2}.$$

Therefore,

$$P(t) = P_0 e^{-t \frac{\ln 2}{5.2}}.$$

Consequently, only 4% of the original amount will remain at time t_0 where

$$\frac{4}{100}P_0 = P_0 e^{-t_0 \frac{\ln 2}{5.2}} \implies t_0 = 5.2 \frac{\ln 25}{\ln 2} \approx 24.15 \text{ years.}$$

17. Maple, or even a TI 92 plus, has no problem in solving these equations.

18. (a) The Malthusian model of population predicts that $P(t) = 151.3e^{kt}$. Since $P(1) = 179.4$, then $179.4 = 151.3e^{10k} \implies k = \frac{1}{10} \ln \frac{179.4}{151.3}$. Hence, $P(t) = 151.3e^{\frac{t \ln(179.4/151.1)}{10}}$.

(b) According to the logistic model, $P(t) = \frac{151.3C}{151.3 + (C - 151.3)e^{-rt}}$. Imposing the initial conditions $P(10) = 179.4$ and $P(20) = 203.3$ gives the pair of equations

$$179.4 = 151.3e^{\frac{10 \ln(179.4/151.1)}{10}} \quad \text{and} \quad 203.3 = 151.3e^{\frac{20 \ln(179.4/151.1)}{10}},$$

whose solution is $C \approx 263.95$ and $r \approx 0.046$. Using these values for C and r gives

$$P(t) = \frac{39935.6}{151.3 + 112.65e^{-0.046t}}.$$

(c)

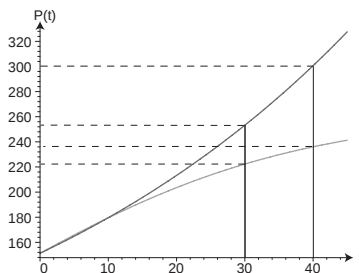


Figure 45: Figure for Problem 18(c)

Malthusian model: $P(30) \approx 253$ million; $P(40) \approx 300$ million.

Logistic model: $P(30) \approx 222$ million; $P(40) \approx 236$ million.

The logistics model fits the data better than the Malthusian model, but still gives a significant underestimate of the 1990 population.

19. The logistic population model predicts that $P(t) = \frac{50C}{50 + (C - 50)e^{-rt}}$, where C and r are to-be-determined. Imposing the conditions $P(5) = 100$ and $P(15) = 250$ gives the pair of equations

$$100 = \frac{50C}{50 + (C - 50)e^{-5r}} \quad \text{and} \quad 250 = \frac{50C}{50 + (C - 50)e^{-15r}},$$

whose positive solutions are $C \approx 370.32$ and $r \approx 0.17$. Using these values for C and r gives

$$P(t) = \frac{18500}{50 + 18450e^{-0.17t}}.$$

From the figure we see that it will take approximately 52 years to reach 95% of the carrying capacity.

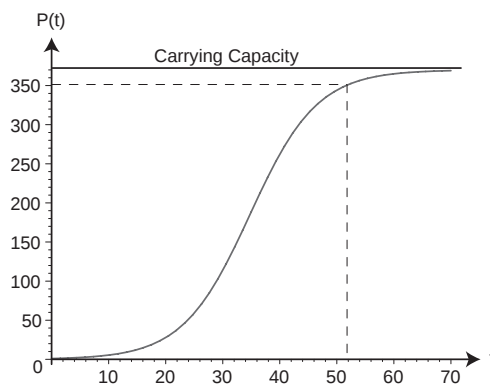


Figure 46: Figure for Problem 19

Solutions to Section 1.6

True-False Review:

1. FALSE. Any solution to the differential equation (1.6.7) serves as an integrating factor for the differential equation. There are infinitely many solutions to (1.6.7), taking the form $I(x) = c_1 e^{\int p(x) dx}$, where c_1 is an arbitrary constant.

2. TRUE. Any solution to the differential equation (1.6.7) serves as an integrating factor for the differential equation. There are infinitely many solutions to (1.6.7), taking the form $I(x) = c_1 e^{\int p(x) dx}$, where c_1 is an arbitrary constant. The most natural choice is $c_1 = 1$, giving the integrating factor $I(x) = e^{\int p(x) dx}$.

3. TRUE. Multiplying $y' + p(x)y = q(x)$ by $I(x)$ yields $y'I + pIy = qI$. Assuming that $I' = pI$, the requirement on the integrating factor, we have $y'I + I'y = qI$, or by the product rule, $(I \cdot y)' = qI$, as requested.

4. FALSE. Before determining an integrating factor, the equation must be rewritten as

$$\frac{dy}{dx} - x^2 y = \sin x,$$

and with $p(x) = -x^2$, we have integrating factor $I(x) = e^{\int -x^2 dx}$, not $e^{\int x^2 dx}$.

5. TRUE. Rewriting the differential equation as

$$\frac{dy}{dx} + \frac{1}{x}y = x,$$

we have $p(x) = \frac{1}{x}$, and so an integrating factor must have the form $I(x) = e^{\int p(x)dx} = e^{\int \frac{dx}{x}} = e^{\ln x + c} = e^c x$. Since $5x$ does indeed have this form, it is an integrating factor.

Problems:

In this section the function $I(x) = e^{\int p(x)dx}$ will represent the integrating factor for a differential equation of the form $y' + p(x)y = q(x)$.

1. An integrating factor is $I(x) = e^{-\int dx} = e^{-x}$. Therefore, $\frac{d(e^{-x}y)}{dx} = e^x \implies e^{-x}y = e^x + c \implies y(x) = e^x(e^x + c)$.

2. We rewrite the differential equation in standard form: $y' - \frac{4}{x}y = x^5 \sin x$. An integrating factor is $I(x) = x^{-4}$. Therefore, $\frac{d(x^{-4}y)}{dx} = x \sin x \implies x^{-4}y = \sin x - x \cos x + c \implies y(x) = x^4(\sin x - x \cos x + c)$.

3. An integrating factor is $I(x) = e^2 \int x dx = e^{x^2}$. Therefore,

$$\frac{d}{dx}(e^{x^2}y) = 2e^{x^2}x^3 \implies e^{x^2}y = 2 \int e^{x^2}x^3 dx \implies e^{x^2}y = e^{x^2}(x^2 - 1) + c \implies y(x) = x^2 - 1 + ce^{-x^2}.$$

4. An integrating factor for the differential equation is

$$I(x) = e^{\int \frac{2x}{1-x^2} dx} = e^{-\ln(1-x^2)} = \frac{1}{1-x^2}.$$

Therefore,

$$\frac{d}{dx} \left(\frac{y}{1-x^2} \right) = \frac{4x}{1-x^2} \implies \frac{y}{1-x^2} = -\ln(1-x^2)^2 + c \implies y(x) = (1-x^2)[- \ln(1-x^2)^2 + c].$$

5. An integrating factor for the differential equation is $I(x) = e^{\int \frac{2x}{1+x^2} dx} = 1+x^2$. Therefore,

$$\frac{d}{dx}[(1+x^2)y] = \frac{4}{1+x^2} \implies (1+x^2)y = 4 \int \frac{dx}{1+x^2} \implies (1+x^2)y = 4 \tan^{-1} x + c \implies y(x) = \frac{1}{1+x^2}(4 \tan^{-1} x + c).$$

6. We rewrite the differential equation in standard form: $y' + \frac{\sin 2x}{2 \cos^2 x}y = 2 \cos^2 x$. An integrating factor for the differential equation is

$$I(x) = e^{\int \frac{\sin 2x}{2 \cos^2 x} dx} = e^{\int \frac{\sin x \cos x}{\cos^2 x} dx} = e^{\int \tan x dx} = e^{-\ln(\cos x)} = \frac{1}{\cos x}.$$

Therefore,

$$\frac{d}{dx}(y \sec x) = 2 \cos x \implies y(x) = \cos x(2 \sin x + c) \implies y(x) = \sin 2x + c \cos x.$$

7. An integrating factor is $I(x) = e^{\int \frac{dx}{x \ln x}} = e^{\ln(\ln x)} = \ln x$. Therefore,

$$\frac{d}{dx}(y \ln x) = 9 \int x^2 \ln x dx \implies y \ln x = 3x^3 \ln x - x^3 + c \implies y(x) = \frac{3x^3 \ln x - x^3 + c}{\ln x}.$$

8. An integrating factor is $I(x) = e^{\int -\tan x dx} = e^{\ln(\cos x)} = \cos x$. Therefore,

$$\frac{d}{dx}(y \cos x) = 8 \cos x \sin^3 x \implies y \cos x = 8 \int \cos x \sin^3 x dx \implies y \cos x = 2 \sin^4 x + c \implies y(x) = \frac{2 \sin^4 x + c}{\cos x}.$$

9. We rewrite the differential equation in standard form: $x' + \frac{2}{t}x = \frac{4e^t}{t}$. An integrating factor is $I(t) = e^{2 \int \frac{dt}{t}} = t^2$. Therefore,

$$\frac{d}{dt}(t^2 x) = 4te^t \implies t^2 x = 4 \int te^t dt \implies t^2 x = 4e^t(t - 1) + c \implies x(t) = \frac{4e^t(t - 1) + c}{t^2}.$$

10. We rewrite the differential equation in standard form: $y' - (\tan x)y = -2 \sin x$. An integrating factor is $I(x) = e^{\int -\tan x dx} = e^{\ln(\cos x)} = \cos x$. Therefore,

$$\frac{d}{dx}(y \cos x) = -2 \sin x \cos x = -\sin 2x \implies y \cos x = \frac{1}{2} \cos 2x + c \implies y(x) = \sec x \left(\frac{1}{2} \cos 2x + c \right).$$

11. We rewrite the differential equation in standard form: $y' + (\tan x)y = \sec x$. An integrating factor is $I(x) = e^{\int \tan x dx} = e^{-\ln(\cos x)} = \sec x$. Therefore,

$$\frac{d}{dx}(y \sec x) = \sec^2 x \implies y \sec x = \int \sec^2 x dx = \tan x + c \implies y(x) = \cos x(\tan x + c) \implies y(x) = \sin x + c \cos x.$$

12. An integrating factor is $I(x) = e^{-\int \frac{1}{x} dx} = x^{-1}$. Therefore,

$$\frac{d}{dx}(x^{-1}y) = 2x \ln x \implies x^{-1}y = 2 \int x \ln x dx = \frac{1}{2}x^2(2 \ln x - 1) + c.$$

Hence, $y(x) = \frac{1}{2}x^3(2 \ln x - 1) + cx$.

13. An integrating factor is $I(x) = e^{\alpha \int dx} = e^{\alpha x}$. Therefore,

$$\frac{d}{dx}(e^{\alpha x}y) = e^{(\alpha+\beta)x} \implies e^{\alpha x}y = \int e^{(\alpha+\beta)x} dx.$$

Now we consider two cases:

Case 1: $\alpha + \beta = 0$. In this case, $e^{\alpha x}y = x + c$, so that $y(x) = e^{-\alpha x}(x + c)$.

Case 2: $\alpha + \beta \neq 0$. In this case, $e^{\alpha x}y = \frac{e^{(\alpha+\beta)x}}{\alpha + \beta} + c$, so that $y(x) = \frac{e^{\beta x}}{\alpha + \beta} + ce^{-\alpha x}$.

14. An integrating factor is $I(x) = e^{\int mx^{-1}dx} = e^{m \ln x} = x^m$. Therefore,

$$\frac{d}{dx}(x^m y) = x^m \ln x \implies x^m y = \int x^m \ln x dx.$$

Now we consider two cases:

Case 1: $m = -1$. In this case, $x^m y = \frac{(\ln x)^2}{2} + c \implies y(x) = x \left[\frac{(\ln x)^2}{2} + c \right]$.

Case 2: $m \neq -1$. In this case, $x^m y = \frac{x^{m+1}}{m+1} \ln x - \frac{x^{m+1}}{(m+1)^2} + c \implies y(x) = \frac{x}{m+1} \ln x - \frac{x}{(m+1)^2} + \frac{c}{x^m}$.

15. An integrating factor is $I(x) = e^{2 \int \frac{dx}{x}} = e^{2 \ln x} = x^2$. Therefore,

$$\frac{d}{dx}(x^2 y) = 4x^3 \implies x^2 y = 4 \int x^3 dx + c \implies x^2 y = x^4 + c,$$

and since $y(1) = 2$, we have $c = 1$. Therefore, $y(x) = \frac{x^4 + 1}{x^2}$.

16. We begin by rewriting the differential equation $y' \sin x - y \cos x = \sin 2x$ as $y' - y \cot x = 2 \cos x$. Now, an integrating factor for this differential equation is

$$I(x) = e^{\int -\cot x dx} = e^{-\ln(\sin x)} = \csc x.$$

Therefore,

$$\frac{d}{dx}(y \csc x) = 2 \csc x \cos x \implies y \csc x = 2 \ln(\sin x) + c,$$

and since $y(\frac{\pi}{2}) = 2$, we have $c = 2$. Therefore, $y(x) = 2 \sin x [\ln(\sin x) + 1]$.

17. An integrating factor is $I(t) = e^{2 \int \frac{dt}{4-t}} = e^{-2 \ln(4-t)} = (4-t)t^{-2}$. Therefore,

$$\frac{d}{dt}((4-t)^{-2}x) = 5(4-t)^{-2} \implies (4-t)^{-2}x = 5 \int (4-t)^{-2} dt + c \implies (4-t)^{-2}x = 5(4-t)^{-1} + c,$$

and since $x(0) = 4$, we have $c = -1$. Therefore, $x(t) = (4-t)^2[5(4-t)^{-1} - 1] = (4-t)(1+t)$.

18. First we rewrite the differential equation $(y - e^{-x})dx + dy = 0$ as $y' + y = e^x$. An integrating factor is $I(x) = e^x$, so that

$$\frac{d}{dx}(e^x y) = e^{2x} \implies e^x y = \frac{e^{2x}}{2} + c.$$

Since $y(0) = 1$, we have $c = \frac{1}{2}$. Therefore, $y(x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$.

19. An integrating factor for this differential equation is $I(x) = e^{\int dx} = e^x$. Therefore,

$$\begin{aligned}\frac{d}{dx}(e^x y) &= e^x f(x) \implies [e^x y]_0^x = \int_0^x e^x f(x) dx \\ &\implies e^x y - y(0) = \int_0^x e^x f(x) dx \\ &\implies e^x y - 3 = \int_0^x e^x dx \\ &\implies y(x) = e^{-x} \left[3 + \int_0^x e^x f(x) dx \right].\end{aligned}$$

$$\text{If } x \leq 1, \int_0^x e^x f(x) dx = \int_0^x e^x dx = e^x - 1 \implies y(x) = e^{-x}(2 + e^x)$$

$$\text{If } x > 1, \int_0^x e^x f(x) dx = \int_0^x e^x dx = e - 1 \implies y(x) = e^{-x}(2 + e).$$

20. An integrating factor for this differential equation is $I(x) = e^{-\int 2dx} = e^{-2x}$. Therefore,

$$\begin{aligned}\frac{d}{dx}(e^{-2x} y) &= e^{-2x} f(x) \implies [e^{-2x} y]_0^x = \int_0^x e^{-2x} f(x) dx \\ &\implies e^{-2x} y - y(0) = \int_0^x e^{-2x} f(x) dx \\ &\implies e^{-2x} y - 1 = \int_0^x e^{-2x} f(x) dx \\ &\implies y(x) = e^{2x} \left[1 + \int_0^x e^{-2x} f(x) dx \right].\end{aligned}$$

If $x < 1$,

$$\int_0^x e^{-2x} f(x) dx = \int_0^x e^{-2x} (1 - x) dx = \frac{1}{4} e^{-2x} (2x - 1 + e^{2x}),$$

so that

$$y(x) = e^{2x} \left[1 + \frac{1}{4} e^{-2x} (2x - 1 + e^{2x}) \right] = \frac{1}{4} (5e^{2x} + 2x - 1).$$

If $x \geq 1$,

$$\int_0^x e^{-2x} f(x) dx = \int_0^x e^{-2x} (1 - x) dx = \frac{1}{4} (1 + e^{-2}) \implies y(x) = e^{2x} \left[1 + \frac{1}{4} (1 + e^{-2}) \right] = \frac{1}{4} e^{2x} (5 + e^{-2}).$$

21. On $(-\infty, 1)$, the differential equation is $y' - y = 1$, which implies that an integrating factor is $I(x) = e^{-x}$. Therefore, $y(x) = c_1 e^x - 1$. Imposing the initial condition $y(0) = 0$ requires $c_1 = 1$, so that $y(x) = e^x - 1$, for $x < 1$.

On $[1, \infty)$, the differential equation is $y' - y = 2 - x$, which implies that an integrating factor is $I(x) = e^{-x}$. Therefore,

$$\frac{d}{dx}(e^{-x} y) = (2 - x)e^{-x} \implies y(x) = x - 1 + c_2 e^{-x}.$$

Continuity at $x = 1$ requires that $\lim_{x \rightarrow 1} y(x) = y(1)$. Consequently, we must choose c_2 to satisfy $c_2 e = e - 1$, so that $c_2 = 1 - e^{-1}$. Hence, for $x \geq 1$, we have $y(x) = x - 1 + (1 - e^{-1})e^x$.

22. Let $u = \frac{dy}{dx}$, so that $\frac{du}{dx} = \frac{d^2 y}{dx^2}$. The first equation becomes $\frac{du}{dx} + \frac{1}{x}u = 9x$, which is first-order linear. An integrating factor for this is $I(x) = e^{\int \frac{1}{x} dx} = x$, so

$$\frac{d}{dx}(xu) = 9x^2 \implies xu = 3x^3 + c_1 \implies u = 3x^2 + c_1 x^{-1},$$

but since $u = \frac{dy}{dx}$, we conclude that

$$\frac{dy}{dx} = 3x^2 + c_1 x^{-1} \implies y = \int (3x^2 + c_1 x^{-1}) dx \implies y(x) = x^3 + c_1 \ln x + c_2.$$

23. The differential equation for Newton's Law of Cooling is $\frac{dT}{dt} = -k(T - T_m)$. We can re-write this equation in the form of a first-order linear differential equation: $\frac{dT}{dt} + kT = kT_m$. An integrating factor for this differential equation is $I = e^{\int k dt} = e^{kt}$. Thus, $\frac{d}{dt}(Te^{kt}) = kT_m e^{kt}$. Integrating both sides, we get $Te^{kt} = T_m e^{kt} + c$, and hence, $T = T_m + ce^{-kt}$, which is the solution to Newton's Law of Cooling.

24. We are given that $\frac{dT_m}{dt} = \alpha$. Therefore, $T_m = \alpha t + c_1$, so that Newton's Law of Cooling in this case reads

$$\frac{dT}{dt} = -k(T - \alpha t - c_1) \implies \frac{dT}{dt} + kT = k(\alpha t + c_1).$$

An integrating factor for this differential equation is $I(t) = e^{\int k dt} = e^{kt}$. Thus,

$$\begin{aligned} \frac{d}{dt}(e^{kt}T) &= ke^{kt}(\alpha t + c_1) \implies e^{kt}T = e^{kt}\left(\alpha t - \frac{\alpha}{k} + c_1\right) + c_2 \\ &\implies T(t) = \alpha t - \frac{\alpha}{k} + c_1 + c_2 e^{-kt}. \end{aligned}$$

25. We are given that $\frac{dT_m}{dt} = 10$. Therefore, $T_m = 10t + c_1$, for some constant c_1 . Since $T_m = 65$ when $t = 0$ (at 8 a.m.), we find that $c_1 = 65$. Therefore, $T_m = 10t + 65$, and Newton's Law of Cooling in this case reads

$$\frac{dT}{dt} = -k(T - T_m) \implies \frac{dT}{dt} = -k(T - 10t - 65).$$

Since $\frac{dT}{dt}(1) = 5$, we have $5 = -k(35 - 10 - 65)$, so that we obtain $k = \frac{1}{8}$. Now, the last differential equation can then be written

$$\frac{dT}{dt} + kT = k(10t + 65) \implies \frac{d}{dt}(e^{kt}T) = 5ke^{kt}(2t + 13).$$

Therefore,

$$e^{kt}T = 5ke^{kt}\left(\frac{2}{k}t - \frac{2}{k^2} + \frac{13}{k}\right) + c \implies T(t) = 5\left(2t - \frac{2}{k} + 13\right) + ce^{-kt} = 5(2t - 3) + ce^{-\frac{t}{8}}.$$

Since $T(1) = 35$, we have $c = 40e^{\frac{1}{8}}$. Thus, $T(t) = 10t - 15 + 40e^{\frac{1}{8}(1-t)}$.

26. (a) In this case, Newton's law of cooling is $\frac{dT}{dt} = -\frac{1}{40}(T - 80e^{-t/20})$. This linear differential equation has standard form $\frac{dT}{dt} + \frac{1}{40}T = 2e^{-t/20}$, with integrating factor $I(t) = e^{t/40}$. Consequently the differential equation can be written in the integrable form $\frac{d}{dt}(e^{t/40}T) = 2e^{-t/40}$, so that $T(t) = -80e^{-t/20} + ce^{-t/40}$. Then $T(0) = 0 \implies c = 80$, so that $T(t) = 80(e^{-t/40} - e^{-t/20})$.

(b) We see that $\lim_{t \rightarrow \infty} T(t) = 0$. This is a reasonable result since the temperature of the surrounding medium also approaches zero as $t \rightarrow \infty$. We would expect the temperature of the object to approach to the temperature of the surrounding medium at late times.

(c) The maximum temperature must occur at a critical point of the function $T(t)$ derived in part (a). To determine critical points, we set

$$0 = \frac{dT}{dt} = 80 \left(-\frac{1}{40}e^{-t/40} + \frac{1}{20}e^{-t/20} \right).$$

We find only one critical value of t , namely $t = 40 \ln 2$. Since $T(0) = 0$ and $\lim_{t \rightarrow \infty} T(t) = 0$, the function assumes a *maximum* value at $t_{max} = 40 \ln 2$. At this time, the temperature of the object is $T(t_{max}) = 80(e^{-\ln 2} - e^{-2 \ln 2}) = 20^\circ\text{F}$, and the temperature of the surrounding medium is $T_m(t_{max}) = 80e^{-2 \ln 2} = 20^\circ\text{F}$.

(d) The behavior of $T(t)$ and $T_m(t)$ is given in the accompanying figure.

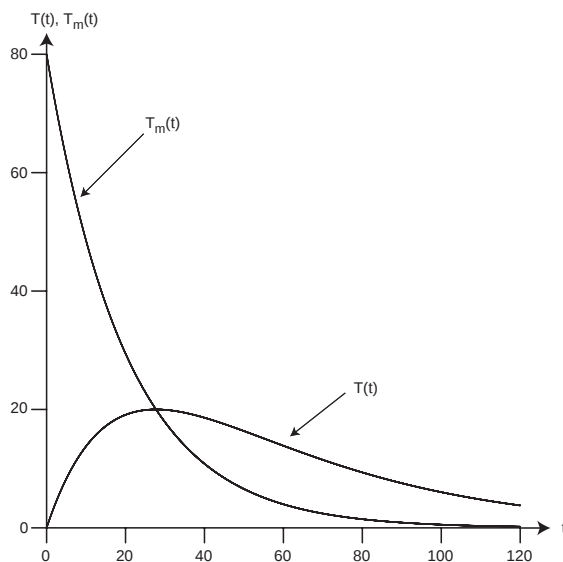


Figure 47: Figure for Problem 26(d)

27. (a) The temperature varies from a minimum of $A - B$ at $t = 0$ to a maximum of $A + B$ when $t = 12$.

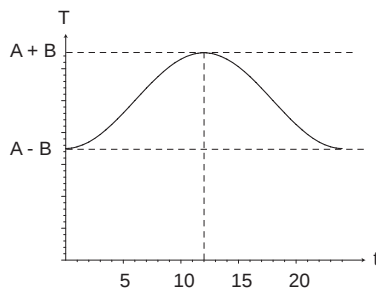


Figure 48: Figure for Problem 27(a)

(b) First write the differential equation in the linear form $\frac{dT}{dt} + k_1 T = k_1(A - B \cos \omega t) + T_0$. Multiplying by the integrating factor $I(t) = e^{k_1 t}$ reduces this differential equation to the integrable form

$$\frac{d}{dt}(e^{k_1 t} T) = k_1 e^{k_1 t} (A - B \cos \omega t) + T_0 e^{k_1 t}.$$

Consequently,

$$e^{k_1 t} T(t) = A e^{k_1 t} - B k_1 \int e^{k_1 t} \cos \omega t dt + \frac{T_0}{k_1} e^{k_1 t} + c,$$

so that

$$T(t) = A + \frac{T_0}{k_1} - \frac{B k_1}{k_1^2 + \omega^2} (k_1 \cos \omega t + \omega \sin \omega t) + c e^{-k_1 t}.$$

This can be written in the equivalent form

$$T(t) = A + \frac{T_0}{k_1} - \frac{B k_1}{\sqrt{k_1^2 + \omega^2}} \cos(\omega t - \alpha) + c e^{-k_1 t}$$

for an appropriate phase constant α .

28. (a) We solve $\frac{dy}{dx} + p(x)y = 0$ via separation of variables:

$$\frac{dy}{y} = -p(x)dx \implies \int \frac{dy}{y} = - \int p(x)dx \implies \ln |y| = - \int p(x)dx \implies y_H = c_1 e^{- \int p(x)dx}.$$

(b) Replace c_1 in part (a) by $u(x)$ and let $v = e^{- \int p(x)dx}$. In order for $y = uv$ to be a solution to (1.6.15), we must have $\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$. Substituting this last result into the original differential equation, $\frac{dy}{dx} + p(x)y = q(x)$, we obtain $u \frac{dv}{dx} + v \frac{du}{dx} + p(x)y = q(x)$. But since $\frac{dv}{dx} = -vp$, the last equation reduces to $v \frac{du}{dx} = q(x)$. That is, $\frac{du}{dx} = v^{-1}(x)q(x)$. Integrating, we obtain $u(x) = \int v^{-1}(x)q(x)dx$. Substituting the values for u and v into $y = uv$, we obtain the general solution

$$y = e^{- \int p(x)dx} \left[\int \frac{q(x)}{e^{- \int p(x)dx}} dx \right].$$

29. The associated homogeneous equation is $\frac{dy}{dx} + x^{-1}y = 0$, with solution $y_H = cx^{-1}$ (see Problem 28(a)). According to Problem 28 we determine the function $u(x)$ such that $y(x) = x^{-1}u(x)$ is a solution to the given differential equation. We have $\frac{dy}{dx} = x^{-1}\frac{du}{dx} - x^{-2}u$. Substituting into $\frac{dy}{dx} + x^{-1}y = \cos x$ yields $x^{-1}\frac{du}{dx} - \frac{1}{x^2}u + x^{-1}(x^{-1}u) = \cos x$, so that $\frac{du}{dx} = x \cos x$. Integrating we obtain $u = x \sin x + \cos x + c$, so that $y(x) = x^{-1}(x \sin x + \cos x + c)$.

30. The associated homogeneous equation is $\frac{dy}{dx} + y = 0$, with solution $y_H = ce^{-x}$ (see Problem 28(a)). According to Problem 28 we determine the function $u(x)$ such that $y(x) = e^{-x}u(x)$ is a solution to the given differential equation. We have $\frac{dy}{dx} = \frac{du}{dx}e^{-x} - e^{-x}u$. Substituting into $\frac{dy}{dx} + y = e^{-2x}$ yields $\frac{du}{dx}e^{-x} - e^{-x}u + e^{-x}u(x) = e^{-2x}$, so that $\frac{du}{dx} = e^{-x}$. Integrating we obtain $u = -e^{-x} + c$, so that $y(x) = e^{-x}(-e^{-x} + c)$.

31. The associated homogeneous equation is $\frac{dy}{dx} + \cot x \cdot y = 0$, with solution $y_H = c \csc x$ (see Problem 28(a)). According to Problem 28 we determine the function $u(x)$ such that $y(x) = \csc x \cdot u(x)$ is a solution to the given differential equation. We have $\frac{dy}{dx} = \csc x \cdot \frac{du}{dx} - \csc x \cdot \cot x \cdot u$. Substituting into $\frac{dy}{dx} + \cot x \cdot y = 2 \cos x$ yields $\csc x \cdot \frac{du}{dx} - \csc x \cdot \cot x \cdot u + \csc x \cdot \cot x \cdot u = \cos x$, so that $\frac{du}{dx} = 2 \cos x \sin x$. Integrating we obtain $u = \sin^2 x + c$, so that $y(x) = \csc x(\sin^2 x + c)$.

32. The associated homogeneous equation is $\frac{dy}{dx} - \frac{1}{x}y = 0$, with solution $y_H = cx$ (see Problem 28(a)). We determine the function $u(x)$ such that $y(x) = xu(x)$ is a solution of the given differential equation. We have $\frac{dy}{dx} = x \frac{du}{dx} + u$. Substituting into $\frac{dy}{dx} - \frac{1}{x}y = x \ln x$ and simplifying yields $\frac{du}{dx} = \ln x$, so that $u = x \ln x - x + c$. Consequently, $y(x) = x(x \ln x - x + c)$.

Problems 33-38 are easily solved using a differential equation solver such as the *dsolve* package in Maple.

Solutions to Section 1.7

True-False Review:

- 1. TRUE.** Concentration of chemical is defined as the ratio of mass to volume; that is, $c(t) = \frac{A(t)}{V(t)}$. Therefore, $A(t) = c(t)V(t)$.
- 2. FALSE.** The rate of change of volume is “rate in” – “rate out”, which is $r_1 - r_2$, not $r_2 - r_1$.
- 3. TRUE.** This is reflected in the fact that c_1 is always assumed to be a constant.
- 4. FALSE.** The concentration of chemical leaving the tank is $c_2(t) = \frac{A(t)}{V(t)}$, and since both $A(t)$ and $V(t)$ can be nonconstant, $c_2(t)$ can also be nonconstant.
- 5. FALSE.** Kirchhoff’s second law states that the sum of the voltage drops around a closed circuit is *zero*, not that it is independent of time.
- 6. TRUE.** This is essentially Ohm’s law, (1.7.10).

7. TRUE. Due to the negative exponential in the formula for the transient current, $i_T(t)$, it decays to zero as $t \rightarrow \infty$. Meanwhile, the steady-state current, $i_S(t)$, oscillates with the same frequency ω as the alternating current, albeit with a phase shift.

8. TRUE. The amplitude is given in (1.7.19) as $A = \frac{E_0}{\sqrt{R^2 + \omega^2 L^2}}$, and so as ω gets larger, the amplitude A gets smaller.

Problems:

1. We have $\Delta V = r_1 \Delta t - r_2 \Delta t = \Delta t$. Therefore,

$$\frac{dV}{dt} = 1 \implies V(t) = t + 10,$$

where we have used the initial condition $V(0) = 10$. Now, $\Delta A \approx c_1 r_1 \Delta t - c_2 r_2 \Delta t$, so

$$\frac{dA}{dt} = 8 - c_2 = 8 - \frac{A}{V} = 8 - \frac{A}{t+10} \implies \frac{dA}{dt} + \frac{1}{t+10} A = 8.$$

This differential equation is first-order linear, with integrating factor $I(t) = e^{\int \frac{1}{t+10} dt} = t + 10$. Therefore, $(t+10)A = 4(t+10)^2 + c_1$. Since $A(0) = 20$, we have $c_1 = -200$. We conclude that

$$A(t) = \frac{4}{t+10}[(t+10)^2 - 50].$$

Therefore, after 40 minutes, we have $A(40) = 196$ g.

2. We need to find $\frac{A(60)}{V(60)}$, where t is measured in minutes. We have $\Delta V = r_1 \Delta t - r_2 \Delta t = 3 \Delta t$. Therefore,

$$\frac{dV}{dt} = 3 \implies V(t) = 3(t+200),$$

where we have used $V(0) = 600$. Now, $\Delta A \approx c_1 r_1 \Delta t - c_2 r_2 \Delta t$, so

$$\frac{dA}{dt} = 30 - 3c_2 = 30 - 3\frac{A}{V} = 30 - \frac{A}{t+200}.$$

This differential equation is first-order linear, with integrating factor $I(t) = e^{\int \frac{1}{t+200} dt} = t + 200$. Hence, $(t+200)A = 15(t+200)^2 + c$. Since $A(0) = 1500$, we have $c = -300000$. We conclude that

$$A(t) = \frac{15}{t+200}[(t+200)^2 - 20000].$$

Thus $\frac{A(60)}{V(60)} = \frac{596}{169}$ g/L.

3. We have $\Delta V = r_1 \Delta t - r_2 \Delta t = 2 \Delta t$. Therefore,

$$\frac{dV}{dt} = 2 \implies V(t) = 2(t+10),$$

where we have used $V(0) = 20$. Thus $V(t) = 40$ for $t = 10$, so we must find $A(10)$. Now, $\Delta A \approx c_1 r_1 \Delta t - c_2 r_2 \Delta t$, so

$$\begin{aligned}\frac{dA}{dt} &= 40 - 2c_2 = 40 - \frac{2A}{V} = 40 - \frac{A}{t+10} \implies \frac{dA}{dt} + \frac{1}{t+10}A = 40 \\ &\implies \frac{d}{dt}[(t+10)A] = 40(t+10)dt \\ &\implies (t+10)A = 20(t+10)^2 + c.\end{aligned}$$

Since $A(0) = 0$, we have $c = -2000$. Thus, we conclude that

$$A(t) = \frac{20}{t+10}[(t+10)^2 - 100] \quad \text{and} \quad A(10) = 300 \text{ g.}$$

4. We have $\Delta V = r_1 \Delta t - r_2 \Delta t = 2 \Delta t$. Therefore,

$$\frac{dV}{dt} = 2 \implies V(t) = 2(t+50),$$

where we have used $V(0) = 100$. Now, $\Delta A \approx c_1 r_1 \Delta t - c_2 r_2 \Delta t$, so

$$\frac{dA}{dt} = 3 - 4c_2 = 3 - \frac{4A}{V} = 3 - \frac{2A}{t+50}.$$

Hence,

$$\begin{aligned}\frac{dA}{dt} + \frac{4A}{2(t+50)} &= 3 \implies \frac{dA}{dt} + \frac{2A}{t+50} = 3 \\ &\implies \frac{d}{dt}[(t+50)^2 A] = 3(t+50)^2 \\ &\implies (t+50)^2 A = (t+50)^3 + c.\end{aligned}$$

Since $A(0) = 100$, we have $c = 125000$, and therefore,

$$A(t) = t + 50 + \frac{125000}{(t+50)^2}.$$

The tank is full when $V(t) = 200$, that is, when $2(t+50) = 200$ so that $t = 50$ min. Therefore the concentration just before the tank overflows is: $\frac{A(50)}{V(50)} = \frac{9}{16}$ g/L.

5.

(a) We have $\Delta V = r_1 \Delta t - r_2 \Delta t$. Therefore,

$$\frac{dV}{dt} = 1 \implies V(t) = t + 10,$$

where we have used $V(0) = 10$. Now, $\Delta A \approx c_1 r_1 \Delta t - c_2 r_2 \Delta t$, so that

$$\frac{dA}{dt} = -2c_2 = -2\frac{A}{V} = -\frac{2A}{t+10}.$$

We solve this by separation of variables:

$$\frac{dA}{A} = -\frac{2dt}{t+10} \implies \ln|A| = -2\ln|t+10| + c \implies A(t) = k(t+10)^{-2},$$

where k is a constant. Since $V(5) = 15$, we have $A(5) = 3$ since $\frac{A(5)}{V(5)} = 0.2$. Thus, $k = 675$ and $A(t) = \frac{675}{(t+10)^2}$. In particular, $A(0) = 6.75$ g.

(b) From part (a), $A(t) = \frac{675}{(t+10)^2}$ and $V(t) = t+10$. Therefore,

$$\frac{A(t)}{V(t)} = \frac{675}{(t+10)^3}.$$

Setting $\frac{A(t)}{V(t)} = 0.1$, we have $(t+10)^3 = 6750 \implies t+10 = 15\sqrt[3]{2}$ so $V(t) = t+10 = 15\sqrt[3]{2}$ L.

6. (a) We have $\Delta V = r_1 \Delta t - r_2 \Delta t = \Delta t$. Therefore,

$$\frac{dV}{dt} = 1 \implies V = t + 20,$$

where we have used $V(0) = 20$. Now, $\Delta A \approx c_1 r_1 \Delta t - c_2 r_2 \Delta t$, so that

$$\frac{dA}{dt} = 3 - 2c_2 = 3 - \frac{2A}{V} = 3 - \frac{2A}{t+20}.$$

Hence,

$$\frac{dA}{dt} + \frac{2}{t+20}A = 3 \implies \frac{d}{dt}[(t+20)^2 A] = 3(t+20)^2 \implies A(t) = \frac{(20+t)^3 + c}{(t+20)^2}.$$

Since $A(0) = 0$, we have $c = -20^3$ which means that

$$A(t) = \frac{(t+20)^3 - 20^3}{(t+20)^2}.$$

(b) The concentration of chemical in the tank, c_2 , is given by $c_2 = \frac{A(t)}{V(t)}$ or $c_2 = \frac{A(t)}{t+20}$ so from part (a), $c_2 = \frac{(t+20)^3 - 20^3}{(t+20)^3}$. Therefore $c_2 = \frac{1}{2}$ g/L when $\frac{1}{2} = \frac{(t+20)^3 - 20^3}{(t+20)^3}$, and we conclude that $t = 20(\sqrt[3]{2} - 1)$ minutes.

7. (a) We have $\Delta V = r_1 \Delta t - r_2 \Delta t = 0$, since $r_1 = r_2 = r$. Therefore,

$$\frac{dV}{dt} = 0 \implies V(t) = V(0) = w$$

for all t . Now, $\Delta A = c_1 r_1 \Delta t - c_2 r_2 \Delta t$, so that

$$\frac{dA}{dt} = kr - r \frac{A}{V} = kr - \frac{r}{w}A,$$

so we have the first-order linear differential equation

$$\frac{dA}{dt} + \frac{r}{w}A = kr \implies \frac{d}{dt}(e^{rt/w}A) = kre^{rt/w} \implies A(t) = kw + ce^{-rt/w}.$$

Since $A(0) = A_0$, we have $c = A_0 - kw$. Therefore,

$$A(t) = e^{-rt/w} [kw(e^{rt/w} - 1) + A_0].$$

(b) We have

$$\lim_{t \rightarrow \infty} \frac{A(t)}{V(t)} = \lim_{t \rightarrow \infty} \frac{e^{-rt/w}}{w} [kw(e^{rt/w} - 1) + A_0] = \lim_{t \rightarrow \infty} \left[k + \left(\frac{A_0}{w} - k \right) e^{-rt/w} \right] = k.$$

This is reasonable since the volume remains constant, and the solution in the tank is gradually mixed with and replaced by the solution of concentration k flowing in.

8. (a) For the top tank we have:

$$\frac{dA_1}{dt} = c_1 r_1 - c_2 r_2 \implies \frac{dA_1}{dt} = c_1 r_1 - r_2 \frac{A_1}{V_1} \implies \frac{dA_1}{dt} = c_1 r_1 - \frac{r_2}{(r_1 - r_2)t + V_1} A_1(t).$$

Rearranging this equation, we obtain

$$\frac{dA_1}{dt} + \frac{r_2}{(r_1 - r_2)t + V_1} A_1 = c_1 r_1.$$

For the bottom tank we have:

$$\begin{aligned} \frac{dA_2}{dt} &= c_2 r_2 - c_3 r_3 \implies \frac{dA_2}{dt} = r_2 \frac{A_1}{(r_1 - r_2)t + V_1} - r_3 \frac{A_2(t)}{V_2(t)} \\ &\implies \frac{dA_2}{dt} = r_2 \frac{A_1}{(r_1 - r_2)t + V_1} - r_3 \frac{A_2(t)}{(r_2 - r_3)t + V_2} \\ &\implies \frac{dA_2}{dt} + \frac{r_3}{(r_1 - r_2)t + V_2} A_2 = \frac{r_2 A_1}{(r_1 - r_2)t + V_1}. \end{aligned}$$

(b) From part (a),

$$\begin{aligned} \frac{dA_1}{dt} + \frac{r_2}{(r_1 - r_2)t + V_1} A_1 &= c_1 r_1 \implies \frac{dA_1}{dt} + \frac{4}{2t + 40} A_1 = 3 \\ &\implies \frac{dA_1}{dt} + \frac{2}{t + 20} A_1 = 3 \\ &\implies \frac{d}{dt} [(t + 20)^2 A_1] = 3(t + 20)^2 \\ &\implies A_1 = t + 20 + \frac{c}{(t + 20)^2}. \end{aligned}$$

Since $A_1(0) = 4$, we have $c = -6400$. Consequently,

$$A_1(t) = t + 20 - \frac{6400}{(t + 20)^2}.$$

Next,

$$\begin{aligned} \frac{dA_2}{dt} + \frac{3}{t + 20} A_2 &= \frac{2}{t + 20} \left[t + 20 - \frac{6400}{(t + 20)^2} \right] \implies \frac{dA_2}{dt} + \frac{3}{t + 20} A_2 = \frac{2[(t + 20)^3 - 6400]}{(t + 20)^3} \\ &\implies \frac{d}{dt} [(t + 20)^3 A_2] = (t + 20)^3 \left\{ \frac{2[(t + 20)^3 - 6400]}{(t + 20)^3} \right\} \\ &\implies A_2(t) = \frac{t + 20}{2} - \frac{12800t}{(t + 20)^3} + \frac{k}{(t + 20)^3}. \end{aligned}$$

Since $A_2(0) = 20$, we find that $k = 80000$. Thus

$$A_2(t) = \frac{t+20}{2} - \frac{12800t}{(t+20)^3} + \frac{80000}{(t+20)^3},$$

and in particular, $A_2(10) = \frac{119}{9} \approx 13.2$ g.

9. Plugging the given values into the differential equation $\frac{di}{dt} + \frac{R}{L}i = \frac{1}{L}E(t)$ yields the first-order linear differential equation $\frac{di}{dt} + 40i = 200$, which has integrating factor $I(t) = e^{40t}$. Therefore,

$$\frac{d}{dt}(e^{40t}i) = 200e^{40t} \implies i(t) = 5 + ce^{-40t}.$$

Since $i(0) = 0$, we have $c = -5$. Consequently, $i(t) = 5(1 - e^{-40t})$.

10. Plugging the given values into the differential equation $\frac{dq}{dt} + \frac{1}{RC}q = \frac{E}{R}$ yields the first-order linear differential equation $\frac{dq}{dt} + 10q = 20$, which has integrating factor $I(t) = e^{10t}$. Therefore,

$$\frac{d}{dt}(qe^{10t}) = 20e^{10t} \implies q(t) = 2 + ce^{-10t}.$$

Since $q(0) = 0$, we have $c = -2$. Consequently, $q(t) = 2(1 - e^{-10t})$.

11. Plugging the given values into the differential equation $\frac{di}{dt} + \frac{R}{L}i = \frac{1}{L}E(t)$ yields the first-order linear differential equation $\frac{di}{dt} + 3i = 15 \sin 4t$, which has integrating factor $I(t) = e^{3t}$. Therefore,

$$\frac{d}{dt}(e^{3t}i) = 15e^{3t} \sin 4t \implies e^{3t}i = \frac{3e^{3t}}{5}(3 \sin 4t - 4 \cos 4t) + c \implies i(t) = 3\left(\frac{3}{5} \sin 4t - \frac{4}{5} \cos 4t\right) + ce^{-3t}.$$

Since $i(0) = 0$, we find that $c = \frac{12}{5}$. Therefore, $i(t) = \frac{3}{5}(3 \sin 4t - 4 \cos 4t + 4e^{-3t})$.

12. Plugging the given values into the differential equation $\frac{dq}{dt} + \frac{1}{RC}q = \frac{E}{R}$ yields the first-order linear differential equation $\frac{dq}{dt} + 4q = 5 \cos 3t$, which has integrating factor $I(t) = e^{4t}$. Therefore,

$$\frac{d}{dt}(e^{4t}q) = 5e^{4t} \cos 3t \implies e^{4t}q = \frac{e^{4t}}{5}(4 \cos 3t + 3 \sin 3t) + c \implies q(t) = \frac{1}{5}(4 \cos 3t + 3 \sin 3t) + ce^{-4t}.$$

Since $q(0) = 1$, we find that $c = \frac{1}{5}$. Thus,

$$q(t) = \frac{1}{5}(4 \cos 3t + 3 \sin 3t + e^{-4t}) \quad \text{and} \quad i(t) = \frac{dq}{dt} = \frac{1}{5}(9 \cos 3t - 12 \sin 3t - 4e^{-4t}).$$

13. In an RC circuit for $t > 0$ the differential equation is given by $\frac{dq}{dt} + \frac{1}{RC}q = \frac{E}{R}$. If $E(t) = 0$ then

$$\frac{dq}{dt} + \frac{1}{RC}q = 0 \implies \frac{d}{dt}(e^{t/RC}q) = 0 \implies q = ce^{t/RC},$$

and if $q(0) = 5$, then $q(t) = 5e^{-t/RC}$. Then $\lim_{t \rightarrow \infty} q(t) = 0$. Yes, this is reasonable. As the time increases and $E(t) = 0$, the charge will dissipate to zero.

14. The differential equation governing the charge $q(t)$ in this case is given by $i + \frac{1}{RC}q = \frac{E_0}{R}$. Differentiating this equation with respect to t , we obtain $\frac{di}{dt} + \frac{1}{RC}\frac{dq}{dt} = 0$. Since $\frac{dq}{dt} = i$,

$$\frac{di}{dt} + \frac{1}{RC}i = 0 \implies \frac{d}{dt}(e^{t/RC}i) = 0 \implies i(t) = ce^{-t/RC}.$$

Since $q(0) = 0$, we see that $i(0) = \frac{E_0}{R}$. Therefore,

$$i(t) = \frac{E_0}{R}e^{-t/RC}.$$

Integrating, we find that $q(t) = \frac{E_0}{R}(-RCe^{-t/RC} + k)$, for some constant k . Using $q(0) = 0$, we obtain $0 = \frac{E_0}{R}(k - RC)$, so that $k = RC$. Hence,

$$q(t) = E_0C(1 - e^{-t/RC}).$$

Then $\lim_{t \rightarrow \infty} q(t) = E_0C$, and $\lim_{t \rightarrow \infty} i(t) = 0$.

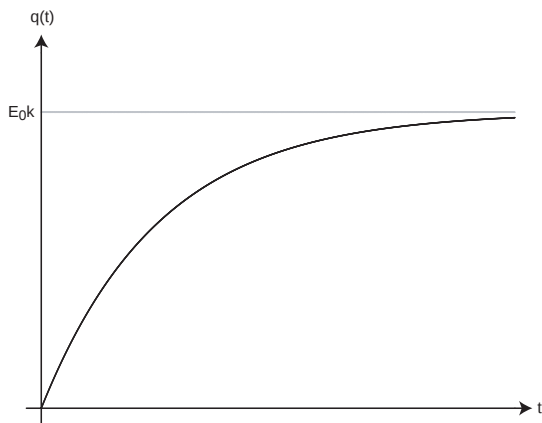


Figure 49: Figure for Problem 14

15. The differential equation governing the current in an RL circuit, $\frac{di}{dt} + \frac{R}{L}i = \frac{E(t)}{L}$, becomes $\frac{di}{dt} + \frac{R}{L}i = \frac{E_0}{L}\sin\omega t$, since $E(t) = E_0\sin\omega t$. An integrating factor here is $I(t) = e^{Rt/L}$, so that

$$\frac{d}{dt}(e^{Rt/L}i) = \frac{E_0}{L}e^{Rt/L}\sin\omega t \implies i(t) = \frac{E_0}{R^2 + L^2\omega^2}[R\sin\omega t - \omega L\cos\omega t] + Ae^{-Rt/L}.$$

We can write this as

$$i(t) = \frac{E_0}{\sqrt{R^2 + L^2\omega^2}} \left[\frac{R}{\sqrt{R^2 + L^2\omega^2}} \sin \omega t - \frac{\omega L}{\sqrt{R^2 + L^2\omega^2}} \cos \omega t \right] + Ae^{-Rt/L}.$$

Defining the phase ϕ by

$$\cos \phi = \frac{R}{\sqrt{R^2 + L^2\omega^2}} \quad \text{and} \quad \sin \phi = \frac{\omega L}{\sqrt{R^2 + L^2\omega^2}},$$

we have

$$i(t) = \frac{E_0}{\sqrt{R^2 + L^2\omega^2}} [\cos \phi \sin \omega t - \sin \phi \cos \omega t] + Ae^{-Rt/L}.$$

That is,

$$i(t) = \frac{E_0}{\sqrt{R^2 + L^2\omega^2}} \sin(\omega t - \phi) + Ae^{-Rt/L}.$$

Transient part of the solution: $i_T(t) = Ae^{-Rt/L}$.

Steady state part of the solution: $i_S(t) = \frac{E_0}{\sqrt{R^2 + L^2\omega^2}} \sin(\omega t - \phi)$.

16. We must solve the initial value problem $\frac{di}{dt} + ai = \frac{E_0}{L}$, $i(0) = 0$, where $a = \frac{R}{L}$, and E_0 denotes the constant EMF. An integrating factor for the differential equation is $I = e^{at}$, so that the differential equation can be written in the form $\frac{d}{dt}(e^{at}i) = \frac{E_0}{L}e^{at}$. Integrating yields $i(t) = \frac{E_0}{aL} + c_1e^{-at}$. The given initial condition requires $c_1 + \frac{E_0}{aL} = 0$, so that $c_1 = -\frac{E_0}{aL}$. Hence $i(t) = \frac{E_0}{aL}(1 - e^{-at}) = \frac{E_0}{R}(1 - e^{-at})$.

17. Substituting $E(t) = E_0e^{-at}$ into the differential equation $\frac{dq}{dt} + \frac{1}{RC}q = \frac{E(t)}{R}$, we obtain $\frac{dq}{dt} + \frac{1}{RC}q = \frac{E_0}{R}e^{-at}$, with integrating factor $I(t) = e^{t/RC}$. Thus,

$$\frac{d}{dt}(e^{t/RC}q) = \frac{E_0}{R}e^{(1/RC-a)t} \implies q(t) = e^{-t/RC} \left[\frac{E_0C}{1-aRC}e^{(1/RC-a)t} + k \right] \implies q(t) = \frac{E_0C}{1-aRC}e^{-at} + ke^{-t/RC}.$$

Imposing the initial condition $q(0) = 0$ (capacitor initially uncharged) requires $k = -\frac{E_0C}{1-aRC}$, so that

$$q(t) = \frac{E_0C}{1-aRC}(e^{-at} - e^{-t/RC}) \quad \text{and} \quad i(t) = \frac{dq}{dt} = \frac{E_0C}{1-aRC} \left(\frac{1}{RC}e^{-t/RC} - ae^{-at} \right).$$

18. From the given second-order differential equation, we use $i(t) = \frac{dq}{dt}$ and $\frac{di}{dt} = \frac{di}{dq} \cdot \frac{dq}{dt} = i \frac{di}{dq}$ to obtain $i \frac{di}{dq} + \frac{1}{LC}q = 0$. Then

$$i di = -\frac{1}{LC}q dq \implies i^2 = -\frac{1}{LC}q^2 + k,$$

for some constant k . Since $q(0) = q_0$ and $i(0) = 0$, we have $k = \frac{q_0^2}{LC}$. Thus,

$$\begin{aligned}
 i^2 &= -\frac{1}{LC}q^2 + \frac{q_0^2}{LC} \implies i = \pm \frac{\sqrt{q_0^2 - q^2}}{\sqrt{LC}} \\
 &\implies \frac{dq}{dt} = \pm \frac{\sqrt{q_0^2 - q^2}}{\sqrt{LC}} \\
 &\implies \frac{dq}{\sqrt{q_0^2 - q^2}} = \pm \frac{dt}{\sqrt{LC}} \\
 &\implies \sin^{-1}\left(\frac{q}{q_0}\right) = \pm \frac{t}{\sqrt{LC}} + k_1 \\
 &\implies q = q_0 \sin\left(\pm \frac{t}{\sqrt{LC}} + k_1\right).
 \end{aligned}$$

Since $q(0) = q_0$, we have $q_0 = q_0 \sin k_1$, so that $k_1 = \frac{\pi}{2} + 2n\pi$, where n is an integer. Hence,

$$q(t) = q_0 \sin\left(\pm \frac{t}{\sqrt{LC}} + \frac{\pi}{2}\right) = q_0 \cos\left(\frac{t}{\sqrt{LC}}\right)$$

and

$$i(t) = \frac{dq}{dt} = -\frac{q_0}{\sqrt{LC}} \sin\left(\frac{t}{\sqrt{LC}}\right).$$

19. In this case, we have the second-order differential equation $\frac{d^2q}{dt^2} + \frac{1}{LC}q = \frac{E_0}{L}$. Since $i = \frac{dq}{dt}$, then

$$\frac{d^2q}{dt^2} = \frac{di}{dt} = \frac{di}{dq} \frac{dq}{dt} = i \frac{di}{dq}.$$

Hence the original differential equation can be written as

$$i \frac{di}{dq} + \frac{1}{LC}q = \frac{E_0}{L} \implies i di + \frac{1}{LC}q dq = \frac{E_0}{L} dq \implies \frac{i^2}{2} + \frac{q^2}{2LC} = \frac{E_0 q}{L} + A,$$

for some constant A . Since $i(0) = 0$ and $q(0) = q_0$, we find that $A = \frac{q_0^2}{2LC} - \frac{E_0 q_0}{L}$. From $\frac{i^2}{2} + \frac{q^2}{2LC} = \frac{E_0 q}{L} + A$ we get that

$$i = \left[2A + \frac{2E_0 q}{L} - \frac{q^2}{LC}\right]^{1/2} \implies i = \left[2A + \frac{(2E_0 C)^2}{LC} - \frac{(q - E_0 C)^2}{LC}\right]^{1/2}.$$

We let $D^2 = 2A + \frac{(E_0 C)^2}{LC}$, so that

$$i \frac{dq}{dt} = D \left[1 - \left(\frac{q - E_0 C}{D\sqrt{LC}}\right)^2\right]^{1/2} \implies \sqrt{LC} \sin^{-1}\left(\frac{q - E_0 C}{D\sqrt{LC}}\right) = t + B.$$

Then since $q(0) = 0$, we have $B = \sqrt{LC} \sin^{-1}\left(\frac{q - E_0 C}{D\sqrt{LC}}\right)$ and therefore

$$\frac{q - E_0 C}{D\sqrt{LC}} = \sin\left(\frac{t + B}{\sqrt{LC}}\right),$$

which implies that

$$q(t) = D\sqrt{LC} \sin\left(\frac{t+B}{\sqrt{LC}}\right) + E_0c \implies i = \frac{dq}{dt} = D \cos\left(\frac{t+B}{\sqrt{LC}}\right).$$

Since $D^2 = \frac{2A + (E_0C)^2}{LC}$ and $A = \frac{q_0^2}{2LC} - \frac{E_0q_0}{L}$, we can substitute to eliminate A and obtain $D = \pm \frac{|q_0 - E_0C|}{\sqrt{LC}}$. Thus

$$q(t) = \pm |q_0 - E_0C| \sin\left(\frac{t+B}{\sqrt{LC}}\right) + E_0c.$$

Solutions to Section 1.8

True-False Review:

1. TRUE. We have

$$f(tx, ty) = \frac{2(tx)(ty) - (tx)^2}{2(tx)(ty) + (ty)^2} = \frac{2xyt^2 - x^2t^2}{2xyt^2 + y^2t^2} = \frac{2xy - x^2}{2xy + y^2} = f(x, y),$$

so f is homogeneous of degree zero.

2. FALSE. We have

$$f(tx, ty) = \frac{(ty)^2}{(tx) + (ty)^2} = \frac{y^2t^2}{xt + y^2t^2} = \frac{y^2t}{x + y^2t} \neq \frac{y^2}{x + y^2},$$

so f is not homogeneous of degree zero.

3. FALSE. Setting $f(x, y) = \frac{1+xy^2}{1+x^2y}$, we have

$$f(tx, ty) = \frac{1 + (tx)(ty)^2}{1 + (tx)^2(ty)} = \frac{1 + xy^2t^3}{1 + x^2yt^3} \neq f(x, y),$$

so f is not homogeneous of degree zero. Therefore, the differential equation is not homogeneous.

4. TRUE. Setting $f(x, y) = \frac{x^2y^2}{x^4+y^4}$, we have

$$f(tx, ty) = \frac{(tx)^2(ty)^2}{(tx)^4 + (ty)^4} = \frac{x^2y^2t^4}{x^4t^4 + y^4t^4} = \frac{x^2y^2}{x^4 + y^4} = f(x, y).$$

Therefore, f is homogeneous of degree zero, and therefore, the differential equation is homogeneous.

5. TRUE. This is verified in the calculation leading to Theorem 1.8.5.

6. TRUE. This is verified in the calculation leading to (1.8.12).

7. TRUE. We can rewrite the equation as

$$y' - \sqrt{xy} = \sqrt{xy}^{1/2},$$

which is the proper form for a Bernoulli equation, with $p(x) = -\sqrt{x}$, $q(x) = \sqrt{x}$, and $n = 1/2$.

8. FALSE. The presence of an exponential e^{xy} involving y prohibits this equation from having the proper form for a Bernoulli equation.

9. TRUE. This is a Bernoulli equation with $p(x) = x$, $q(x) = x^2$, and $n = 2/3$.

Unless otherwise indicated in this section $V = \frac{y}{x}$, $\frac{dy}{dx} = V + x \frac{dV}{dx}$, and $t > 0$.

Problems:

1. We have $f(tx, ty) = \frac{(tx)^2 - (ty)^2}{(tx)(ty)} = \frac{x^2 - y^2}{xy} = f(x, y)$. Thus f is homogeneous of degree zero. We write

$$f(x, y) = \frac{x^2 - y^2}{xy} = \frac{1 - \left(\frac{y}{x}\right)^2}{\frac{y}{x}} = \frac{1 - V^2}{V} = F(V).$$

2. We have $f(tx, ty) = (tx) - (ty) = t(x - y) = tf(x, y)$. Thus f is homogeneous of degree one.

3. We have $f(tx, ty) = \frac{(tx) \sin\left(\frac{tx}{ty}\right) - (ty) \cos\left(\frac{ty}{tx}\right)}{\frac{y}{x}} = \frac{x \sin \frac{x}{y} - y \cos \frac{y}{x}}{y} = f(x, y)$. Thus f is homogeneous of degree zero. We write

$$f(x, y) = \frac{\sin \frac{x}{y} - \frac{y}{x} \cos \frac{y}{x}}{\frac{y}{x}} = \frac{\sin \frac{1}{V} - V \cos V}{V} = F(V).$$

4. We have $f(tx, ty) = \frac{\sqrt{(tx)^2 + (ty)^2}}{tx - ty} = \frac{|t|\sqrt{x^2 + y^2}}{t(x - y)} = \frac{\sqrt{x^2 + y^2}}{x - y} = f(x, y)$. Thus f is homogeneous of degree zero. We write

$$f(x, y) = \frac{\sqrt{1 + \left(\frac{y}{x}\right)^2}}{1 - \frac{y}{x}} = \frac{\sqrt{1 + V^2}}{1 - V} = F(V).$$

5. We have $f(tx, ty) = \frac{ty}{tx - 1}$. Thus f is not homogeneous.

6. We have $f(tx, ty) = \frac{tx - 3}{ty} + \frac{5(ty) + 9}{3(ty)} = \frac{3(tx) + 5(ty)}{3(ty)} = \frac{3x + 5y}{3y} = f(x, y)$. Thus f is homogeneous of degree zero. We write

$$f(x, y) = \frac{3x + 5y}{3y} = \frac{3 + 5\frac{y}{x}}{3\frac{y}{x}} = \frac{3 + 5V}{3V} = F(V).$$

7. We have $f(tx, ty) = \frac{\sqrt{(tx)^2 + (ty)^2}}{tx} = \frac{\sqrt{x^2 + y^2}}{x} = f(x, y)$. Thus f is homogeneous of degree zero. We write

$$f(x, y) = \frac{\sqrt{x^2 + y^2}}{x} = \frac{|x|\sqrt{1 + \left(\frac{y}{x}\right)^2}}{x} = -\sqrt{1 + \left(\frac{y}{x}\right)^2} = -\sqrt{1 + V^2} = F(V).$$

8. We have $f(tx, ty) = \frac{\sqrt{(tx)^2 + 4(ty)^2} - (tx) + (ty)}{(tx) + 3(ty)} = \frac{\sqrt{x^2 + 4y^2} - x + y}{x + 3y} = f(x, y)$. Thus f is homoge-

neous of degree zero. We write

$$f(x, y) = \frac{\sqrt{x^2 + 4y^2} - x + y}{x + 3y} = \frac{\sqrt{1 + 4(\frac{y}{x})^2} - 1 + \frac{y}{x}}{1 + 3\frac{y}{x}} = \frac{\sqrt{1 + 4V^2} - 1 + V}{1 + 3V} = F(V).$$

9. We divide both sides of the given differential equation through by x to obtain $(3 - \frac{2y}{x})\frac{dy}{dx} = \frac{3y}{x}$. Substituting $V = y/x$ and replacing $\frac{dy}{dx}$ with $\frac{d}{dx}(xV) = V + x\frac{dV}{dx}$, we obtain $(3 - 2V)\left(V + x\frac{dV}{dx}\right) = 3V$. Thus, $x\frac{dV}{dx} = \frac{3V}{3 - 2V} - V$. A short algebraic manipulation allows us to separate variables as follows: $\frac{3 - 2V}{2V^2}dV = \frac{dx}{x}$. Integrating on each side, we obtain $-\frac{3}{2V} - \ln|V| = \ln|x| + c_1$. Finally, replacing V with y/x , we conclude that

$$-\frac{3x}{2y} - \ln\left|\frac{y}{x}\right| = \ln|x| + c_1 \implies \ln y = -\frac{3x}{2y} + c_2 \implies y^2 = ce^{-3x/y}.$$

10. We divide the given differential equation through by x^2 on the top and bottom of the right-hand side to obtain $\frac{dy}{dx} = \frac{1}{2}\left(1 + \frac{y}{x}\right)^2$. Substituting $V = y/x$ and replacing $\frac{dy}{dx}$ with $\frac{d}{dx}(xV) = V + x\frac{dV}{dx}$, we obtain $V + x\frac{dV}{dx} = \frac{1}{2}(1 + V)^2$. Using a short algebraic manipulation to separate the variables, we find that $\frac{1}{V^2 + 1} = \frac{1}{2x}dx$. Integrating on each side, we obtain $\tan^{-1} V = \frac{1}{2}\ln|x| + c$. Finally, replacing V with y/x , we conclude that

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\ln|x| + c.$$

11. We divide both sides of the given differential equation through x to obtain $\sin\left(\frac{y}{x}\right)\left(\frac{dy}{dx} - \frac{y}{x}\right) = \cos\left(\frac{y}{x}\right)$. Substituting $V = y/x$ and replacing $\frac{dy}{dx}$ with $\frac{d}{dx}(xV) = V + x\frac{dV}{dx}$, we obtain $\sin V\left(V + x\frac{dV}{dx} - V\right) = \cos V$. Therefore, $\sin V\left(x\frac{dV}{dx}\right) = \cos V$. Separating variables, we find that $\tan V dV = \frac{dx}{x}$. Integrating on each side, we obtain $-\ln|\cos V| = \ln|x| + c_1$, or, upon replacing V with y/x and rearranging, $\ln\left[|x| \cdot \left|\cos\left(\frac{y}{x}\right)\right|\right] = -c_1$. Exponentiation and removal of absolute value yields

$$x \cos\left(\frac{y}{x}\right) = c_2 \implies y(x) = x \cos^{-1}\left(\frac{c}{x}\right).$$

12. We divide both sides of the given differential equation through by x to obtain

$$\frac{dy}{dx} = \frac{\sqrt{16x^2 - y^2} + y}{x} \implies \frac{dy}{dx} = \sqrt{16 - \left(\frac{y}{x}\right)^2} + \frac{y}{x}.$$

Substituting $V = y/x$ and replacing $\frac{dy}{dx}$ with $\frac{d}{dx}(xV) = V + x\frac{dV}{dx}$, we obtain $V + x\frac{dV}{dx} = \sqrt{16 - V^2} + V$,

or $x \frac{dV}{dx} = \sqrt{16 - V^2}$. Separating variables, we have $\frac{dV}{\sqrt{16 - V^2}} = \frac{dx}{x}$. Integrating on each side, we obtain $\sin^{-1}(\frac{V}{4}) = \ln|x| + c$. Finally, replacing V with y/x , we conclude that $\sin^{-1}(\frac{y}{4x}) = \ln|x| + c$.

13. We first rewrite the given differential equation in the equivalent form $y' = \frac{\sqrt{(9x^2 + y^2)} + y}{x}$. Factoring out an x^2 from the square root yields $y' = \frac{|x|\sqrt{9 + (\frac{y}{x})^2} + y}{x}$. Since we are told to solve the differential equation on the interval $x > 0$ we have $|x| = x$, so that $y' = 9 + (\frac{y}{x})^2 + \frac{y}{x}$, which we recognize as being homogeneous. We therefore let $y = xV$, so that $y' = xV' + V$. Substitution into the preceding differential equation yields $xV' + V = \sqrt{9 + V^2} + V$. That is, $xV' = \sqrt{9 + V^2}$. Separating the variables in this equation we obtain $\frac{1}{\sqrt{9 + V^2}} dV = \frac{1}{x} dx$. Integrating we obtain $\ln(V + \sqrt{9 + V^2}) = \ln x + c$. Exponentiating both sides yields $V + \sqrt{9 + V^2} = c_1 x$. Substituting $\frac{y}{x} = V$ and multiplying through by x yields the general solution $y + \sqrt{9x^2 + y^2} = c_1 x^2$.

14. The given differential equation can be written in the equivalent form

$$\frac{dy}{dx} = \frac{y(x^2 - y^2)}{x(x^2 + y^2)},$$

which we recognize as being first-order homogeneous. The substitution $y = xV$ yields

$$V + x \frac{dV}{dx} = \frac{V(1 - V^2)}{1 + V^2} \implies x \frac{dV}{dx} = -\frac{2V^3}{1 + V^2},$$

so that

$$\int \frac{1 + V^2}{V^3} dv = -2 \int \frac{dx}{x} \implies -\frac{V^{-2}}{2} + \ln|V| = -2 \ln|x| + c_1.$$

Consequently,

$$-\frac{x^2}{2y^2} + \ln|xy| = c_1.$$

15. We rearrange the given differential equation to read $\frac{dy}{dx} = \frac{y}{x} \ln\left(\frac{y}{x}\right)$. Substituting $V = y/x$ and replacing $\frac{dy}{dx}$ with $\frac{d}{dx}(xV) = V + x \frac{dV}{dx}$, we obtain $V + x \frac{dV}{dx} = V \ln V$. Separating the variables, we have $\frac{dV}{V(\ln V - 1)} = \frac{dx}{x}$. Integrating on each side, we have $\ln|\ln V - 1| = \ln|x| + c_1$. That is, $\ln\left|\frac{\ln V - 1}{x}\right| = c_1$. Exponentiation on both sides gives $\ln V - 1 = c_2 x$, or $\ln V = 1 + c_2 x$. Exponentiating once more gives $V = e^{1+c_2 x}$. Therefore, $y(x) = x e^{1+c_2 x}$.

16. Substituting $V = y/x$ and replacing $\frac{dy}{dx}$ with $\frac{d}{dx}(xV) = V + x \frac{dV}{dx}$, we obtain

$$V + x \frac{dV}{dx} = \frac{V^2 + 2V - 2}{1 - V + V^2} \implies x \frac{dV}{dx} = \frac{-V^3 + 2V^2 + V - 2}{V^2 - V + 1} \implies \frac{V^2 - V + 1}{V^3 - 2V^2 - V + 2} dV = -\frac{dx}{x}.$$

We can use a partial fractions decomposition on the left-hand side:

$$\frac{V^2 - V + 1}{V^3 - 2V^2 - V + 2} dv = \frac{V^2 - V + 1}{(V - 1)(V + 2)(V + 1)} dv = \left[\frac{1}{V - 2} - \frac{1}{2(V - 1)} + \frac{1}{2(V + 1)} \right] dV.$$

Therefore,

$$\left[\frac{1}{V-2} - \frac{1}{2(V-1)} + \frac{1}{2(V+1)} \right] dV = -\frac{dx}{x}.$$

Integrating on both sides, we have

$$\ln|V-2| - \frac{1}{2}\ln|V-1| + \frac{1}{2}\ln|V+1| = -\ln|x| + c_1 \implies \ln\left|\frac{(V-2)^2(V+1)}{V-1}\right| = -2\ln|x| + c_2 \implies \frac{(y-2x)^2(y+x)}{y-x} = c.$$

17. Dividing the given differential equation through on both sides by x^2 yields

$$2\frac{y}{x}\frac{dy}{dx} - \left(e^{-y^2/x^2} + 2\left(\frac{y}{x}\right)^2 \right) = 0.$$

Substituting $V = y/x$ and replacing $\frac{dy}{dx}$ with $\frac{d}{dx}(xV) = V + x\frac{dV}{dx}$, we obtain

$$2V\left(V + x\frac{dV}{dx}\right) - (e^{-V^2} + 2V^2) = 0.$$

Simplifying, we obtain

$$2Vx\frac{dV}{dx} = e^{-V^2} \implies e^{V^2}(2VdV) = \frac{dx}{x}.$$

Integrating on each side, we conclude that

$$e^{V^2} = \ln|x| + c_1 \implies e^{y^2/x^2} = \ln(cx) \implies y^2 = x^2 \ln(\ln(cx)).$$

18. Dividing the given differential equation through on both sides by x^2 yields $\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + 3\frac{y}{x} + 1$.

Substituting $V = y/x$ and replacing $\frac{dy}{dx}$ with $\frac{d}{dx}(xV) = V + x\frac{dV}{dx}$, we obtain

$$v + x\frac{dv}{dx} = v^2 + 3v + 1 \implies x\frac{dv}{dx} = (v+1)^2 \implies \frac{dv}{(v+1)^2} = \frac{dx}{x}.$$

Integrating on each side, we conclude that

$$\begin{aligned} -\frac{1}{v+1} &= \ln|x| + c_1 \implies -\frac{1}{\frac{y}{x}+1} = \ln|x| + c_1 \\ &\implies -\left(\frac{y}{x}+1\right) = \frac{1}{\ln|x|+c_1} \\ &\implies -\frac{y}{x} = 1 + \frac{1}{\ln|x|+c_1} \\ &\implies y = -x\left[1 + \frac{1}{\ln|x|+c_1}\right]. \end{aligned}$$

19. Dividing the given differential equation through on both sides by y and factoring x from the right-hand side, we obtain

$$\frac{dy}{dx} = \frac{\sqrt{1 + \left(\frac{y}{x}\right)^2} - 1}{\frac{y}{x}}.$$

Substituting $V = y/x$ and replacing $\frac{dy}{dx}$ with $\frac{d}{dx}(xV) = V + x\frac{dV}{dx}$, we obtain $V + x\frac{dV}{dx} = \frac{\sqrt{1+V^2}-1}{V}$. Therefore, $x\frac{dV}{dx} = \frac{\sqrt{1+V^2}-1-V^2}{V}$, and separating variables, we find that $\frac{V}{\sqrt{1+V^2}-1-V^2}dV = \frac{dx}{x}$. Integrating on both sides, we have $\ln|1-V| = \ln|x| + c_1 \implies |x(1-V)| = c_2 \implies 1-V = \frac{c}{x} \implies V^2 = \frac{c^2}{x^2} - 2\frac{c}{x} + 1 \implies V^2 = \frac{c^2}{x^2} - 2\frac{c}{x} \implies y^2 = c^2 - 2cx$.

20. Dividing through by x^2 on both sides of the given differential equation, we find that

$$2\left(\frac{y}{x} + 2\right)\frac{dy}{dx} = \frac{y}{x}\left(4 - \frac{y}{x}\right).$$

Substituting $V = y/x$ and replacing $\frac{dy}{dx}$ with $\frac{d}{dx}(xV) = V + x\frac{dV}{dx}$, we obtain

$$2(V+2)\left(V + x\frac{dV}{dx}\right) = V(4-V).$$

A short algebraic manipulations yields

$$2x\frac{dV}{dx} = -\frac{3V^2}{V+2}.$$

Separating variables, we obtain

$$2\frac{V+2}{V^2}dV = -3\frac{dx}{x}.$$

Integrating on both sides, we have

$$2\ln|V| - \frac{4}{V} = -3\ln|x| + c_1 \implies xy^2 = ce^{4x/y}.$$

21. We rewrite the differential equation as $\frac{dy}{dx} = \tan(y/x) + \frac{y}{x}$. Substituting $V = y/x$ and replacing $\frac{dy}{dx}$ with $\frac{d}{dx}(xV) = V + x\frac{dV}{dx}$, we obtain

$$V + x\frac{dV}{dx} = \tan V + V \implies x\frac{dV}{dx} = \tan V \implies \cot V dV = \frac{dx}{x}.$$

Integrating on both sides, we find that

$$\ln|\sin V| = \ln|x| + c_1 \implies \sin V = cx \implies V = \sin^{-1}(cx) \implies y(x) = x \sin^{-1}(cx).$$

22. We rewrite the differential equation as $\frac{dy}{dx} = \sqrt{\left(\frac{x}{y}\right)^2 + 1} + \frac{y}{x}$. Substituting $V = y/x$ and replacing $\frac{dy}{dx}$ with $\frac{d}{dx}(xV) = V + x\frac{dV}{dx}$, we obtain

$$V + x\frac{dV}{dx} = \sqrt{\left(\frac{1}{V}\right)^2 + 1} + V \implies x\frac{dV}{dx} = \sqrt{\left(\frac{1}{V}\right)^2 + 1} \implies \frac{dV}{\sqrt{\left(\frac{1}{V}\right)^2 + 1}} = \frac{dx}{x} \implies \frac{VdV}{\sqrt{1+V^2}} = \frac{dx}{x}.$$

Next, we integrate on both sides to obtain $\sqrt{1+V^2} = \ln|x| + c$. A short algebraic manipulation and replacing V with y/x yields the implicit solution

$$y^2 = x^2 [(\ln|x| + c)^2 - 1].$$

23. The given differential equation can be written as $(x-4y)dy = (4x+y)dx$. Converting to polar coordinates we have $x = r \cos \theta \implies dx = \cos \theta dr - r \sin \theta d\theta$ and $y = r \sin \theta \implies dy = \sin \theta dr + r \cos \theta d\theta$. Substituting these results into the preceding differential equation and simplifying yields the separable equation $4r^{-1}dr = d\theta$ which can be integrated directly to yield $4 \ln r = \theta + c$, so that $r = c_1 e^{\theta/4}$.

24. THE INITIAL CONDITION GIVEN IN THE TEXT DOES NOT WORK. The given differential equation can be rewritten as

$$\frac{dy}{dx} = \frac{2\left(\frac{2y}{x} - 1\right)}{1 + \frac{y}{x}}.$$

Substituting $V = y/x$ and replacing $\frac{dy}{dx}$ with $\frac{d}{dx}(xV) = V + x \frac{dV}{dx}$, we obtain

$$V + x \frac{dV}{dx} = \frac{2(2V-1)}{1+V}.$$

Therefore,

$$x \frac{dV}{dx} = \frac{-V^2 + 3V - 2}{V+1} = -\frac{(V-2)(V-1)}{V+1}.$$

Separating variables, we have

$$\frac{V+1}{(V-2)(V-1)} dV = -\frac{dx}{x}.$$

Using a partial fractions decomposition on the left-hand side, this becomes

$$\left(\frac{3}{V-2} - \frac{2}{V-1}\right) dV = -\frac{dx}{x}.$$

Integrating on each side, this becomes

$$3 \ln|V-2| - 2 \ln|V-1| = -\ln|x| + c.$$

Substituting $V = y/x$, we recover

$$3 \ln\left|\frac{y}{x} - 2\right| - 2 \ln\left|\frac{y}{x} - 1\right| = -\ln|x| + c.$$

25. The given differential equation can be written as $\frac{dy}{dx} = \frac{2 - \frac{y}{x}}{1 + 4\frac{y}{x}}$. Substituting $V = y/x$, replacing $\frac{dy}{dx}$

with $\frac{d}{dx}(xV) = V + x \frac{dV}{dx}$, and separating variables, we obtain

$$V + x \frac{dV}{dx} = \frac{2-V}{1+4V} \implies x \frac{dV}{dx} = \frac{2-2V-4V^2}{1+4V} \implies \frac{1}{2} \int \frac{1+4V}{2V^2+V-1} dV = -\int \frac{dx}{x}.$$

Integrating on each side of the equation, we obtain

$$\frac{1}{2} \ln |2v^2 + v - 1| = -\ln |x| + c \implies \frac{1}{2} \ln |x^2(2v^2 + v - 1)| = c \implies \frac{1}{2} \ln |2y^2 + yx - x^2| = c.$$

Since $y(1) = 1$, we have $c = \frac{1}{2} \ln 2$. Thus $\frac{1}{2} \ln |2y^2 + yx - x^2| = \frac{1}{2} \ln 2$, which implies that $2y^2 + yx - x^2 = 2$.

26. The given differential equation can be written as $\frac{dy}{dx} = \frac{y}{x} - \sqrt{1 + \frac{y^2}{x^2}}$. Substituting $V = y/x$ and replacing $\frac{dy}{dx}$ with $\frac{d}{dx}(xV) = V + x\frac{dV}{dx}$, we obtain

$$V + x\frac{dV}{dx} = V - \sqrt{1 + V^2}.$$

Therefore, $x\frac{dV}{dx} = -\sqrt{1 + V^2}$. Separating variables, we have $\frac{dV}{\sqrt{1 + V^2}} = -\frac{dx}{x}$. Integrating on each side of this equation, we obtain $\ln |V + \sqrt{1 + V^2}| = -\ln |x| + c$. That is,

$$\ln \left| \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} \right| = -\ln |x| + c.$$

Since $y(3) = 4$, we find that $c = 2 \ln 3 = \ln 9$. Hence, we have

$$\ln \left| \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} \right| = -\ln |x| + \ln 9.$$

A short algebraic manipulation gives

$$\ln |y + \sqrt{x^2 + y^2}| = \ln 9.$$

Therefore, $y + \sqrt{x^2 + y^2} = 9$.

27. Dividing the given differential equation through by x , we have $\frac{dy}{dx} - \frac{y}{x} = \sqrt{4 - \left(\frac{y}{x}\right)^2}$. Substituting $V = y/x$ and replacing $\frac{dy}{dx}$ with $\frac{d}{dx}(xV) = V + x\frac{dV}{dx}$, we obtain

$$V + x\frac{dV}{dx} = V + \sqrt{4 - V^2}.$$

Separating variables, we have

$$\frac{dv}{\sqrt{4 - v^2}} = \frac{dx}{x},$$

and integration on both sides yields

$$\sin^{-1} \left(\frac{v}{2} \right) = \ln |x| + c \implies \sin^{-1} \left(\frac{y}{2x} \right) = \ln x + c,$$

since $x > 0$.

28. (a) Substituting $y = xV$ and simplifying yields $x \frac{dV}{dx} = \frac{1+V^2}{a-V}$. Separating the variables and integrating, we obtain $a \tan^{-1} V - \frac{1}{2} \ln(1+V^2) = \ln x + \ln c$, or equivalently, $a \tan^{-1} \frac{y}{x} - \frac{1}{2} \ln(x^2 + y^2) = \ln c$. Substituting $x = r \cos \theta$ and $y = r \sin \theta$ yields $a\theta - \ln r = \ln c$. Exponentiating then gives $r = ke^{a\theta}$.

(b) The initial condition $y(1) = 1$ corresponds to $r(\frac{\pi}{4}) = \sqrt{2}$. Imposing this condition on the polar form of the solution obtained in (a) yields $k = \sqrt{2}e^{-\pi/8}$. Hence, the solution to the initial value problem is $r = \sqrt{2}e^{(\theta-\pi/4)/2}$. When $a = \frac{1}{2}$, the differential equation is $\frac{dy}{dx} = \frac{2x+y}{x-2y}$. Consequently, every solution curve has a vertical tangent line at points of intersection with the line $y = \frac{x}{2}$. The maximum interval of existence for the solution of the initial value problem can be obtained by determining where $y = \frac{x}{2}$ intersects the curve $r = \sqrt{2}e^{(\theta-\pi/4)/2}$. The line $y = \frac{x}{2}$ has a polar equation $\tan \theta = \frac{1}{2}$. The corresponding values of θ are $\theta = \theta_1 = \tan^{-1} \frac{1}{2} \approx 0.464$, $\theta = \theta_2 = \theta_1 + \pi \approx 3.61$. Consequently, the x -coordinates of the intersection points are $x_1 = r \cos \theta_1 = \sqrt{2}e^{(\theta_1-\pi/4)/2} \cos \theta_1 \approx 1.08$, $x_2 = r \cos \theta_2 = \sqrt{2}e^{(\theta_2-\pi/4)/2} \cos \theta_2 \approx -5.18$. Hence the maximum interval of existence for the solution is approximately $(-5.18, 1.08)$.

(c)

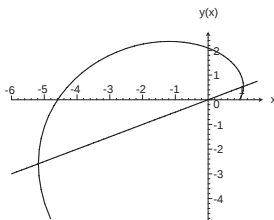


Figure 50: Figure for Problem 28(c)

29. The given family of curves satisfies $x^2 + y^2 = 2cy$, so implicit differentiation gives

$$2x + 2y \frac{dy}{dx} = 2c \frac{dy}{dx}.$$

Since $c = \frac{x^2 + y^2}{2y}$, we have

$$\frac{dy}{dx} = \frac{x}{c-y} = \frac{2xy}{x^2 - y^2}.$$

Therefore, the orthogonal trajectories satisfy $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$. Let $y = xV$ so that

$$\frac{dy}{dx} = V + x \frac{dV}{dx}.$$

Substituting these results into the last equation yields

$$x \frac{dV}{dx} = -\frac{V^2 + 1}{2V} \implies \ln |V^2 + 1| = -\ln |x| + c_1 \implies \frac{y^2}{x^2} + 1 = \frac{c_2}{x} \implies x^2 + y^2 = 2c_2x.$$

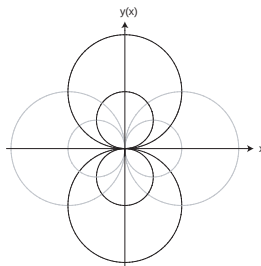


Figure 51: Figure for Problem 29

30. The given family of curves satisfies $(x - c)^2 + (y - c)^2 = 2c^2$, so implicit differentiation gives $2(x - c) + 2(y - c)\frac{dy}{dx} = 0$. Since $c = \frac{x^2 + y^2}{2(x + y)}$, we have

$$\frac{dy}{dx} = \frac{c - x}{y - c} = \frac{y^2 - 2xy - x^2}{y^2 + 2xy - x^2}.$$

Therefore, the orthogonal trajectories satisfy $\frac{dy}{dx} = \frac{y^2 + 2xy - x^2}{x^2 + 2xy - y^2}$. Let $y = xV$ so that $\frac{dy}{dx} = V + x\frac{dV}{dx}$.

Substituting these results into the last equation yields $V + x\frac{dV}{dx} = \frac{V^2 + 2V - 1}{1 + 2V - V^2}$. Separating the variables, we have $\frac{1 + 2V - V^2}{V^3 - V^2 + V - 1}dV = \frac{1}{x}dx$. A partial fractions decomposition can be used on the left-hand side to give

$$\left(\frac{1}{V - 1} - \frac{2V}{V^2 + 1} \right) dV = \frac{1}{x}dx.$$

Integration gives $\ln|V - 1| - \ln|V^2 + 1| = \ln|x| + c_1$, for some constant c_1 . That is, $\frac{V - 1}{V^2 + 1} = c_2x$. Replacing V with y/x and doing a short algebraic manipulation gives the equations of the orthogonal trajectories:

$$x^2 + y^2 = 2k(x - y) \implies (x - k)^2 + (y + k)^2 = 2k^2.$$

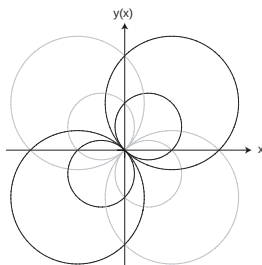


Figure 52: Figure for Problem 30

31. (a) Let r represent the radius of one of the circles with center at (a, ma) and passing through $(0, 0)$. Then we have $r = \sqrt{(a - 0)^2 + (ma - 0)^2} = |a|\sqrt{1 + m^2}$. Thus, the circle's equation can be written as $(x - a)^2 + (y - ma)^2 = (|a|\sqrt{1 + m^2})^2$, or $(x - a)^2 + (y - ma)^2 = a^2(1 + m^2)$.

(b) From part (a), we have $(x - a)^2 + (y - ma)^2 = a^2(1 + m^2)$. Therefore, $a = \frac{x^2 + y^2}{2(x + my)}$. Differentiating the first equation with respect x and solving we obtain $\frac{dy}{dx} = \frac{a - x}{y - ma}$. Substituting for a and simplifying yields $\frac{dy}{dx} = \frac{y^2 - x^2 - 2mxy}{my^2 - mx^2 + 2xy}$. Therefore, the orthogonal trajectories satisfy

$$\frac{dy}{dx} = \frac{mx^2 - my^2 - 2xy}{y^2 - x^2 - 2mxy} = \frac{m - m(\frac{y}{x})^2 - 2\frac{y}{x}}{(\frac{y}{x})^2 - 1 - 2m\frac{y}{x}}.$$

Let $y = xV$, so that $\frac{dy}{dx} = V + x\frac{dV}{dx}$. Substituting these results into the last equation yields

$$\begin{aligned} V + x\frac{dV}{dx} &= \frac{m - mV^2 - 2V}{V^2 - 1 - 2mV} \Rightarrow \frac{x dV}{dx} = \frac{(m - V)(1 + V^2)}{V^2 - 2mV - 1} \\ &\Rightarrow \int \frac{V^2 - 2mV - 1}{(m - V)(1 + V^2)} dV = \int \frac{dx}{x} \\ &\Rightarrow \int \frac{dV}{V - m} - \int \frac{2V}{1 + V^2} dV = \int \frac{dx}{x} \\ &\Rightarrow \ln|V - m| - \ln(1 + V^2) = \ln|x| + c_1 \\ &\Rightarrow V - m = c_2 x(1 + V^2) \\ &\Rightarrow y - mx = c_2 x^2 + c_2 y^2 \\ &\Rightarrow x^2 + y^2 + cmx - cy = 0. \end{aligned}$$

Completing the square we obtain $(x + cm/2)^2 + (y - c/2)^2 = c^2/4(m^2 + 1)$. Now letting $b = c/2$, the last equation becomes $(x + bm)^2 + (y - b)^2 = b^2(m^2 + 1)$ which is a family of circles lying on the line $x = -my$ and passing through the origin.

(c)

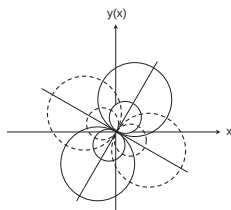


Figure 53: Figure for Problem 31(c)

32. Implicit differentiation of $x^2 + y^2 = c$ with respect to x yields $\frac{dy}{dx} = -\frac{x}{y} = m_2$. Now

$$m_1 = \frac{m_2 - \tan(\frac{\pi}{4})}{1 + m_2 \tan(\frac{\pi}{4})} = \frac{-x/y - 1}{1 - x/y} = \frac{x + y}{x - y}.$$

Let $y = xV$, so that $\frac{dy}{dx} = V + x\frac{dV}{dx}$. Substituting these results into the last equation yields

$$\begin{aligned} V + x\frac{dV}{dx} &= \frac{1+V}{1-V} \implies \frac{1-V}{1+V^2}dV = \frac{dx}{x} \\ &\implies \int \left(-\frac{V}{1+V^2} + \frac{1}{V^2+1} \right) dV = \int \frac{dx}{x} \\ &\implies -\frac{1}{2}\ln(1+V^2) + \tan^{-1}V = \ln|x| + c_1. \end{aligned}$$

Therefore, the equations of the oblique trajectories are

$$\ln(x^2 + y^2) - 2\tan^{-1}(y/x) = c_2.$$

33. We have $\frac{dy}{dx} = 6y/x = m_2$. Now

$$m_1 = \frac{m_2 - \tan(\frac{\pi}{4})}{1 + m_2 \tan(\frac{\pi}{4})} = \frac{6y/x - 1}{1 + 6y/x} = \frac{6y - x}{6y + x}.$$

Let $y = xV$, so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substitute these results into the last equation yields

$$\begin{aligned} V + x\frac{dV}{dx} &= \frac{6V-1}{6V+1} \implies x\frac{dV}{dx} = \frac{(3V-1)(1-2V)}{6V+1} \\ &\implies \int \left(\frac{9}{3V-1} - \frac{8}{2V-1} \right) dV = \int \frac{dx}{x} \\ &\implies 3\ln|3V-1| - 4\ln|2V-1| = \ln|x| + c_1. \end{aligned}$$

Therefore, the equations of the oblique trajectories are

$$(3y - x)^3 = k(2y - x)^4.$$

34. Implicit differentiation of $x^2 + y^2 = 2cx$ with respect to x gives $2x + 2y\frac{dy}{dx} = 2c$, so that $\frac{dy}{dx} = \frac{c-x}{y}$.

Since $c = \frac{x^2 + y^2}{2x}$, we find that $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = m_2$. Now

$$m_1 = \frac{m_2 - \tan(\frac{\pi}{4})}{1 + m_2 \tan(\frac{\pi}{4})} = \frac{\frac{y^2 - x^2}{2xy} - 1}{1 + \frac{y^2 - x^2}{2xy}} = \frac{y^2 - x^2 - 2xy}{y^2 - x^2 + 2xy}.$$

Let $y = xV$, so that $\frac{dy}{dx} = V + x\frac{dV}{dx}$. Substituting these results into the last equation yields

$$\begin{aligned} V + x\frac{dV}{dx} &= \frac{V^2 - 2V - 1}{V^2 + 2V - 1} \implies x\frac{dV}{dx} = \frac{-V^3 - V^2 - V - 1}{V^2 + 2V - 1} \\ &\implies \frac{-V^2 - 2V + 1}{V^3 + V^2 + V + 1} dV = \frac{dx}{x} \\ &\implies \int \left(\frac{1}{V+1} - \frac{2V}{V^2+1} \right) dV = \int \frac{dx}{x} \\ &\implies \ln|V+1| - \ln(V^2+1) = \ln|x| + c_1 \\ &\implies \ln|y+x| = \ln|y^2+x^2| + c_1. \end{aligned}$$

Therefore, the equations of the oblique trajectories are $x^2 + y^2 = 2k(x+y)$ or, equivalently,

$$(x-k)^2 + (y-k)^2 = 2k^2.$$

35. (a) From $y = cx^{-1}$, we find that $\frac{dy}{dx} = -cx^{-2} = -\frac{y}{x}$. Therefore,

$$m_1 = \frac{m_2 - \tan \alpha_0}{1 + m_2 \tan \alpha_0} = \frac{-(y/x) - \tan \alpha_0}{1 - (y/x) \tan \alpha_0}.$$

Let $y = xV$, so that $\frac{dy}{dx} = V + x\frac{dV}{dx}$. Substituting these results into the last equation yields

$$\begin{aligned} V + x\frac{dV}{dx} &= \frac{\tan \alpha_0 + V}{V \tan \alpha_0 - 1} \implies \frac{2V \tan \alpha_0 - 2}{V^2 \tan \alpha_0 - 2V - \tan \alpha_0} dV = -\frac{2dx}{x} \\ &\implies \ln|V^2 \tan \alpha_0 - 2V - \tan \alpha_0| = -2 \ln|x| + c_1 \\ &\implies (y^2 - x^2) \tan \alpha_0 - 2xy = k. \end{aligned}$$

(b) See accompanying figure.

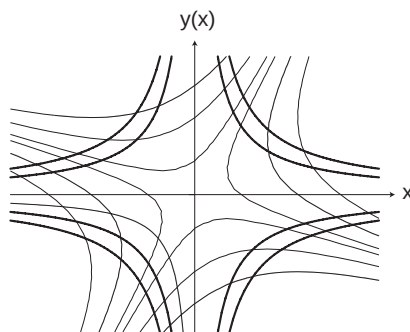


Figure 54: Figure for Problem 35(b)

36. (a) From $x^2 + y^2 = c$, we find that $\frac{dy}{dx} = -\frac{x}{y}$. Therefore,

$$m_1 = \frac{m_2 - \tan \alpha_0}{1 + m_2 \tan \alpha_0} = \frac{-x/y - m}{1 - (x/y)m} = \frac{x + my}{mx - y}.$$

Let $y = xV$, so that $\frac{dy}{dx} = V + x\frac{dV}{dx}$. Substituting these results into the last equation yields

$$\begin{aligned} V + x\frac{dV}{dx} &= \frac{1+mV}{m-V} \implies x\frac{dV}{dx} = \frac{1+V^2}{m-V} \\ &\implies \frac{V-m}{1+V^2}dV = -\frac{dx}{x} \\ &\implies \int \left(\frac{V}{1+V^2} - \frac{m}{1+V^2} \right) dV = -\int \frac{dx}{x} \\ &\implies \frac{1}{2} \ln(1+V^2) - m \tan V = -\ln|x| + c_1. \end{aligned}$$

In polar coordinates, $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} y/x$, so this result becomes $\ln r - m\theta = c_1 \implies r = e^{m\theta}$, where k is an arbitrary constant.

(b) See accompanying figure.

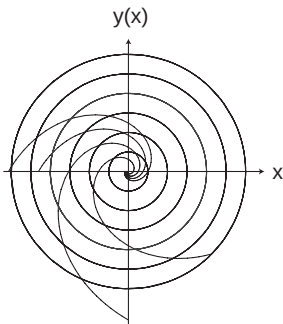


Figure 55: Figure for Problem 36(b)

37. This is a Bernoulli equation. Multiplying both sides by y results in $y\frac{dy}{dx} - \frac{1}{x}y^2 = 4x^2 \cos x$. Let $u = y^2$, so $\frac{du}{dx} = 2y\frac{dy}{dx}$ or $y\frac{dy}{dx} = \frac{1}{2}\frac{du}{dx}$. Substituting these results into $y\frac{dy}{dx} - \frac{1}{x}y^2 = 4x^2 \cos x$ yields $\frac{du}{dx} - \frac{2}{x}u = 8x^2 \cos x$, which has an integrating factor $I(x) = x^{-2}$. Therefore,

$$\begin{aligned} \frac{d}{dx}(x^{-2}u) &= 8 \cos x \implies x^{-2}u = 8 \int \cos x dx + c \\ &\implies x^{-2}u = 8 \sin x + c \\ &\implies u = x^2(8 \sin x + c) \\ &\implies y^2 = x^2(8 \sin x + c) \\ &\implies y = x\sqrt{8 \sin x + c}. \end{aligned}$$

38. This is a Bernoulli equation. Let $u = y^{-2}$, so $\frac{du}{dx} = -2y^{-3}\frac{dy}{dx}$ or $y^{-3}\frac{dy}{dx} = \frac{1}{2}\frac{du}{dx}$. Substituting these results into the last equation yields $\frac{du}{dx} - u \tan x = -4 \sin x$. An integrating factor for this equation is

$I(x) = \cos x$. Thus,

$$\begin{aligned}\frac{d}{dx}(u \cos x) &= -4 \cos x \sin x \implies u \cos x = 4 \int \cos x \sin x dx \\ \implies u &= \frac{1}{\cos x} (\cos^2 x + c) \\ \implies y^{-2} &= 2 \cos x + \frac{c}{\cos x} \\ \implies y(x) &= \sqrt{\frac{1}{2 \cos x + \frac{c}{\cos x}}}.\end{aligned}$$

39. This is a Bernoulli equation. Let $u = y^{2/3}$, so that we have $\frac{du}{dx} - \frac{1}{x}u = 4x^2 \ln x$. An integrating factor for this equation is $I(x) = \frac{1}{x}$. Thus,

$$\begin{aligned}\frac{d}{dx}(x^{-1}u) &= 4x \ln x \implies x^{-1}u = 4 \int x \ln x dx + c \\ \implies x^{-1}u &= 2x^2 \ln x - x^2 + c \\ \implies u &= x(2x^2 \ln x - x^2 + c) \\ \implies y^{2/3} &= x(2x^2 \ln x - x^2 + c) \\ \implies y(x) &= (x(2x^2 \ln x - x^2 + c))^{3/2}.\end{aligned}$$

40. This is a Bernoulli equation. Let $u = y^{1/2}$, so that we have $\frac{du}{dx} + \frac{1}{x}u = 3\sqrt{1+x^2}$. An integrating factor for this equation is $I(x) = x$, so

$$\begin{aligned}\frac{d}{dx}(xu) &= 3x\sqrt{1+x^2} \implies xu = 3 \int x\sqrt{1+x^2} dx + c \\ \implies xu &= (1+x^2)^{3/2} + c \\ \implies u &= \frac{1}{x}(1+x^2)^{3/2} + \frac{c}{x} \\ \implies y^{1/2} &= \frac{1}{x}(1+x^2)^{3/2} + \frac{c}{x} \\ \implies y(x) &= \left(\frac{1}{x}(1+x^2)^{3/2} + \frac{c}{x} \right)^2.\end{aligned}$$

41. This is a Bernoulli equation. Let $u = y^{-1}$, so that we have $\frac{du}{dx} - \frac{2}{x}u = -6x^4$. An integrating factor for this equation is $I(x) = x^{-2}$, so that

$$\begin{aligned}\frac{d}{dx}(x^{-2}u) &= -6x^2 \implies x^{-2}u = -2x^3 + c \\ \implies u &= -2x^5 + cx^2 \\ \implies y^{-1} &= -2x^5 + cx^2 \\ \implies y(x) &= \frac{1}{x^2(c - 2x^3)}.\end{aligned}$$

42. Rewriting this equation as $y' + \frac{y}{2x} = -y^3x^2$, we see that it is Bernoulli. Let $u = y^{-2}$, so that we have $\frac{du}{dx} - \frac{1}{x}u = 2x^2$. An integrating factor for this equation is $I(x) = \frac{1}{x}$, so

$$\begin{aligned}\frac{d}{dx}(x^{-1}u) &= 2x \implies x^{-1}u = x^2 + c \\ \implies u &= x^3 + cx \\ \implies y^{-2} &= x^3 + cx \\ \implies y(x) &= \sqrt{\frac{1}{x^3 + cx}}.\end{aligned}$$

43. Rewriting this equation as $y' - \frac{2(b-a)}{(x-a)(x-b)}y = \sqrt{y}$, we see that it is Bernoulli. Let $u = y^{1/2}$, so that we have $\frac{du}{dx} - \frac{(b-a)}{(x-a)(x-b)}u = \frac{1}{2}$. An integrating factor for this equation is $I(x) = \frac{x-a}{x-b}$, so that

$$\begin{aligned}\frac{d}{dx}\left(\frac{x-a}{x-b}u\right) &= \frac{x-a}{2(x-b)} \implies \frac{x-a}{x-b}u = \frac{1}{2}[x + (b-a)\ln|x-b| + c] \\ \implies y^{1/2} &= \frac{x-b}{2(x-a)}[x + (b-a)\ln|x-b| + c] \\ \implies y(x) &= \frac{1}{4}\left(\frac{x-b}{x-a}\right)^2 [x + (b-a)\ln|x-b| + c]^2.\end{aligned}$$

44. This is a Bernoulli equation. Let $u = y^{1/3}$, so that we have $\frac{du}{dx} + \frac{2}{x}u = \frac{\cos x}{x}$. An integrating factor for this equation is $I(x) = x^2$, so

$$\begin{aligned}\frac{d}{dx}(x^2u) &= x \cos x \implies x^2u = \cos x + x \sin x + c \\ \implies y^{1/3} &= \frac{\cos x + x \sin x + c}{x^2} \\ \implies y(x) &= \left(\frac{\cos x + x \sin x + c}{x^2}\right)^3.\end{aligned}$$

45. This is a Bernoulli equation. Let $u = y^{1/2}$, so that we have $\frac{du}{dx} + 2xu = 2x^3$. An integrating factor for this equation is $I(x) = e^{x^2}$, so

$$\begin{aligned}\frac{d}{dx}(e^{x^2}u) &= 2e^{x^2}x^3 \implies e^{x^2}u = e^{x^2}(x^2 - 1) + c \\ \implies y^{1/2} &= x^2 - 1 + ce^{-x^2} \\ \implies y(x) &= [(x^2 - 1) + ce^{-x^2}]^2.\end{aligned}$$

46. This is a Bernoulli equation. Let $u = y^{-2}$, so that we have $\frac{du}{dx} + \frac{1}{x \ln x}u = -4x$. An integrating factor

for this equation is $I(x) = \ln x$, so

$$\begin{aligned}\frac{d}{dx}(u \ln x) &= -4x \ln x \implies u \ln x = x^2 - 2x^2 \ln x + c \\ \implies y^2 &= \frac{\ln x}{x^2(1 - 2 \ln x) + c}.\end{aligned}$$

47. This is a Bernoulli equation. Let $u = y^{1-\pi}$, so that we have $\frac{du}{dx} + \frac{1}{x}u = 3x$. An integrating factor for this equation is $I(x) = x$, so

$$\begin{aligned}\frac{d}{dx}(xu) &= 3x^2 \implies xu = x^3 + c \\ \implies y^{1-\pi} &= \frac{x^3 + c}{x} \\ \implies y(x) &= \left(\frac{x^3 + c}{x}\right)^{1/(1-\pi)}.\end{aligned}$$

48. This is a Bernoulli equation. Let $u = y^2$, so that we have $\frac{du}{dx} + u \cot x = 8 \cos^3 x$. An integrating factor for this equation is $I(x) = \sin x$, so

$$\begin{aligned}\frac{d}{dx}(u \sin x) &= 8 \cos^3 x \sin x \implies u \sin x = -2 \cos^4 x + c \\ \implies y^2 &= \frac{-2 \cos^4 x + c}{\sin x}.\end{aligned}$$

49. This is a Bernoulli equation. Let $u = y^{1-\sqrt{3}}$, so that we have $\frac{du}{dx} + u \sec x = \sec x$. An integrating factor for this equation is $I(x) = \sec x + \tan x$, so that

$$\begin{aligned}\frac{d}{dx}[(\sec x + \tan x)u] &= \sec x(\sec x + \tan x) \implies (\sec x + \tan x)u = \tan x + \sec x + c \\ \implies y^{1-\sqrt{3}} &= 1 + \frac{1}{\sec x + \tan x} \\ \implies y(x) &= \left(1 + \frac{c}{\sec x + \tan x}\right)^{1/(1-\sqrt{3})}.\end{aligned}$$

50. This is a Bernoulli equation. Let $u = y^{-1}$, so that we have $\frac{du}{dx} - \frac{2x}{1+x^2}u = -x$. An integrating factor

for this equation is $I(x) = \frac{1}{1+x^2}$, so that

$$\begin{aligned}\frac{d}{dx} \left(\frac{u}{1+x^2} \right) &= -\frac{x}{1+x^2} \implies \frac{u}{1+x^2} = -\int \frac{x dx}{1+x^2} + c \\ &\implies \frac{u}{1+x^2} = -\frac{1}{2} \ln(1+x^2) + c \\ &\implies u = (1+x^2) \left(-\frac{1}{2} \ln(1+x^2) + c \right) \\ &\implies y^{-1} = (1+x^2) \left(-\frac{1}{2} \ln(1+x^2) + c \right).\end{aligned}$$

Since $y(0) = 1$, we find that $c = 1$. Therefore

$$\frac{1}{y} = (1+x^2) \left(-\frac{1}{2} \ln(1+x^2) + 1 \right).$$

Hence,

$$y(x) = \frac{1}{(1+x^2) \left(-\frac{1}{2} \ln(1+x^2) + 1 \right)}.$$

51. This is a Bernoulli equation. Let $u = y^{-2}$, so that we have $\frac{du}{dx} - 2u \cot x = -2 \sin^3 x$. An integrating factor for this equation is $I(x) = \csc^2 x$, so that

$$\frac{d}{dx}(u \csc^2 x) = -\sin x \implies u \csc^2 x = 2 \cos x + c.$$

Since $y(\pi/2) = 1$, we find that $c = 1$. Thus $y^2 = \frac{1}{\sin^2 x (2 \cos x + 1)}$. Hence,

$$y(x) = \sqrt{\frac{1}{\sin^2 x (2 \cos x + 1)}}.$$

52. Let $V = ax + by + c$, so that $\frac{dV}{dx} = a + b \frac{dy}{dx}$. Hence, $b \frac{dy}{dx} = \frac{dV}{dx} - a$, and thus, $\frac{dy}{dx} = \frac{1}{b} \left(\frac{dV}{dx} - a \right) = F(V)$.

We conclude that

$$\frac{dV}{dx} - a = bF(V) \implies \frac{dV}{dx} = bF(V) + a \implies \frac{dV}{bF(V) + a} = dx.$$

53. Let $V = 9x - y$, so that $\frac{dy}{dx} = 9 - \frac{dV}{dx}$. Hence, $\frac{dV}{dx} = 9 - V^2$, and so

$$\int \frac{dV}{9 - V^2} = \int dx \implies \frac{1}{3} \tanh^{-1}(V/3) = x + c_1.$$

However, $y(0) = 0$, so $c = 0$. Thus, $\tanh^{-1} \left(\frac{9x - y}{3} \right) = 3x$, or $y(x) = 3(3x - \tanh 3x)$.

54. Let $V = 4x + y + 2$, so that

$$\begin{aligned}\frac{dV}{dx} &= 4 + \frac{dy}{dx} = 4 + V^2 \implies \frac{dV}{V^2 + 4} = dx \\ &\implies \int \frac{dV}{V^2 + 4} = \int dx \\ &\implies \frac{1}{2} \tan^{-1}(V/2) = x + c_1 \\ &\implies \tan^{-1}(2x + y/2 + 1) = 2x + c \\ &\implies y(x) = 2[\tan(2x + c) - 2x - 1].\end{aligned}$$

55. Let $V = 3x - 3y + 1$, so that

$$\begin{aligned}\frac{dy}{dx} &= 1 - \frac{1}{3} \frac{dV}{dx} \implies 1 - \frac{1}{3} \frac{dV}{dx} = \sin^2 V \\ &\implies \frac{dV}{dx} = 3 \cos^2 V \\ &\implies \int \sec^2 V dV = 3 \int dx \\ &\implies \tan V = 3x + c \\ &\implies \tan(3x - 3y + 1) = 3x + c \\ &\implies y(x) = \frac{1}{3}[3x - \tan^{-1}(3x + c) + 1].\end{aligned}$$

56. Since $V = xy$, we have $V' = xy' + y$, so that $y' = (V' - y)/x$. Substitution into the differential equation yields

$$(V' - y)/x = yF(V)/x \implies V' = y[F(V) + 1] \implies V' = V[F(V) + 1]/x,$$

so that $\frac{1}{V[F(V) + 1]} \frac{dV}{dx} = \frac{1}{x}$.

57. Substituting into $\frac{1}{V[F(V) + 1]} \frac{dV}{dx} = \frac{1}{x}$ for $F(V) = \ln V - 1$ yields

$$\frac{1}{V \ln V} dV = \frac{1}{x} dx \implies \ln(\ln V) = \ln cx \implies V = e^{cx} \implies y(x) = \frac{1}{x} e^{cx}.$$

58. (a) Since $x = u - 1$ and $y = v + 1$, we have that $\frac{dy}{dx} = \frac{dv}{du}$. Substitution into the given differential equation yields $\frac{dv}{du} = \frac{u + 2v}{2u - v}$.

(b) The differential equation obtained in (a) is first-order homogeneous. We therefore let $W = v/u$ and substitute into the differential equation to obtain $W'u + W = \frac{1 + 2W}{2 - W} \implies W'u = \frac{1 + W^2}{2 - W}$. Separating the variables yields $\left(\frac{2}{1 + W^2} - \frac{W}{1 + W^2} \right) dW = \frac{1}{u} du$. This can be integrated directly to obtain

$$2 \tan^{-1} W - \frac{1}{2} \ln(1 + W^2) = \ln u + \ln c.$$

Simplifying, we obtain $cu^2(1+W^2) = e^{4\tan^{-1}W} \implies c(u^2+v^2) = e^{\tan^{-1}(v/u)}$. Substituting back in for x and y yields $c[(x+1)^2 + (y-1)^2] = e^{\tan^{-1}[(y-1)/(x+1)]}$.

59. (a) Taking the derivative of $y = Y(x) + v^{-1}(x)$ with respect to x gives $y' = Y'(x) - v^{-2}(x)v'(x)$. Now substitute into the given differential equation and simplify algebraically to obtain

$$Y'(x) + p(x)Y(x) + q(x)Y^2(x) - v^{-2}(x)v'(x) + v^{-1}(x)p(x) + q(x)[2Y(x)v^{-1}(x) + v^{-2}(x)] = r(x).$$

We are told that $Y(x)$ is a particular solution to the given differential equation, and therefore

$$Y'(x) + p(x)Y(x) + q(x)Y^2(x) = r(x).$$

Consequently, the transformed differential equation reduces to

$$-v^{-2}(x)v'(x) + v^{-1}p(x) + q(x)[2Y(x)v^{-1}(x) + v^{-2}(x)] = 0,$$

or equivalently,

$$v' - [p(x) + 2Y(x)q(x)]v = q(x).$$

(b) The given differential equation can be written as $y' - x^{-1}y - y^2 = x^{-2}$, which is a Riccati differential equation with $p(x) = -x^{-1}$, $q(x) = -1$, and $r(x) = x^{-2}$. Since $y(x) = -x^{-1}$ is a solution to the given differential equation, we make the substitution $y(x) = -x^{-1} + v^{-1}(x)$. According to the result from part (a), the given differential equation then reduces to $v' - (-x^{-1} + 2x^{-1})v = -1$, or equivalently $v' - x^{-1}v = -1$.

This linear differential equation has an integrating factor $I(x) = x^{-1}$, so that v must satisfy $\frac{d}{dx}(x^{-1}v) = -x^{-1} \implies v(x) = x(c - \ln x)$. Hence the solution to the original equation is

$$y(x) = -\frac{1}{x} + \frac{1}{x(c - \ln x)} = \frac{1}{x} \left(\frac{1}{c - \ln x} - 1 \right).$$

60. (a) If $y = ax^r$, then $y' = arx^{r-1}$. Substituting these expressions into the given differential equation yields $arx^{r-1} + 2ax^{r-1} - a^2x^{2r} = -2x^{-2}$. For this to hold for all $x > 0$, the powers of x must match up on either side of the equation. Hence, $r = -1$. Then a is determined from the quadratic $-a + 2a - a^2 = -1 \iff a^2 - a - 2 = 0 \iff (a-2)(a+1) = 0$. Consequently, $a = 2$ or $a = -1$ in order for us to have a solution to the given differential equation. Therefore, two solutions to the differential equation are $y_1(x) = 2x^{-1}$, $y_2(x) = -x^{-1}$.

(b) Taking $Y(x) = 2x^{-1}$ and using the result from part (a) of the previous problem, we now substitute $y(x) = 2x^{-1} + v^{-1}$ into the given Riccati equation. The result is

$$(-2x^{-2} - v^{-2}v') + 2x^{-1}(2x^{-1} + v^{-1}) - (4x^{-2} + 4x^{-1}v^{-1} + v^{-2}) = -2x^{-2}.$$

Simplifying this equation yields the linear equation $v' + 2x^{-1}v = -1$. Multiplying by the integrating factor $I(x) = e^{\int 2x^{-1}dx} = x^2$ results in the integrable differential equation $\frac{d}{dx}(x^2v) = -x^2$. Integrating this differential equation we obtain

$$v(x) = x^2 \left(-\frac{1}{3}x^3 + c \right) = \frac{1}{3}x^2(c_1 - x^3).$$

Consequently, the general solution to the Riccati equation is

$$y(x) = \frac{2}{x} + \frac{3}{x^2(c_1 - x^3)}.$$

61. (a) From $y = x^{-1} + w(x)$, we have $y' = -x^{-2} + w'$. Substituting into the given differential equation yields

$$(-x^{-2} + w') + 7x^{-1}(x^{-1} + w) - 3(x^{-2} + 2x^{-1}w + w^2) = 3x^{-2},$$

which simplifies to $w' + x^{-1}w - 3w^2 = 0$.

(b) The preceding equation can be written in the equivalent form $w^{-2}w' + x^{-1}w^{-1} = 3$. We let $u = w^{-1}$, so that $u' = -w^{-2}w'$. Substitution into the differential equation gives, after simplification, $u' - x^{-1}u = -3$. An integrating factor for this linear differential equation is $I(x) = x^{-1}$, so that the differential equation can be written in the integrable form $\frac{d}{dx}(x^{-1}u) = -3x^{-1}$. Integrating, we obtain

$$u(x) = x(-3 \ln x + c) \quad \text{and so} \quad w(x) = \frac{1}{x(c - 3 \ln x)}.$$

Consequently, the solution to the original Riccati equation is

$$y(x) = \frac{1}{x} \left(1 + \frac{1}{c - 3 \ln x} \right).$$

62. If we let $u = \ln y$, then $\frac{du}{dx} = \frac{1}{y} \frac{dy}{dx}$ and the given equation becomes $\frac{du}{dx} + p(x)u = q(x)$, which is first-order linear and has a solution of the form

$$u = e^{-\int p(x)dx} \left[\int e^{\int p(x)dx} q(x)dx + c \right].$$

Substituting

$$\ln y = e^{-\int p(x)dx} \left[\int e^{\int p(x)dx} q(x)dx + c \right]$$

into $u = \ln y$, we obtain

$$y(x) = e^{I^{-1} \left[\int I(t)q(t)dt + c \right]},$$

where $I(x) = e^{\int p(t)dt}$ and c is an arbitrary constant.

63. We let $u = \ln y$ and use the technique of the preceding problem: $\frac{du}{dx} - \frac{2}{x}u = \frac{1 - 2 \ln x}{x}$ and

$$u = e^{2 \int \frac{dx}{x}} \left[\int e^{-2 \int \frac{dx}{x}} \left(\frac{1 - 2 \ln x}{x} \right) dx + c_1 \right] = x^2 \left[\int \left(\frac{1 - 2 \ln x}{x^3} \right) dx + c_1 \right] = \ln x + cx^2,$$

and since $u = \ln y$, we have $\ln y = \ln x + cx^2$. Now $y(1) = e$ so $c = 1$. Thus, the solution to the initial-value problem is $y(x) = xe^{x^2}$.

64. If $u = f(y)$, then $\frac{du}{dx} = f'(y) \frac{dy}{dx}$, and the given equation becomes $\frac{du}{dx} + p(x)u = q(x)$, which is first-order linear with a solution of the form

$$u(x) = e^{-\int p(x)dx} \left[\int e^{\int p(x)dx} q(x)dx + c \right].$$

Substituting

$$f(y) = e^{-\int p(x)dx} \left[\int e^{\int p(x)dx} q(x)dx + c \right]$$

into $u = f(y)$ and using the fact that f is invertible, we obtain

$$y(x) = f^{-1} \left[I^{-1} \left(\int I(t)q(t)dt \right) + c \right],$$

where $I(x) = e^{\int p(t)dt}$ and c is an arbitrary constant.

65. Let $u = \tan y$ so that $\frac{du}{dx} = \sec^2 y \frac{dy}{dx}$ and the given equation becomes first-order linear:

$$\frac{du}{dx} + \frac{1}{2\sqrt{1+x}}u = \frac{1}{2\sqrt{1+x}}.$$

An integrating factor for this equation is $I(x) = e^{\sqrt{1+x}}$, so that

$$\begin{aligned} \frac{d}{dx}(e^{\sqrt{1+x}}u) &= \frac{e^{\sqrt{1+x}}}{2\sqrt{1+x}} \implies e^{\sqrt{1+x}}u = \int \frac{e^{\sqrt{1+x}}}{2\sqrt{1+x}} \\ &\implies e^{\sqrt{1+x}}u = e^{\sqrt{1+x}} + c \\ &\implies u = 1 + ce^{-\sqrt{1+x}}. \end{aligned}$$

But $u = \tan y$, so that $\tan y = 1 + ce^{-\sqrt{1+x}}$ or

$$y(x) = \tan^{-1}(1 + ce^{-\sqrt{1+x}}).$$

Solutions to Section 1.9

True-False Review:

- 1. FALSE.** The requirement, as stated in Theorem 1.9.4, is that $M_y = N_x$, not $M_x = N_y$, as stated.
- 2. FALSE.** A potential function $\phi(x, y)$ is not an equation. The general solution to an exact differential equation takes the form $\phi(x, y) = c$, where $\phi(x, y)$ is a potential function.
- 3. FALSE.** According to Definition 1.9.2, $M(x)dx + N(y)dy = 0$ is only exact if there exists a function $\phi(x, y)$ such that $\phi_x = M$ and $\phi_y = N$ for all (x, y) in a region R of the xy -plane.
- 4. TRUE.** This is the content of part 1 of Theorem 1.9.11.
- 5. FALSE.** If $\phi(x, y)$ is a potential function for $M(x, y)dx + N(x, y)dy = 0$, then so is $\phi(x, y) + c$ for any constant c .
- 6. TRUE.** We have

$$M_y = 2e^{2x} - \cos y \quad \text{and} \quad N_x = 2e^{2x} - \cos y,$$

and so since $M_y = N_x$, this equation is exact.

- 7. FALSE.** We have

$$M_y = \frac{(x^2 + y)^2(-2x) + 4xy(x^2 + y)}{(x^2 + y)^4}$$

and

$$N_x = \frac{(x^2 + y)^2(2x) - 2x^2(x^2 + y)(2x)}{(x^2 + y)^4}.$$

Thus, $M_y \neq N_x$, and so this equation is not exact.

8. FALSE. We have

$$M_y = 2y \quad \text{and} \quad N_x = 2y^2,$$

and since $M_y \neq N_x$, we conclude that this equation is not exact.

9. FALSE. We have

$$M_y = e^{x \sin y} \cos y + x e^{x \sin y} \cos y \quad \text{and} \quad N_x = \cos y \sin y e^{x \sin y},$$

and since $M_y \neq N_x$, we conclude that this equation is not exact.

Problems:

1. We have $M = y + 3x^2$ and $N = x$. Thus, $M_y = 1$ and $N_x = 1$. Since $M_y = N_x$, the differential equation is exact.

2. We have $M = \cos(xy) - xy \sin(xy)$ and $N = -x^2 \sin(xy)$. Thus, $M_y = -2x \sin(xy) - x^2 y \cos(xy)$ and $N_x = -2x \sin(xy) - x^2 y \cos(xy)$. Since $M_y = N_x$, the differential equation is exact.

3. We have $M = ye^{xy}$ and $N = 2y - xe^{-xy}$. Thus, $M_y = yxe^{xy} + e^{xy}$ and $N_x = xye^{-xy} - e^{-xy}$. Since $M_y \neq N_x$, the differential equation is not exact.

4. We have $M = 2xy$ and $N = x^2 + 1$. Thus, $M_y = 2x$ and $N_x = 2x$. Since $M_y = N_x$, the differential equation is exact so there exists a potential function ϕ such that (a) $\frac{\partial \phi}{\partial x} = 2xy$ and (b) $\frac{\partial \phi}{\partial x} = 2xy + \frac{dh(x)}{dx}$. From (a), $2xy = 2xy + \frac{dh(x)}{dx} \implies \frac{dh(x)}{dx} = 0 \implies h(x)$ is a constant. Since we need just one potential function, let $h(x) = 0$. Thus, $\phi(x, y) = (x^2 + 1)y$; hence, $(x^2 + 1)y = c$.

5. We have $M = y^2 + \cos x$ and $N = 2xy + \sin y$. Thus, $M_y = 2y$ and $N_x = 2y$. Since $M_y = N_x$, the differential equation is exact so there exists a potential function ϕ such that (a) $\frac{\partial \phi}{\partial x} = y^2 + \cos x$ and (b) $\frac{\partial \phi}{\partial y} = 2xy + \sin y$. From (a), $\phi(x, y) = xy^2 + \sin x + h(y) \implies \frac{\partial \phi}{\partial y} = 2xy + \frac{dh(y)}{dy}$, and so from (b), $2xy + \frac{dh(y)}{dy} = 2xy + \sin y \implies \frac{dh(y)}{dy} = \sin y \implies h(y) = -\cos y$, where the constant of integration has been set to zero since we just need one potential function. Therefore, we have $\phi(x, y) = xy^2 + \sin x - \cos y$; hence, $xy^2 + \sin x - \cos y = c$.

6. Given $\frac{xy-1}{x}dx + \frac{xy+1}{y}dy = 0$, we have $M_y = N_x = 1$. Therefore, the differential equation is exact, and so there exists a potential function ϕ such that (a) $\frac{\partial \phi}{\partial x} = \frac{xy-1}{x}$ and (b) $\frac{\partial \phi}{\partial y} = \frac{xy+1}{y}$. From (a), $\phi(x, y) = xy - \ln|x| + h(y) \implies \frac{\partial \phi}{\partial y} = x + \frac{dh(y)}{dy}$, and so from (b), $x + \frac{dh(y)}{dy} = \frac{xy+1}{y} \implies \frac{dh(y)}{dy} = y^{-1} \implies h(y) = \ln|y|$, where the constant of integration has been set to zero since we need just one potential function. Therefore, we have $\phi(x, y) = xy + \ln|y/x|$; hence, $xy + \ln|y/x| = c$.

7. Given $(4e^{2x} + 2xy - y^2)dx + (x - y)^2dy = 0$, we have $M_y = N_x = 2y$. Therefore, the differential equation is exact, and so there exists a potential function ϕ such that (a) $\frac{\partial\phi}{\partial x} = 4e^{2x} + 2xy - y^2$ and (b) $\frac{\partial\phi}{\partial y} = (x - y)^2$. From (b), $\phi(x, y) = x^2y - xy^2 + \frac{y^3}{3} + h(x) \implies \frac{\partial\phi}{\partial x} = 2xy - y^2 + \frac{dh(x)}{dx}$, and so from (a), $2xy - y^2 + \frac{dh(x)}{dx} = 4e^{2x} + 2xy - y^2 \implies \frac{dh(x)}{dx} = 4e^{2x} \implies h(x) = 2e^{2x}$, where the constant of integration has been set to zero since we need just one potential function. Therefore, we have $\phi(x, y) = x^2y - xy^2 + \frac{y^3}{3} + 2e^{2x}$; hence,

$$x^2y - xy^2 + \frac{y^3}{3} + 2e^{2x} = c_1 \implies 6e^{2x} + 3x^2y - 3xy^2 + y^3 = c.$$

8. Given $(y^2 - 2x)dx + 2xydy = 0$, we have $M_y = N_x = 2xy$. Therefore, the differential equation is exact and there exists a potential function ϕ such that (a) $\frac{\partial\phi}{\partial x} = y^2 - 2x$ and (b) $\frac{\partial\phi}{\partial y} = 2xy$. From (b), $\phi(x, y) = xy^2 + h(x) \implies \frac{\partial\phi}{\partial x} = y^2 + \frac{dh(x)}{dx}$, and so from (a), $y^2 + \frac{dh(x)}{dx} = y^2 - 2x \implies \frac{dh(x)}{dx} = -2x \implies h(x) = -x^2$, where the constant of integration has been set to zero since we just need one potential function. Therefore, we have $\phi(x, y) = xy^2 - x^2$; hence, $xy^2 - x^2 = c$.

9. Given $\left(\frac{1}{x} - \frac{y}{x^2 + y^2}\right)dx + \frac{x}{x^2 + y^2}dy = 0$, we have $M_y = N_x = \frac{y^2 - x^2}{(x^2 + y^2)^2}$. Therefore, the differential equation is exact and there exists a potential function ϕ such that (a) $\frac{\partial\phi}{\partial x} = \frac{1}{x} - \frac{y}{x^2 + y^2}$ and (b) $\frac{\partial\phi}{\partial y} = \frac{x}{x^2 + y^2}$. From (b), we have $\phi(x, y) = \tan^{-1}(y/x) + h(x) \implies \frac{\partial\phi}{\partial x} = -\frac{y}{x^2 + y^2} + \frac{dh(x)}{dx}$, and so from (a), $-\frac{y}{x^2 + y^2} + \frac{dh(x)}{dx} = \frac{1}{x} - \frac{y}{x^2 + y^2} \implies \frac{dh(x)}{dx} = x^{-1} \implies h(x) = \ln|x|$, where the constant of integration is set to zero since we only need one potential function. Therefore, we have $\phi(x, y) = \tan^{-1}(y/x) + \ln|x|$; hence, $\tan^{-1}(y/x) + \ln|x| = c$.

10. Given $[1 + \ln(xy)]dx + \frac{x}{y}dy = 0$, we have $M_y = N_x = y^{-1}$. Therefore, the differential equation is exact and there exists a potential function ϕ such that (a) $\frac{\partial\phi}{\partial x} = 1 + \ln(xy)$ and (b) $\frac{\partial\phi}{\partial y} = \frac{x}{y}$. From (b), we have $\phi(x, y) = x \ln y + h(x)$. Therefore, $\frac{\partial\phi}{\partial x} = \ln y + \frac{dh(x)}{dx}$, and so from (a), $\ln y + \frac{dh(x)}{dx} = 1 + \ln(xy)$. Hence, $\frac{dh(x)}{dx} = 1 + \ln x \implies h(x) = x \ln x$, where the constant of integration is set to zero since we only need one potential function. Therefore, we have $\phi(x, y) = x \ln y + x \ln x$; hence, $x \ln y + x \ln x = c$, or $x \ln(xy) = c$.

11. Given $[y \cos(xy) - \sin x]dx + x \cos(xy)dy = 0$, we have $M_y = N_x = -xy \sin(xy) + \cos(xy)$. Therefore, the differential equation is exact and there exists a potential function ϕ such that (a) $\frac{\partial\phi}{\partial x} = y \cos(xy) - \sin x$ and (b) $\frac{\partial\phi}{\partial y} = x \cos(xy)$. From (b), we have that $\phi(x, y) = \sin(xy) + h(x) \implies \frac{\partial\phi}{\partial x} = y \cos(xy) + \frac{dh(x)}{dx}$. Therefore, from (a), we have $y \cos(xy) + \frac{dh(x)}{dx} = y \cos(xy) - \sin x$. We conclude that $\frac{dh(x)}{dx} = -\sin x \implies h(x) = \cos x$, where the constant of integration is set to zero since we only need one potential function. Therefore, we have

$\phi(x, y) = \sin(xy) + \cos x$; hence, $\sin(xy) + \cos x = c$.

12. Given $(2xy + \cos y)dx + (x^2 - x \sin y - 2y)dy = 0$, we have $M_y = N_x = 2x - \sin y$. Therefore, the differential equation is exact and there exists a potential function ϕ such that (a) $\frac{\partial \phi}{\partial x} = 2xy + \cos y$ and (b) $\frac{\partial \phi}{\partial y} = x^2 - x \sin y - 2y$. From (a), we have $\phi(x, y) = x^2y + x \cos y + h(y) \Rightarrow \frac{\partial \phi}{\partial y} = x^2 - x \sin y + \frac{dh(y)}{dy}$. Therefore, from (b), we have $x^2 - x \sin y + \frac{dh(y)}{dy} = x^2 - x \sin y - 2y \Rightarrow \frac{dh}{dy} = -2y \Rightarrow h(y) = -y^2$, where the constant of integration has been set to zero since we only need one potential function. Therefore, we have $\phi(x, y) = x^2y + x \cos y - y^2$; hence, $x^2y + x \cos y - y^2 = c$.

13. Given $(3x^2 \ln x + x^2 - y)dx - xdy = 0$, we have $M_y = N_x = -1$. Therefore, the differential equation is exact and there exists a potential function ϕ such that (a) $\frac{\partial \phi}{\partial x} = 3x^2 \ln x + x^2 - y$ and (b) $\frac{\partial \phi}{\partial y} = -x$. From (b), we have $\phi(x, y) = -xy + h(x) \Rightarrow \frac{\partial \phi}{\partial x} = -y + \frac{dh(x)}{dx}$. Therefore, from (a), we have $-y + \frac{dh(x)}{dx} = 3x^2 \ln x + x^2 - y \Rightarrow \frac{dh(x)}{dx} = 3x^2 \ln x + x^2 \Rightarrow h(x) = x^3 \ln x$, where the constant of integration has been set to zero since we only need one potential function. Therefore, we have $\phi(x, y) = -xy + x^3 \ln x$; hence, $-xy + x^3 \ln x = c$. Now since $y(1) = 5$, we have $c = -5$. Thus, $x^3 \ln x - xy = -5$, or $y(x) = \frac{x^3 \ln x + 5}{x}$.

14. Given $2x^2 \frac{dy}{dx} + 4xy = 3 \sin x \Rightarrow (4xy - 3 \sin x)dx + 2x^2 dy = 0$, we have $M_y = N_x = 4x$. Therefore, the differential equation is exact, and so there exists a potential function ϕ such that (a) $\frac{\partial \phi}{\partial x} = 4xy - 3 \sin x$ and (b) $\frac{\partial \phi}{\partial y} = 2x^2$. From (b), we have $\phi(x, y) = 2x^2y + h(x) \Rightarrow \frac{\partial \phi}{\partial x} = 4xy + \frac{dh(x)}{dx}$, and so from (a), we have $4xy + \frac{dh(x)}{dx} = 4xy - 3 \sin x \Rightarrow \frac{dh(x)}{dx} = -3 \sin x \Rightarrow h(x) = 3 \cos x$, where the constant of integration has been set to zero since we only need one potential function. Therefore, we have $\phi(x, y) = 2x^2y + 3 \cos x$; hence $2x^2y + 3 \cos x = c$. Now since $y(2\pi) = 0$, we have $c = 3$. Therefore, $2x^2y + 3 \cos x = 3$, or $y(x) = \frac{3 - 3 \cos x}{2x^2}$.

15. Given $(ye^{xy} + \cos x)dx + xe^{xy}dy = 0$, we have $M_y = N_x = xye^{xy} + e^{xy}$. Therefore, the differential equation is exact, and so there exists a potential function ϕ such that (a) $\frac{\partial \phi}{\partial x} = ye^{xy} + \cos x$ and (b) $\frac{\partial \phi}{\partial y} = xe^{xy}$. From (b), we have $\phi(x, y) = e^{xy} + h(x) \Rightarrow \frac{\partial \phi}{\partial x} = ye^{xy} + \frac{dh(x)}{dx}$, and so from (a), we have $ye^{xy} + \cos x \Rightarrow \frac{dh(x)}{dx} = \cos x \Rightarrow h(x) = \sin x$, where the constant of integration is set to zero since we only need one potential function. Therefore, we have $\phi(x, y) = e^{xy} + \sin x$; hence, $e^{xy} + \sin x = c$. Now since $y(\pi/2) = 0$, we have $c = 2$. Thus, $e^{xy} + \sin x = 2$, and hence, $y(x) = \frac{\ln(2 - \sin x)}{x}$.

16. If $\phi(x, y)$ is a potential function for $M dx + N dy = 0$, then $d(\phi(x, y)) = 0$. Therefore, $d(\phi(x, y) + c) = d(\phi(x, y)) + d(c) = 0 + 0 = 0$, which implies that $\phi(x, y) + c$ is also a potential function.

17. We have

$$M = \cos(xy)[\tan(xy) + xy] \quad \text{and} \quad N = x^2 \cos(xy),$$

so that

$$M_y = 2x \cos(xy) - x^2 y \sin(xy) = N_x.$$

Thus, $M dx + N dy = 0$ is exact. We conclude that $I(x, y) = \cos(xy)$ is an integrating factor for $[\tan(xy) + xy]dx + x^2 dy = 0$.

18. We have

$$M = \sec x [2x - (x^2 + y^2) \tan x] \quad \text{and} \quad N = 2y \sec x,$$

so that

$$M_y = -2y \sec x \tan x \quad \text{and} \quad N_x = 2y \sec x \tan x.$$

Therefore, $M_y \neq N_x$. We conclude that $M dx + N dy = 0$ is not exact, and $I(x) = \sec x$ is not an integrating factor for $[2x - (x^2 + y^2) \tan x]dx + 2y dy = 0$.

19. We have

$$M = e^{-x/y}(x^2 y^{-1} - 2x) \quad \text{and} \quad N = -e^{-x/y} x^3 y^{-2},$$

so that

$$M_y = e^{-x/y}(x^3 y^{-3} - 3x^2 y^{-2}) = N_x.$$

Thus, $M dx + N dy = 0$ is exact. We conclude that $I(x, y) = y^{-2} e^{-x/y}$ is an integrating factor for $y[x^2 - 2xy]dx - x^3 dy = 0$.

20. Given $(xy - 1)dx + x^2 dy = 0$, we have $M = xy - 1$ and $N = x^2$. Thus $M_y = x$ and $N_x = 2x$, so $\frac{M_y - N_x}{N} = -x^{-1} = f(x)$ is a function of x alone so $I(x) = e^{\int f(x)dx} = x^{-1}$ is an integrating factor for the given equation. Multiplying the given equation by $I(x)$ results in the exact equation $(y - x^{-1})dx + x dy = 0$. We find that $\phi(x, y) = xy - \ln|x|$ and hence, the general solution of our differential equation is $xy - \ln|x| = c$.

21. Given $ydx - (2x + y^4)dy = 0$, we have $M = y$ and $N = -(2x + y^4)$. Thus $M_y = 1$ and $N_x = -2$, so $\frac{M_y - N_x}{M} = 3y^{-1} = g(y)$ is a function of y alone so $I(y) = e^{-\int g(y)dy} = 1/y^3$ is an integrating factor for the given differential equation. Multiplying the given equation by $I(y)$ results in the exact equation $y^{-2}dx - (2xy^{-3} + y)dy = 0$. We find that $\phi(x, y) = xy^{-2} - y^2/2$, and hence, the general solution of our differential equation is $xy^{-2} - y^2/2 = c_1 \implies 2x - y^4 = cy^2$.

22. Given $x^2 y dx + y(x^3 + e^{-3y} \sin y)dy = 0$, we have $M = x^2 y$ and $N = y(x^3 + e^{-3y} \sin y)$. Thus $M_y = x^2$ and $N_x = 3x^2 y$, so $\frac{M_y - N_x}{M} = y^{-1} - 3 = g(y)$ is a function of y alone so $I(y) = e^{\int g(y)dy} = e^{3y}/y$ is an integrating factor for the given equation. Multiplying the given equation by $I(y)$ results in the exact equation $x^2 e^{3y} dx + e^{3y}(x^3 + e^{-3y} \sin y)dy = 0$. We find that $\phi(x, y) = \frac{x^3 e^{3y}}{3} - \cos y$, and hence, the general solution of our differential equation is $\frac{x^3 e^{3y}}{3} - \cos y = c$.

23. Given $(y - x^2)dx + 2x dy = 0$, we have $M = y - x^2$ and $N = 2x$. Thus $M_y = 1$ and $N_x = 2$, so $\frac{M_y - N_x}{N} = -\frac{1}{2x} = f(x)$ is a function of x alone so $I(x) = e^{\int f(x)dx} = \frac{1}{\sqrt{x}}$ is an integrating factor for the given equation. Multiplying the given equation by $I(x)$ results in the exact equation $(x^{-1/2}y - x^{3/2})dx + 2x^{1/2}dy = 0$. We find

that $\phi(x, y) = 2x^{1/2}y - \frac{2x^{5/2}}{5}$, and hence, the general solution of our differential equation is $2x^{1/2}y - \frac{2x^{5/2}}{5} = c$ or $y(x) = \frac{c + 2x^{5/2}}{10\sqrt{x}}$.

24. Given $xy[2\ln(xy) + 1]dx + x^2dy = 0$, we have $M = xy[2\ln(xy) + 1]$ and $N = x^2$. Thus $M_y = 3x + 2x\ln(xy)$ and $N_x = 2x$, so $\frac{M_y - N_x}{M} = y^{-1} = g(y)$ is a function of y only so $I(y) = e^{\int g(y)dy} = \frac{1}{y}$ is an integrating factor for the given equation. Multiplying the given equation by $I(y)$ results in the exact equation $x[2\ln(xy) + 1]dx + x^2y^{-1}dy = 0$. We find that $\phi(x, y) = x^2\ln y + x^2\ln x$, and hence, the general solution of our differential equation is $x^2\ln y + x^2\ln x = c$ or $y(x) = xe^{c/x^2}$.

25. Given

$$\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{1}{(1+x^2)^2} \implies (2xy + 2x^3y - 1)dx + (1+x^2)^2dy = 0,$$

we have $M = 2xy + 2x^3y - 1$ and $N = (1+x^2)^2$. Thus $M_y = 2x + 2x^3$ and $N_x = 4x(1+x^2)$, so that $\frac{M_y - N_x}{N} = -\frac{2x}{1+x^2} = f(x)$ is a function of x alone so $I(x) = e^{\int f(x)dx} = \frac{1}{1+x^2}$ is an integrating factor for the given equation. Multiplying the given equation by $I(x)$ yields the exact equation $\left(2xy - \frac{1}{1+x^2}\right)dx + (1+x^2)dy = 0$. We find that $\phi(x, y) = (1+x^2)y - \tan^{-1}x$, and hence, the general solution of our differential equation is $(1+x^2)y - \tan^{-1}x = c$ or $y(x) = \frac{\tan^{-1}x + c}{1+x^2}$.

26. Given $(3xy - 2y^{-1})dx + x(x + y^{-2})dy = 0$, we have $M = 3xy - 2y^{-1}$ and $N = x(x + y^{-2})$. Thus $M_y = 3x + 2y^{-2}$ and $N_x = 2x + y^{-2}$, so that $\frac{M_y - N_x}{N} = \frac{1}{x} = f(x)$ is a function of x alone. Therefore, $I(x) = e^{\int f(x)dx} = x$ is an integrating factor for the given equation. Multiplying the given equation by $I(x)$ results in the exact equation $(3x^2y - 2xy^{-1})dx + x^2(x + y^{-2})dy = 0$. We find that $\phi(x, y) = x^3y - x^2y^{-1}$, and hence, the general solution of our differential equation is $x^3y - x^2y^{-1} = c$.

27. We are given

$$(y^{-1} - x^{-1})dx + (xy^{-2} - 2y^{-1})dy = 0 \implies x^r y^s (y^{-1} - x^{-1})dx + x^r y^s (xy^{-2} - 2y^{-1})dy = 0 \implies (x^r y^{s-1} - x^{r-1} y^s)dx + (x^{r+1} y^{s-2} - 2x^r y^{s-1})dy = 0$$

Therefore, $M = x^r y^{s-1} - x^{r-1} y^s$ and $N = x^{r+1} y^{s-2} - 2x^r y^{s-1}$, so that $M_y = x^r(s-1)y^{s-2} - x^{r-1}sy^{s-1}$ and $N_x = (r+1)x^r y^{s-2} - 2rx^{r-1}y^{s-1}$. The equation is exact if and only if $M_y = N_x$, which in turn requires that

$$x^r y^{s-1} - x^{r-1} y^s = (r+1)x^r y^{s-2} - 2rx^{r-1} y^{s-1} \implies \frac{s-1}{y^2} - \frac{s}{xy} = \frac{r+1}{y^2} - \frac{2r}{xy} \implies \frac{s-r-2}{y^2} = \frac{s-2r}{xy}.$$

From the last equation, exactness requires that $s-r-2=0$ and $s-2r=0$. Solving this system yields $r=2$ and $s=4$.

28. We are given

$$y(5xy^2 + 4)dx + x(xy^2 - 1)dy = 0 \implies x^r y^s y(5xy^2 + 4)dx + x^r y^s x(xy^2 - 1)dy = 0.$$

Therefore, $M = x^r y^{s+1}(5xy^2 + 4)$ and $N = x^{r+1}y^s(xy^2 - 1)$, so that $M_y = 5(s+3)x^{r+1}y^{s+2} + 4(s+1)x^r y^s$ and $N_x = (r+2)x^{r+1}y^{s-2} - (r+1)x^r y^s$. The equation is exact if and only if $M_y = N_x$, which in turn

requires that

$$5(s+3)x^{r+1}y^{s+2} + 4(s+1)x^r y^s = (r+2)x^{r+1}y^{s+2} - (r+1)x^r y^s \implies 5(s+3)xy^2 + 4(s+1) = (r+2)xy^2 - (r+1).$$

From the last equation, exactness requires that $5(s+3) = r+2$ and $4(s+1) = -(r+1)$. Solving this system yields $r = 3$ and $s = -2$.

29. We are given

$$2y(y+2x^2)dx + x(4y+3x^2)dy = 0 \implies x^r y^s 2y(y+2x^2)dx + x^r y^s x(4y+3x^2)dy = 0.$$

Therefore,

$$M = 2x^r y^{s+2} + 4x^{r+2} y^{s+1} \quad \text{and} \quad N = 4x^{r+1} y^{s+1} + 3x^{r+3} y^s,$$

so that

$$M_y = 2x^r(s+2)y^{s+1} + 4x^{r+2}(s+1)y^s \quad \text{and} \quad N_x = 4(r+1)x^r y^{s+1} + 3(r+3)x^{r+2} y^s.$$

The equation is exact if and only if $M_y = N_x$, which in turn requires that

$$2x^r(s+2)y^{s+1} + 4x^{r+2}(s+1)y^s = 4(r+1)x^r y^{s+1} + 3(r+3)x^{r+2} y^s \implies 2(s+2)y + 4x^2(s+1) = 4(r+1)y + 3(r+3)x^2.$$

From this last equation, exactness requires that $2(s+2) = 4(r+1)$ and $4(s+1) = 3(r+3)$. Solving this system yields $r = 1$ and $s = 2$.

30. Suppose that $\frac{M_y - N_x}{M} = g(y)$ is a function of y only. Dividing Equation (1.9.21) by M , it follows that I is an integrating factor for $M(x, y)dx + N(x, y)dy = 0$ if and only if I is a solution of the differential equation

$$\frac{N}{M} \frac{\partial I}{\partial x} - \frac{\partial I}{\partial y} = Ig(y).$$

We must show that this differential equation has a solution $I = I(y)$. However, if $I = I(y)$, then the differential equation reduces to $\frac{dI}{dy} = -Ig(y)$, which is a separable equation with solution $I(y) = e^{-\int g(t)dt}$.

31.

(a) Note that $\frac{dy}{dx} + py = q$ can be written in the differential form as $(py - q)dx + dy = 0$. This has $M = py - q$ and $N = 1$, so that $\frac{M_y - N_x}{N} = p(x)$. Consequently, an integrating factor is $I(x) = e^{\int^x p(t)dt}$.

(b) Multiplying $(py - q)dx + dy = 0$ by $I(x) = e^{\int^x p(t)dt}$ yields the exact equation $e^{\int^x p(t)dt}(py - q)dx + e^{\int^x p(t)dt}dy = 0$. Hence, there exists a potential function ϕ such that (a) $\frac{\partial \phi}{\partial x} = e^{\int^x p(t)dt}(py - q)$ and (b) $\frac{\partial \phi}{\partial y} = e^{\int^x p(t)dt}$. From (i), we have

$$p(x)ye^{\int^x p(t)dt} + \frac{dh(x)}{dx} = e^{\int^x p(t)dt}(py - q) \implies \frac{dh(x)}{dx} = -q(x)e^{\int^x p(t)dt} \implies h(x) = -\int q(x)e^{\int^x p(t)dt}dx,$$

where the constant of integration has been set to zero since we just need one potential function. Consequently, $\phi(x, y) = ye^{\int^x p(t)dt} - \int q(x)e^{\int^x p(t)dt}dx$, so that $y(x) = I^{-1}(\int^x Iq(t)dt + c)$, where $I(x) = e^{\int^x p(t)dt}$.

Solutions to Section 1.10

True-False Review:

1. **TRUE.** This is well-illustrated by the calculations shown in Example 1.10.1.
2. **TRUE.** The equation

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

is the tangent line to the curve $\frac{dy}{dx} = f(x, y)$ at the point (x_0, y_0) . Once the point (x_1, y_1) is determined, the procedure can be iterated over and over at the new points obtained to carry out Euler's method.

3. **FALSE.** It is possible, depending on the circumstances, for the errors associated with Euler's method to decrease from one step to the next.

Problems:

1. Applying Euler's method with $y' = 4y - 1$, $x_0 = 0$, $y_0 = 1$, and $h = 0.05$, we have $y_{n+1} = y_n + 0.05(4y_n - 1)$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.05	1.15
2	0.10	1.33
3	0.15	1.546
4	0.20	1.805
5	0.25	2.116
6	0.30	2.489
7	0.35	2.937
8	0.40	3.475
9	0.45	4.120
10	0.50	4.894

Consequently the Euler approximation to $y(0.5)$ is $y_{10} = 4.894$. (Actual value: $y(.05) = 5.792$ rounded to 3 decimal places).

2. Applying Euler's method with $y' = -\frac{2xy}{1+x^2}$, $x_0 = 0$, $y_0 = 1$, and $h = 0.1$, we have $y_{n+1} = y_n - 0.2\frac{x_n y_n}{1+x_n^2}$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.1	1
2	0.2	0.980
3	0.3	0.942
4	0.4	0.891
5	0.5	0.829
6	0.6	0.763
7	0.7	0.696
8	0.8	0.610
9	0.9	0.569
10	1.0	0.512

Consequently the Euler approximation to $y(1)$ is $y_{10} = 0.512$. (Actual value: $y(1) = 0.5$).

3. Applying Euler's method with $y' = x - y^2$, $x_0 = 0$, $y_0 = 2$, and $h = 0.05$, we have $y_{n+1} = y_n + 0.05(x_n - y_n^2)$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.05	1.80
2	0.10	1.641
3	0.15	1.511
4	0.20	1.404
5	0.25	1.316
6	0.30	1.242
7	0.35	1.180
8	0.40	1.127
9	0.45	1.084
10	0.50	1.048

Consequently the Euler approximation to $y(0.5)$ is $y_{10} = 1.048$. (Actual value: $y(.05) = 1.0477$ rounded to four decimal places).

4. Applying Euler's method with $y' = -x^2y$, $x_0 = 0$, $y_0 = 1$, and $h = 0.2$, we have $y_{n+1} = y_n - 0.2x_n^2y_n$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.2	1
2	0.4	0.992
3	0.6	0.960
4	0.8	0.891
5	1.0	0.777

Consequently the Euler approximation to $y(1)$ is $y_5 = 0.777$. (Actual value: $y(1) = 0.717$ rounded to 3 decimal places).

5. Applying Euler's method with $y' = 2xy^2$, $x_0 = 0$, $y_0 = 1$, and $h = 0.1$, we have $y_{n+1} = y_n + 0.1x_ny_n^2$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.1	0.5
2	0.2	0.505
3	0.3	0.515
4	0.4	0.531
5	0.5	0.554
6	0.6	0.584
7	0.7	0.625
8	0.8	0.680
9	0.9	0.754
10	1.0	0.858

Consequently the Euler approximation to $y(1)$ is $y_{10} = 0.856$. (Actual value: $y(1) = 1$).

6. Applying the modified Euler method with $y' = 4y - 1$, $x_0 = 0$, $y_0 = 1$, and $h = 0.05$, we have $y_{n+1}^* = y_n + 0.05(4y_n - 1)$
 $y_{n+1} = y_n + 0.025(4y_n - 1 + 4y_{n+1}^* - 1)$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.05	1.165
2	0.10	1.3663
3	0.15	1.6119
4	0.20	1.9115
5	0.25	2.2770
6	0.30	2.7230
7	0.35	3.2670
8	0.40	3.9308
9	0.45	4.7406
10	0.50	5.7285

Consequently the modified Euler approximation to $y(0.5)$ is $y_{10} = 5.7285$. (Actual value: $y(.05) = 5.7918$ rounded to 4 decimal places).

7. Applying the modified Euler method with $y' = -\frac{2xy}{1+x^2}$, $x_0 = 0$, $y_0 = 1$, and $h = 0.1$, we have $y_{n+1}^* = y_n - 0.2\frac{x_n y_n}{1+x_n^2}$
 $y_{n+1} = y_n + 0.05\left[-\frac{x_n y_n}{1+x_n^2} - 2\frac{x_{n+1} y_{n+1}^*}{1+x_{n+1}^2}\right]$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.1	0.9900
2	0.2	0.9616
3	0.3	0.9177
4	0.4	0.8625
5	0.5	0.8007
6	0.6	0.7163
7	0.7	0.6721
8	0.8	0.6108
9	0.9	0.5536
10	1.0	0.5012

Consequently the modified Euler approximation to $y(1)$ is $y_{10} = 0.5012$. (Actual value: $y(1) = 0.5$).

8. Applying the modified Euler method with $y' = x - y^2$, $x_0 = 0$, $y_0 = 2$, and $h = 0.05$, we have $y_{n+1}^* = y_n - 0.05(x_n - y_n^2)$
 $y_{n+1} = y_n + 0.025(x_n - y_n^2 + x_{n+1} - (y_{n+1}^*)^2)$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.05	1.8203
2	0.10	1.6725
3	0.15	1.5497
4	0.20	1.4468
5	0.25	1.3600
6	0.30	1.2866
7	0.35	1.2243
8	0.40	1.1715
9	0.45	1.1269
10	0.50	1.0895

Consequently the modified Euler approximation to $y(0.5)$ is $y_{10} = 1.0895$. (Actual value: $y(.05) = 1.0878$ rounded to 4 decimal places).

9. Applying the modified Euler method with $y' = -x^2y$, $x_0 = 0$, $y_0 = 1$, and $h = 0.2$, we have $y_{n+1}^* = y_n - 0.2x_n^2y_n$
 $y_{n+1} = y_n - 0.1[x_n^2y_n + x_{n+1}^2y_{n+1}^*]$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.2	0.9960
2	0.4	0.9762
3	0.6	0.9266
4	0.8	0.8382
5	1.0	0.7114

Consequently the modified Euler approximation to $y(1)$ is $y_5 = 0.7114$. (Actual value: $y(1) = 0.7165$ rounded to 4 decimal places).

10. Applying the modified Euler method with $y' = 2xy^2$, $x_0 = 0$, $y_0 = 1$, and $h = 0.1$, we have $y_{n+1}^* = y_n + 0.1x_ny_n^2$
 $y_{n+1} = y_n + 0.05[x_ny_n^2 + x_{n+1}(y_{n+1}^*)^2]$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.1	0.5025
2	0.2	0.5102
3	0.3	0.5235
4	0.4	0.5434
5	0.5	0.5713
6	0.6	0.6095
7	0.7	0.6617
8	0.8	0.7342
9	0.9	0.8379
10	1.0	0.9941

Consequently the modified Euler approximation to $y(1)$ is $y_{10} = 0.9941$. (Actual value: $y(1) = 1$).

11. We have $y' = 4y - 1$, $x_0 = 0$, $y_0 = 1$, and $h = 0.05$. So,

$$k_1 = 0.05(4y_n - 1), \quad k_2 = 0.05[4(y_n + \frac{1}{2}k_1) - 1], \quad k_3 = 0.05[4(y_n + \frac{1}{2}k_2) - 1], \quad k_4 = 0.05[4(y_n + \frac{1}{2}k_3) - 1].$$

Using $y_{n+1} = y_n + \frac{1}{6}(k_1 + k_2 + k_3 + k_4)$, we generate the sequence of approximants given in the table below (computations rounded to five decimal places).

n	x_n	y_n
1	0.05	1.16605
2	0.10	1.36886
3	0.15	1.61658
4	0.20	1.91914
5	0.25	2.28868
6	0.30	2.74005
7	0.35	3.29135
8	0.40	3.96471
9	0.45	4.78714
10	0.50	5.79167

Consequently the Runge-Kutta approximation to $y(0.5)$ is $y_{10} = 5.79167$. (Actual value: $y(.05) = 5.79179$ rounded to 5 decimal places).

12. We have $y' = -2\frac{xy}{1+x^2}$, $x_0 = 0$, $y_0 = 1$, and $h = 0.1$. So,

$$k_1 = -0.2\frac{x_n y_n}{1+x_n^2}, \quad k_2 = -0.2\frac{(x_n + 0.05)(y_n + \frac{k_1}{2})}{[1+(x_n + 0.05)^2]}, \quad k_3 = -0.2\frac{(x_n + 0.05)(y_n + \frac{k_2}{2})}{[1+(x_n + 0.05)^2]}, \quad k_4 = -0.2\frac{x_{n+1}(y_n + k_3)}{[1+(x_{n+1})^2]}.$$

Using $y_{n+1} = y_n + \frac{1}{6}(k_1 + k_2 + k_3 + k_4)$, we generate the sequence of approximants given in the table below (computations rounded to seven decimal places).

n	x_n	y_n
1	0.1	0.9900990
2	0.2	0.9615383
3	0.3	0.9174309
4	0.4	0.8620686
5	0.5	0.7999996
6	0.6	0.7352937
7	0.7	0.6711406
8	0.8	0.6097558
9	0.9	0.5524860
10	1.0	0.4999999

Consequently the Runge-Kutta approximation to $y(1)$ is $y_{10} = 0.4999999$. (Actual value: $y(.05) = 0.5$).

13. We have $y' = x - y^2$, $x_0 = 0$, $y_0 = 2$, and $h = 0.05$. So,

$$k_1 = 0.05(x_n - y_n^2), \quad k_2 = 0.05[x_n + 0.025 - (y_n + \frac{k_1}{2})^2], \quad k_3 = 0.05[x_n + 0.025 - (y_n + \frac{k_2}{2})^2], \quad k_4 = 0.05[x_{n+1} - (y_n + k_3)^2].$$

Using $y_{n+1} = y_n + \frac{1}{6}(k_1 + k_2 + k_3 + k_4)$, we generate the sequence of approximants given in the table below (computations rounded to six decimal places).

n	x_n	y_n
1	0.05	1.1.81936
2	0.10	1.671135
3	0.15	1.548079
4	0.20	1.445025
5	0.25	1.358189
6	0.30	1.284738
7	0.35	1.222501
8	0.40	1.169789
9	0.45	1.125263
10	0.50	1.087845

Consequently the Runge-Kutta approximation to $y(0.5)$ is $y_{10} = 1.087845$. (Actual value: $y(0.5) = 1.087845$ rounded to 6 decimal places).

14. We have $y' = -x^2y$, $x_0 = 0$, $y_0 = 1$, and $h = 0.2$. So,

$$k_1 = -0.2x_n^2y_n, \quad k_2 = -0.2(x_n + 0.1)^2(y_n + \frac{k_1}{2}), \quad k_3 = -0.2(x_n + 0.1)^2(y_n + \frac{k_2}{2}), \quad k_4 = -0.2(x_{n+1})^2(y_n + k_3).$$

Using $y_{n+1} = y_n + \frac{1}{6}(k_1 + k_2 + k_3 + k_4)$, we generate the sequence of approximants given in the table below (computations rounded to six decimal places).

n	x_n	y_n
1	0.2	0.997337
2	0.4	0.978892
3	0.6	0.930530
4	0.8	0.843102
5	1.0	0.716530

Consequently the Runge-Kutta approximation to $y(1)$ is $y_{10} = 0.716530$. (Actual value: $y(1) = 0.716531$ rounded to 6 decimal places).

15. We have $y' = 2xy^2$, $x_0 = 0$, $y_0 = 1$, and $h = 0.1$. So,

$$k_1 = 0.2x_n - y_n^2, \quad k_2 = 0.2(x_n + 0.05)(y_n + \frac{k_1}{2})^2, \quad k_3 = 0.2(x_n + 0.05)(y_n + \frac{k_2}{2})^2, \quad k_4 = 0.2x_{n+1}(y_n + k_3)^2.$$

Using $y_{n+1} = y_n + \frac{1}{6}(k_1 + k_2 + k_3 + k_4)$, we generate the sequence of approximants given in the table below (computations rounded to six decimal places).

n	x_n	y_n
1	0.1	0.502513
2	0.2	0.510204
3	0.3	0.523560
4	0.4	0.543478
5	0.5	0.571429
6	0.6	0.609756
7	0.7	0.662252
8	0.8	0.735295
9	0.9	0.840336
10	1.0	0.999996

Consequently the Runge-Kutta approximation to $y(1)$ is $y_{10} = 0.999996$. (Actual value: $y(1) = 1$).

16. We have $y' + \frac{1}{10}y = e^{-x/10} \cos x, x_0 = 0, y_0 = 0$, and $h = 0.5$ Hence,

$$\begin{aligned} k_1 &= 0.5 \left(-\frac{1}{10}y_n + e^{x_n/10} \cos x_n \right), \\ k_2 &= 0.5 \left[-\frac{1}{10} \left(y_n + \frac{1}{2}k_1 \right) + e^{(-x_n+0.25)/10} \cos (x_n + 0.25) \right], \\ k_3 &= 0.5 \left[-\frac{1}{10} \left(y_n + \frac{1}{2}k_2 \right) + e^{(-x_n+0.25)/10} \cos (x_n + 0.25) \right], \\ k_4 &= 0.5 \left[-\frac{1}{10}(y_n + k_3) + e^{-x_{n+1}/10} \cos x_{n+1} \right]. \end{aligned}$$

Using

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

we generate the sequence of approximants plotted in the accompanying figure. We see that the solution appears to be oscillating with a diminishing amplitude. Indeed, the exact solution to the initial value problem is $y(x) = e^{-x/10} \sin x$. The corresponding solution curve is also given in the figure.

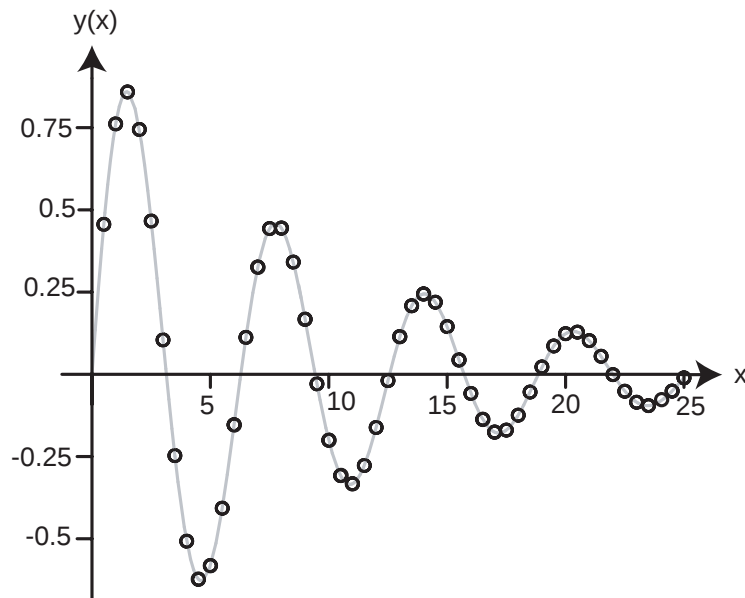


Figure 56: Figure for Problem 16

Solutions to Section 1.11

Problems:

1. Rewriting the equation, we have $\frac{d^2y}{dx^2} = \frac{2}{x} \frac{dy}{dx} + 4x^2$. Let $u = \frac{dy}{dx}$, so that $\frac{du}{dx} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $\frac{du}{dx} = \frac{2}{x}u + 4x^2 \implies \frac{du}{dx} - \frac{2}{x}u = 4x^2$. An appropriate integrating factor for this equation is

$$I(x) = e^{-2 \int \frac{dx}{x}} = x^{-2}.$$

Therefore,

$$\begin{aligned} d(x^{-2}u) &= 4 \implies x^{-2}u = 4 \int dx \\ &\implies x^{-2}u = 4x + c_1 \\ &\implies u = 4x^3 + c_1x^2 \\ &\implies \frac{dy}{dx} = 4x^3 + c_1x^2 \\ &\implies y(x) = c_2x^3 + x^4 + c_3. \end{aligned}$$

2. Rewriting the equation, we have $\frac{d^2y}{dx^2} = \frac{1}{(x-1)(x-2)} \left[\frac{dy}{dx} - 1 \right]$. Let $u = \frac{dy}{dx}$, so that $\frac{du}{dx} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $\frac{du}{dx} = \frac{1}{(x-1)(x-2)}(u-1)$, or

$$\frac{du}{dx} - \frac{1}{(x-1)(x-2)}u = -\frac{1}{(x-1)(x-2)}.$$

An appropriate integrating factor for this equation is

$$I(x) = e^{-\int \frac{1}{(x-1)(x-2)} dx} = \frac{x-1}{x-2}.$$

Therefore,

$$\begin{aligned} \frac{d}{dx} \left(\frac{x-1}{x-2} u \right) &= -\frac{1}{(x-2)^2} \implies \frac{x-1}{x-2} u = -\int (x-2)^{-2} dx = \frac{1}{x-2} + c \\ &\implies u = \frac{1}{x-1} + c_1 \frac{x-2}{x-1} \\ &\implies \frac{dy}{dx} = \frac{1}{x-1} + c_1 \frac{x-2}{x-1} = \frac{1}{x-1} + c_1 - \frac{c_1}{x-1} \\ &\implies y(x) = \ln|x-1| + c_1x - c_1 \ln|x-1| + c_2. \end{aligned}$$

3. Rewriting the equation, we have $\frac{d^2y}{dx^2} + \frac{2}{y} \left(\frac{dy}{dx} \right)^2 = \frac{dy}{dx}$. Let $u = \frac{dy}{dx}$, so that $\frac{du}{dx} = u \frac{du}{dy} = \frac{d^2y}{dx^2}$.

Substituting these results into the first equation yields $u \frac{du}{dy} + \frac{2}{y} u^2 = u$, which implies that either $u = 0$ or

$\frac{du}{dy} + \frac{2}{y}u = 1$. If $u = 0$, then $\frac{du}{dx} = 0$, which implies that y is constant. Therefore, a constant function is a solution to the equation. Turning to the second possibility, an appropriate integrating factor for the latter

equation is $I(y) = e^{\int \frac{2}{y} dy} = y^2$. Therefore,

$$\begin{aligned}\frac{d}{dy}(y^2 u) &= y^2 \implies y^2 u = \int y^2 dy \\ &\implies y^2 u = \frac{y^3}{3} + c_1 \\ &\implies \frac{dy}{dx} = \frac{y}{3} + \frac{c_1}{y^2}.\end{aligned}$$

This is a Bernoulli equation, which can be solved by previous techniques to yield

$$\ln |y^3 + c_2| = x + c_3 \implies y(x) = \sqrt[3]{c_4 e^x + c_5}.$$

Note that the first possibility (y constant) is included here in the case when $c_4 = 0$.

4. Rewriting the equation, we have $\frac{d^2 y}{dx^2} = \left(\frac{dy}{dx}\right)^2 \tan y$. Let $u = \frac{dy}{dx}$, so that $\frac{du}{dx} = u \frac{du}{dy} = \frac{d^2 y}{dx^2}$. Substituting these results into the first equation yields $u \frac{du}{dy} = u^2 \tan y$. If $u = 0$ then $\frac{du}{dx} = 0 \implies y$ equals a constant and this is a solution to the equation. Now suppose that $u \neq 0$. Then

$$\frac{du}{dy} = u \tan y \implies \int \frac{du}{u} = \int \tan y dy \implies u = c_1 \sec y \implies \frac{dy}{dx} = c_1 \sec y \implies y(x) = \sin^{-1}(c_1 x + c_2).$$

5. Rewriting the equation, we have $\frac{d^2 y}{dx^2} + \tan x \frac{dy}{dx} = \left(\frac{dy}{dx}\right)^2$. Let $u = \frac{dy}{dx}$, so that $\frac{du}{dx} = \frac{d^2 y}{dx^2}$. Substituting these results into the first equation yields $\frac{du}{dx} + \tan x u = u^2$, which is a Bernoulli equation. Letting $z = u^{-1}$ gives $\frac{1}{u^2} = \frac{du}{dx} = -\frac{dz}{dx}$. Substituting these results into the last equation yields $\frac{dz}{dx} - \tan x z = -1$. Then an integrating factor for this equation is $I(x) = e^{-\int \tan x dx} = \cos x$. Therefore,

$$\begin{aligned}\frac{d}{dx}(z \cos x) &= -\cos x \implies z \cos x = -\int \cos x dx \\ &\implies z = \frac{-\sin x + c_1}{\cos x} \\ &\implies u = \frac{\cos x}{c_1 - \sin x} \\ &\implies \frac{dy}{dx} = \frac{\cos x}{c_1 - \sin x} \\ &\implies y(x) = c_2 - \ln |c_1 - \sin x|.\end{aligned}$$

6. Rewriting the equation, we have $\frac{d^2 x}{dt^2} = \left(\frac{dx}{dt}\right)^2 + 2\frac{dx}{dt}$. Let $u = \frac{dx}{dt}$, so that $\frac{du}{dt} = \frac{d^2 x}{dt^2}$. Substituting these results into the first equation yields $\frac{du}{dt} = u^2 + 2u \implies \frac{du}{dt} - 2u = u^2$, which is a Bernoulli equation. If $u = 0$, then x is a constant function. Such a function satisfies the given differential equation. Now suppose

that $u \neq 0$. Let $z = u^{-1}$, so that $\frac{dz}{dt} = -\frac{1}{u^2} \frac{du}{dt}$. Substituting these results into the last equation yields $\frac{dz}{dt} + 2z = -1$. An integrating factor for this equation is $I(t) = e^{2t}$. Therefore,

$$\begin{aligned}\frac{d}{dt}(e^{2t}z) &= -e^{2t} \implies z = ce^{-2t} - \frac{1}{2} \\ \implies u &= \frac{2e^{2t}}{2c - e^{2t}} \\ \implies x(t) &= \int \frac{2e^{2t}}{2c - e^{2t}} dt \\ \implies x(t) &= c_2 - \ln|c_1 - e^{2t}|.\end{aligned}$$

7. Rewriting the equation, we have $\frac{d^2y}{dx^2} - \frac{2}{x} \cdot \frac{dy}{dx} = 6x^4$. Let $u = \frac{dy}{dx}$, so that $\frac{du}{dx} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $\frac{du}{dx} - \frac{2}{x}u = 6x^4$. An appropriate integrating factor for this equation is $I(x) = e^{-2 \int \frac{dx}{x}} = x^{-2}$. Therefore,

$$\begin{aligned}\frac{d}{dx}(x^{-2}u) &= 6x^2 \implies x^{-2}u = 6 \int x^2 dx \\ \implies u &= 2x^5 + cx^2 \\ \implies \frac{dy}{dx} &= 2x^5 + cx^2 \\ \implies y(x) &= \frac{1}{3}x^6 + c_1x^3 + c_2.\end{aligned}$$

8. Rewriting the equation, we have $t \frac{d^2x}{dt^2} = 2 \left(t + \frac{dx}{dt} \right)$. Let $u = \frac{dx}{dt}$, so that $\frac{du}{dt} = \frac{d^2x}{dt^2}$. Substituting these results into the first equation yields $\frac{du}{dt} - \frac{2}{t}u = 2$. An integrating factor for this equation is $I(t) = t^{-2}$. Therefore,

$$\begin{aligned}\frac{d}{dt}(t^{-2}u) &= 2t^{-2} \implies u = -2t + ct^2 \\ \implies \frac{dx}{dt} &= -2t + ct^2 \\ \implies x(t) &= c_1t^3 - t^2 + c_2.\end{aligned}$$

9. Rewriting the equation, we have $\frac{d^2y}{dx^2} - \alpha \left(\frac{dy}{dx} \right)^2 - \beta \frac{dy}{dx} = 0$. Let $u = \frac{dy}{dx}$, so that $\frac{du}{dx} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $\frac{du}{dx} - \beta u = \alpha u^2$, which is a Bernoulli equation. If $u = 0$ then y is a constant function, and such a function satisfies the differential equation. Now suppose that $u \neq 0$. Let $z = u^{-1}$ so that $\frac{dz}{dx} = -u^{-2} \frac{du}{dx}$. Substituting these results into the last equation yields $\frac{dz}{dx} + \beta z = -\alpha$. An

integrating factor for this equation is $I(x) = e^{\beta \int dx} = e^{\beta x}$. Therefore,

$$\begin{aligned} e^{\beta x} z &= -\alpha \int e^{\beta x} dx \implies z = -\frac{\alpha}{\beta} + ce^{-\beta x} \\ \implies u &= \frac{\beta e^{\beta x}}{c\beta - \alpha e^{\beta x}} \\ \implies \frac{dy}{dx} &= \frac{\beta e^{\beta x}}{c\beta - \alpha e^{\beta x}} \\ \implies y(x) &= \int \frac{\beta e^{\beta x}}{c\beta - \alpha e^{\beta x}} dx \\ \implies y(x) &= -\frac{1}{\alpha} \ln |c_1 + c_2 e^{\beta x}|. \end{aligned}$$

10. Rewriting the equation, we have $\frac{d^2 y}{dx^2} - \frac{2}{x} \frac{dy}{dx} = 18x^4$. Let $u = \frac{dy}{dx}$, so that $\frac{du}{dx} = \frac{d^2 y}{dx^2}$. Substituting these results into the first equation yields $\frac{du}{dx} - \frac{2}{x} u = 18x^4$, which is a first-order linear differential equation. An integrating factor for this equation is $I(x) = x^{-2}$. Therefore,

$$\begin{aligned} \frac{d}{dx}(x^{-2}u) &= 18x^2 \implies u = 6x^5 + cx^2 \\ \implies \frac{dy}{dx} &= 6x^5 + cx^2 \\ \implies y(x) &= x^6 + c_1 x^3 + c_2. \end{aligned}$$

11. Rewriting the equation, we have $\frac{d^2 y}{dx^2} = -\frac{2x}{1+x^2} \frac{dy}{dx}$. Let $u = \frac{dy}{dx}$, so that $\frac{du}{dx} = \frac{d^2 y}{dx^2}$. If $u = 0$, then y is a constant function, and such a function satisfies the differential equation. Now suppose that $u \neq 0$. Substituting these results into the first equation yields $\frac{du}{dx} = -\frac{2x}{1+x^2} u$, a separable differential equation. Separating variables and integrating both sides, we obtain

$$\begin{aligned} \ln |u| &= -\ln(1+x^2) + c \implies u = \frac{c_1}{1+x^2} \\ \implies \frac{dy}{dx} &= \frac{c_1}{1+x^2} \\ \implies y(x) &= c_1 \tan^{-1} x + c_2. \end{aligned}$$

12. Rewriting the equation, we have $\frac{d^2 y}{dx^2} + \frac{1}{y} \left(\frac{dy}{dx} \right)^2 = ye^{-y} \left(\frac{dy}{dx} \right)^3$. Let $u = \frac{dy}{dx}$, so that $\frac{du}{dx} = u \frac{du}{dy} = \frac{d^2 y}{dx^2}$. Substituting these results into the first equation yields $u \frac{du}{dy} + \frac{1}{y} u^2 = ye^{-y} u^3$. If $u = 0$, then y is a constant function, and such a function satisfies the differential equation. Now suppose that $u \neq 0$. Dividing the differential equation for u in terms of y by u gives $\frac{du}{dy} + \frac{u}{y} = ye^{-y} u^2$, which is a Bernoulli equation. Let $v = u^{-1}$ so that $\frac{dv}{dy} = -u^{-2} \frac{du}{dy}$. Substituting these results into the last equation in the usual manner for a

Bernoulli equation yields $\frac{dv}{dy} - \frac{v}{y} = -ye^{-y}$, a first-order linear differential equation for v as a function of y . An integrating factor for this differential equation is $I(y) = y^{-1}$. Therefore,

$$\begin{aligned}\frac{d}{dy}(y^{-1}v) &= -e^{-y} \implies v = y(e^{-y} + c) \\ \implies u &= \frac{e^y}{y + cye^y} \\ \implies \frac{dy}{dx} &= \frac{e^y}{y + cye^y} \\ \implies (ye^{-y} + cy)dy &= dx \\ \implies e^{-y}(y + 1) + c_1y^2 &= x.\end{aligned}$$

13. Rewriting the equation, we have $\frac{d^2y}{dx^2} - \tan x \frac{dy}{dx} = 1$. Let $u = \frac{dy}{dx}$, so that $\frac{du}{dx} = u \frac{du}{dy} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $\frac{du}{dx} - u \tan x = 1$. An appropriate integrating factor for this equation is $I(x) = e^{-\int \tan x dx} = \cos x$. Therefore,

$$\begin{aligned}\frac{d}{dx}(u \cos x) &= \cos x \implies u \cos x = \sin x + c \\ \implies u(x) &= \tan x + c \sec x \\ \implies \frac{dy}{dx} &= \tan x + c \sec x \\ \implies y(x) &= \ln \sec x + c_1 \ln(\sec x + \tan x) + c_2.\end{aligned}$$

14. Rewriting the equation, we have $y \frac{d^2y}{dx^2} = 2 \left(\frac{dy}{dx} \right)^2 + y^2$. Let $u = \frac{dy}{dx}$, so that $\frac{du}{dx} = u \frac{du}{dy} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $u \frac{du}{dy} - \frac{2}{y}u^2 = y$, a Bernoulli equation. Let $z = u^2$ so that $u \frac{du}{dy} = \frac{1}{2} \frac{dz}{dy}$. Substituting these results into the last equation yields $\frac{dz}{dy} - \frac{4}{y}z = 2y$, which has an integrating factor $I(y) = y^{-4}$. Therefore,

$$\begin{aligned}\frac{d}{dy}(y^{-4}z) &= 2y^{-3} \implies z = c_1y^4 - y^2 \\ \implies u^2 &= c_1y^4 - y^2 \\ \implies u &= \pm \sqrt{c_1y^4 - y^2} \\ \implies \frac{dy}{dx} &= \pm \sqrt{c_1y^4 - y^2} \\ \implies \cos^{-1} \left(\frac{1}{y\sqrt{c_1}} \right) &= \pm x + c_2.\end{aligned}$$

Using the facts that $y(0) = 1$ and $y'(0) = 0$, we find that $c_1 = 1$ and $c_2 = 0$. Thus, $y(x) = \sec x$.

15. We have $\frac{d^2y}{dx^2} = \omega^2 y$, where $\omega > 0$. Let $u = \frac{dy}{dx}$, so that $\frac{du}{dx} = u \frac{du}{dy} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation yields $u \frac{du}{dy} = \omega^2 y \implies u^2 = \omega^2 y^2 + c_2$. Using the given that $y(0) = a$ and $y'(0) = 0$, we find that $c_2 = -a^2 \omega^2$. Then

$$\frac{dy}{dx} = \pm \omega \sqrt{y^2 - a^2},$$

which is a separable differential equation. Separating variables, we obtain

$$\begin{aligned} \frac{1}{\omega} \cosh^{-1}(y/a) &= \pm x + c \implies y(x) = a \cosh[\omega(c \pm x)] \\ &\implies y'(x) = \pm a \omega \sinh[\omega(c \pm x)], \end{aligned}$$

and since $y'(0) = 0$, $c = 0$. Hence, $y(x) = a \cosh(\omega x)$.

16. Let $u = \frac{dy}{dx}$, so that $u \frac{du}{dx} = \frac{d^2y}{dx^2}$. Substituting these results into the differential equation yields $u \frac{du}{dy} = \frac{1}{a} \sqrt{1+u^2}$. Separating the variables and integrating we obtain $\sqrt{1+u^2} = \frac{1}{a}y + c$. Imposing the initial conditions $y(0) = a$, $\frac{dy}{dx}(0) = 0$ gives $c = 0$. Hence, $\sqrt{1+u^2} = \frac{1}{a}y$ so that $1+u^2 = \frac{1}{a^2}y^2$ or equivalently, $u = \pm \sqrt{y^2/a^2 - 1}$. Substituting $u = \frac{dy}{dx}$ and separating the variables gives $\frac{1}{\sqrt{y^2 - a^2}} = \pm \frac{1}{|a|} dx$ which can be integrated to obtain $\cosh^{-1}(y/a) = \pm x/a + c_1$ so that $y = a \cosh(\pm x/a + c_1)$. Imposing the initial condition $y(0) = a$ gives $c_1 = 0$ so that $y(x) = a \cosh(x/a)$.

17. Let $u = \frac{dy}{dx}$, so that $\frac{du}{dx} = \frac{d^2y}{dx^2}$. Substituting these results into the first equation gives us the equivalent differential equation $\frac{du}{dx} + p(x)u = q(x)$. This has a solution

$$u(x) = e^{-\int p(x)dx} \left[\int e^{\int p(x)dx} q(x)dx + c_1 \right],$$

so that

$$\frac{dy}{dx} = e^{-\int p(x)dx} \left[\int e^{\int p(x)dx} q(x)dx + c_1 \right].$$

Thus

$$y(x) = \int \left\{ e^{-\int p(x)dx} \left[\int e^{\int p(x)dx} q(x)dx + c_1 dx \right] \right\} + c_2$$

is a solution to the original differential equation.

18.

(a) We have

$$u_1 = y \implies u_2 = \frac{du_1}{dx} = \frac{dy}{dx} \implies u_3 = \frac{du_2}{dx} = \frac{d^2y}{dx^2} \implies \frac{du_3}{dx} = \frac{d^3y}{dx^3}.$$

Thus, $\frac{d^3y}{dx^3} = F\left(x, \frac{d^2y}{dx^2}\right)$, since the latter equation is equivalent to $\frac{du_3}{dx} = F(x, u_3)$.

(b) We can replace $\frac{d^3y}{dx^3} = \frac{1}{x} \left(\frac{d^2y}{dx^2} - 1 \right)$ by the equivalent first order system:

$$\frac{du_1}{dx} = u_2, \quad \frac{du_2}{dx} = u_3, \quad \text{and} \quad \frac{du_3}{dx} = \frac{1}{x}(u_3 - 1).$$

Therefore,

$$\begin{aligned} \int \frac{du_3}{u_3 - 1} &= \int \frac{dx}{x} \implies u_3 = Kx + 1 \\ &\implies \frac{du_2}{dx} = Kx + 1 \\ &\implies u_2 = \frac{K}{2}x^2 + x + c_2 \\ &\implies \frac{du_1}{dx} = \frac{K}{2}x^2 + x + c_2 \\ &\implies u_1 = \frac{K}{6}x^3 + \frac{1}{2}x^2 + c_2x + c_3 \\ &\implies y(x) = u_1 = c_1x^3 + \frac{1}{2}x^2 + c_2x + c_3. \end{aligned}$$

19.

(a) Let $u = \frac{d\theta}{dt}$ so that

$$\frac{du}{dt} = \frac{d^2\theta}{dt^2} = \frac{du}{d\theta} \frac{d\theta}{dt} = u \frac{du}{d\theta}.$$

Substituting these results into the given linear differential equation yields $u \frac{du}{d\theta} + \frac{g}{L}\theta = 0$, and integrating this with respect to θ gives $u^2 = -\frac{g}{L}\theta^2 + c_1$, for some constant c_1 . But we are given the initial conditions $\frac{d\theta}{dt}(0) = 0$ and $\theta(0) = \theta_0$, from which it follows that $c_1 = \frac{g}{L}\theta_0^2$. Therefore,

$$\begin{aligned} u^2 &= \frac{g}{L}(\theta_0^2 - \theta^2) \implies u = \pm \sqrt{\frac{g}{L}} \sqrt{\theta_0^2 - \theta^2} \\ &\implies \sin^{-1} \left(\frac{\theta}{\theta_0} \right) = \pm \sqrt{\frac{g}{L}} t + c_2 \end{aligned}$$

. However, since $\theta(0) = \theta_0$, we find that $c_2 = \frac{\pi}{2}$. Thus,

$$\begin{aligned} \sin^{-1} \left(\frac{\theta}{\theta_0} \right) &= \frac{\pi}{2} \pm \sqrt{\frac{g}{L}} t \implies \theta = \theta_0 \sin \left(\frac{\pi}{2} \pm \sqrt{\frac{g}{L}} t \right) \\ &\implies \theta = \theta_0 \cos \left(\sqrt{\frac{g}{L}} t \right). \end{aligned}$$

Yes, the predicted motion is reasonable.

(b) Let $u = \frac{d\theta}{dt}$ so that

$$\frac{du}{dt} = \frac{d^2\theta}{dt^2} = \frac{du}{d\theta} \frac{d\theta}{dt} = u \frac{du}{d\theta}.$$

Substituting these results into (1.11.28) yields $u \frac{du}{d\theta} + \frac{g}{L} \sin \theta = 0$, and integrating this with respect to θ gives $u^2 = \frac{2g}{L} \cos \theta + c$, for some constant c . Since $\theta(0) = \theta_0$ and $\frac{d\theta}{dt}(0) = 0$, then $c = -\frac{2g}{L} \cos \theta_0$. Therefore,

$$\begin{aligned} u^2 = \frac{2g}{L} \cos \theta - \frac{2g}{L} \cos \theta_0 &\implies \frac{d\theta}{dt} = \pm \sqrt{\frac{2g}{L} \cos \theta - \frac{2g}{L} \cos \theta_0} \\ &\implies \frac{d\theta}{dt} = \pm \sqrt{\frac{2g}{L} [\cos \theta - \cos \theta_0]}^{1/2}. \end{aligned}$$

(c) From part (b), $\sqrt{\frac{L}{2g}} \frac{d\theta}{[\cos \theta - \cos \theta_0]^{1/2}} = \pm dt$. When the pendulum goes from $\theta = \theta_0$ to $\theta = 0$ (which corresponds to one quarter of a period) $\frac{d\theta}{dt}$ is negative; hence, choose the negative sign. Thus,

$$T = -\sqrt{\frac{L}{2g}} \int_{\theta_0}^0 \frac{d\theta}{[\cos \theta - \cos \theta_0]^{1/2}} \implies T = \sqrt{\frac{L}{2g}} \int_0^{\theta_0} \frac{d\theta}{[\cos \theta - \cos \theta_0]^{1/2}}.$$

(d) From $T = \sqrt{\frac{L}{2g}} \int_0^{\theta_0} \frac{d\theta}{[\cos \theta - \cos \theta_0]^{1/2}}$, we have

$$T = \sqrt{\frac{L}{2g}} \int_0^{\theta_0} \frac{d\theta}{\left[2 \sin^2 \left(\frac{\theta_0}{2}\right) - 2 \sin^2 \left(\frac{\theta}{2}\right)\right]^{1/2}} = \frac{1}{2} \sqrt{\frac{L}{2g}} \int_0^{\theta_0} \frac{d\theta}{\left[\sin^2 \left(\frac{\theta_0}{2}\right) - \sin^2 \left(\frac{\theta}{2}\right)\right]^{1/2}}.$$

Let $k = \sin \left(\frac{\theta_0}{2}\right)$ so that

$$T = \frac{1}{2} \sqrt{\frac{L}{2g}} \int_0^{\theta_0} \frac{d\theta}{\left[k^2 - \sin^2 \left(\frac{\theta}{2}\right)\right]^{1/2}}.$$

We now make a change of variables in this equation as follows: Let $\sin \theta/2 = k \sin u$. When $\theta = 0, u = 0$, and when $\theta = \theta_0, u = \pi/2$. Moreover,

$$\begin{aligned} d\theta = \frac{2k \cos(u) du}{\cos(\theta/2)} &\implies d\theta = \frac{2k \sqrt{1 - \sin^2(u)} du}{\sqrt{1 - \sin^2(\theta/2)}} \\ &\implies d\theta = \frac{2\sqrt{k^2 - (k \sin(u))^2} du}{\sqrt{1 - k^2 \sin^2(u)}} \\ &\implies d\theta = \frac{2\sqrt{k^2 - \sin^2(\theta/2)} du}{\sqrt{1 - k^2 \sin^2(u)}}. \end{aligned}$$

Making this change of variables in the equation above, we obtain

$$T = \sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{du}{\sqrt{1 - k^2 \sin^2(u)}},$$

where k is defined as above.

Solutions to Section 1.12

Problems:

1. The acceleration of gravity is $a = 32$ ft/sec². Integrating, we find that the vertical component of the velocity of the ball is $v(t) = 16t + c_1$. Since the ball is initially hit horizontally, we have $v(0) = 0$, so that $c_1 = 0$. Hence, $v(t) = 16t$. Integrating again, we find the position $s(t) = 8t^2 + c_2$. Setting $s = 0$ at two feet above the ground, we have $s(0) = 0$ so that $c_2 = 0$. Thus, $s(t) = 8t^2$. The ball hits the ground when $s(t) = 2$, so that $t^2 = \frac{1}{4}$. Therefore, $t = \frac{1}{2}$. Since 80 miles per hour equates to over 117 ft/sec. In one-half second, the horizontal change in position of the ball is therefore more than $\frac{117}{2} = 58.5$ feet, more than enough to span the necessary 40 feet for the ball to reach the front wall. Therefore, the ball does reach the front wall before hitting the ground.

2. The acceleration of gravity is $a = 9.8$ meters/sec². Integrating, we find that the vertical component of the velocity of the rocket is $v(t) = 4.9t + c_1$. We are given that $v(0) = -10$, so that $c_1 = -10$. Thus, $v(t) = 4.9t - 10$. Integrating again, we find the position $s(t) = 2.495t^2 - 10t + c_2$. Setting $s = 0$ at two meters above the ground, we have $s(0) = 0$ so that $s(t) = 2.495t^2 - 10t$.

(a) The highest point above the ground is obtained when $v(t) = 0$. That is, $t = \frac{10}{4.9} \approx 2.04$ seconds. Thus, the highest point is approximately $s(2.04) = 2.495 \cdot (2.04)^2 - 10(2.04) \approx -10.02$, which is 12.02 meters above the ground.

(b) The rocket hits the ground when $s(t) = 2$. That is $2.495t^2 - 10t - 2 = 0$. Solving for t with the quadratic formula, we find that $t = -0.19$ or $t = 4.27$. Since we must report a positive answer, we conclude that the rocket hits the ground 4.27 seconds after launch.

3. We first determine the slope of the given family at the point (x, y) . Differentiating $y = cx^3$ with respect to x yields $\frac{dy}{dx} = 3cx^2$. We substitute $c = \frac{y}{x^3}$ into the latter equation to obtain

$$\frac{dy}{dx} = \frac{3y}{x}.$$

Consequently, the differential equation for the orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{x}{3y}.$$

Separating the variables and integrating gives

$$\frac{3}{2}y^2 = -\frac{1}{2}x^2 + C,$$

which can be written in the equivalent form

$$x^2 + 3y^2 = k.$$

4. We first determine the slope of the given family at the point (x, y) . Differentiating $y^2 = cx^3$ with respect to x yields $2y \frac{dy}{dx} = 3cx^2$, so that $\frac{dy}{dx} = \frac{3cx^2}{2y}$. Substituting $c = \frac{y^2}{x^3}$ into this latter equation yields

$$\frac{dy}{dx} = \frac{3y}{2x}.$$

Consequently, the differential equation for the orthogonal trajectories is

$$\frac{dy}{dx} = -\frac{2x}{3y}.$$

Separating the variables and integrating gives

$$\frac{3}{2}y^2 = -x^2 + C,$$

which can be written in the equivalent form

$$2x^2 + 3y^2 = k.$$

5. We first determine the slope of the given family at the point (x, y) . Differentiating $y = \ln(cx)$ with respect to x yields $\frac{dy}{dx} = \frac{1}{x}$. Consequently, the differential equation for the orthogonal trajectories is $\frac{dy}{dx} = -x$. This can be integrated directly to obtain

$$y = -\frac{1}{2}x^2 + k.$$

6. We first determine the slope of the given family at the point (x, y) . Differentiating $x^4 + y^4 = c$ with respect to x yields $4x^3 + 4y^3 \frac{dy}{dx} = 0$. Therefore, $\frac{dy}{dx} = -\frac{x^3}{y^3}$. Consequently, the differential equation for the orthogonal trajectories is

$$\frac{dy}{dx} = \frac{y^3}{x^3}.$$

Separating the variables and integrating gives

$$-\frac{1}{2}y^{-2} = -\frac{1}{2}x^{-2} + C,$$

which can be written in the equivalent form

$$y^2 - x^2 = kx^2y^2.$$

7.

(a) We first determine the slope of the given family at the point (x, y) . Differentiating $x^2 + 3y^2 = 2cy$ with respect to x yields $2x + 6y \frac{dy}{dx} = 2c \frac{dy}{dx}$, so that $\frac{dy}{dx} = \frac{x}{c-3y}$. Substituting $c = \frac{x^2+3y^2}{2y}$ into the latter equation yields $\frac{dy}{dx} = \frac{x}{\frac{x^2+3y^2}{2y}-3y} = \frac{2xy}{x^2-3y^2}$, as required.

(b) It follows that the differential equation for the orthogonal trajectories is

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy}.$$

This differential equation is first-order homogeneous. Substituting $y = xV$ into the preceding differential equation gives

$$x \frac{dV}{dx} + V = \frac{3V^2 - 1}{2V}$$

which simplifies to

$$\frac{dV}{dx} = \frac{V^2 - 1}{2V}.$$

Separating the variables and integrating, we obtain $\ln(V^2 - 1) = \ln x + C$, or, upon exponentiation, $V^2 - 1 = kx$. Inserting $V = y/x$ into the preceding equation yields $\frac{y^2}{x^2} - 1 = kx$. That is,

$$y^2 - x^2 = kx^3.$$

8. See accompanying figure.

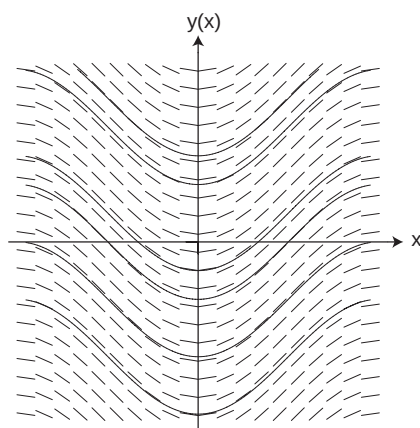


Figure 57: Figure for Problem 8

9. See accompanying figure.

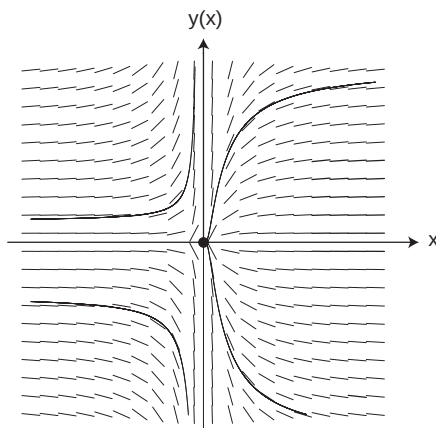


Figure 58: Figure for Problem 9

10.

(a) If $v(t) = 25$, then

$$\frac{dv}{dt} = 0 = \frac{1}{2}(25 - v).$$

(b) The accompanying figure suggests that

$$\lim_{t \rightarrow \infty} v(t) = 25.$$

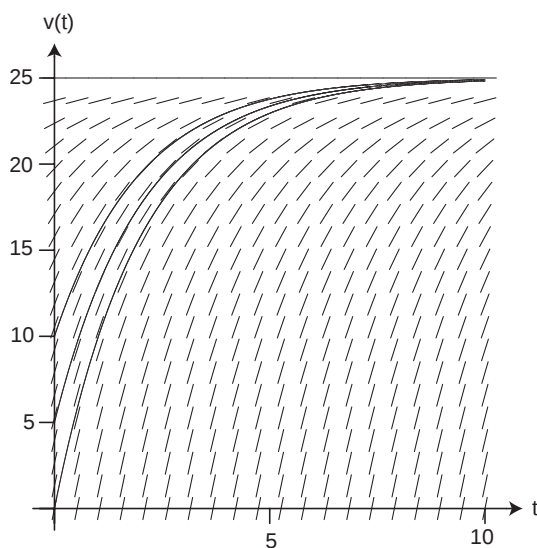


Figure 59: Figure for Problem 10

11.

(a) Separating the variables in Equation (1.12.6) yields

$$\frac{mv}{mg - kv^2} \frac{dv}{dy} = 1$$

which can be integrated to obtain

$$-\frac{m}{2k} \ln(mg - kv^2) = y + c.$$

Multiplying both sides of this equation by -1 and exponentiating gives

$$mg - kv^2 = c_1 e^{-\frac{2k}{m}y}.$$

The initial condition $v(0) = 0$ requires that $c_1 = mg$, which, when inserted into the preceding equation yields

$$mg - kv^2 = mge^{-\frac{2k}{m}y},$$

or equivalently,

$$v^2 = \frac{mg}{k} \left(1 - e^{-\frac{2k}{m}y} \right),$$

as required.

(b) See accompanying figure.

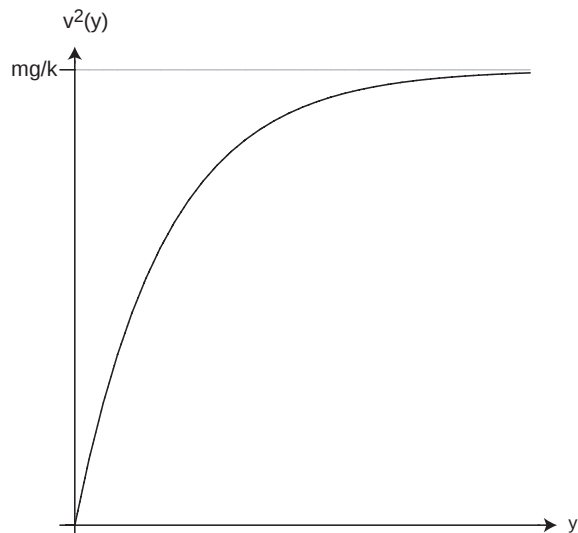


Figure 60: Figure for Problem 11

12. The given differential equation is separable. Separating the variables gives

$$y \frac{dy}{dx} = 2 \frac{\ln x}{x},$$

which can be integrated directly to obtain

$$\frac{1}{2}y^2 = (\ln x)^2 + c,$$

or, equivalently,

$$y^2 = 2(\ln x)^2 + c_1.$$

13. The given differential equation is first-order linear. We first divide by x to put the differential equation in standard form:

$$\frac{dy}{dx} - \frac{2}{x}y = 2x \ln x. \quad (0.0.1)$$

An integrating factor for this equation is $I = e^{\int (-2/x) dx} = x^{-2}$. Multiplying Equation (0.0.1) by x^{-2} reduces it to

$$\frac{d}{dx}(x^{-2}y) = 2x^{-1} \ln x,$$

which can be integrated to obtain

$$x^{-2}y = (\ln x)^2 + c$$

so that

$$y(x) = x^2[(\ln x)^2 + c].$$

14. We first re-write the given differential equation in the differential form

$$2xy \, dx + (x^2 + 2y)dy = 0. \quad (0.0.2)$$

Then

$$M_y = 2x = N_x$$

so that the differential equation is exact. Consequently, there exists a potential function ϕ satisfying

$$\frac{\partial \phi}{\partial x} = 2xy, \quad \frac{\partial \phi}{\partial y} = x^2 + 2y.$$

Integrating these two equations in the usual manner yields

$$\phi(x, y) = x^2y + y^2.$$

Therefore Equation (0.0.2) can be written in the equivalent form

$$d(x^2y + y^2) = 0$$

with general solution

$$x^2y + y^2 = c.$$

15. We first rewrite the given differential equation as

$$\frac{dy}{dx} = \frac{y^2 + 3xy + x^2}{x^2},$$

which is first order homogeneous. Substituting $y = xV$ into the preceding equation yields

$$x \frac{dV}{dx} + V = V^2 + 3V + 1$$

so that

$$x \frac{dV}{dx} = V^2 + 2V + 1 = (V + 1)^2,$$

or, in separable form,

$$\frac{1}{(V + 1)^2} \frac{dV}{dx} = \frac{1}{x}.$$

This equation can be integrated to obtain

$$-(V + 1)^{-1} = \ln x + c$$

so that

$$V + 1 = \frac{1}{c_1 - \ln x}.$$

Inserting $V = y/x$ into the preceding equation yields

$$\frac{y}{x} + 1 = \frac{1}{c_1 - \ln x},$$

so that

$$y(x) = \frac{x}{c_1 - \ln x} - x.$$

16. We first rewrite the given differential equation in the equivalent form

$$\frac{dy}{dx} + y \cdot \tan x = y^2 \sin x,$$

which is a Bernoulli equation. Dividing this equation by y^2 yields

$$y^{-2} \frac{dy}{dx} + y^{-1} \tan x = \sin x. \quad (0.0.3)$$

Now make the change of variables $u = y^{-1}$ in which case $\frac{du}{dx} = -y^{-2} \frac{dy}{dx}$. Substituting these results into Equation (0.0.3) gives the linear differential equation

$$-\frac{du}{dx} + u \cdot \tan x = \sin x$$

or, in standard form,

$$\frac{du}{dx} - u \cdot \tan x = -\sin x. \quad (0.0.4)$$

An integrating factor for this differential equation is $I = e^{-\int \tan x \, dx} = \cos x$. Multiplying Equation (0.0.4) by $\cos x$ reduces it to

$$\frac{d}{dx}(u \cdot \cos x) = -\sin x \cos x$$

which can be integrated directly to obtain

$$u \cdot \cos x = \frac{1}{2} \cos^2 x + c,$$

so that

$$u = \frac{\cos^2 x + c_1}{\cos x}.$$

Inserting $u = y^{-1}$ into the preceding equation and rearranging yields

$$y(x) = \frac{2 \cos x}{\cos^2 x + c_1}.$$

17. The given differential equation is linear with integrating factor

$$I(x) = e^{\int \frac{2e^{2x}}{1+e^{2x}} dx} = e^{\ln(1+e^{2x})} = 1 + e^{2x}.$$

Multiplying the given differential equation by $1 + e^{2x}$ yields

$$\frac{d}{dx} [(1 + e^{2x})y] = \frac{e^{2x} + 1}{e^{2x} - 1} = -1 + \frac{2e^{2x}}{e^{2x} - 1}$$

which can be integrated directly to obtain

$$(1 + e^{2x})y = -x + \ln |e^{2x} - 1| + c,$$

so that

$$y(x) = \frac{-x + \ln |e^{2x} - 1| + c}{1 + e^{2x}}.$$

18. We first rewrite the given differential equation in the equivalent form

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 - y^2}}{x},$$

which we recognize as being first order homogeneous. Inserting $y = xV$ into the preceding equation yields

$$x \frac{dV}{dx} + V = V + \frac{|x|}{x} \sqrt{1 - V^2},$$

that is,

$$\frac{1}{\sqrt{1 - V^2}} \frac{dV}{dx} = \pm \frac{1}{x}.$$

Integrating we obtain

$$\sin^{-1} V = \pm \ln |x| + c,$$

so that

$$V = \sin(c \pm \ln |x|).$$

Inserting $V = y/x$ into the preceding equation yields

$$y(x) = x \sin(c \pm \ln |x|).$$

19. We first rewrite the given differential equation in the equivalent form

$$(\sin y + y \cos x + 1)dx - (1 - x \cos y - \sin x)dy = 0.$$

Then

$$M_y = \cos y + \cos x = N_x$$

so that the differential equation is exact. Consequently, there is a potential function satisfying

$$\frac{\partial \phi}{\partial x} = \sin y + y \cos x + 1, \quad \frac{\partial \phi}{\partial y} = -(1 - x \cos y - \sin x).$$

Integrating these two equations in the usual manner yields

$$\phi(x, y) = x - y + x \sin y + y \sin x,$$

so that the differential equation can be written as

$$d(x - y + x \sin y + y \sin x) = 0,$$

and therefore has general solution

$$x - y + x \sin y + y \sin x = c.$$

20. Writing the given differential equation as

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{25}{2}y^{-1}x^2 \ln x,$$

we see that it is a Bernoulli equation with $n = -1$. We therefore divide the equation by y^{-1} to obtain

$$y \frac{dy}{dx} + \frac{1}{x}y^2 = \frac{25}{2}x^2 \ln x.$$

We now make the change of variables $u = y^2$, in which case, $\frac{du}{dx} = 2y \frac{dy}{dx}$. Inserting these results into the preceding differential equation yields

$$\frac{1}{2} \frac{du}{dx} + \frac{1}{x}u = \frac{25}{2}x^2 \ln x,$$

or, in standard form,

$$\frac{du}{dx} + \frac{2}{x}u = 25x^2 \ln x.$$

An integrating factor for this linear differential equation is $I = e^{\int (2/x)dx} = x^2$. Multiplying the previous differential equation by x^2 reduces it to

$$\frac{d}{dx}(x^2u) = 25x^4 \ln x$$

which can be integrated directly to obtain

$$x^2u = 25 \left(\frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 \right) + c$$

so that

$$u = x^3(5 \ln x - 1) + cx^{-2}.$$

Making the replacement $u = y^2$ in this equation gives

$$y^2 = x^3(5 \ln x - 1) + cx^{-2}.$$

21. The given differential equation can be written in the equivalent form

$$\frac{dy}{dx} = \frac{e^{x-y}}{e^{2x+y}} = e^{-x}e^{-2y},$$

which we recognize as being separable. Separating the variables gives

$$e^{2y} \frac{dy}{dx} = e^{-x}$$

which can be integrated to obtain

$$\frac{1}{2}e^{2y} = -e^{-x} + c$$

so that

$$y(x) = \frac{1}{2} \ln(c_1 - 2e^{-x}).$$

22. The given differential equation is linear with integrating factor $I = e^{\int \cot x \, dx} = \sin x$. Multiplying the given differential equation by $\sin x$ reduces it to

$$\frac{d}{dx}(y \sin x) = \frac{\sin x}{\cos x}$$

which can be integrated directly to obtain

$$y \sin x = -\ln(\cos x) + c,$$

so that

$$y(x) = \frac{c - \ln(\cos x)}{\sin x}.$$

23. Writing the given differential equation as

$$\frac{dy}{dx} + \frac{2e^x}{1+e^x}y = 2y^{\frac{1}{2}}e^{-x},$$

we see that it is a Bernoulli equation with $n = 1/2$. We therefore divide the equation by $y^{\frac{1}{2}}$ to obtain

$$y^{-\frac{1}{2}} \frac{dy}{dx} + \frac{2e^x}{1+e^x}y^{\frac{1}{2}} = 2e^{-x}.$$

We now make the change of variables $u = y^{\frac{1}{2}}$, in which case, $\frac{du}{dx} = \frac{1}{2}y^{-\frac{1}{2}} \frac{dy}{dx}$. Inserting these results into the preceding differential equation yields

$$2 \frac{du}{dx} + \frac{2e^x}{1+e^x}u = 2e^{-x},$$

or, in standard form,

$$\frac{du}{dx} + \frac{e^x}{1+e^x}u = e^{-x}.$$

An integrating factor for this linear differential equation is

$$I = e^{\int \frac{e^x}{1+e^x} dx} = e^{\ln(1+e^x)} = 1 + e^x.$$

Multiplying the previous differential equation by $1 + e^x$ reduces it to

$$\frac{d}{dx}[(1+e^x)u] = e^{-x}(1+e^x) = e^{-x} + 1$$

which can be integrated directly to obtain

$$(1+e^x)u = -e^{-x} + x + c$$

so that

$$u = \frac{x - e^{-x} + c}{1 + e^x}.$$

Making the replacement $u = y^{\frac{1}{2}}$ in this equation gives

$$y^{\frac{1}{2}} = \frac{x - e^{-x} + c}{1 + e^x}.$$

24. We first rewrite the given differential equation in the equivalent form

$$\frac{dy}{dx} = \frac{y}{x} \left[\ln \left(\frac{y}{x} \right) + 1 \right].$$

The function appearing on the right of this equation is homogeneous of degree zero, and therefore the differential equation itself is first order homogeneous. We therefore insert $y = xV$ into the differential equation to obtain

$$x \frac{dV}{dx} + V = V(\ln V + 1),$$

so that

$$x \frac{dV}{dx} = V \ln V.$$

Separating the variables yields

$$\frac{1}{V \ln V} \frac{dV}{dx} = \frac{1}{x}$$

which can be integrated to obtain

$$\ln(\ln V) = \ln x + c.$$

Exponentiating both side of this equation gives

$$\ln V = c_1 x,$$

or equivalently,

$$V = e^{c_1 x}.$$

Inserting $V = y/x$ in the preceding equation yields

$$y = x e^{c_1 x}.$$

25. For the given differential equation we have

$$M(x, y) = 1 + 2xe^y \quad \text{and} \quad N(x, y) = -(e^y + x),$$

so that

$$\frac{M_y - N_x}{M} = \frac{1 + 2xe^y}{1 + 2xe^y} = 1.$$

Consequently, an integrating factor for the given differential equation is

$$I = e^{-\int dy} = e^{-y}.$$

Multiplying the given differential equation by e^{-y} yields the exact differential equation

$$(2x + e^{-y})dx - (1 + xe^{-y})dy = 0. \quad (0.0.5)$$

Therefore, there exists a potential function ϕ satisfying

$$\frac{\partial \phi}{\partial x} = 2x + e^{-y}, \quad \frac{\partial \phi}{\partial y} = -(1 + xe^{-y}).$$

Integrating these two equations in the usual manner yields

$$\phi(x, y) = x^2 - y + xe^{-y}.$$

Therefore Equation (0.0.5) can be written in the equivalent form

$$d(x^2 - y + xe^{-y}) = 0$$

with general solution

$$x^2 - y + xe^{-y} = c.$$

26. The given differential equation is first-order linear. However, it can also be written in the equivalent form

$$\frac{dy}{dx} = (1 - y) \sin x$$

which is separable. Separating the variables and integrating yields

$$-\ln |1 - y| = -\cos x + c,$$

so that

$$1 - y = c_1 e^{\cos x}.$$

Hence,

$$y(x) = 1 - c_1 e^{\cos x}.$$

27. For the given differential equation we have

$$M(x, y) = 3y^2 + x^2 \quad \text{and} \quad N(x, y) = -2xy,$$

so that

$$\frac{M_y - N_x}{N} = -\frac{4}{x}.$$

Consequently, an integrating factor for the given differential equation is

$$I = e^{-\int \frac{4}{x} dx} = x^{-4}.$$

Multiplying the given differential equation by x^{-4} yields the exact differential equation

$$(3y^2 x^{-4} + x^{-2})dx - 2yx^{-3}dy = 0. \quad (0.0.6)$$

Therefore, there exists a potential function ϕ satisfying

$$\frac{\partial \phi}{\partial x} = 3y^2 x^{-4} + x^{-2}, \quad \frac{\partial \phi}{\partial y} = -2yx^{-3}.$$

Integrating these two equations in the usual manner yields

$$\phi(x, y) = -y^2 x^{-3} - x^{-1}.$$

Therefore Equation (0.0.6) can be written in the equivalent form

$$d(-y^2 x^{-3} - x^{-1}) = 0$$

with general solution

$$-y^2 x^{-3} - x^{-1} = c,$$

or equivalently,

$$x^2 + y^2 = c_1 x^3.$$

Notice that the given differential equation can be written in the equivalent form

$$\frac{dy}{dx} = \frac{3y^2 + x^2}{2xy},$$

which is first-order homogeneous. Another equivalent way of writing the given differential equation is

$$\frac{dy}{dx} - \frac{3}{2x}y = \frac{1}{2}xy^{-1},$$

which is a Bernoulli equation.

28. The given differential equation can be written in the equivalent form

$$\frac{dy}{dx} - \frac{1}{2x \ln x}y = -\frac{9}{2}x^2y^3,$$

which is a Bernoulli equation with $n = 3$. We therefore divide the equation by y^3 to obtain

$$y^{-3}\frac{dy}{dx} - \frac{1}{2x \ln x}y^{-2} = -\frac{9}{2}x^2.$$

We now make the change of variables $u = y^{-2}$, in which case, $\frac{du}{dx} = -2y^{-3}\frac{dy}{dx}$. Inserting these results into the preceding differential equation yields

$$-\frac{1}{2}\frac{du}{dx} - \frac{1}{2x \ln x}u = -\frac{9}{2}x^2,$$

or, in standard form,

$$\frac{du}{dx} + \frac{1}{x \ln x}u = 9x^2.$$

An integrating factor for this linear differential equation is

$$I = e^{\int \frac{1}{x \ln x} dx} = e^{\ln(\ln x)} = \ln x.$$

Multiplying the previous differential equation by $\ln x$ reduces it to

$$\frac{d}{dx}(\ln x \cdot u) = 9x^2 \ln x$$

which can be integrated to obtain

$$\ln x \cdot u = x^3(3 \ln x - 1) + c$$

so that

$$u = \frac{x^3(3 \ln x - 1) + c}{\ln x}.$$

Making the replacement $u = y^3$ in this equation gives

$$y^3 = \frac{x^3(3 \ln x - 1) + c}{\ln x}.$$

29. Separating the variables in the given differential equation yields

$$\frac{1}{y} \frac{dy}{dx} = \frac{2+x}{1+x} = 1 + \frac{1}{1+x},$$

which can be integrated to obtain

$$\ln |y| = x + \ln |1+x| + c.$$

Exponentiating both sides of this equation gives

$$y(x) = c_1(1+x)e^x.$$

30. The given differential equation can be written in the equivalent form

$$\frac{dy}{dx} + \frac{2}{x^2-1}y = 1 \quad (0.0.7)$$

which is first-order linear. An integrating factor is

$$I = e^{\int \frac{2}{x^2-1} dx} = e^{\int (\frac{1}{x-1} - \frac{1}{x+1}) dx} = e^{[\ln(x-1) - \ln(x+1)]} = \frac{x-1}{x+1}.$$

Multiplying (0.0.7) by $(x-1)/(x+1)$ reduces it to the integrable form

$$\frac{d}{dx} \left(\frac{x-1}{x+1} \cdot y \right) = \frac{x-1}{x+1} = 1 - \frac{2}{x+1}.$$

Integrating both sides of this differential equation yields

$$\left(\frac{x-1}{x+1} \cdot y \right) = x - 2 \ln(x+1) + c$$

so that

$$y(x) = \left(\frac{x+1}{x-1} \right) [x - 2 \ln(x+1) + c].$$

31. The given differential equation can be written in the equivalent form

$$[y \sec^2(xy) + 2x]dx + x \sec^2(xy)dy = 0$$

Then

$$M_y = \sec^2(xy) + 2xy \sec^2(x) \tan(xy) = N_x$$

so that the differential equation is exact. Consequently, there is a potential function satisfying

$$\frac{\partial \phi}{\partial x} = y \sec^2(xy) + 2x, \quad \frac{\partial \phi}{\partial y} = x \sec^2(xy).$$

Integrating these two equations in the usual manner yields

$$\phi(x, y) = x^2 + \tan(xy),$$

so that the differential equation can be written as

$$d(x^2 + \tan(xy)) = 0,$$

and therefore has general solution

$$x^2 + \tan(xy) = c,$$

or equivalently,

$$y(x) = \frac{\tan^{-1}(c - x^2)}{x}.$$

32. CHANGE PROBLEM IN TEXT TO

$$\frac{dy}{dx} + 4xy = 4x\sqrt{y}.$$

The given differential equation is a Bernoulli equation with $n = \frac{1}{2}$. We therefore divide the equation by $y^{\frac{1}{2}}$ to obtain

$$y^{-\frac{1}{2}} \frac{dy}{dx} - 4xy^{\frac{1}{2}} = 4x.$$

We now make the change of variables $u = y^{\frac{1}{2}}$, in which case, $\frac{du}{dx} = \frac{1}{2}y^{-\frac{1}{2}} \frac{dy}{dx}$. Inserting these results into the preceding differential equation yields

$$2 \frac{du}{dx} + 4xu = 4x,$$

or, in standard form,

$$\frac{du}{dx} + 2xu = 2x.$$

An integrating factor for this linear differential equation is

$$I = e^{\int 2x dx} = e^{x^2}.$$

Multiplying the previous differential equation by e^{x^2} reduces it to

$$\frac{d}{dx} (e^{x^2} u) = 2xe^{x^2}.$$

which can be integrated directly to obtain

$$e^{x^2} u = e^{x^2} + c$$

so that

$$u = 1 + ce^{x^2}.$$

Making the replacement $u = y^{\frac{1}{2}}$ in this equation gives

$$y^{\frac{1}{2}} = 1 + ce^{x^2}.$$

33. CHANGE PROBLEM IN TEXT TO

$$\frac{dy}{dx} = \frac{x^2}{x^2 + y^2} + \frac{y}{x}$$

then the answer is correct. The given differential equation is first-order homogeneous. Inserting $y = xV$ into the given equation yields

$$x \frac{dV}{dx} + V = \frac{1}{1 + V^2} + V,$$

that is,

$$(1 + V^2) \frac{dV}{dx} = \frac{1}{x}.$$

Integrating we obtain

$$V + \frac{1}{3}V^3 = \ln|x| + c.$$

Inserting $V = y/x$ into the preceding equation yields

$$\frac{y}{x} + \frac{y^3}{3x^3} = \ln|x| + c.$$

34. For the given differential equation we have

$$M_y = \frac{1}{y} = N_x$$

so that the differential equation is exact. Consequently, there is a potential function satisfying

$$\frac{\partial \phi}{\partial x} = \ln(xy) + 1, \quad \frac{\partial \phi}{\partial y} = \frac{x}{y} + 2y.$$

Integrating these two equations in the usual manner yields

$$\phi(x, y) = x \ln(xy) + y^2,$$

so that the differential equation can be written as

$$d[x \ln(xy) + y^2] = 0,$$

and therefore has general solution

$$x \ln(xy) + y^2 = c.$$

35. The given differential equation is a Bernoulli equation with $n = -1$. We therefore divide the equation by y^{-1} to obtain

$$y \frac{dy}{dx} + \frac{1}{x} y^2 = \frac{25 \ln x}{2x^3}.$$

We now make the change of variables $u = y^2$, in which case, $\frac{du}{dx} = 2y \frac{dy}{dx}$. Inserting these results into the preceding differential equation yields

$$\frac{1}{2} \frac{du}{dx} + \frac{1}{x} u = \frac{25 \ln x}{2x^3},$$

or, in standard form,

$$\frac{du}{dx} + \frac{2}{x} u = 25x^{-3} \ln x.$$

An integrating factor for this linear differential equation is

$$I = e^{\int \frac{2}{x} dx} = x^2.$$

Multiplying the previous differential equation by x^2 reduces it to

$$\frac{d}{dx}(x^2 u) = 25x^{-1} \ln x,$$

which can be integrated directly to obtain

$$x^2 u = \frac{25}{2} (\ln x)^2 + c$$

so that

$$u = \frac{25(\ln x)^2 + c}{2x^2}.$$

Making the replacement $u = y^2$ in this equation gives

$$y^2 = \frac{25(\ln x)^2 + c}{2x^2}.$$

36. The problem as written is separable, but the integration does not work. CHANGE PROBLEM TO:

$$(1 + y) \frac{dy}{dx} = x e^{x-y}.$$

The given differential equation can be written in the equivalent form

$$e^y (1 + y) \frac{dy}{dx} = x e^x$$

which is separable. Integrating both sides of this equation gives

$$y e^y = e^x (x - 1) + c.$$

37. The given differential equation can be written in the equivalent form

$$\frac{dy}{dx} - \frac{\cos x}{\sin x} y = -\cos x$$

which is first order linear with integrating factor

$$I = e^{-\int \frac{\cos x}{\sin x} dx} = e^{-\ln(\sin x)} = \frac{1}{\sin x}.$$

Multiplying the preceding differential equation by $\frac{1}{\sin x}$ reduces it to

$$\frac{d}{dx} \left(\frac{1}{\sin x} \cdot y \right) = -\frac{\cos x}{\sin x}$$

which can be integrated directly to obtain

$$\frac{1}{\sin x} \cdot y = -\ln(\sin x) + c$$

so that

$$y(x) = \sin x [c - \ln(\sin x)].$$

38. The given differential equation is linear, and therefore can be solved using an appropriate integrating factor. However, if we rearrange the terms in the given differential equation then it can be written in the equivalent form

$$\frac{1}{1+y} \frac{dy}{dx} = x^2$$

which is separable. Integrating both sides of the preceding differential equation yields

$$\ln(1+y) = \frac{1}{3}x^3 + c$$

so that

$$y(x) = c_1 e^{\frac{1}{3}x^3} - 1.$$

Imposing the initial condition $y(0) = 5$ we find $c_1 = 6$. Therefore the solution to the initial-value problem is

$$y(x) = 6e^{\frac{1}{3}x^3} - 1.$$

39. The given differential equation can be written in the equivalent form

$$e^{-6y} \frac{dy}{dx} = -e^{-4x}$$

which is separable. Integrating both sides of the preceding equation yields

$$-\frac{1}{6}e^{-6y} = \frac{1}{4}e^{-4x} + c$$

so that

$$y(x) = -\frac{1}{6} \ln \left(c_1 - \frac{3}{2}e^{-4x} \right).$$

Imposing the initial condition $y(0) = 0$ requires that

$$0 = \ln \left(c_1 - \frac{3}{2} \right).$$

Hence, $c_1 = \frac{5}{2}$, and so

$$y(x) = -\frac{1}{6} \ln \left(\frac{5 - 3e^{-4x}}{2} \right).$$

40. For the given differential equation we have

$$M_y = 4xy = N_x$$

so that the differential equation is exact. Consequently, there is a potential function satisfying

$$\frac{\partial \phi}{\partial x} = 3x^2 + 2xy^2, \quad \frac{\partial \phi}{\partial y} = 2x^2y.$$

Integrating these two equations in the usual manner yields

$$\phi(x, y) = x^2y^2 + x^3,$$

so that the differential equation can be written as

$$d(x^2y^2 + x^3) = 0,$$

and therefore has general solution

$$x^2y^2 + x^3 = c.$$

Imposing the initial condition $y(1) = 3$ yields $c = 10$. Therefore,

$$x^2 y^2 + x^3 = 10$$

so that

$$y^2 = \frac{10 - x^3}{x^2}.$$

Note that the given differential equation can be written in the equivalent form

$$\frac{dy}{dx} + \frac{1}{x}y = -\frac{3}{2}y^{-1},$$

which is a Bernoulli equation with $n = -1$. Consequently, the Bernoulli technique could also have been used to solve the differential equation.

41. The given differential equation is linear with integrating factor

$$I = e^{-\int \sin x \, dx} = e^{\cos x}.$$

Multiplying the given differential equation by $e^{\cos x}$ reduces it to the integrable form

$$\frac{d}{dx}(e^{\cos x} \cdot y) = 1,$$

which can be integrated directly to obtain

$$e^{\cos x} \cdot y = x + c.$$

Hence,

$$y(x) = e^{-\cos x}(x + c).$$

Imposing the given initial condition $y(0) = \frac{1}{e}$ requires that $c = 1$. Consequently,

$$y(x) = e^{-\cos x}(x + 1).$$

42.

(a) For the given differential equation we have

$$M_y = my^{m-1}, \quad N_x = -nx^{n-1}y^3.$$

We see that the only values for m and n for which $M_y = N_x$ are $m = n = 0$. Consequently, these are the only values of m and n for which the differential equation is exact.

(b) We rewrite the given differential equation in the equivalent form

$$\frac{dy}{dx} = \frac{x^5 + y^m}{x^n y^3}, \tag{0.0.8}$$

from which we see that the differential equation is separable provided $m = 0$. In this case there are no restrictions on n .

(c) From Equation (0.0.8) we see that the only values of m and n for which the differential equation is first-order homogeneous are $m = 5$ and $n = 2$.

(d) We now rewrite the given differential equation in the equivalent form

$$\frac{dy}{dx} - x^{-n}y^{m-3} = x^{5-n}y^{-3}. \quad (0.0.9)$$

Due to the y^{-3} term on the right-hand side of the preceding differential equation, it follows that there are no values of m and n for which the equation is linear.

(e) From Equation (0.0.9) we see that the differential equation is a Bernoulli equation whenever $m = 4$. There are no constraints on n in this case.

43. In Newton's Law of Cooling we have

$$T_m = 180^\circ\text{F}, \quad T(0) = 80^\circ\text{F}, \quad T(3) = 100^\circ\text{F}.$$

We need to determine the time, t_0 when $T(t_0) = 140^\circ\text{F}$. The temperature of the sandals at time t is governed by the differential equation

$$\frac{dT}{dt} = -k(T - 180).$$

This separable differential equation is easily integrated to obtain

$$T(t) = 180 + ce^{-kt}.$$

Since $T(0) = 80$ we have

$$80 = 180 + c \implies c = -100.$$

Hence,

$$T(t) = 180 - 100e^{-kt}.$$

Imposing the condition $T(3) = 100$ requires

$$100 = 180 - 100e^{-3k}.$$

Solving for k we find $k = \frac{1}{3} \ln\left(\frac{5}{4}\right)$. Inserting this value for k into the preceding expression for $T(t)$ yields

$$T(t) = 180 - 100e^{-\frac{t}{3} \ln\left(\frac{5}{4}\right)}.$$

We need to find t_0 such that

$$140 = 180 - 100e^{-\frac{t_0}{3} \ln\left(\frac{5}{4}\right)}.$$

Solving for t_0 we find

$$t_0 = 3 \frac{\ln\left(\frac{5}{2}\right)}{\ln\left(\frac{5}{4}\right)} \approx 12.32 \text{ min.}$$

44. In Newton's Law of Cooling we have

$$T_m = 70^\circ\text{F}, \quad T(0) = 150^\circ\text{F}, \quad T(10) = 125^\circ\text{F}.$$

We need to determine the time, t_0 when $T(t_0) = 100^\circ\text{F}$. The temperature of the plate at time t is governed by the differential equation

$$\frac{dT}{dt} = -k(T - 70).$$

This separable differential equation is easily integrated to obtain

$$T(t) = 70 + ce^{-kt}.$$

Since $T(0) = 150$ we have

$$150 = 70 + c \implies c = 80.$$

Hence,

$$T(t) = 70 + 80e^{-kt}.$$

Imposing the condition $T(10) = 125$ requires

$$125 = 70 + 80e^{-10k}.$$

Solving for k we find $k = \frac{1}{10} \ln\left(\frac{16}{11}\right)$. Inserting this value for k into the preceding expression for $T(t)$ yields

$$T(t) = 70 + 80e^{-\frac{t}{10} \ln\left(\frac{16}{11}\right)}.$$

We need to find t_0 such that

$$100 = 70 + 80e^{-\frac{t_0}{10} \ln\left(\frac{16}{11}\right)}.$$

Solving for t_0 we find

$$t_0 = 10 \frac{\ln\left(\frac{8}{3}\right)}{\ln\left(\frac{16}{11}\right)} \approx 26.18 \text{ min.}$$

45. Let $T(t)$ denote the temperature of the object at time t , and let T_m denote the temperature of the surrounding medium. Then we must solve the initial-value problem

$$\frac{dT}{dt} = k(T - T_m)^2, \quad T(0) = T_0,$$

where k is a constant. The differential equation can be written in separated form as

$$\frac{1}{(T - T_m)^2} \frac{dT}{dt} = k.$$

Integrating both sides of this differential equation yields

$$-\frac{1}{T - T_m} = kt + c$$

so that

$$T(t) = T_m - \frac{1}{kt + c}.$$

Imposing the initial condition $T(0) = T_0$ we find that

$$c = \frac{1}{T_m - T_0}$$

which, when substituted back into the preceding expression for $T(t)$ yields

$$T(t) = T_m - \frac{1}{kt + \frac{1}{T_m - T_0}} = T_m - \frac{T_m - T_0}{k(T_m - T_0)t + 1}.$$

As $t \rightarrow \infty$, $T(t)$ approaches T_m .

46. We are given the differential equation

$$\frac{dT}{dt} = -k(T - 5 \cos 2t) \quad (0.0.10)$$

together with the initial conditions

$$T(0) = 0; \quad \frac{dT}{dt}(0) = 5. \quad (0.0.11)$$

(a) Setting $t = 0$ in (0.0.10) and using (0.0.11) yields

$$5 = -k(0 - 5)$$

so that $k = 1$.

(b) Substituting $k = 1$ into the differential equation (0.0.10) and rearranging terms yields

$$\frac{dT}{dt} + T = 5 \cos t.$$

An integrating factor for this linear differential equation is $I = e^{\int dt} = e^t$. Multiplying the preceding differential equation by e^t reduces it to

$$\frac{d}{dt}(e^t \cdot T) = 5e^t \cos 2t$$

which upon integration yields

$$e^t \cdot T = e^t(\cos 2t + 2 \sin 2t) + c,$$

so that

$$T(t) = ce^{-t} + \cos 2t + 2 \sin 2t.$$

Imposing the initial condition $T(0) = 0$ we find that $c = -1$. Hence,

$$T(t) = \cos 2t + 2 \sin 2t - e^{-t}.$$

(c) For large values of t we have

$$T(t) \approx \cos 2t + 2 \sin 2t,$$

which can be written in phase-amplitude form as

$$T(t) \approx \sqrt{5} \cos(2t - \phi),$$

where $\tan \phi = 2$. consequently, for large t , the temperature is approximately oscillatory with period π and amplitude $\sqrt{5}$.

47. If we let $C(t)$ denote the number of sandhill cranes in the Platte River valley t days after April 1, then $C(t)$ is governed by the differential equation

$$\frac{dC}{dt} = -kC \quad (0.0.12)$$

together with the auxiliary conditions

$$C(0) = 500,000; \quad C(15) = 100,000. \quad (0.0.13)$$

Separating the variables in the differential equation (0.0.12) yields

$$\frac{1}{C} \frac{dC}{dt} = -k,$$

which can be integrated directly to obtain

$$\ln C = -kt + c.$$

Exponentiation yields

$$C(t) = c_0 e^{-kt}.$$

The initial condition $C(0) = 500,000$ requires $c_0 = 500,000$, so that

$$C(t) = 500,000 e^{-kt}. \quad (0.0.14)$$

Imposing the auxiliary condition $C(15) = 100,000$ yields

$$100,000 = 500,000 e^{-15k}.$$

Taking the natural logarithm of both sides of the preceding equation and simplifying we find that $k = \frac{1}{15} \ln 5$. Substituting this value for k into (0.0.14) gives

$$C(t) = 500,000 e^{-\frac{t}{15} \ln 5}. \quad (0.0.15)$$

(a) $C(3) = 500,000 e^{-2 \ln 5} = 500,000 \cdot \frac{1}{25} = 20,000$ sandhile cranes.

(b) $C(35) = 500,000 e^{-\frac{35}{15} \ln 5} \approx 11696$ sandhile cranes.

(c) We need to determine t_0 such that

$$1000 = 500,000 e^{-\frac{t_0}{15} \ln 5}$$

that is,

$$e^{-\frac{t_0}{15} \ln 5} = \frac{1}{500}.$$

Taking the natural logarithm of both sides of this equation and simplifying yields

$$t_0 = 15 \cdot \frac{\ln 500}{\ln 5} \approx 57.9 \text{ days after April 1.}$$

48. Substituting $P_0 = 200,000$ into Equation (1.5.3) in the text (page 45) yields

$$P(t) = \frac{200,000C}{200,000 + (C - 200,000)e^{-rt}}. \quad (0.0.16)$$

We are given

$$P(3) = P(t_1) = 230,000, \quad P(6) = P(t_2) = 250,000.$$

Since $t_2 = 2t_1$ we can use the formulas (1.5.5) and (1.5.6) on page 47 of the text to obtain r and C directly as follows:

$$r = \frac{1}{3} \ln \left[\frac{25(23-20)}{20(25-23)} \right] = \frac{1}{3} \ln \left(\frac{15}{8} \right) \approx 0.21.$$

$$C = \frac{230,000[(23)(45) - (40)(25)]}{(23)^2 - (20)(25)} = 277586.$$

Substituting these values for r and C into (0.0.16) yields

$$P(t) = \frac{55517200000}{200,000 + (77586)e^{-0.21t}}.$$

Therefore,

$$P(10) = \frac{55517200000}{200,000 + (77586)e^{-2.1}} \approx 264,997,$$

and

$$P(20) = \frac{55517200000}{200,000 + (77586)e^{-4.2}} \approx 275981.$$

49. The differential equation for determining $q(t)$ is

$$\frac{dq}{dt} + \frac{5}{4}q = \frac{3}{2} \cos 2t,$$

which has integrating factor $I = e^{\int \frac{5}{4} dt} = e^{\frac{5}{4}t}$. Multiplying the preceding differential equation by $e^{\frac{5}{4}t}$ reduces it to the integrable form

$$\frac{d}{dt} \left(e^{\frac{5}{4}t} \cdot q \right) = \frac{3}{2} e^{\frac{5}{4}t} \cos 2t.$$

Integrating and simplifying we find

$$q(t) = \frac{6}{89} (5 \cos 2t + 8 \sin 2t) + ce^{-\frac{5}{4}t}. \quad (0.0.17)$$

The initial condition $q(0) = 3$ requires

$$3 = \frac{30}{89} + c,$$

so that $c = \frac{237}{89}$. Making this replacement in (0.0.17) yields

$$q(t) = \frac{6}{89} (5 \cos 2t + 8 \sin 2t) + \frac{237}{89} e^{-\frac{5}{4}t}.$$

The current in the circuit is

$$i(t) = \frac{dq}{dt} = \frac{12}{89} (8 \cos 2t - 5 \sin 2t) - \frac{1185}{356} e^{-\frac{5}{4}t}.$$

Answer in text has incorrect exponent.

50. The current in the circuit is governed by the differential equation

$$\frac{di}{dt} + 10i = \frac{100}{3},$$

which has integrating factor $I = e^{\int 10 dt} = e^{10t}$. Multiplying the preceding differential equation by e^{10t} reduces it to the integrable form

$$\frac{d}{dt} (e^{10t} \cdot i) = \frac{100}{3} e^{10t}.$$

Integrating and simplifying we find

$$i(t) = \frac{10}{3} + ce^{-10t}. \quad (0.0.18)$$

The initial condition $i(0) = 3$ requires

$$3 = \frac{10}{3} + c,$$

so that $c = -\frac{1}{3}$. Making this replacement in (0.0.18) yields

$$i(t) = \frac{1}{3}(10 - e^{-10t}).$$

51. We are given:

$$r_1 = 6 \text{ L/min}, \quad c_1 = 3 \text{ g/L}, \quad r_2 = 4 \text{ L/min}, \quad V(0) = 30 \text{ L}, \quad A(0) = 0 \text{ g},$$

and we need to determine the amount of salt in the tank when $V(t) = 60\text{L}$. Consider a small time interval Δt . Using the preceding information we have:

$$\Delta V = 6\Delta t - 4\Delta t = 2\Delta t,$$

and

$$\Delta A \approx 18\Delta t - 4\frac{A}{V}\Delta t.$$

Dividing both of these equations by Δt and letting $\Delta t \rightarrow 0$ yields

$$\frac{dV}{dt} = 2. \quad (0.0.19)$$

$$\frac{dA}{dt} + 4\frac{A}{V} = 18. \quad (0.0.20)$$

Integrating (0.0.19) and imposing the initial condition $V(0) = 30$ yields

$$V(t) = 2(t + 15). \quad (0.0.21)$$

We now insert this expression for $V(t)$ into (0.0.20) to obtain

$$\frac{dA}{dt} + \frac{2}{t + 15}A = 18.$$

An integrating factor for this differential equation is $I = e^{\int \frac{2}{t+15} dt} = (t + 15)^2$. Multiplying the preceding differential equation by $(t + 15)^2$ reduces it to the integrable form

$$\frac{d}{dt} [(t + 15)^2 A] = 18(t + 15)^2.$$

Integrating and simplifying we find

$$A(t) = \frac{6(t + 15)^3 + c}{(t + 15)^2}.$$

Imposing the initial condition $A(0) = 0$ requires

$$0 = \frac{6(15)^3 + c}{(15)^2},$$

so that $c = -20250$. Consequently,

$$A(t) = \frac{6(t+15)^3 - 20250}{(t+15)^2}.$$

We need to determine the time when the solution overflows. Since the tank can hold 60 L of solution, from (0.0.21) overflow will occur when

$$60 = 2(t+15) \implies t = 15.$$

The amount of chemical in the tank at this time is

$$A(15) = \frac{6(30)^3 - 20250}{(30)^2} \approx 157.5 \text{ g.}$$

52. Applying Euler's method with $y' = x^2 + 2y^2$, $x_0 = 0$, $y_0 = -3$, and $h = 0.1$ we have $y_{n+1} = y_n + 0.1(x_n^2 + 2y_n^2)$. This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.1	-1.2
2	0.2	-0.911
3	0.3	-0.74102
4	0.4	-0.62219
5	0.5	-0.52877
6	0.6	-0.44785
7	0.7	-0.371736
8	0.8	-0.29510
9	0.9	-0.21368
10	1.0	-0.12355

Consequently the Euler approximation to $y(1)$ is $y_{10} = -0.12355$.

53. Applying Euler's method with $y' = \frac{3x}{y} + 2$, $x_0 = 1$, $y_0 = 2$, and $h = 0.05$ we have

$$y_{n+1} = y_n + 0.05 \left(\frac{3x_n}{y_n} + 2 \right).$$

This generates the sequence of approximants given in the table below.

n	x_n	y_n
1	1.05	2.1750
2	1.10	2.34741
3	1.15	2.51770
4	1.20	2.68622
5	1.25	2.85323
6	1.30	3.01894
7	1.35	3.18353
8	1.40	3.34714
9	1.45	3.50988
10	1.50	3.67185

Consequently, the Euler approximation to $y(1.5)$ is $y_{10} = 3.67185$.

54. Applying the modified Euler method with $y' = x^2 + 2y^2$, $x_0 = 0$, $y_0 = -3$, and $h = 0.1$ generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.1	-1.9555
2	0.2	-1.42906
3	0.3	-1.11499
4	0.4	-0.90466
5	0.5	-0.74976
6	0.6	-0.62555
7	0.7	-0.51778
8	0.8	-0.41723
9	0.9	-0.31719
10	1.0	-0.21196

Consequently, the modified Euler approximation to $y(1)$ is $y_{10} = -0.21196$. Comparing this to the corresponding Euler approximation from Problem 52 we have

$$|y_{\text{ME}} - y_{\text{E}}| = |0.21196 - 0.12355| = 0.8841.$$

55. Applying the modified Euler method with $y' = \frac{3x}{y} + 2$, $x_0 = 1$, $y_0 = 2$, and $h = 0.05$ generates the sequence of approximants given in the table below.

n	x_n	y_n
1	1.05	2.17371
2	1.10	2.34510
3	1.15	2.51457
4	1.20	2.68241
5	1.25	2.84886
6	1.30	3.01411
7	1.35	3.17831
8	1.40	3.34159
9	1.45	3.50404
10	1.50	3.66576

Consequently, the modified Euler approximation to $y(1.5)$ is $y_{10} = 3.66576$. Comparing this to the corresponding Euler approximation from Problem 53 we have

$$|y_{\text{ME}} - y_{\text{E}}| = |3.66576 - 3.67185| = 0.00609.$$

56. Applying the Runge-Kutta method with $y' = x^2 + 2y^2$, $x_0 = 0$, $y_0 = -3$, and $h = 0.1$ generates the sequence of approximants given in the table below.

n	x_n	y_n
1	0.1	-1.87392
2	0.2	-1.36127
3	0.3	-1.06476
4	0.4	-0.86734
5	0.5	-0.72143
6	0.6	-0.60353
7	0.7	-0.50028
8	0.8	-0.40303
9	0.9	-0.30541
10	1.0	-0.20195

Consequently the Runge-Kutta approximation to $y(1)$ is $y_{10} = -0.20195$. Comparing this to the corresponding Euler approximation from Problem 52 we have

$$|y_{\text{RK}} - y_{\text{E}}| = |0.20195 - 0.12355| = 0.07840.$$

57. Applying the Runge-Kutta method with $y' = \frac{3x}{y} + 2$, $x_0 = 1$, $y_0 = 2$, and $h = 0.05$ generates the sequence of approximants given in the table below.

n	x_n	y_n
1	1.05	2.17369
2	1.10	2.34506
3	1.15	2.51452
4	1.20	2.68235
5	1.25	2.84880
6	1.30	3.01404
7	1.35	3.17823
8	1.40	3.34151
9	1.45	3.50396
10	1.50	3.66568

Consequently the Runge-Kutta approximation to $y(1.5)$ is $y_{10} = 3.66568$. Comparing this to the corresponding Euler approximation from Problem 53 we have

$$|y_{\text{RK}} - y_{\text{E}}| = |3.66568 - 3.67185| = 0.00617.$$

Last digit in answer the text needs changing.

Solutions to Section 2.1

True-False Review:

1. TRUE. A diagonal matrix has no entries below the main diagonal, so it is upper triangular. Likewise, it has no entries above the main diagonal, so it is also lower triangular.

2. FALSE. An $m \times n$ matrix has m row vectors and n column vectors.

- 3. TRUE.** Since A is symmetric, $A = A^T$. Thus, $(A^T)^T = A = A^T$, so A^T is symmetric.
- 4. FALSE.** The trace of a matrix is the *sum* of the entries along the main diagonal.
- 5. TRUE.** If A is skew-symmetric, then $A^T = -A$. But A and A^T contain the same entries along the main diagonal, so for $A^T = -A$, both A and $-A$ must have the same main diagonal. This is only possible if all entries along the main diagonal are 0.
- 6. TRUE.** If A is both symmetric and skew-symmetric, then $A = A^T = -A$, and $A = -A$ is only possible if all entries of A are zero.
- 7. TRUE.** Both matrix functions are defined for values of t such that $t > 0$.
- 8. FALSE.** The $(3, 2)$ -entry contains a function that is not defined for values of t with $t \leq 3$. So for example, this matrix functions is not defined for $t = 2$.
- 9. TRUE.** Each numerical entry of the matrix function is a constant function, which has domain $\mathbb{R}$.
- 10. FALSE.** For instance, the matrix function $A(t) = [t]$ and $B(t) = [t^2]$ satisfy $A(0) = B(0)$, but A and B are not the same matrix function.

Problems:

1. $a_{31} = 0, a_{24} = -1, a_{14} = 2, a_{32} = 2, a_{21} = 7, a_{34} = 4$.

2. $\begin{bmatrix} 1 & 5 \\ -1 & 3 \end{bmatrix}$; 2×2 matrix.

3. $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 4 & -2 \end{bmatrix}$; 2×3 matrix.

4. $\begin{bmatrix} -1 \\ 1 \\ 1 \\ -5 \end{bmatrix}$; 4×1 matrix.

5. $\begin{bmatrix} 1 & -3 & -2 \\ 3 & 6 & 0 \\ 2 & 7 & 4 \\ -4 & -1 & 5 \end{bmatrix}$; 4×3 matrix.

6. $\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$; 3×3 matrix.

7. $\text{tr}(A) = 1 + 3 = 4$.

8. $\text{tr}(A) = 1 + 2 + (-3) = 0$.

9. $\text{tr}(A) = 2 + 2 + (-5) = -1$.

10. Column vectors: $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \end{bmatrix}$.

Row vectors: $[1 \ -1], [3 \ 5]$.

11. Column vectors: $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 7 \end{bmatrix}$.

Row vectors: $[1 \ 3 \ -4], [-1 \ -2 \ 5], [2 \ 6 \ 7]$.

12. Column vectors: $\begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 10 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix}$. Row vectors: $[2 \ 10 \ 6], [5 \ -1 \ 3]$.

13. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 1 \end{bmatrix}$. Column vectors: $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$.

14. $B = \begin{bmatrix} 2 & 5 & 0 & 1 \\ -1 & 7 & 0 & 2 \\ 4 & -6 & 0 & 3 \end{bmatrix}$. Row vectors: $[2 \ 5 \ 0 \ 1], [-1 \ 7 \ 0 \ 2], [4 \ -6 \ 0 \ 3]$.

15. $A = [\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_p]$ has p columns and each column q -vector has q rows, so the resulting matrix has dimensions $q \times p$.

16. One example: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

17. One example: $\begin{bmatrix} 2 & 3 & 1 & 2 \\ 0 & 5 & 6 & 2 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

18. One example: $\begin{bmatrix} 1 & 3 & -1 & 2 \\ -3 & 0 & 4 & -3 \\ 1 & -4 & 0 & 1 \\ -2 & 3 & -1 & 0 \end{bmatrix}$.

19. One example: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

20. The only possibility here is the zero matrix: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

21. One example: $\begin{bmatrix} \frac{1}{\sqrt{3-t}} & \sqrt{t+2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

22. One example: $\begin{bmatrix} t^2 - t & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

23. One example: $\begin{bmatrix} t^2 + 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

24. One example: $\begin{bmatrix} \frac{1}{t^2+1} \\ 0 \end{bmatrix}$.

25. One example: Let A and B be 1×1 matrix functions given by

$$A(t) = [t] \quad \text{and} \quad B(t) = [t^2].$$

26. Let A be a symmetric upper-triangular matrix. Then all elements below the main diagonal are zeros. Consequently, since A is symmetric, all elements above the main diagonal must also be zero. Hence, the

only nonzero entries can occur along the main diagonal. That is, A is a diagonal matrix.

27. Since A is skew-symmetric, $a_{11} = a_{22} = a_{33} = 0$. Further, $a_{12} = -a_{21} = -1$, $a_{13} = -a_{31} = -3$, and $a_{32} = -a_{23} = 1$. Consequently, $A = \begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -1 \\ 3 & 1 & 0 \end{bmatrix}$.

Solutions to Section 2.2

True-False Review:

1. FALSE. The correct statement is $(AB)C = A(BC)$, the associative law. A counterexample to the particular statement given in this review item can be found in Problem 7.

2. TRUE. Multiplying from left to right, we note that AB is an $m \times p$ matrix, and right multiplying AB by the $p \times q$ matrix C , we see that ABC is an $m \times q$ matrix.

3. TRUE. We have $(A + B)^T = A^T + B^T = A + B$, so $A + B$ is symmetric.

4. FALSE. For example, let $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix}$. Then A and B are skew-symmetric, but $AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$ is not symmetric.

5. FALSE. The correct equation is $(A + B)^2 = A^2 + AB + BA + B^2$. The statement is false since $AB + BA$ does not necessarily equal $2AB$. For instance, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $(A + B)^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \neq (A + B)^2$.

6. FALSE. For example, let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $AB = 0$ even though $A \neq 0$ and $B \neq 0$.

7. FALSE. For example, let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then A is not upper triangular, despite the fact that AB is the zero matrix, hence automatically upper triangular.

8. FALSE. For instance, the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is neither the zero matrix nor the identity matrix, and yet $A^2 = A$.

9. TRUE. The derivative of each entry of the matrix is zero, since in each entry, we take the derivative of a constant, thus obtaining zero for each entry of the derivative of the matrix.

10. FALSE. The correct statement is given in Problem 41. The problem with the statement as given is that the second term should be $\frac{dA}{dt}B$, not $B\frac{dA}{dt}$.

11. FALSE. For instance, the matrix function $A = \begin{bmatrix} 2e^t & 0 \\ 0 & 3e^t \end{bmatrix}$ satisfies $A = \frac{dA}{dt}$, but A does not have the form $\begin{bmatrix} ce^t & 0 \\ 0 & ce^t \end{bmatrix}$.

12. TRUE. This follows by exactly the same proof as given in the text for matrices of numbers (see part 3 of Theorem 2.2.21).

Problems:

1. $2A = \begin{bmatrix} 2 & 4 & -2 \\ 6 & 10 & 4 \end{bmatrix}, -3B = \begin{bmatrix} -6 & 3 & -9 \\ -3 & -12 & -15 \end{bmatrix},$

$$\begin{aligned} A - 2B &= \begin{bmatrix} 1 & 2 & -1 \\ 3 & 5 & 2 \end{bmatrix} - \begin{bmatrix} 4 & -2 & 6 \\ 2 & 8 & 10 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 4 & -7 \\ 1 & -3 & -8 \end{bmatrix} \end{aligned}$$

,

$$\begin{aligned} 3A + 4B &= \begin{bmatrix} 3 & 6 & -3 \\ 9 & 15 & 6 \end{bmatrix} + \begin{bmatrix} 8 & -4 & 12 \\ 4 & 16 & 20 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 2 & 9 \\ 13 & 31 & 26 \end{bmatrix}. \end{aligned}$$

2. Solving for D , we have

$$\begin{aligned} 2A + B - 3C + 2D &= A + 4C \\ 2D &= -A - B + 7C \\ D &= \frac{1}{2}(-A - B + 7C). \end{aligned}$$

When appropriate substitutions are made for A, B , and C , we obtain:

$$D = \begin{bmatrix} -5 & -2.5 & 2.5 \\ 1.5 & 6.5 & 9 \\ -2.5 & 2.5 & -0.5 \end{bmatrix}.$$

3.

$$AB = \begin{bmatrix} 5 & 10 & -3 \\ 27 & 22 & 3 \end{bmatrix}, \quad BC = \begin{bmatrix} 9 \\ 8 \\ -6 \end{bmatrix}, \quad DC = [10],$$

$$DB = [6 \ 14 \ -4], \quad CD = \begin{bmatrix} 2 & -2 & 3 \\ -2 & 2 & -3 \\ 4 & -4 & 6 \end{bmatrix}.$$

CA and AD cannot be computed.

4.

$$\begin{aligned} AB &= \begin{bmatrix} 2-i & 1+i \\ -i & 2+4i \end{bmatrix} \begin{bmatrix} i & 1-3i \\ 0 & 4+i \end{bmatrix} \\ &= \begin{bmatrix} (2-i)i + (1+i)0 & (2-i)(1-3i) + (1+i)(4+i) \\ -i(i) + (2+4i)0 & -i(1-3i) + (2+4i)(4+i) \end{bmatrix} \\ &= \begin{bmatrix} 1+2i & 2-2i \\ 1 & 1+17i \end{bmatrix}. \end{aligned}$$

5.

$$\begin{aligned}
 AB &= \begin{bmatrix} 3+2i & 2-4i \\ 5+i & -1+3i \end{bmatrix} \begin{bmatrix} -1+i & 3+2i \\ 4-3i & 1+i \end{bmatrix} \\
 &= \begin{bmatrix} (3+2i)(-1+i) + (2-4i)(4-3i) & (3+2i)(3+2i) + (2-4i)(1+i) \\ (5+i)(-1+i) + (-1+3i)(4-3i) & (5+i)(3+2i) + (-1+3i)(1+i) \end{bmatrix} \\
 &= \begin{bmatrix} -9-21i & 11+10i \\ -1+19i & 9+15i \end{bmatrix}.
 \end{aligned}$$

6.

$$\begin{aligned}
 AB &= \begin{bmatrix} 3-2i & i \\ -i & 1 \end{bmatrix} \begin{bmatrix} -1+i & 2-i & 0 \\ 1+5i & 0 & 3-2i \end{bmatrix} \\
 &= \begin{bmatrix} (3-2i)(-1+i) + i(1+5i) & (3-2i)(2-i) + i \cdot 0 & (3-2i)0 + i(3-2i) \\ -i(-1+i) + 1(1+5i) & -i(2-i) + 1 \cdot 0 & -i \cdot 0 + 1(3-2i) \end{bmatrix} \\
 &= \begin{bmatrix} -6+6i & 4-7i & 2+3i \\ 2+6i & -1-2i & 3-2i \end{bmatrix}
 \end{aligned}$$

7.

$$\begin{aligned}
 ABC &= \begin{bmatrix} 1 & -1 & 2 & 3 \\ -2 & 3 & 4 & 6 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 5 \\ 4 & -3 \\ -1 & 6 \end{bmatrix} C \\
 &= \begin{bmatrix} 7 & 9 \\ 7 & 35 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} -12 & -22 \\ 14 & -126 \end{bmatrix}. \\
 CAB &= \begin{bmatrix} -3 & 2 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & 3 \\ -2 & 3 & 4 & 6 \end{bmatrix} B \\
 &= \begin{bmatrix} -7 & 9 & 2 & 3 \\ 9 & -13 & -14 & -21 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 5 \\ 4 & -3 \\ -1 & 6 \end{bmatrix} = \begin{bmatrix} -7 & 43 \\ -21 & -131 \end{bmatrix}.
 \end{aligned}$$

8.

$$\begin{aligned}
 (2A-3B)C &= \left(2 \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} - 3 \begin{bmatrix} -1 & 2 \\ 5 & 3 \end{bmatrix} \right) C \\
 &= \begin{bmatrix} 5 & -10 \\ -9 & -7 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 25 \\ -20 \end{bmatrix}.
 \end{aligned}$$

9.

$$A\mathbf{c} = \begin{bmatrix} 1 & 3 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ -2 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ -5 \end{bmatrix} + (-2) \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -38 \end{bmatrix}.$$

10.

$$A\mathbf{c} = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 1 & 5 \\ 7 & -6 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \\ -6 \end{bmatrix} + (-4) \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -13 \\ -13 \\ -16 \end{bmatrix}.$$

11.

$$A\mathbf{c} = \begin{bmatrix} -1 & 2 \\ 4 & 7 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = 5 \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} -7 \\ 13 \\ 29 \end{bmatrix}.$$

12. The dimensions of B should be $n \times r$ in order that ABC is defined. The elements of the i th row of A are $a_{i1}, a_{i2}, \dots, a_{in}$ and the elements of the j th column of BC are

$$\sum_{m=1}^r b_{1m}c_{mj}, \sum_{m=1}^r b_{2m}c_{mj}, \dots, \sum_{m=1}^r b_{nm}c_{mj},$$

so the element in the i th row and j th column of $ABC = A(BC)$ is

$$\begin{aligned} & a_{i1} \sum_{m=1}^r b_{1m}c_{mj} + a_{i2} \sum_{m=1}^r b_{2m}c_{mj} + \dots + a_{in} \sum_{m=1}^r b_{nm}c_{mj} \\ &= \sum_{k=1}^n a_{ik} \left(\sum_{m=1}^r b_{km}c_{mj} \right) = \sum_{k=1}^n \left(\sum_{m=1}^r a_{ik}b_{km} \right) c_{mj}. \end{aligned}$$

13. (a):

$$\begin{aligned} A^2 &= AA = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix}. \\ A^3 &= A^2A = \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -9 & -11 \\ 22 & 13 \end{bmatrix}. \\ A^4 &= A^3A = \begin{bmatrix} -9 & -11 \\ 22 & 13 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -31 & -24 \\ 48 & 17 \end{bmatrix}. \end{aligned}$$

(b):

$$\begin{aligned} A^2 &= AA = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & -1 \end{bmatrix}. \\ A^3 &= A^2A = \begin{bmatrix} -2 & 0 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 0 \\ 6 & 4 & -3 \\ -12 & 3 & 4 \end{bmatrix}. \\ A^4 &= A^3A = \begin{bmatrix} 4 & -3 & 0 \\ 6 & 4 & -3 \\ -12 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 4 & -3 \\ -20 & 9 & 4 \\ 10 & -16 & 3 \end{bmatrix}. \end{aligned}$$

14. (a):

$$(A+B)^2 = (A+B)(A+B) = A(A+B) + B(A+B) = A^2 + AB + BA + B^2.$$

(b):

$$\begin{aligned} (A-B)^2 &= (A+(-1)B)^2 \\ &= A^2 + A(-1)B + (-1)BA + [(-1)B]^2 \quad \text{by part (a)} \\ &= A^2 - AB - BA + B^2. \end{aligned}$$

15.

$$\begin{aligned} A^2 - 2A - 8I_2 &= \begin{bmatrix} 14 & -2 \\ -10 & 6 \end{bmatrix} - 2 \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix} - 8 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 14 & -2 \\ -10 & 6 \end{bmatrix} + \begin{bmatrix} -6 & 2 \\ 10 & 2 \end{bmatrix} + \begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix} = 0_2. \end{aligned}$$

16.

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Substituting $A = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$ for A , we have

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

that is,

$$\begin{bmatrix} 1 & 2x & 2z + xy \\ 0 & 1 & 2y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since corresponding elements of equal matrices are equal, we obtain the following implications:

$$\begin{aligned} 2y &= 1 \implies y = 1/2, \\ 2x &= 1 \implies x = 1/2, \\ 2z + xy &= 0 \implies 2z + (1/2)(1/2) = 0 \implies z = -1/8. \end{aligned}$$

Thus, $A = \begin{bmatrix} 1 & 1/2 & -1/8 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}.$

17. In order that $A^2 = A$, we require $\begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix} \begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix} = \begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix}$, that is, $\begin{bmatrix} x^2 - 2 & x + y \\ -2x - 2y & -2 + y^2 \end{bmatrix} = \begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix}$, or equivalently, $\begin{bmatrix} x^2 - x - 2 & x + y - 1 \\ -2x - 2y + 2 & y^2 - y - 2 \end{bmatrix} = 0_2$. Since corresponding elements of equal matrices are equal, it follows that

$$\begin{aligned} x^2 - x - 2 &= 0 \implies x = -1 \text{ or } x = 2, \text{ and} \\ y^2 - y - 2 &= 0 \implies y = -1 \text{ or } y = 2. \end{aligned}$$

Two cases arise from $x + y - 1 = 0$:

(a): If $x = -1$, then $y = 2$.

(b): If $x = 2$, then $y = -1$. Thus,

$$A = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}.$$

18.

$$\begin{aligned} \sigma_1 \sigma_2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = i \sigma_3, \\ \sigma_2 \sigma_3 &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = i \sigma_1, \\ \sigma_3 \sigma_1 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = i \sigma_2. \end{aligned}$$

19.

$$\begin{aligned}
[A, B] &= AB - BA \\
&= \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & -1 \\ 10 & 4 \end{bmatrix} - \begin{bmatrix} 5 & -2 \\ 8 & -2 \end{bmatrix} \\
&= \begin{bmatrix} -6 & 1 \\ 2 & 6 \end{bmatrix} \neq 0_2.
\end{aligned}$$

20.

$$\begin{aligned}
[A_1, A_2] &= A_1 A_2 - A_2 A_1 \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0_2, \quad \text{thus } A_1 \text{ and } A_2 \text{ commute.}
\end{aligned}$$

$$\begin{aligned}
[A_1, A_3] &= A_1 A_3 - A_3 A_1 \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0_2, \quad \text{thus } A_1 \text{ and } A_3 \text{ commute.}
\end{aligned}$$

$$\begin{aligned}
[A_2, A_3] &= A_2 A_3 - A_3 A_2 \\
&= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \neq 0_2.
\end{aligned}$$

Then $[A_3, A_2] = -[A_2, A_3] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \neq 0_2$. Thus, A_2 and A_3 do not commute.

21.

$$\begin{aligned}
[A_1, A_2] &= A_1 A_2 - A_2 A_1 \\
&= \frac{1}{4} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \\
&= \frac{1}{4} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \frac{1}{4} \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\
&= \frac{1}{4} \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = A_3.
\end{aligned}$$

$$\begin{aligned}
[A_2, A_3] &= A_2 A_3 - A_3 A_2 \\
&= \frac{1}{4} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \frac{1}{4} \begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
&= \frac{1}{4} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \\
&= \frac{1}{4} \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = A_1.
\end{aligned}$$

$$\begin{aligned}
[A_3, A_1] &= A_3 A_1 - A_1 A_3 \\
&= \frac{1}{4} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \\
&= \frac{1}{4} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \frac{1}{4} \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = A_2.
\end{aligned}$$

22.

$$\begin{aligned}
&[A, [B, C]] + [B, [C, A]] + [C, [A, B]] \\
&= [A, BC - CB] + [B, CA - AC] + [C, AB - BA] \\
&= A(BC - CB) - (BC - CB)A + B(CA - AC) - (CA - AC)B + C(AB - BA) - (AB - BA)C \\
&= ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB + CAB - CBA - ABC + BAC = 0.
\end{aligned}$$

23.

Proof that $A(BC) = (AB)C$: Let $A = [a_{ij}]$ be of size $m \times n$, $B = [b_{jk}]$ be of size $n \times p$, and $C = [c_{kl}]$ be of size $p \times q$. Consider the (i, j) -element of $(AB)C$:

$$[(AB)C]_{ij} = \sum_{k=1}^p \left(\sum_{h=1}^n a_{ih} b_{hk} \right) c_{kj} = \sum_{h=1}^n a_{ih} \left(\sum_{k=1}^p b_{hk} c_{kj} \right) = [A(BC)]_{ij}.$$

Proof that $A(B + C) = AB + AC$: We have

$$\begin{aligned}
[A(B + C)]_{ij} &= \sum_{k=1}^n a_{ik} (b_{kj} + c_{kj}) \\
&= \sum_{k=1}^n (a_{ik} b_{kj} + a_{ik} c_{kj}) \\
&= \sum_{k=1}^n a_{ik} b_{kj} + \sum_{k=1}^n a_{ik} c_{kj} \\
&= [AB + AC]_{ij}.
\end{aligned}$$

24.

Proof that $(A^T)^T = A$: Let $A = [a_{ij}]$. Then $A^T = [a_{ji}]$, so $(A^T)^T = [a_{ji}]^T = a_{ij} = A$, as needed.

Proof that $(A + C)^T = A^T + C^T$: Let $A = [a_{ij}]$ and $C = [c_{ij}]$. Then $[(A + C)^T]_{ij} = [A + C]_{ji} = [A]_{ji} + [C]_{ji} = a_{ji} + c_{ji} = [A^T]_{ij} + [C^T]_{ij} = [A^T + C^T]_{ij}$. Hence, $(A + C)^T = A^T + C^T$.

25. We have

$$(IA)_{ij} = \sum_{k=1}^m \delta_{ik} a_{kj} = \delta_{ii} a_{ij} = a_{ij},$$

for $1 \leq i \leq m$ and $1 \leq j \leq p$. Thus, $I_m A_{m \times p} = A_{m \times p}$.

26. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ matrices. Then

$$\text{tr}(AB) = \sum_{k=1}^n \left(\sum_{i=1}^n a_{ki} b_{ik} \right) = \sum_{k=1}^n \left(\sum_{i=1}^n b_{ik} a_{ki} \right) = \sum_{i=1}^n \left(\sum_{k=1}^n b_{ik} a_{ki} \right) = \text{tr}(BA).$$

27.

$$\begin{aligned}
A^T &= \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 1 & 2 & -1 \\ 4 & -3 & 0 \end{bmatrix}, \quad B^T = \begin{bmatrix} 0 & -1 & 1 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix}, \\
AA^T &= \begin{bmatrix} 1 & -1 & 1 & 4 \\ 2 & 0 & 2 & -3 \\ 3 & 4 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 1 & 2 & -1 \\ 4 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 19 & -8 & -2 \\ -8 & 17 & 4 \\ -2 & 4 & 26 \end{bmatrix}, \\
AB &= \begin{bmatrix} 1 & -1 & 1 & 4 \\ 2 & 0 & 2 & -3 \\ 3 & 4 & -1 & 0 \\ 3 & 4 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 2 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ -4 & 1 \\ -5 & 10 \end{bmatrix}, \\
B^T A^T &= \begin{bmatrix} 0 & -1 & 1 & 2 \\ 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 1 & 2 & -1 \\ 4 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 10 & -4 & -5 \\ 4 & 1 & 10 \end{bmatrix}.
\end{aligned}$$

28.

(a): We have

$$S = [\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3] = \begin{bmatrix} -x & -y & z \\ 0 & y & 2z \\ x & -y & z \end{bmatrix},$$

so

$$AS = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -x & -y & z \\ 0 & y & 2z \\ x & -y & z \end{bmatrix} = \begin{bmatrix} -x & -y & 7z \\ 0 & y & 14z \\ x & -y & 7z \end{bmatrix} = [\mathbf{s}_1, \mathbf{s}_2, 7\mathbf{s}_3].$$

(b):

$$S^T AS = S^T (AS) = \begin{bmatrix} -x & 0 & x \\ -y & y & -y \\ z & 2z & z \end{bmatrix} \begin{bmatrix} -x & -y & 7z \\ 0 & y & 14z \\ x & -y & 7z \end{bmatrix} = \begin{bmatrix} 2x^2 & 0 & 0 \\ 0 & 3y^2 & 0 \\ 0 & 0 & 42z^2 \end{bmatrix},$$

but $S^T AS = \text{diag}(1, 1, 7)$, so we have the following

$$\begin{aligned}
2x^2 &= 1 \implies x = \pm \frac{\sqrt{2}}{2} \\
3y^2 &= 1 \implies y = \pm \frac{\sqrt{3}}{3} \\
6z^2 &= 1 \implies z = \pm \frac{\sqrt{6}}{6}.
\end{aligned}$$

29.

$$(a): \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

$$(b): \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

30. Suppose A is an $n \times n$ scalar matrix with trace k . If $A = aI_n$, then $\text{tr}(A) = na = k$, so we conclude that $a = k/n$. So $A = \frac{k}{n}I_n$, a uniquely determined matrix.

31. We have

$$S^T = \left[\frac{1}{2}(A + A^T) \right]^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + A) = S$$

and

$$T^T = \left[\frac{1}{2}(A - A^T) \right]^T = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -T.$$

Thus, S is symmetric and T is skew-symmetric.

32.

$$S = \frac{1}{2} \left(\begin{bmatrix} 1 & -5 & 3 \\ 3 & 2 & 4 \\ 7 & -2 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 7 \\ -5 & 2 & -2 \\ 3 & 4 & 6 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 2 & -2 & 10 \\ -2 & 4 & 2 \\ 10 & 2 & 12 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 6 \end{bmatrix}.$$

$$T = \frac{1}{2} \left(\begin{bmatrix} 1 & -5 & 3 \\ 3 & 2 & 4 \\ 7 & -2 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 7 \\ -5 & 2 & -2 \\ 3 & 4 & 6 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} 0 & -8 & -4 \\ 8 & 0 & 6 \\ 4 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 & -2 \\ 4 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}.$$

33. If A is an $n \times n$ symmetric matrix, then $A^T = A$, so it follows that

$$T = \frac{1}{2}(A - A^T) = \frac{1}{2}(A - A) = 0_n.$$

If A is an $n \times n$ skew-symmetric matrix, then $A^T = -A$ and it follows that

$$S = \frac{1}{2}(A + A^T) = \frac{1}{2}(A + (-A)) = 0_n.$$

34. Let A be any $n \times n$ matrix. Then

$$A = \frac{1}{2}(2A) = \frac{1}{2}(A + A^T + A - A^T) = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T),$$

a sum of a symmetric and skew-symmetric matrix, respectively, by Problem 31.

35. If $A = [a_{ij}]$ and $D = \text{diag}(d_1, d_2, \dots, d_n)$, then we must show that the (i, j) -entry of DA is $d_i a_{ij}$. In index notation, we have

$$(DA)_{ij} = \sum_{k=1}^n d_i \delta_{ik} a_{kj} = d_i \delta_{ii} a_{ij} = d_i a_{ij}.$$

Hence, DA is the matrix obtained by multiplying the i th row vector of A by d_i , where $1 \leq i \leq n$.

36.

(a): We have $(AA^T)^T = (A^T)^T A^T = AA^T$, so that AA^T is symmetric.

(b): We have $(ABC)^T = [(AB)C]^T = C^T(AB)^T = C^T(B^T A^T) = C^T B^T A^T$, as needed.

37. $A'(t) = \begin{bmatrix} -2e^{-2t} \\ \cos t \end{bmatrix}.$

$$38. A'(t) = \begin{bmatrix} 1 & \cos t \\ -\sin t & 4 \end{bmatrix}.$$

$$39. A'(t) = \begin{bmatrix} e^t & 2e^{2t} & 2t \\ 2e^t & 8e^{2t} & 10t \end{bmatrix}.$$

$$40. A'(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 1 \\ 0 & 3 & 0 \end{bmatrix}.$$

41. We show that the (i, j) -entry of both sides of the equation agree. First, recall that the (i, j) -entry of AB is $\sum_{k=1}^n a_{ik}b_{kj}$, and therefore, the (i, j) -entry of $\frac{d}{dt}(AB)$ is (by the product rule)

$$\sum_{k=1}^n a'_{ik}b_{kj} + a_{ik}b'_{kj} = \sum_{k=1}^n a'_{ik}b_{kj} + \sum_{k=1}^n a_{ik}b'_{kj}.$$

The former term is precise the (i, j) -entry of the matrix $\frac{dA}{dt}B$, while the latter term is precise the (i, j) -entry of the matrix $A\frac{dB}{dt}$. Thus, the (i, j) -entry of $\frac{d}{dt}(AB)$ is precisely the sum of the (i, j) -entry of $\frac{dA}{dt}B$ and the (i, j) -entry of $A\frac{dB}{dt}$. Thus, the equation we are proving follows immediately.

42. We have

$$\int_0^{\pi/2} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} dt = \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix} \Big|_0^{\pi/2} = \begin{bmatrix} \sin(\pi/2) \\ -\cos(\pi/2) \end{bmatrix} - \begin{bmatrix} \sin 0 \\ -\cos 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

43. We have

$$\int_0^1 \begin{bmatrix} e^t & e^{-t} \\ 2e^t & 5e^{-t} \end{bmatrix} dt = \begin{bmatrix} e^t & -e^{-t} \\ 2e^t & -5e^{-t} \end{bmatrix} \Big|_0^1 = \begin{bmatrix} e & -1/e \\ 2e & -5/e \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} e-1 & 1-1/e \\ 2e-2 & 5-5/e \end{bmatrix}.$$

44. We have

$$\begin{aligned} \int_0^1 \begin{bmatrix} e^{2t} & \sin 2t \\ t^2 - 5 & te^t \end{bmatrix} dt &= \begin{bmatrix} \frac{1}{2}e^{2t} & -\frac{1}{2}\cos 2t \\ \frac{t^3}{3} - 5t & te^t - e^t \end{bmatrix} \Big|_0^1 \\ &= \begin{bmatrix} \frac{e^2}{2} & -\frac{\cos 2}{2} \\ -14/3 & 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{e^2-1}{2} & \frac{1-\cos 2}{2} \\ -14/3 & 1 \end{bmatrix}. \end{aligned}$$

45. We have

$$\begin{aligned} \int_0^1 \begin{bmatrix} e^t & e^{2t} & t^2 \\ 2e^t & 4e^{2t} & 5t^2 \end{bmatrix} dt &= \begin{bmatrix} e^t & \frac{1}{2}e^{2t} & \frac{t^3}{3} \\ 2e^t & 2e^{2t} & \frac{5}{3}t^3 \end{bmatrix} \Big|_0^1 \\ &= \begin{bmatrix} e & e^2/2 & 1/3 \\ 2e & 2e^2 & 5/3 \end{bmatrix} - \begin{bmatrix} 1 & 1/2 & 0 \\ 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} e-1 & \frac{e^2-1}{2} & 1/3 \\ 2e-2 & 2e^2-2 & 5/3 \end{bmatrix}. \end{aligned}$$

$$46. \int \begin{bmatrix} 2t \\ 3t^2 \end{bmatrix} dt = \begin{bmatrix} t^2 \\ t^3 \end{bmatrix}.$$

$$47. \int \begin{bmatrix} \sin t & \cos t & 0 \\ -\cos t & \sin t & t \\ 0 & 3t & 1 \end{bmatrix} dt = \begin{bmatrix} -\cos t & \sin t & 0 \\ -\sin t & -\cos t & t^2/2 \\ 0 & 3t^2/2 & t \end{bmatrix}.$$

$$48. \int \begin{bmatrix} e^t & e^{-t} \\ 2e^t & 5e^{-t} \end{bmatrix} dt = \begin{bmatrix} e^t & -e^{-t} \\ 2e^t & -5e^{-t} \end{bmatrix}.$$

$$49. \int \begin{bmatrix} e^{2t} & \sin 2t \\ t^2 - 5 & te^t \\ \sec^2 t & 3t - \sin t \end{bmatrix} dt = \begin{bmatrix} \frac{1}{2}e^{2t} & -\frac{1}{2}\cos 2t \\ \frac{t^3}{3} - 5t & te^t - e^t \\ \tan t & \frac{3}{2}t^2 + \cos t \end{bmatrix}.$$

Solutions to Section 2.3

True-False Review:

1. **FALSE.** The last column of the augmented matrix corresponds to the constants on the right-hand side of the linear system, so if the augmented matrix has n columns, there are only $n - 1$ unknowns under consideration in the system.

2. **FALSE.** Three distinct planes can intersect in a line (e.g. Figure 2.3.1, lower right picture). For instance, the xy -plane, the xz -plane, and the plane $y = z$ intersect in the x -axis.

3. **FALSE.** The right-hand side vector must have m components, not n components.

4. **TRUE.** If a linear system has two distinct solutions $\mathbf{x}_1$ and $\mathbf{x}_2$, then any point on the line containing $\mathbf{x}_1$ and $\mathbf{x}_2$ is also a solution, giving us infinitely many solutions, not exactly two solutions.

5. **TRUE.** The augmented matrix for a linear system has one additional column (containing the constants on the right-hand side of the equation) beyond the matrix of coefficients.

6. **FALSE.** For instance, if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then solutions to $A\mathbf{x} = \mathbf{0}$ take the form $\begin{bmatrix} t \\ 0 \end{bmatrix}$, while solutions to $A^T\mathbf{x} = \mathbf{0}$ take the form $\begin{bmatrix} 0 \\ t \end{bmatrix}$. The solution sets are not the same.

Problems:

1.

$$2 \cdot 1 - 3(-1) + 4 \cdot 2 = 13,$$

$$1 + (-1) - 2 = -2,$$

$$5 \cdot 1 + 4(-1) + 2 = 3.$$

2.

$$2 + (-3) - 2 \cdot 1 = -3,$$

$$3 \cdot 2 - (-3) - 7 \cdot 1 = 2,$$

$$2 + (-3) + 1 = 0,$$

$$2 \cdot 2 + 2(-3) - 4 \cdot 1 = -6.$$

3.

$$(1 - t) + (2 + 3t) + (3 - 2t) = 6,$$

$$(1 - t) - (2 + 3t) - 2(3 - 2t) = -7,$$

$$5(1-t) + (2+3t) - (3-2t) = 4.$$

4.

$$s + (s-2t) - (2s+3t) + 5t = 0,$$

$$2(s-2t) - (2s+3t) + 7t = 0,$$

$$4s + 2(s-2t) - 3(2s+3t) + 13t = 0.$$

5. The lines $2x + 3y = 1$ and $2x + 3y = 2$ are parallel in the xy -plane, both with slope $-2/3$; thus, the system has no solution.

$$6. A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -5 \\ 7 & 2 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, A^\# = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 2 & 4 & -5 & 2 \\ 7 & 2 & -1 & 3 \end{array} \right].$$

$$7. A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 4 & -3 & 7 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, A^\# = \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 2 & 4 & -3 & 7 & 2 \end{array} \right].$$

$$8. A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -2 \\ 5 & 6 & -5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, A^\# = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 3 & -2 & 0 \\ 5 & 6 & -5 & 0 \end{array} \right].$$

9. It is acceptable to use any variable names. We will use x_1, x_2, x_3, x_4 :

$$x_1 - x_2 + 2x_3 + 3x_4 = 1,$$

$$x_1 + x_2 - 2x_3 + 6x_4 = -1,$$

$$3x_1 + x_2 + 4x_3 + 2x_4 = 2.$$

10. It is acceptable to use any variable names. We will use x_1, x_2, x_3 :

$$2x_1 + x_2 + 3x_3 = 3,$$

$$4x_1 - x_2 + 2x_3 = 1,$$

$$7x_1 + 6x_2 + 3x_3 = -5.$$

11. Given $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$, and an arbitrary constant c ,

(a): we have

$$A\mathbf{z} = A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

and

$$A\mathbf{w} = A(c\mathbf{x}) = c(A\mathbf{x}) = c\mathbf{0} = \mathbf{0}.$$

(b): No, because

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{b} + \mathbf{b} = 2\mathbf{b} \neq \mathbf{b},$$

and

$$A(c\mathbf{x}) = c(A\mathbf{x}) = c\mathbf{b} \neq \mathbf{b}$$

in general.

$$12. \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4t \\ t^2 \end{bmatrix}.$$

$$13. \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} t^2 & -t \\ -\sin t & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$$14. \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} 0 & e^{2t} \\ -\sin t & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$15. \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} 0 & -\sin t & 1 \\ -e^t & 0 & t^2 \\ -t & t^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} t \\ t^3 \\ 1 \end{bmatrix}.$$

16. We have

$$\mathbf{x}'(t) = \begin{bmatrix} 4e^{4t} \\ -2(4e^{4t}) \end{bmatrix} = \begin{bmatrix} 4e^{4t} \\ -8e^{4t} \end{bmatrix}$$

and

$$A\mathbf{x} + \mathbf{b} = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} e^{4t} \\ -2e^{4t} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^{4t} + (-1)(-2e^{4t}) + 0 \\ -2e^{4t} + 3(-2e^{4t}) + 0 \end{bmatrix} = \begin{bmatrix} 4e^{4t} \\ -8e^{4t} \end{bmatrix}.$$

17. We have

$$\mathbf{x}'(t) = \begin{bmatrix} 4(-2e^{-2t}) + 2\cos t \\ 3(-2e^{-2t}) + \sin t \end{bmatrix} = \begin{bmatrix} -8e^{-2t} + 2\cos t \\ -6e^{-2t} + \sin t \end{bmatrix}$$

and

$$\begin{aligned} A\mathbf{x} + \mathbf{b} &= \begin{bmatrix} 1 & -4 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 4e^{-2t} + 2\sin t \\ 3e^{-2t} - \cos t \end{bmatrix} + \begin{bmatrix} -2(\cos t + \sin t) \\ 7\sin t + 2\cos t \end{bmatrix} \\ &= \begin{bmatrix} 4e^{-2t} + 2\sin t - 4(3e^{-2t} - \cos t) - 2(\cos t + \sin t) \\ -3(4e^{-2t} + 2\sin t) + 2(3e^{-2t} - \cos t) + 7\sin t + 2\cos t \end{bmatrix} = \begin{bmatrix} -8e^{-2t} + 2\cos t \\ -6e^{-2t} + \sin t \end{bmatrix}. \end{aligned}$$

Solutions to Section 2.4

True-False Review:

1. **TRUE.** The precise row-echelon form obtained for a matrix depends on the particular elementary row operations (and their order). However, Theorem 2.4.15 states that there is a unique reduced row-echelon form for a matrix.

2. **FALSE.** Upper triangular matrices could have pivot entries that are not 1. For instance, the following matrix is upper triangular, but not in row echelon form: $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

3. **TRUE.** The pivots in a row-echelon form of an $n \times n$ matrix must move down and to the right as we look from one row to the next beneath it. Thus, the pivots must occur on or to the right of the main diagonal of the matrix, and thus all entries below the main diagonal of the matrix are zero.

4. **FALSE.** This would not be true, for example, if A was a zero matrix with 5 rows and B was a nonzero matrix with 4 rows.

5. **FALSE.** If A is a nonzero matrix and $B = -A$, then $A + B = 0$, so $\text{rank}(A + B) = 0$, but $\text{rank}(A)$, $\text{rank}(B) \geq 1$ so $\text{rank}(A) + \text{rank}(B) \geq 2$.

6. **FALSE.** For example, if $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $AB = 0$, so $\text{rank}(AB) = 0$, but $\text{rank}(A) + \text{rank}(B) = 1 + 1 = 2$.

7. TRUE. A matrix of rank zero cannot have any pivots, hence no nonzero rows. It must be the zero matrix.

8. TRUE. The matrices A and $2A$ have the same reduced row-echelon form, since we can move between the two matrices by multiplying the rows of one of them by 2 or $1/2$, a matter of carrying out elementary row operations. If the two matrices have the same reduced row-echelon form, then they have the same rank.

9. TRUE. The matrices A and $2A$ have the same reduced row-echelon form, since we can move between the two matrices by multiplying the rows of one of them by 2 or $1/2$, a matter of carrying out elementary row operations.

Problems:

1. Row-echelon form.

2. Neither.

3. Reduced row-echelon form.

4. Neither.

5. Reduced row-echelon form.

6. Row-echelon form.

7. Reduced row-echelon form.

8. Reduced row-echelon form.

9.

$$\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -3 \\ 0 & 7 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}, \text{Rank } (A) = 2.$$

1. P_{12}	2. $A_{12}(-2)$	3. $M_2(\frac{1}{7})$
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10.

$$\begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -2 \\ -4 & 8 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}, \text{Rank } (A) = 1.$$

1. $M_1(\frac{1}{2})$	2. $A_{12}(4)$
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11.

$$\begin{bmatrix} 2 & 1 & 4 \\ 2 & -3 & 4 \\ 3 & -2 & 6 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 3 & -2 & 6 \\ 2 & -3 & 4 \\ 2 & 1 & 4 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & 4 \\ 2 & 1 & 4 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & 0 \\ 0 & -1 & 0 \end{bmatrix} \stackrel{4}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & -5 & 0 \end{bmatrix} \\ \stackrel{5}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & -5 & 0 \end{bmatrix} \stackrel{6}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{Rank } (A) = 2.$$

1. P_{13}	2. $A_{21}(-1)$	3. $A_{12}(-2), A_{13}(-3)$	4. P_{23}	5. $M_2(-1)$	6. $A_{32}(5)$
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12.

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 3 & 5 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{Rank } (A) = 2.$$

$$\boxed{\mathbf{1. A}_{12}(-1), \mathbf{A}_{13}(-3) \quad \mathbf{2. A}_{23}(-4)}$$

13.

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \\ 2 & 5 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 3 & 2 \\ 2 & -1 \\ 2 & 5 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 2 & 5 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 3 \\ 0 & -7 \\ 0 & -1 \end{bmatrix} \stackrel{4}{\sim} \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 0 & -7 \end{bmatrix} \stackrel{5}{\sim} \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{Rank } (A) = 2.$$

$$\boxed{\mathbf{1. P}_{12} \quad \mathbf{2. A}_{21}(-1) \quad \mathbf{3. A}_{12}(-2), \mathbf{A}_{13}(-2) \quad \mathbf{4. P}_{23} \quad \mathbf{5. M}_2(-1), \mathbf{A}_{23}(7)}.$$

14.

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 3 & 1 & -2 \\ 2 & -1 & 3 \\ 2 & -2 & 1 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 2 & -1 & 3 \\ 0 & -1 & -2 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & -5 & 13 \\ 0 & -1 & -2 \end{bmatrix} \stackrel{4}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & -1 & -2 \\ 0 & -5 & 13 \end{bmatrix} \\ \stackrel{5}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 2 \\ 0 & -5 & 13 \end{bmatrix} \stackrel{6}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 23 \end{bmatrix} \stackrel{7}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \text{Rank } (A) = 3.$$

$$\boxed{\mathbf{1. P}_{12} \quad \mathbf{2. A}_{21}(-1), \mathbf{A}_{23}(-1) \quad \mathbf{3. A}_{12}(-2) \quad \mathbf{4. P}_{23} \quad \mathbf{5. M}_2(-1) \quad \mathbf{6. A}_{23}(5) \quad \mathbf{7. M}_3(1/23)}.$$

15.

$$\begin{bmatrix} 2 & -1 & 3 & 4 \\ 1 & -2 & 1 & 3 \\ 1 & -5 & 0 & 5 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 1 & -5 & 0 & 5 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{Rank } (A) = 2.$$

$$\boxed{\mathbf{1. P}_{12} \quad \mathbf{2. A}_{12}(-2), \mathbf{A}_{13}(-1) \quad \mathbf{3. M}_2(1/3)}$$

16.

$$\begin{bmatrix} 2 & -2 & -1 & 3 \\ 3 & -2 & 3 & 1 \\ 1 & -1 & 1 & 0 \\ 2 & -1 & 2 & 2 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 3 & -2 & 3 & 1 \\ 2 & -2 & -1 & 3 \\ 2 & -1 & 2 & 2 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & 3 \\ 0 & 1 & 0 & 2 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ \stackrel{4}{\sim} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{Rank } (A) = 4.$$

$$\boxed{\mathbf{1. P}_{13} \quad \mathbf{2. A}_{12}(-3), \mathbf{A}_{13}(-2), \mathbf{A}_{14}(-2) \quad \mathbf{3. A}_{24}(-1) \quad \mathbf{4. M}_3(1/3)}$$

17.

$$\begin{bmatrix} 4 & 7 & 4 & 7 \\ 3 & 5 & 3 & 5 \\ 2 & -2 & 2 & -2 \\ 5 & -2 & 5 & -2 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 5 & 3 & 5 \\ 2 & -2 & 2 & -2 \\ 5 & -2 & 5 & -2 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -1 & 0 & -1 \\ 0 & -6 & 0 & -6 \\ 0 & -12 & 0 & -12 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & -6 & 0 & -6 \\ 0 & -12 & 0 & -12 \end{bmatrix}$$

$$\stackrel{4}{\sim} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{Rank } (A) = 2.$$

$$\boxed{\text{1. } A_{21}(-1) \quad \text{2. } A_{12}(-3), A_{13}(-2), A_{14}(-5) \quad \text{3. } M_2(-1) \quad \text{4. } A_{23}(6), A_{24}(12)}$$

18.

$$\begin{bmatrix} 2 & 1 & 3 & 4 & 2 \\ 1 & 0 & 2 & 1 & 3 \\ 2 & 3 & 1 & 5 & 7 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 2 & 1 & 3 & 4 & 2 \\ 2 & 3 & 1 & 5 & 7 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -1 & 2 & -4 \\ 0 & 3 & -3 & 3 & 1 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -1 & 2 & -4 \\ 0 & 0 & 0 & -3 & 1 \end{bmatrix}$$

$$\stackrel{4}{\sim} \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -1 & 2 & -4 \\ 0 & 0 & 0 & 1 & -\frac{1}{3} \end{bmatrix}, \text{Rank } (A) = 3.$$

$$\boxed{\text{1. } P_{12} \quad \text{2. } A_{12}(-2), A_{13}(-2), \quad \text{3. } A_{23}(-3) \quad \text{4. } M_3(-\frac{1}{3})}$$

19.

$$\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \stackrel{4}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \text{Rank } (A) = 2.$$

$$\boxed{\text{1. } P_{12} \quad \text{2. } A_{12}(-3) \quad \text{3. } M_2(\frac{1}{5}) \quad \text{4. } A_{21}(1)}$$

20.

$$\begin{bmatrix} 3 & 7 & 10 \\ 2 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 3 & 7 & 10 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 1 & 7 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 7 \end{bmatrix} \stackrel{4}{\sim} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\stackrel{5}{\sim} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \stackrel{6}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3, \text{Rank } (A) = 3.$$

$$\boxed{\text{1. } P_{13} \quad \text{2. } A_{12}(-2), A_{13}(-3) \quad \text{3. } M_2(-1) \quad \text{4. } A_{21}(-2), A_{23}(-1) \quad \text{5. } M_3(\frac{1}{4}) \quad \text{6. } A_{31}(5), A_{32}(-3)}$$

21.

$$\begin{bmatrix} 3 & -3 & 6 \\ 2 & -2 & 4 \\ 6 & -6 & 12 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{Rank } (A) = 1.$$

$$\boxed{\text{1. } M_1(\frac{1}{3}), A_{12}(-2), A_{13}(-6)}$$

22.

$$\begin{bmatrix} 3 & 5 & -12 \\ 2 & 3 & -7 \\ -2 & -1 & 1 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & -1 & 3 \\ 0 & 3 & -9 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & -3 \\ 0 & 3 & -9 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}, \text{Rank } (A) = 2.$$

1. $A_{21}(-1), A_{12}(-2), A_{13}(2)$	2. $M_2(-1)$	3. $A_{21}(-2), A_{23}(-3)$
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23.

$$\begin{aligned}
 \begin{bmatrix} 1 & -1 & -1 & 2 \\ 3 & -2 & 0 & 7 \\ 2 & -1 & 2 & 4 \\ 4 & -2 & 3 & 8 \end{bmatrix} &\stackrel{1}{\sim} \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 2 & 7 & 0 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \\
 &\stackrel{4}{\sim} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \stackrel{5}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4, \text{Rank}(A) = 4.
 \end{aligned}$$

1. $A_{12}(-3), A_{13}(-2), A_{14}(-4)$	2. $A_{21}(1), A_{23}(-1), A_{24}(-2)$	3. $A_{31}(-2), A_{32}(-3), A_{34}(-1)$
4. $M_4(-1)$ 5. $A_{41}(-5), A_{42}(-4), A_{43}(1)$		

24.

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 3 & -6 & 2 & 7 \\ 4 & -8 & 3 & 10 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -2 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{Rank}(A) = 2.$$

1. $A_{12}(-3), A_{13}(-4)$	2. $M_2(-1)$	3. $A_{21}(-1), A_{23}(1)$
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25.

$$\begin{aligned}
 \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 2 & 0 & 1 \end{bmatrix} &\stackrel{1}{\sim} \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & -6 & -2 \\ 0 & 0 & -4 & -1 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & -4 & -1 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 1/3 \end{bmatrix} \\
 &\stackrel{4}{\sim} \begin{bmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \stackrel{5}{\sim} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{Rank}(A) = 3.
 \end{aligned}$$

1. $A_{12}(-3), A_{13}(-2)$	2. $M_2(-\frac{1}{6})$	3. $A_{21}(-2), A_{23}(4)$	4. $M_3(3)$	5. $A_{32}(-\frac{1}{3}), A_{31}(-\frac{1}{3})$
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Solutions to Section 2.5

True-False Review:

1. **FALSE.** This process is known as Gaussian elimination. Gauss-Jordan elimination is the process by which a matrix is brought to *reduced* row echelon form via elementary row operations.
2. **TRUE.** A homogeneous linear system always has the trivial solution $\mathbf{x} = \mathbf{0}$, hence it is consistent.
3. **TRUE.** The columns of the row-echelon form that contain leading 1s correspond to leading variables, while columns of the row-echelon form that do not contain leading 1s correspond to free variables.
4. **TRUE.** If the last column of the row-reduced augmented matrix for the system does not contain a pivot, then the system can be solved by back-substitution. On the other hand, if this column does contain

a pivot, then that row of the row-reduced matrix containing the pivot in the last column corresponds to the impossible equation $0 = 1$.

5. FALSE. The linear system $x = 0$, $y = 0$, $z = 0$ has a solution in $(0, 0, 0)$ even though none of the variables here is free.

6. FALSE. The columns containing the leading 1s correspond to the *leading* variables, not the free variables.

Problems:

For the problems of this section, A will denote the coefficient matrix of the given system, and $A^\#$ will denote the augmented matrix of the given system.

1. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & 5 & 1 & 3 \\ 2 & 6 & 7 & 1 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 2 & 5 & -1 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 5 & -1 \end{array} \right] \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{array} \right].$$

1. $A_{12}(-3), A_{13}(-2)$	2. $M_2(-1)$	3. $A_{23}(-2)$
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The last augmented matrix results in the system:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1, \\ x_2 + 2x_3 &= 0, \\ x_3 &= -1. \end{aligned}$$

By back substitution we obtain the solution $(-2, 2, -1)$.

2. Converting the given system of equations to an augmented matrix and using Gaussian elimination, we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 3 & -1 & 0 & 1 \\ 2 & 1 & 5 & 4 \\ 7 & -5 & -8 & -3 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -5 & -3 \\ 2 & 1 & 5 & 4 \\ 7 & -5 & -8 & -3 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -5 & -3 \\ 0 & 5 & 15 & 10 \\ 0 & 9 & 27 & 18 \end{array} \right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -5 & -3 \\ 0 & 1 & 3 & 2 \\ 0 & 9 & 27 & 18 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

1. $A_{21}(-1)$	2. $A_{12}(-2), A_{13}(-7)$	3. $M_2(\frac{1}{5})$	4. $A_{21}(2), A_{23}(-9)$
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The last augmented matrix results in the system:

$$\begin{aligned} x_1 + x_3 &= 1, \\ x_2 + 3x_3 &= 2. \end{aligned}$$

Let the free variable $x_3 = t$, a real number. By back substitution we find that the system has the solution set $\{(1 - t, 2 - 3t, t) : \text{for all real numbers } t\}$.

3. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 3 & 5 & -1 & 14 \\ 1 & 2 & 1 & 3 \\ 2 & 5 & 6 & 2 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & 5 & -1 & 4 \\ 2 & 5 & 6 & 2 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & -4 & -5 \\ 0 & 1 & 4 & -4 \end{array} \right] \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 1 & 4 & -4 \end{array} \right]$$

$$\stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & -9 \end{array} \right] \stackrel{5}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 1 \end{array} \right].$$

1. P_{12}	2. $A_{12}(-3), A_{13}(-2)$	3. $M_2(-1)$	4. $A_{23}(-1)$	5. $M_4(-\frac{1}{9})$
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This system of equations is inconsistent since $2 = \text{rank}(A) < \text{rank}(A^\#) = 3$.

4. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 6 & -3 & 3 & 12 \\ 2 & -1 & 1 & 4 \\ -4 & 2 & -2 & -8 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & 2 \\ 2 & -1 & 1 & 4 \\ -4 & 2 & -2 & -8 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

1. $M_1(\frac{1}{6})$	2. $A_{12}(-2), A_{13}(4)$
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Since x_2 and x_3 are free variables, let $x_2 = s$ and $x_3 = t$. The single equation obtained from the augmented matrix is given by $x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 = 2$. Thus, the solution set of our system is given by

$$\{(2 + \frac{s}{2} - \frac{t}{2}, s, t) : s, t \text{ any real numbers}\}.$$

5. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 14 \\ 3 & 1 & -2 & -1 \\ 7 & 2 & -3 & 3 \\ 5 & -1 & -2 & 5 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 3 & 1 & -2 & -1 \\ 2 & -1 & 3 & 14 \\ 7 & 2 & -3 & 3 \\ 5 & -1 & -2 & 5 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -5 & -15 \\ 2 & -1 & 3 & -14 \\ 7 & 2 & -3 & 3 \\ 5 & -1 & -2 & 5 \end{array} \right] \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -5 & -15 \\ 0 & -5 & 13 & 44 \\ 0 & -12 & 32 & 108 \\ 0 & -11 & 23 & 80 \end{array} \right]$$

$$\stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -5 & -15 \\ 0 & -12 & 32 & 108 \\ 0 & -5 & 13 & 44 \\ 0 & -11 & 23 & 80 \end{array} \right] \stackrel{5}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -5 & -15 \\ 0 & -1 & 9 & 28 \\ 0 & -5 & 13 & 44 \\ 0 & -11 & 23 & 80 \end{array} \right] \stackrel{6}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -5 & -15 \\ 0 & 1 & -9 & -28 \\ 0 & -5 & 13 & 44 \\ 0 & -11 & 23 & 80 \end{array} \right]$$

$$\stackrel{7}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -5 & -15 \\ 0 & 1 & -9 & -28 \\ 0 & 0 & -32 & -96 \\ 0 & 0 & -76 & -228 \end{array} \right] \stackrel{8}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -5 & -15 \\ 0 & 1 & -9 & -28 \\ 0 & 0 & 32 & 96 \\ 0 & 0 & -76 & -228 \end{array} \right] \stackrel{9}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -5 & -15 \\ 0 & 1 & -9 & -28 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

1. P_{12}	2. $A_{21}(-1)$	3. $A_{12}(-2), A_{13}(-7), A_{14}(-5)$	4. P_{23}
5. $A_{42}(-1)$	6. $M_2(-1)$	7. $A_{23}(5), A_{24}(11)$	8. $M_3(-1)$
		9. $M_3(\frac{1}{32}), A_{34}(76)$	

The last augmented matrix results in the system of equations:

$$\begin{aligned}x_1 - 2x_2 - 5x_3 &= -15, \\x_2 - 9x_3 &= -28, \\x_3 &= 3.\end{aligned}$$

Thus, using back substitution, the solution set for our system is given by $\{(2, -1, 3)\}$.

6. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\begin{aligned}&\left[\begin{array}{ccc|c}2 & -1 & -4 & 5 \\3 & 2 & -5 & 8 \\5 & 6 & -6 & 20 \\1 & 1 & -3 & -3\end{array}\right] \xrightarrow{1} \left[\begin{array}{ccc|c}1 & 1 & -3 & -3 \\3 & 2 & -5 & 8 \\5 & 6 & -6 & 20 \\2 & -1 & -4 & -5\end{array}\right] \xrightarrow{2} \left[\begin{array}{ccc|c}1 & 1 & -3 & -3 \\0 & -1 & 4 & 17 \\0 & 1 & 9 & 35 \\0 & -3 & 2 & 11\end{array}\right] \xrightarrow{3} \left[\begin{array}{ccc|c}1 & 1 & -3 & -3 \\0 & 1 & -4 & -17 \\0 & 1 & 9 & 35 \\0 & -3 & 2 & 11\end{array}\right] \\&\xrightarrow{4} \left[\begin{array}{ccc|c}1 & 1 & -3 & -3 \\0 & 1 & -4 & -17 \\0 & 0 & 13 & 52 \\0 & 0 & -10 & -40\end{array}\right] \xrightarrow{5} \left[\begin{array}{ccc|c}1 & 1 & -3 & -3 \\0 & 1 & -4 & -17 \\0 & 0 & 1 & 4 \\0 & 0 & -10 & -40\end{array}\right] \xrightarrow{6} \left[\begin{array}{ccc|c}1 & 1 & -3 & -3 \\0 & 1 & -4 & -17 \\0 & 0 & 1 & 4 \\0 & 0 & 0 & 0\end{array}\right].\end{aligned}$$

1. P_{14}	2. $A_{12}(-3), A_{13}(-5), A_{14}(-2)$	3. $M_2(-1)$	4. $A_{23}(-1), A_{24}(3)$	5. $M_3(\frac{1}{13})$	6. $A_{34}(10)$
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The last augmented matrix results in the system of equations:

$$\begin{aligned}x_1 + x_2 - 3x_3 &= -3, \\x_2 - 4x_3 &= -17, \\x_3 &= 4.\end{aligned}$$

By back substitution, we obtain the solution set $\{(10, -1, 4)\}$.

7. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cccc|c}1 & 2 & -1 & 1 & 1 \\2 & 4 & -2 & 2 & 2 \\5 & 10 & -5 & 5 & 5\end{array}\right] \xrightarrow{1} \left[\begin{array}{cccc|c}1 & 2 & -1 & 1 & 1 \\0 & 0 & 0 & 0 & 0 \\0 & 0 & 0 & 0 & 0\end{array}\right].$$

1. $A_{12}(-2), A_{13}(-5)$

The last augmented matrix results in the equation $x_1 + 2x_3 - x_3 + x_4 = 1$. Now x_2, x_3 , and x_4 are free variables, so we let $x_2 = r$, $x_3 = s$, and $x_4 = t$. It follows that $x_1 = 1 - 2r + s - t$. Consequently, the solution set of the system is given by $\{(1 - 2r + s - t, r, s, t) : r, s, t \text{ and real numbers}\}$.

8. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cccc|c}1 & 2 & -1 & 1 & 1 \\2 & -3 & 1 & -1 & 2 \\1 & -5 & 2 & -2 & 1 \\4 & 1 & -1 & 1 & 3\end{array}\right] \xrightarrow{1} \left[\begin{array}{cccc|c}1 & 2 & -1 & 1 & 1 \\0 & -7 & 3 & -3 & 0 \\0 & -7 & 3 & -3 & 0 \\0 & -7 & 3 & -3 & -1\end{array}\right] \xrightarrow{2} \left[\begin{array}{cccc|c}1 & 2 & -1 & 1 & 1 \\0 & 1 & -\frac{3}{7} & \frac{3}{7} & 0 \\0 & -7 & 3 & -3 & 0 \\0 & -7 & 3 & -3 & -1\end{array}\right]$$

$$\sim_3 \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -\frac{3}{7} & \frac{3}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array} \right] \sim_4 \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -\frac{3}{7} & \frac{3}{7} & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \sim_5 \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -\frac{3}{7} & \frac{3}{7} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

$$\boxed{\text{1. } A_{12}(-2), A_{13}(-1), A_{14}(-4) \quad \text{2. } M_2(-\frac{1}{7}) \quad \text{3. } A_{23}(7), A_{24}(7) \quad \text{4. } P_{34} \quad \text{5. } M_3(-1)}$$

The given system of equations is inconsistent since $2 = \text{rank}(A) < \text{rank}(A^\#) = 3$.

9. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & -2 & 3 \\ 0 & 0 & 1 & 4 & -3 & 2 \\ 2 & 4 & -1 & -10 & 5 & 0 \end{array} \right] \sim_1 \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & -2 & 3 \\ 0 & 0 & 1 & 4 & -3 & 2 \\ 0 & 0 & -3 & -12 & 9 & -6 \end{array} \right] \sim_2 \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 1 & -2 & 3 \\ 0 & 0 & 1 & 4 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

$$\boxed{\text{1. } A_{13}(-2) \quad \text{2. } A_{23}(3)}$$

The last augmented matrix indicates that the first two equations of the initial system completely determine its solution. We see that x_4 and x_5 are free variables, so let $x_4 = s$ and $x_5 = t$. Then $x_3 = 2 - 4x_4 + 3x_5 = 2 - 4s + 3t$. Moreover, x_2 is a free variable, say $x_2 = r$, so then $x_1 = 3 - 2r - (2 - 4s + 3t) - s + 2t = 1 - 2r + 3s - t$. Hence, the solution set for the system is

$$\{(1 - 2r + 3s - t, r, 2 - 4s + 3t, s, t) : r, s, t \text{ any real numbers}\}.$$

10. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 2 & -1 & -2 & 2 \\ 4 & 3 & -2 & -1 \\ 1 & 4 & 1 & 4 \end{array} \right] \sim_1 \left[\begin{array}{ccc|c} 1 & 4 & 1 & 4 \\ 4 & 3 & -2 & -1 \\ 2 & -1 & -1 & 2 \end{array} \right] \sim_2 \left[\begin{array}{ccc|c} 1 & 4 & 1 & 4 \\ 0 & -13 & -6 & -17 \\ 0 & -9 & -3 & -6 \end{array} \right] \sim_3 \left[\begin{array}{ccc|c} 1 & 4 & 1 & 4 \\ 0 & -9 & -3 & -6 \\ 0 & -13 & -6 & -17 \end{array} \right]$$

$$\sim_4 \left[\begin{array}{ccc|c} 1 & 4 & 1 & 4 \\ 0 & 12 & 4 & 8 \\ 0 & -13 & -6 & -17 \end{array} \right] \sim_5 \left[\begin{array}{ccc|c} 1 & 4 & 1 & 4 \\ 0 & 12 & 4 & 8 \\ 0 & -1 & -2 & -9 \end{array} \right] \sim_6 \left[\begin{array}{ccc|c} 1 & 4 & 1 & 4 \\ 0 & -1 & -2 & -9 \\ 0 & 12 & 4 & 8 \end{array} \right] \sim_7 \left[\begin{array}{ccc|c} 1 & 4 & 1 & 4 \\ 0 & 1 & 2 & 9 \\ 0 & 12 & 4 & 8 \end{array} \right]$$

$$\sim_8 \left[\begin{array}{ccc|c} 1 & 0 & -7 & -32 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & -20 & -100 \end{array} \right] \sim_9 \left[\begin{array}{ccc|c} 1 & 0 & -7 & -32 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & 1 & 5 \end{array} \right] \sim_{10} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{array} \right].$$

$$\boxed{\text{1. } P_{13} \quad \text{2. } A_{12}(-4), A_{13}(-2) \quad \text{3. } P_{23} \quad \text{4. } M_2(-\frac{4}{3}) \quad \text{5. } A_{23}(1) \\ \text{6. } P_{23} \quad \text{7. } M_2(-1) \quad \text{8. } A_{21}(-4), A_{23}(-12) \quad \text{9. } M_3(-\frac{1}{20}) \quad \text{10. } A_{31}(7), A_{32}(-2)}$$

The last augmented matrix results in the solution $(3, -1, 5)$.

11. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 3 & 1 & 5 & 2 \\ 1 & 1 & -1 & 1 \\ 2 & 1 & 2 & 3 \end{array} \right] \sim_1 \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & 1 & 5 & 2 \\ 2 & 1 & 2 & 3 \end{array} \right] \sim_2 \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 8 & -1 \\ 0 & -1 & 4 & 1 \end{array} \right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -4 & \frac{1}{2} \\ 0 & -1 & 4 & 1 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -4 & 1/2 \\ 0 & 0 & 0 & 3/2 \end{array} \right].$$

We can stop here, since we see from this last augmented matrix that the system is inconsistent. In particular, $2 = \text{rank}(A) < \text{rank}(A^\#) = 3$.

1. P_{12}	2. $A_{12}(-3), A_{13}(-2)$	3. $M_2(-\frac{1}{2})$	4. $A_{23}(1)$
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12. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 3 & -2 & 4 & -9 \\ 1 & -4 & 2 & -3 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & -2 & 2 & 0 \\ 0 & -4 & 4 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & -1 & 0 \\ 0 & -4 & 4 & 0 \end{array} \right] \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

1. $A_{12}(-3), A_{13}(-1)$	2. $M_2(-\frac{1}{2})$	3. $A_{23}(4)$
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The last augmented matrix results in the following system of equations:

$$x_1 - 2x_3 = -3 \quad \text{and} \quad x_2 - x_3 = 0.$$

Since x_3 is free, let $x_3 = t$. Thus, from the system we obtain the solutions $\{(2t-3, t, t) : t \text{ any real number}\}$.

13. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cccc|c} 2 & -1 & 3 & -1 & 3 \\ 3 & 2 & 1 & -5 & -6 \\ 1 & -2 & 3 & 1 & 6 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 6 \\ 3 & 2 & 1 & -5 & -6 \\ 2 & -1 & 3 & -1 & 3 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 6 \\ 0 & 8 & -8 & -8 & -24 \\ 0 & 3 & -3 & -3 & -9 \end{array} \right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 6 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 3 & -3 & -3 & -9 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

1. P_{13}	2. $A_{12}(-3), A_{13}(-2)$	3. $M_2(\frac{1}{8})$	4. $A_{21}(2), A_{23}(-3)$
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The last augmented matrix results in the following system of equations:

$$x_1 + x_3 - x_4 = 0 \quad \text{and} \quad x_2 - x_3 - x_4 = -3.$$

Since x_3 and x_4 are free variables, we can let $x_3 = s$ and $x_4 = t$, where s and t are real numbers. It follows that the solution set of the system is given by $\{(t-s, s+t-3, s, t) : s, t \text{ any real numbers}\}$.

14. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 4 \\ 1 & -1 & -1 & -1 & 2 \\ 1 & 1 & -1 & 1 & -2 \\ 1 & -1 & 1 & 1 & -8 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 4 \\ 0 & -2 & -2 & 0 & -2 \\ 0 & 0 & -2 & 2 & -6 \\ 0 & -2 & 0 & 2 & -12 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 4 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 1 & 0 & -1 & 6 \end{array} \right]$$

$$\begin{aligned} & \sim^3 \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 3 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & -1 & 5 \end{array} \right] \sim^4 \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & -2 & 8 \end{array} \right] \sim^5 \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 & -4 \end{array} \right] \sim^6 \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -4 \end{array} \right]. \\ & \boxed{\begin{array}{lll} \mathbf{1.} \ A_{12}(-1), A_{13}(-1), A_{14}(-1) & \mathbf{2.} \ M_2(-\frac{1}{2}), M_3(-\frac{1}{2}), M_4(-\frac{1}{2}) & \mathbf{3.} \ A_{24}(-1) \\ \mathbf{4.} \ A_{32}(-1), A_{34}(1) & \mathbf{5.} \ M_4(-\frac{1}{2}) & \mathbf{6.} \ A_{41}(1), A_{42}(-1), A_{43}(1) \end{array}} \end{aligned}$$

It follows from the last augmented matrix that the solution to the system is given by $(-1, 2, -1, -4)$.

15. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\begin{aligned} & \left[\begin{array}{ccccc|c} 2 & -1 & 3 & 1 & -1 & 11 \\ 1 & -3 & -2 & -1 & -2 & 2 \\ 3 & 1 & -2 & -1 & 1 & -2 \\ 1 & 2 & 1 & 2 & 3 & -3 \\ 5 & -3 & -3 & 1 & 2 & 2 \end{array} \right] \sim^1 \left[\begin{array}{ccccc|c} 1 & -3 & -2 & -1 & -2 & 2 \\ 2 & -1 & 3 & 1 & -1 & 11 \\ 3 & 1 & -2 & -1 & 1 & -2 \\ 1 & 2 & 1 & 2 & 3 & -3 \\ 5 & -3 & -3 & 1 & 2 & 2 \end{array} \right] \sim^2 \left[\begin{array}{ccccc|c} 1 & -3 & -2 & -1 & -2 & 2 \\ 0 & 5 & 7 & 3 & 3 & 7 \\ 0 & 10 & 4 & 2 & 7 & -8 \\ 0 & 5 & 3 & 3 & 5 & -5 \\ 0 & 12 & 7 & 6 & 12 & -8 \end{array} \right] \\ & \sim^3 \left[\begin{array}{ccccc|c} 1 & -3 & -2 & -1 & -2 & 2 \\ 0 & 1 & \frac{7}{5} & \frac{3}{5} & \frac{3}{5} & \frac{7}{5} \\ 0 & 10 & 4 & 2 & 7 & -8 \\ 0 & 5 & 3 & 3 & 5 & -5 \\ 0 & 12 & 7 & 6 & 12 & -8 \end{array} \right] \sim^4 \left[\begin{array}{ccccc|c} 1 & 0 & \frac{11}{5} & \frac{4}{5} & -\frac{1}{5} & \frac{31}{5} \\ 0 & 1 & \frac{7}{5} & \frac{3}{5} & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & -10 & -4 & 1 & -22 \\ 0 & 0 & -4 & 0 & 2 & -12 \\ 0 & 0 & -\frac{49}{5} & -\frac{6}{5} & \frac{24}{5} & -\frac{124}{5} \end{array} \right] \sim^5 \left[\begin{array}{ccccc|c} 1 & 0 & \frac{11}{5} & \frac{4}{5} & -\frac{1}{5} & \frac{31}{5} \\ 0 & 1 & \frac{7}{5} & \frac{3}{5} & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{10} & \frac{11}{5} \\ 0 & 0 & -4 & 0 & 2 & -12 \\ 0 & 0 & -\frac{49}{5} & -\frac{6}{5} & \frac{24}{5} & -\frac{124}{5} \end{array} \right] \\ & \sim^6 \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{2}{25} & \frac{1}{37} & \frac{34}{25} \\ 0 & 1 & 0 & -\frac{1}{25} & -\frac{42}{37} & -\frac{25}{25} \\ 0 & 0 & 1 & \frac{1}{25} & -\frac{11}{50} & -\frac{25}{25} \\ 0 & 0 & 0 & -\frac{1}{10} & -\frac{16}{5} & -\frac{16}{5} \\ 0 & 0 & 0 & \frac{68}{25} & \frac{191}{50} & -\frac{81}{25} \end{array} \right] \sim^7 \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{2}{25} & \frac{1}{37} & \frac{34}{25} \\ 0 & 1 & 0 & -\frac{1}{25} & -\frac{42}{37} & -\frac{25}{25} \\ 0 & 0 & 1 & \frac{1}{25} & -\frac{11}{50} & -\frac{25}{25} \\ 0 & 0 & 0 & 1 & -\frac{1}{10} & -\frac{16}{5} \\ 0 & 0 & 0 & \frac{68}{25} & \frac{191}{50} & -\frac{81}{25} \end{array} \right] \sim^8 \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & \frac{1}{10} & \frac{6}{5} \\ 0 & 1 & 0 & 0 & -\frac{1}{10} & -\frac{7}{5} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 3 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & \frac{11}{10} & \frac{11}{5} \end{array} \right] \\ & \sim^9 \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & \frac{1}{10} & \frac{6}{5} \\ 0 & 1 & 0 & 0 & -\frac{1}{10} & -\frac{7}{5} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 3 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array} \right] \sim^{10} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array} \right] \end{aligned}$$

$$\boxed{\begin{array}{llll} \mathbf{1.} \ P_{12} & \mathbf{2.} \ A_{12}(-2), A_{13}(-3), A_{14}(-1), A_{15}(-5) & \mathbf{3.} \ M_2(\frac{1}{5}) & \mathbf{4.} \ A_{21}(3), A_{23}(-10), A_{24}(-5), A_{25}(-12) \\ & \mathbf{5.} \ M_3(-\frac{1}{10}) & \mathbf{6.} \ A_{31}(-\frac{11}{5}), A_{32}(-\frac{7}{5}), A_{34}(4), A_{35}(\frac{49}{5}) & \mathbf{7.} \ M_4(\frac{5}{8}) \\ \mathbf{8.} \ A_{41}(\frac{2}{25}), A_{42}(-\frac{1}{25}), A_{43}(-\frac{2}{5}), A_{45}(-\frac{68}{25}) & \mathbf{9.} \ M_5(\frac{10}{11}) & \mathbf{10.} \ A_{51}(-\frac{1}{10}), A_{52}(-\frac{7}{10}), A_{53}(\frac{1}{2}), A_{54}(-1) \end{array}}$$

It follows from the last augmented matrix that the solution to the system is given by $(1, -3, 4, -4, 2)$.

16. The equation $A\mathbf{x} = \mathbf{b}$ reads

$$\begin{bmatrix} 1 & -3 & 1 \\ 5 & -4 & 1 \\ 2 & 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ -4 \end{bmatrix}.$$

Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 8 \\ 5 & -4 & 1 & 15 \\ 2 & 4 & -3 & -4 \end{array} \right] \sim^1 \left[\begin{array}{ccc|c} 1 & -3 & 1 & 8 \\ 0 & 11 & -4 & -25 \\ 0 & 10 & -5 & -20 \end{array} \right] \sim^2 \left[\begin{array}{ccc|c} 1 & -3 & 1 & 8 \\ 0 & 1 & 1 & -5 \\ 0 & 10 & -5 & -20 \end{array} \right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 4 & -7 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & -15 & 30 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 4 & -7 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 1 & -2 \end{array} \right] \stackrel{5}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{array} \right].$$

1. $A_{12}(-5), A_{13}(-2)$	2. $A_{32}(-1)$	3. $A_{21}(3), A_{23}(-10)$	4. $M_3(-\frac{1}{15})$	5. $A_{31}(-4), A_{32}(-1)$
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Thus, from the last augmented matrix, we see that $x_1 = 1$, $x_2 = -3$, and $x_3 = -2$.

17. The equation $A\mathbf{x} = \mathbf{b}$ reads

$$\begin{bmatrix} 1 & 0 & 5 \\ 3 & -2 & 11 \\ 2 & -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\begin{bmatrix} 1 & 0 & 5 & | & 0 \\ 3 & -2 & 11 & | & 2 \\ 2 & -2 & 6 & | & 2 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 0 & 5 & | & 0 \\ 0 & -2 & -4 & | & 2 \\ 0 & -2 & -4 & | & 2 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 0 & 5 & | & 0 \\ 0 & 1 & 2 & | & -1 \\ 0 & -2 & -4 & | & 2 \end{bmatrix}$$

$$\stackrel{3}{\sim} \begin{bmatrix} 1 & 0 & 5 & | & 0 \\ 0 & 1 & 2 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

1. $A_{12}(-3), A_{13}(-2)$	2. $M_2(-1/2)$	3. $A_{23}(2)$
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Hence, we have $x_1 + 5x_3 = 0$ and $x_2 + 2x_3 = -1$. Since x_3 is a free variable, we can let $x_3 = t$, where t is any real number. It follows that the solution set for the given system is given by $\{(-5t, -2t - 1, t) : t \in \mathbb{R}\}$.

18. The equation $A\mathbf{x} = \mathbf{b}$ reads

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 5 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ 5 \end{bmatrix}.$$

Converting the given system of equations to an augmented matrix using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\begin{bmatrix} 0 & 1 & -1 & | & -2 \\ 0 & 5 & 1 & | & 8 \\ 0 & 2 & 1 & | & 5 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 0 & 1 & -1 & | & -2 \\ 0 & 0 & 6 & | & 18 \\ 0 & 0 & 3 & | & 9 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 0 & 1 & -1 & | & -2 \\ 0 & 0 & 1 & | & 3 \\ 0 & 0 & 3 & | & 9 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 3 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

1. $A_{12}(-5), A_{13}(-2)$	2. $M_2(1/6)$	3. $A_{21}(1), A_{23}(-3)$
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Consequently, from the last augmented matrix it follows that the solution set for the matrix equation is given by $\{(t, 1, 3) : t \in \mathbb{R}\}$.

19. The equation $A\mathbf{x} = \mathbf{b}$ reads

$$\begin{bmatrix} 1 & -1 & 0 & -1 \\ 2 & 1 & 3 & 7 \\ 3 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}.$$

Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 2 \\ 2 & 1 & 3 & 7 & 2 \\ 3 & -2 & 1 & 0 & 4 \end{array} \right] \xrightarrow{1} \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 2 \\ 0 & 3 & 3 & 9 & -2 \\ 0 & 1 & 1 & 3 & -2 \end{array} \right] \xrightarrow{2} \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 3 & -2 \\ 0 & 3 & 3 & 9 & -2 \end{array} \right] \xrightarrow{3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 4 \end{array} \right].$$

$$\boxed{1. A_{12}(-2), A_{13}(-3) \quad 2. P_{23} \quad 3. A_{21}(1), A_{23}(-3)}$$

From the last row of the last augmented matrix, it is clear that the given system is inconsistent.

20. The equation $A\mathbf{x} = \mathbf{b}$ reads

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 3 & 1 & -2 & 3 \\ 2 & 3 & 1 & 1 \\ -2 & 3 & 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 3 \\ -9 \end{bmatrix}.$$

Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 3 & 1 & -2 & 3 & 8 \\ 2 & 3 & 1 & 2 & 3 \\ -2 & 3 & 5 & -2 & -9 \end{array} \right] \xrightarrow{1} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & -2 & -2 & 0 & 2 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 5 & 5 & 0 & -5 \end{array} \right] \xrightarrow{2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & -2 & -2 & 0 & 2 \\ 0 & 5 & 5 & 0 & -5 \end{array} \right] \xrightarrow{3} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 3 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

$$\boxed{1. A_{12}(-3), A_{13}(-2), A_{14}(2) \quad 2. P_{23} \quad 3. A_{21}(-1), A_{23}(2), A_{24}(-5)}$$

From the last augmented matrix, we obtain the system of equations: $x_1 - x_3 + x_4 = 3$, $x_2 + x_3 = -1$. Since both x_3 and x_4 are free variables, we may let $x_3 = r$ and $x_4 = t$, where r and t are real numbers. The solution set for the system is given by $\{(3 + r - t, -r - 1, r, t) : r, t \in \mathbb{R}\}$.

21. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 7 \\ 1 & 1 & -k^2 & -k \end{array} \right] \xrightarrow{1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & 1 - k^2 & -3 - k \end{array} \right] \xrightarrow{2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 4 - k^2 & -2 - k \end{array} \right].$$

$$\boxed{1. A_{12}(-2), A_{13}(-1) \quad 2. A_{23}(1)}$$

(a): If $k = 2$, then the last row of the last augmented matrix reveals an inconsistency; hence the system has no solutions in this case.

(b): If $k = -2$, then the last row of the last augmented matrix consists entirely of zeros, and hence we have only two pivots (first two columns) and a free variable x_3 ; hence the system has infinitely many solutions.

(c): If $k \neq \pm 2$, then the last augmented matrix above contains a pivot for each variable x_1 , x_2 , and x_3 , and can be solved for a unique solution by back-substitution.

22. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\begin{aligned} & \left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 0 \\ 4 & 2 & -1 & 1 & 0 \\ 3 & -1 & 1 & k & 0 \end{array} \right] \xrightarrow{1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 2 & 1 & -1 & 1 & 0 \\ 4 & 2 & -1 & 1 & 0 \\ 3 & -1 & 1 & k & 0 \end{array} \right] \xrightarrow{2} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & -1 & -3 & 3 & 0 \\ 0 & -2 & -5 & 5 & 0 \\ 0 & -4 & -2 & k+3 & 0 \end{array} \right] \\ & \xrightarrow{3} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & -2 & -5 & 5 & 0 \\ 0 & -4 & -2 & k+3 & 0 \end{array} \right] \xrightarrow{4} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 10 & k-9 & 0 \end{array} \right] \xrightarrow{5} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & k+1 & 0 \end{array} \right]. \end{aligned}$$

1. P_{12} 2. $A_{12}(-2), A_{13}(-4), A_{14}(-3)$ 3. $M_2(-1)$ 4. $A_{23}(2), A_{24}(4)$ 5. $A_{34}(-10)$

(a): Note that the trivial solution $(0, 0, 0, 0)$ exists under all circumstances, so there are no values of k for which there is no solution.

(b): From the last row of the last augmented matrix, we see that if $k = -1$, then the variable x_4 corresponds to an unpivoted column, and hence it is a free variable. In this case, therefore, we have infinitely solutions.

(c): Provided that $k \neq -1$, then each variable in the system corresponds to a pivoted column of the last augmented matrix above. Therefore, we can solve the system by back-substitution. The conclusion from this is that there is a unique solution, $(0, 0, 0, 0)$.

23. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 3 & 5 & -4 & 16 \\ 2 & 3 & -a & b \end{array} \right] \xrightarrow{1} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & 4-a & b-8 \end{array} \right] \xrightarrow{2} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 4-a & b-8 \end{array} \right] \xrightarrow{3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3-a & b-10 \end{array} \right].$$

1. $A_{12}(-3), A_{13}(-2)$ 2. $M_2(\frac{1}{2})$ 3. $A_{21}(-1), A_{23}(-1)$

(a): From the last row of the last augmented matrix above, we see that there is no solution if $a = 3$ and $b \neq 10$.

(b): From the last row of the augmented matrix above, we see that there are infinitely many solutions if $a = 3$ and $b = 10$, because in that case, there is no pivot in the column of the last augmented matrix corresponding to the third variable x_3 .

(c): From the last row of the augmented matrix above, we see that if $a \neq 3$, then regardless of the value of b , there is a pivot corresponding to each variable x_1, x_2 , and x_3 . Therefore, we can uniquely solve the corresponding system by back-substitution.

24. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cc|c} 1 & -a & 3 \\ 2 & 1 & 6 \\ -3 & a+b & 1 \end{array} \right] \xrightarrow{1} \left[\begin{array}{cc|c} 1 & -a & 3 \\ 0 & 1+2a & 0 \\ 0 & b-2a & 10 \end{array} \right].$$

From the middle row, we see that if $a \neq -\frac{1}{2}$, then we must have $x_2 = 0$, but this leads to an inconsistency in solving for x_1 (the first equation would require $x_1 = 3$ while the last equation would require $x_1 = -\frac{1}{3}$). Now suppose that $a = -\frac{1}{2}$. Then the augmented matrix on the right reduces to $\left[\begin{array}{ccc|c} 1 & -1/2 & 3 & 3 \\ 0 & b+1 & 10 & 10 \end{array} \right]$. If $b = -1$, then once more we have an inconsistency in the last row. However, if $b \neq -1$, then the row-echelon form obtained has full rank, and there is a unique solution. Therefore, we draw the following conclusions:

(a): There is no solution to the system if $a \neq -\frac{1}{2}$ or if $a = -\frac{1}{2}$ and $b = -1$.

(b): Under no circumstances are there an infinite number of solutions to the linear system.

(c): There is a unique solution if $a = -\frac{1}{2}$ and $b \neq -1$.

25. The corresponding augmented matrix for this linear system can be reduced to row-echelon form via

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & y_1 \\ 2 & 3 & 1 & y_2 \\ 3 & 5 & 1 & y_3 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & y_1 \\ 0 & 1 & -1 & y_2 - 2y_1 \\ 0 & 2 & -2 & y_3 - 3y_1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & y_1 \\ 0 & 1 & -1 & y_2 - 2y_1 \\ 0 & 0 & 0 & y_1 - 2y_2 + y_3 \end{array} \right].$$

$$\boxed{1. A_{12}(-2), A_{13}(-3) \quad 2. A_{23}(-2)}$$

For consistency, we must have $\text{rank}(A) = \text{rank}(A^\#)$, which requires (y_1, y_2, y_3) to satisfy $y_1 - 2y_2 + y_3 = 0$. If this holds, then the system has an infinite number of solutions, because the column of the augmented matrix corresponding to y_3 will be unpivoted, indicating that y_3 is a free variable in the solution set.

26. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following row-equivalent matrices. Since $a_{11} \neq 0$:

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & \frac{a_{22}a_{11} - a_{21}a_{12}}{a_{11}} & \frac{a_{11}b_2 - a_{21}b_1}{a_{11}} \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & \frac{\Delta}{a_{11}} & \frac{\Delta_2}{a_{11}} \end{array} \right].$$

$$\boxed{1. M_1(1/a_{11}), A_{12}(-a_{21}) \quad 2. \text{Definition of } \Delta \text{ and } \Delta_2}$$

(a): If $\Delta \neq 0$, then $\text{rank}(A) = \text{rank}(A^\#) = 2$, so the system has a unique solution (of course, we are assuming $a_{11} \neq 0$ here). Using the last augmented matrix above, $\left(\frac{\Delta}{a_{11}}\right)x_2 = \frac{\Delta_2}{a_{11}}$, so that $x_2 = \frac{\Delta_2}{\Delta}$. Using this, we can solve $x_1 + \frac{a_{12}}{a_{11}}x_2 = \frac{b_1}{a_{11}}$ for x_1 to obtain $x_1 = \frac{\Delta_1}{\Delta}$, where we have used the fact that $\Delta_1 = a_{22}b_1 - a_{12}b_2$.

(b): If $\Delta = 0$ and $a_{11} \neq 0$, then the augmented matrix of the system is $\left[\begin{array}{cc|c} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & 0 & \frac{\Delta_2}{a_{11}} \end{array} \right]$, so it follows that the system has (i) no solution if $\Delta_2 \neq 0$, since $\text{rank}(A) < \text{rank}(A^\#) = 2$, and (ii) an infinite number of solutions if $\Delta_2 = 0$, since $\text{rank}(A^\#) < 2$.

(c): An infinite number of solutions would be represented as one line. No solution would be two parallel lines. A unique solution would be the intersection of two distinct lines at one point.

27. We first use the partial pivoting algorithm to reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & 5 & 1 & 3 \\ 2 & 6 & 7 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 3 & 5 & 1 & 3 \\ 1 & 2 & 1 & 1 \\ 2 & 6 & 7 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 3 & 5 & 1 & 3 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 8/3 & 19/3 & -1 \end{array} \right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccc|c} 3 & 5 & 1 & 3 \\ 0 & 8/3 & 19/3 & -1 \\ 0 & 1/3 & 2/3 & 0 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 3 & 5 & 1 & 3 \\ 0 & 8/3 & 19/3 & -1 \\ 0 & 0 & -1/8 & 1/8 \end{array} \right].$$

$$\boxed{\begin{array}{llll} \mathbf{1.} & \mathbf{P}_{12} & \mathbf{2.} & \mathbf{A}_{12}(-1/3), \mathbf{A}_{13}(-2/3) & \mathbf{3.} & \mathbf{P}_{23} & \mathbf{4.} & \mathbf{A}_{23}(-1/8) \end{array}}$$

Using back substitution to solve the equivalent system yields the unique solution $(-2, 2, -1)$.

28. We first use the partial pivoting algorithm to reduce the augmented matrix of the system:

$$\begin{aligned} & \left[\begin{array}{ccc|c} 2 & -1 & 3 & 14 \\ 3 & 1 & -2 & -1 \\ 7 & 2 & -3 & 3 \\ 5 & -1 & -2 & 5 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 7 & 2 & -3 & 3 \\ 3 & 1 & -2 & -1 \\ 2 & -1 & 3 & 14 \\ 5 & -1 & -2 & 5 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 7 & 2 & -3 & 3 \\ 0 & 1/7 & -5/7 & -16/7 \\ 0 & -11/7 & 27/7 & 92/7 \\ 0 & -17/7 & 1/7 & 20/7 \end{array} \right] \\ & \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 7 & 2 & -3 & 3 \\ 0 & -17/7 & 1/7 & 20/7 \\ 0 & -11/7 & 27/7 & 92/7 \\ 0 & 1/7 & -5/7 & -16/7 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 7 & 2 & -3 & 3 \\ 0 & -17/7 & 1/7 & 20/7 \\ 0 & 0 & 64/17 & 192/17 \\ 0 & 0 & -12/17 & -36/17 \end{array} \right] \\ & \stackrel{5}{\sim} \left[\begin{array}{ccc|c} 7 & 2 & -3 & 3 \\ 0 & -17/7 & 1/7 & 20/7 \\ 0 & 0 & 64/17 & 192/17 \\ 0 & 0 & 0 & 0 \end{array} \right]. \end{aligned}$$

$$\boxed{\begin{array}{llll} \mathbf{1.} & \mathbf{P}_{13} & \mathbf{2.} & \mathbf{A}_{12}(-3/7), \mathbf{A}_{13}(-2/7), \mathbf{A}_{14}(-5/7) & \mathbf{3.} & \mathbf{P}_{24} \\ & & \mathbf{4.} & \mathbf{A}_{23}(-11/17), \mathbf{A}_{24}(1/17) & \mathbf{5.} & \mathbf{A}_{34}(3/16) \end{array}}$$

Using back substitution to solve the equivalent system yields the unique solution $(2, -1, 3)$.

29. We first use the partial pivoting algorithm to reduce the augmented matrix of the system:

$$\begin{aligned} & \left[\begin{array}{ccc|c} 2 & -1 & -4 & 5 \\ 3 & 2 & -5 & 8 \\ 5 & 6 & -6 & 20 \\ 1 & 1 & -3 & -3 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 5 & 6 & -6 & -20 \\ 3 & 2 & -5 & 8 \\ 2 & -1 & -4 & 5 \\ 1 & 1 & -3 & -3 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 5 & 6 & -6 & 20 \\ 0 & -8/5 & -7/5 & -4 \\ 0 & -17/5 & -8/5 & -3 \\ 0 & -1/5 & -9/5 & -7 \end{array} \right] \\ & \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 5 & 6 & -6 & 20 \\ 0 & -17/5 & -8/5 & -3 \\ 0 & -8/5 & -7/5 & -4 \\ 0 & -1/5 & -9/5 & -7 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 5 & 6 & -6 & 20 \\ 0 & -17/5 & -8/5 & -3 \\ 0 & 0 & -11/17 & -44/17 \\ 0 & 0 & -29/17 & -116/17 \end{array} \right] \\ & \stackrel{5}{\sim} \left[\begin{array}{ccc|c} 5 & 6 & -6 & 20 \\ 0 & -17/5 & -8/5 & -3 \\ 0 & 0 & -29/17 & -116/17 \\ 0 & 0 & -11/17 & -44/17 \end{array} \right] \stackrel{6}{\sim} \left[\begin{array}{ccc|c} 5 & 6 & -6 & 20 \\ 0 & -17/5 & -8/5 & -3 \\ 0 & 0 & -29/17 & -116/17 \\ 0 & 0 & 0 & 0 \end{array} \right]. \end{aligned}$$

$$\boxed{\begin{array}{llll} \mathbf{1.} & \mathbf{P}_{13} & \mathbf{2.} & \mathbf{A}_{12}(-3/5), \mathbf{A}_{13}(-2/5), \mathbf{A}_{14}(-1/5) & \mathbf{3.} & \mathbf{P}_{23} \\ & & \mathbf{4.} & \mathbf{A}_{23}(-8/17), \mathbf{A}_{24}(-1/17) & \mathbf{5.} & \mathbf{P}_{34} & \mathbf{6.} & \mathbf{A}_{34}(-11/29) \end{array}}$$

Using back substitution to solve the equivalent system yields the unique solution $(10, -1, 4)$.

30. We first use the partial pivoting algorithm to reduce the augmented matrix of the system:

$$\begin{aligned} \left[\begin{array}{ccc|c} 2 & -1 & -1 & 2 \\ 4 & 3 & -2 & -1 \\ 1 & 4 & 1 & 4 \end{array} \right] &\stackrel{1}{\sim} \left[\begin{array}{ccc|c} 4 & 3 & -2 & -1 \\ 2 & -1 & -1 & 2 \\ 1 & 4 & 1 & 4 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 4 & 3 & -2 & -1 \\ 0 & -5/2 & 0 & 5/2 \\ 0 & 13/4 & 3/2 & 17/4 \end{array} \right] \\ &\stackrel{3}{\sim} \left[\begin{array}{ccc|c} 4 & 3 & -2 & -1 \\ 0 & 13/4 & 3/2 & 17/4 \\ 0 & -5/2 & 0 & 5/2 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 4 & 3 & -2 & -1 \\ 0 & 13/4 & 3/2 & 17/4 \\ 0 & 0 & 15/13 & 75/13 \end{array} \right]. \end{aligned}$$

1. P_{12}	2. $A_{12}(-1/2), A_{13}(-1/4)$	3. P_{23}	4. $A_{23}(10/13)$
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Using back substitution to solve the equivalent system yields the unique solution $(3, -1, 5)$.

31.

(a): Let

$$A^\# = \left[\begin{array}{ccccc|c} a_{11} & 0 & 0 & \dots & 0 & b_1 \\ a_{21} & a_{22} & 0 & \dots & 0 & b_2 \\ a_{31} & a_{32} & a_{33} & \dots & 0 & b_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} & b_n \end{array} \right]$$

represent the corresponding augmented matrix of the given system. Since $a_{11}x_1 = b_1$, we can solve for x_1 easily:

$$x_1 = \frac{b_1}{a_{11}}, \quad (a_{11} \neq 0).$$

Now since $a_{21}x_1 + a_{22}x_2 = b_2$, by using the expression for x_1 we just obtained, we can solve for x_2 :

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22}}.$$

In a similar manner, we can solve for $x_3, x_4, \dots, x_n$.

(b): We solve instantly for x_1 from the first equation: $x_1 = 2$. Substituting this into the middle equation, we obtain $2 \cdot 2 - 3 \cdot x_2 = 1$, from which it quickly follows that $x_2 = 1$. Substituting for x_1 and x_2 in the bottom equation yields $3 \cdot 2 + 1 - x_3 = 8$, from which it quickly follows that $x_3 = -1$. Consequently, the solution of the given system is $(2, 1, -1)$.

32. This system of equations is not linear in x_1, x_2 , and x_3 ; however, the system *is* linear in x_1^3, x_2^2 , and x_3 , so we can first solve for x_1^3, x_2^2 , and x_3 . Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\begin{aligned} \left[\begin{array}{ccc|c} 4 & 2 & 3 & 12 \\ 1 & -1 & 1 & 2 \\ 3 & 1 & -1 & 2 \end{array} \right] &\stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 4 & 2 & 3 & 12 \\ 3 & 1 & -1 & 2 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 6 & -1 & 4 \\ 0 & 4 & -4 & -4 \end{array} \right] \\ &\stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 4 & -4 & -4 \\ 0 & 6 & -1 & 4 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 6 & -1 & 4 \end{array} \right] \stackrel{5}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 5 & 10 \end{array} \right] \\ &\stackrel{6}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array} \right] \stackrel{7}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right]. \end{aligned}$$

1. P_{12}	2. $A_{12}(-4), A_{13}(-3)$	3. P_{23}	4. $M_2(1/4)$
5. $A_{21}(1), A_{23}(-6)$	6. $M_2(1/5)$	7. $A_{32}(1)$	

Thus, taking only real solutions, we have $x_1^3 = 1$, $x_2^2 = 1$, and $x_3 = 2$. Therefore, $x_1 = 1$, $x_2 = \pm 1$, and $x_3 = 2$, leading to the *two* solutions $(1, 1, 2)$ and $(1, -1, 2)$ to the original system of equations. There is no contradiction of Theorem 2.5.9 here since, as mentioned above, this system is *not linear* in x_1 , x_2 , and x_3 .

33. Reduce the augmented matrix of the system:

$$\begin{aligned}
 \left[\begin{array}{ccc|c} 3 & 2 & -1 & 0 \\ 2 & 1 & 1 & 0 \\ 5 & -4 & 1 & 0 \end{array} \right] &\stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -1 & 5 & 0 \\ 0 & -9 & 11 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & -9 & 11 & 0 \end{array} \right] \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & -34 & 0 \end{array} \right] \\
 &\stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \stackrel{5}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right].
 \end{aligned}$$

1. $A_{21}(-1), A_{12}(-2), A_{13}(-5)$	2. $M_2(-1)$	3. $A_{21}(-1), A_{23}(9)$
4. $M_3(-1/34)$	5. $A_{31}(-3), A_{32}(5)$	

Therefore, the unique solution to this system is $x_1 = x_2 = x_3 = 0$: $(0, 0, 0)$.

34. Reduce the augmented matrix of the system:

$$\begin{aligned}
 \left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 3 & -1 & 2 & 0 \\ 1 & -1 & -1 & 0 \\ 5 & 2 & -2 & 0 \end{array} \right] &\stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 3 & -1 & 2 & 0 \\ 2 & 1 & -1 & 0 \\ 5 & 2 & -2 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 7 & 3 & 0 \end{array} \right] \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 7 & 3 & 0 \end{array} \right] \\
 &\stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 7 & 3 & 0 \end{array} \right] \stackrel{5}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 31 & 0 \end{array} \right] \stackrel{6}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 31 & 0 \end{array} \right] \stackrel{7}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].
 \end{aligned}$$

1. P_{13}	2. $A_{12}(-3), A_{13}(-2), A_{14}(-5)$	3. P_{23}	4. $A_{32}(-1)$
5. $A_{21}(1), A_{23}(-2), A_{24}(-7)$	6. $M_3(1/13)$	7. $A_{31}(5), A_{32}(4), A_{34}(-31)$	

Therefore, the unique solution to this system is $x_1 = x_2 = x_3 = 0$: $(0, 0, 0)$.

35. Reduce the augmented matrix of the system:

$$\begin{aligned}
 \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 5 & -1 & 2 & 0 \\ 1 & 1 & 4 & 0 \end{array} \right] &\stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 5 & -1 & 2 & 0 \\ 2 & -1 & -1 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & -6 & -18 & 0 \\ 0 & -3 & -9 & 0 \end{array} \right] \\
 &\stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -3 & -9 & 0 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].
 \end{aligned}$$

1. P_{13}	2. $A_{12}(-5), A_{13}(-2)$	3. $M_2(-1/6)$	4. $A_{21}(-1), A_{23}(3)$
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It follows that $x_1 + x_3 = 0$ and $x_2 + 3x_3 = 0$. Setting $x_3 = t$, where t is a free variable, we get $x_2 = -3t$ and $x_1 = -t$. Thus we have that the solution set of the system is $\{(-t, -3t, t) : t \in \mathbb{R}\}$.

36. Reduce the augmented matrix of the system:

$$\begin{aligned} & \left[\begin{array}{ccc|c} 1+2i & 1-i & 1 & 0 \\ i & 1+i & -i & 0 \\ 2i & 1 & 1+3i & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} i & 1+i & -i & 0 \\ 1+2i & 1-i & 1 & 0 \\ 2i & 1 & 1+3i & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & 1-i & -1 & 0 \\ 1+2i & 1-i & 1 & 0 \\ 2i & 1 & 1+3i & 0 \end{array} \right] \\ & \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 1-i & -1 & 0 \\ 0 & -2-2i & 1+2i & 0 \\ 0 & -1-2i & 1+5i & 0 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & 1-i & -1 & 0 \\ 0 & -2-2i & 1+2i & 0 \\ 0 & 1 & 3i & 0 \end{array} \right] \stackrel{5}{\sim} \left[\begin{array}{ccc|c} 1 & 1-i & -1 & 0 \\ 0 & 0 & -5+8i & 0 \\ 0 & 1 & 3i & 0 \end{array} \right] \\ & \stackrel{6}{\sim} \left[\begin{array}{ccc|c} 1 & 1-i & -1 & 0 \\ 0 & 1 & 3i & 0 \\ 0 & 0 & -5+8i & 0 \end{array} \right] \stackrel{7}{\sim} \left[\begin{array}{ccc|c} 1 & 1-i & -1 & 0 \\ 0 & 1 & 3i & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \stackrel{8}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]. \end{aligned}$$

1. P_{12}	2. $M_1(-i)$	3. $A_{12}(-1-2i), A_{13}(-2i)$	4. $A_{23}(-1)$	5. $A_{32}(2+2i)$
	6. P_{23}	7. $M_3(\frac{1}{-5+8i})$	8. $A_{21}(-1+i), A_{31}(1), A_{32}(-3i)$	

Therefore, the unique solution to this system is $x_1 = x_2 = x_3 = 0$: $(0, 0, 0)$.

37. Reduce the augmented matrix of the system:

$$\begin{aligned} & \left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 6 & -1 & 2 & 0 \\ 12 & 6 & 4 & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 6 & -1 & 2 & 0 \\ 12 & 6 & 4 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -5 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array} \right] \\ & \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]. \end{aligned}$$

1. $M_1(1/3)$	2. $A_{12}(-6), A_{13}(-12)$	3. $M_2(-1/5)$	4. $A_{21}(-2/3), A_{23}(2)$
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From the last augmented matrix, we have $x_1 + \frac{1}{3}x_3 = 0$ and $x_2 = 0$. Since x_3 is a free variable, we let $x_3 = t$, where t is a real number. It follows that the solution set for the given system is given by $\{(t, 0, -3t) : t \in \mathbb{R}\}$.

38. Reduce the augmented matrix of the system:

$$\begin{aligned} & \left[\begin{array}{ccc|c} 2 & 1 & -8 & 0 \\ 3 & -2 & -5 & 0 \\ 5 & -6 & -3 & 0 \\ 3 & -5 & 1 & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 3 & -2 & -5 & 0 \\ 2 & 1 & -8 & 0 \\ 5 & -6 & -3 & 0 \\ 3 & -5 & 1 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 3 & 0 \\ 2 & 1 & -8 & 0 \\ 5 & -6 & -3 & 0 \\ 3 & -5 & 1 & 0 \end{array} \right] \\ & \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 3 & 0 \\ 0 & 7 & -14 & 0 \\ 0 & 9 & -18 & 0 \\ 0 & 4 & -8 & 0 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 9 & -18 & 0 \\ 0 & 4 & -8 & 0 \end{array} \right] \stackrel{5}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]. \end{aligned}$$

1. P_{12}	2. $A_{21}(-1)$	3. $A_{12}(-2), A_{13}(-5), A_{14}(-3)$	4. $M_2(1/7)$	5. $A_{21}(3), A_{23}(-9), A_{24}(-4)$
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From the last augmented matrix we have: $x_1 - 3x_3 = 0$ and $x_2 - 2x_3 = 0$. Since x_3 is a free variable, we let $x_3 = t$, where t is a real number. It follows that $x_2 = 2t$ and $x_1 = 3t$. Thus, the solution set for the given system is given by $\{(3t, 2t, t) : t \in \mathbb{R}\}$.

39. Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 1+i & 1-i & 0 \\ i & 1 & i & 0 \\ 1-2i & -1+i & 1-3i & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & 1+i & 1-i & 0 \\ 0 & 2-i & -1 & 0 \\ 0 & -4+2i & 2 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & 1+i & 1-i & 0 \\ 0 & 2-i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 1+i & 1-i & 0 \\ 0 & 1 & \frac{-2-i}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & \frac{6-2i}{5} & 0 \\ 0 & 1 & \frac{-2-i}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

1. $A_{12}(-i), A_{13}(-1+2i)$	2. $A_{23}(2)$	3. $M_2(\frac{1}{2-i})$	4. $A_{21}(-1-i)$
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From the last augmented matrix we see that x_3 is a free variable. We set $x_3 = 5s$, where $s \in \mathbb{C}$. Then $x_1 = 2(i-3)s$ and $x_2 = (2+i)s$. Thus, the solution set of the system is $\{(2(i-3)s, (2+i)s, 5s) : s \in \mathbb{C}\}$.

40. Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 3 & 0 & -1 & 0 \\ 5 & 1 & -1 & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 6 & -6 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 6 & -6 & 0 \end{array} \right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 5/3 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & -10 & 0 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 5/3 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -10 & 0 \end{array} \right] \stackrel{5}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

1. $A_{13}(-3), A_{14}(-5)$	2. $M_2(1/3)$	3. $A_{21}(1), A_{23}(-3), A_{24}(-6)$
4. $M_3(-1/6)$	5. $A_{31}(-5/3), A_{32}(-2/3), A_{34}(10)$	

Therefore, the unique solution to this system is $x_1 = x_2 = x_3 = 0$: $(0, 0, 0)$.

41. Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 2 & -4 & 6 & 0 \\ 3 & -6 & 9 & 0 \\ 1 & -2 & 3 & 0 \\ 5 & -10 & 15 & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 3 & -6 & 9 & 0 \\ 2 & -4 & 6 & 0 \\ 5 & -10 & 15 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

1. $M_1(1/2)$	2. $A_{12}(-3), A_{13}(-2), A_{14}(-5)$
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From the last matrix we have that $x_1 - 2x_3 + 3x_3 = 0$. Since x_2 and x_3 are free variables, let $x_2 = s$ and let $x_3 = t$, where s and t are real numbers. The solution set of the given system is therefore $\{(2s - 3t, s, t) : s, t \in \mathbb{R}\}$.

42. Reduce the augmented matrix of the system:

$$\left[\begin{array}{cccc|c} 4 & -2 & -1 & -1 & 0 \\ 3 & 1 & -2 & 3 & 0 \\ 5 & -1 & -2 & 1 & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 1 & -4 & 0 \\ 3 & 1 & -2 & 3 & 0 \\ 5 & -1 & -2 & 1 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 1 & -4 & 0 \\ 0 & 10 & -5 & 15 & 0 \\ 0 & 14 & -7 & 21 & 0 \end{array} \right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 1 & -4 & 0 \\ 0 & 2 & -1 & 3 & 0 \\ 0 & 2 & -1 & 3 & 0 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 1 & -4 & 0 \\ 0 & 2 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \stackrel{5}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 1 & -4 & 0 \\ 0 & 1 & -1/2 & 3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

1. $A_{21}(-1)$ **2.** $A_{12}(-3), A_{13}(-5)$ **3.** $M_2(1/5), M_3(1/7)$
4. $A_{23}(-1)$ **5.** $M_2(1/2)$

From the last augmented matrix above we have that $x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 = 0$ and $x_1 - 3x_2 + x_3 - 4x_4 = 0$. Since x_3 and x_4 are free variables, we can set $x_3 = 2s$ and $x_4 = 2t$, where s and t are real numbers. Then $x_2 = s - 3t$ and $x_1 = s - t$. It follows that the solution set of the given system is $\{(s - t, s - 3t, 2s, 2t) : s, t \in \mathbb{R}\}$.

43. Reduce the augmented matrix of the system:

$$\begin{aligned} & \left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 0 \\ 3 & -1 & 1 & -2 & 0 \\ 4 & 2 & -1 & 1 & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 2 & 1 & -1 & 1 & 0 \\ 3 & -1 & 1 & -2 & 0 \\ 4 & 2 & -1 & 1 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & -1 & -3 & 3 & 0 \\ 0 & -4 & -2 & 1 & 0 \\ 0 & -2 & -5 & 5 & 0 \end{array} \right] \\ & \stackrel{3}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & -4 & -2 & 1 & 0 \\ 0 & -2 & -5 & 5 & 0 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 10 & -11 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{array} \right] \stackrel{5}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & -3 & 3 & 0 \\ 0 & 0 & 10 & -11 & 0 \end{array} \right] \\ & \stackrel{6}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 10 & -11 & 0 \end{array} \right] \stackrel{7}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array} \right] \stackrel{8}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right] \stackrel{9}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]. \end{aligned}$$

1. P_{12} **2.** $A_{12}(-2), A_{13}(-3), A_{14}(-4)$ **3.** $M_2(-1)$ **4.** $A_{21}(-1), A_{23}(4), A_{24}(2)$
5. P_{34} **6.** $M_3(-1/3)$ **7.** $A_{31}(2), A_{32}(-3), A_{34}(-10)$ **8.** $M_4(-1)$ **9.** $A_{43}(1)$

From the last augmented matrix, it follows that the solution set to the system is given by $\{(0, 0, 0, 0)\}$.

44. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 3 & 4 & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 3 & 4 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{11}{2} & 0 \end{array} \right] \stackrel{3}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right].$$

1. $M_1(1/2)$ **2.** $A_{12}(-3)$ **3.** $M_2(2/11)$ **4.** $A_{21}(1/2)$

From the last augmented matrix, we see that $x_1 = x_2 = 0$. Hence, the solution set is $\{(0, 0)\}$.

45. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1-i & 2i \\ 1+i & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{cc|c} 1-i & 2i & 0 \\ 1+i & -2 & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{cc|c} 1 & -1+i & 0 \\ 1+i & -2 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{cc|c} 1 & -1+i & 0 \\ 0 & 0 & 0 \end{array} \right].$$

$$\boxed{1. M_1(\frac{1+i}{2}) \quad 2. A_{12}(-1-i)}$$

It follows that $x_1 + (-1+i)x_2 = 0$. Since x_2 is a free variable, we can let $x_2 = t$, where t is a complex number. The solution set to the system is then given by $\{(t(1-i), t) : t \in \mathbb{C}\}$.

46. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1+i & 1-2i \\ -1+i & 2+i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{cc|c} 1+i & 1-2i & 0 \\ -1+i & 2+i & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{1+3i}{2} & 0 \\ -1+i & 2+i & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{1+3i}{2} & 0 \\ 0 & 0 & 0 \end{array} \right].$$

$$\boxed{1. M_1(\frac{1-i}{2}) \quad 2. A_{12}(1-i)}$$

It follows that $x_1 - \frac{1+3i}{2}x_2 = 0$. Since x_2 is a free variable, we can let $x_2 = r$, where r is any complex number. Thus, the solution set to the given system is $\{(\frac{1+3i}{2}r, r) : r \in \mathbb{C}\}$.

47. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array} \right] &\stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -5 & -6 & 0 \\ 0 & -1 & -2 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -5 & -6 & 0 \end{array} \right] \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -5 & -6 & 0 \end{array} \right] \\ &\stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 4 & 0 \end{array} \right] \stackrel{5}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \stackrel{6}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]. \end{aligned}$$

$$\boxed{1. A_{12}(-2), A_{13}(-1) \quad 2. P_{23} \quad 3. M_2(-1) \quad 4. A_{21}(-2), A_{23}(5) \quad 5. M_3(1/4) \quad 6. A_{31}(1), A_{32}(-2)}$$

From the last augmented matrix, we see that the only solution to the given system is $x_1 = x_2 = x_3 = 0$: $\{(0, 0, 0)\}$.

48. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & 0 & -1 & 2 \\ 1 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 1 & 3 & 2 & 2 & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 3 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array} \right] \stackrel{3}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array} \right].$$

$$\boxed{1. A_{12}(1), A_{13}(-1) \quad 2. A_{21}(-1), A_{23}(-2) \quad 3. A_{31}(-1)}$$

From the last augmented matrix, we see that x_4 is a free variable. We set $x_4 = t$, where t is a real number. The last row of the reduced row echelon form above corresponds to the equation $x_3 + x_4 = 0$. Therefore, $x_3 = -t$. The second row corresponds to the equation $x_2 + x_4 = 0$, so we likewise find that $x_2 = -t$. Finally, from the first equation we have $x_1 - 3x_4 = 0$, so that $x_1 = 3t$. Consequently, the solution set of the original system is given by $\{(3t, -t, -t, t) : t \in \mathbb{R}\}$.

49. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 2-3i & 1+i & i-1 \\ 3+2i & -1+i & -1-i \\ 5-i & 2i & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of this system:

$$\left[\begin{array}{ccc|c} 2-3i & 1+i & i-1 & 0 \\ 3+2i & -1+i & -1-i & 0 \\ 5-i & 2i & -2 & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{-1+5i}{13} & \frac{-5-i}{13} & 0 \\ 3+2i & -1+i & -1-i & 0 \\ 5-i & 2i & -2 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{-1+5i}{13} & \frac{-5-i}{13} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

$$\boxed{\text{1. } M_1\left(\frac{2+3i}{13}\right) \quad \text{2. } A_{12}(-3-2i), A_{13}(-5+i)}$$

From the last augmented matrix, we see that $x_1 + \frac{-1+5i}{13}x_2 + \frac{-5-i}{13}x_3 = 0$. Since x_2 and x_3 are free variables, we can let $x_2 = 13r$ and $x_3 = 13s$, where r and s are complex numbers. It follows that the solution set of the system is $\{(r(1-5i) + s(5+i), 13r, 13s) : r, s \in \mathbb{C}\}$.

50. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & 3 & 0 \\ -2 & -3 & 0 \\ 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ -2 & -3 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array} \right] \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

$$\boxed{\text{1. } A_{12}(2), A_{13}(-1) \quad \text{2. } P_{23} \quad \text{3. } A_{21}(-3), A_{23}(-3)}$$

From the last augmented matrix we see that the solution set of the system is $\{(0, 0, t) : t \in \mathbb{R}\}$.

51. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & 0 & 3 \\ 3 & -1 & 7 \\ 2 & 1 & 8 \\ 1 & 1 & 5 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 3 & -1 & 7 & 0 \\ 2 & 1 & 8 & 0 \\ 1 & 1 & 5 & 0 \\ -1 & 1 & -1 & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array} \right] \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

$$\boxed{\text{1. } A_{12}(-3), A_{13}(-2), A_{14}(-1), A_{15}(1) \quad \text{2. } M_2(-1) \quad \text{3. } A_{23}(-1), A_{24}(-1), A_{25}(-1)}$$

From the last augmented matrix, we obtain the equations $x_1 + 3x_3 = 0$ and $x_2 + 2x_3 = 0$. Since x_3 is a free variable, we let $x_3 = t$, where t is a real number. The solution set for the given system is then given by $\{(-3t, -2t, t) : t \in \mathbb{R}\}$.

52. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 3 & -2 & 0 & 5 \\ -1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 3 & -2 & 0 & 5 & 0 \\ -1 & 2 & 0 & 1 & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

$$\boxed{\text{1. } A_{12}(-3), A_{13}(1) \quad \text{2. } A_{21}(1), A_{23}(-1)}$$

From the last augmented matrix we obtain the equations $x_1 + 3x_4 = 0$ and $x_2 + 2x_4 = 0$. Because x_3 and x_4 are free, we let $x_3 = t$ and $x_4 = s$, where s and t are real numbers. It follows that the solution set of the system is $\{(-3s, -2s, t, s) : s, t \in \mathbb{R}\}$.

53. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 3 & 0 & -9 & 0 \\ -2 & 0 & 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 3 & 0 & -9 & 0 & 0 \\ -2 & 0 & 6 & 0 & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

$$\boxed{\text{1. } A_{12}(-3), A_{13}(2)}$$

From the last augmented matrix we obtain $x_1 - 3x_3 = 0$. Therefore, x_2, x_3 , and x_4 are free variables, so we let $x_2 = r$, $x_3 = s$, and $x_4 = t$, where r, s, t are real numbers. The solution set of the given system is therefore $\{(3s, r, s, t) : r, s, t \in \mathbb{R}\}$.

54. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 2+i & i & 3-2i \\ i & 1-i & 4+3i \\ 3-i & 1+i & 1+5i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 2+i & i & 3-2i & 0 \\ i & 1-i & 4+3i & 0 \\ 3-i & 1+i & 1+5i & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} i & 1-i & 4+3i & 0 \\ 2+i & i & 3-2i & 0 \\ 3-i & 1+i & 1+5i & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & -1-i & 3-3i & 0 \\ 2+i & i & 3-2i & 0 \\ 3-i & 1+i & 1+5i & 0 \end{array} \right]$$

$$\begin{aligned}
& \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & -1-i & 3-4i & 0 \\ 0 & 1+4i & -7+3i & 0 \\ 0 & 5+3i & -4+20i & 0 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & -1-i & 3-4i & 0 \\ 0 & 1 & \frac{5+31i}{17} & 0 \\ 0 & 5+3i & -4+20i & 0 \end{array} \right] \stackrel{5}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & \frac{25-32i}{17} & 0 \\ 0 & 1 & \frac{5+31i}{17} & 0 \\ 0 & 0 & 10i & 0 \end{array} \right] \\
& \stackrel{6}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & \frac{25-32i}{17} & 0 \\ 0 & 1 & \frac{5+31i}{17} & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \stackrel{7}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right].
\end{aligned}$$

1. P_{12} 2. $M_1(-i)$ 3. $A_{12}(-2-i), A_{13}(-3+i)$ 4. $M_2(\frac{1-4i}{17})$ 5. $A_{21}(1+i), A_{23}(-5-3i)$
6. $M_3(-i/10)$ 7. $A_{31}(\frac{-25+32i}{17}), A_{32}(\frac{-5-31i}{17})$

From the last augmented matrix above, we see that the only solution to this system is the trivial solution.

Solutions to Section 2.6

True-False Review:

1. **FALSE.** An invertible matrix is also known as a *nonsingular* matrix.
2. **FALSE.** For instance, the matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ does not contain a row of zeros, but fails to be invertible.
3. **TRUE.** If A is invertible, then the unique solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$.
4. **FALSE.** For instance, if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $AB = I_2$, but A is not even a square matrix, hence certainly not invertible.
5. **FALSE.** For instance, if $A = I_n$ and $B = -I_n$, then A and B are both invertible, but $A + B = 0_n$ is not invertible.
6. **TRUE.** We have
$$(AB)B^{-1}A^{-1} = I_n \quad \text{and} \quad B^{-1}A^{-1}(AB) = I_n,$$
and therefore, AB is invertible, with inverse $B^{-1}A^{-1}$.
7. **TRUE.** From $A^2 = A$, we subtract to obtain $A(A - I) = 0$. Left multiplying both sides of this equation by A^{-1} (since A is invertible, A^{-1} exists), we have $A - I = A^{-1}0 = 0$. Therefore, $A = I$, the identity matrix.
8. **TRUE.** From $AB = AC$, we left-multiply both sides by A^{-1} (since A is invertible, A^{-1} exists) to obtain $A^{-1}AB = A^{-1}AC$. Since $A^{-1}A = I$, we obtain $IB = IC$, or $B = C$.
9. **TRUE.** Any 5×5 invertible matrix must have rank 5, not rank 4 (Theorem 2.6.5).
10. **TRUE.** Any 6×6 matrix of rank 6 is invertible (Theorem 2.6.5).

Problems:

1. We have

$$AA^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} (2)(-1) + (-1)(-3) & (2)(1) + (-1)(2) \\ (3)(-1) + (-1)(-3) & (3)(1) + (-1)(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

2. We have

$$AA^{-1} = \begin{bmatrix} 4 & 9 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 7 & -9 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} (4)(7) + (9)(-3) & (4)(-9) + (9)(4) \\ (3)(7) + (7)(-3) & (3)(-9) + (7)(4) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

3. We have

$$\begin{aligned}
 AA^{-1} &= \begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix} \begin{bmatrix} 8 & -29 & 3 \\ -5 & 19 & -2 \\ 2 & -8 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} (3)(8) + (5)(-5) + (1)(2) & (3)(-29) + (5)(19) + (1)(-8) & (3)(3) + (5)(-2) + (1)(1) \\ (1)(8) + (2)(-5) + (1)(2) & (1)(-29) + (2)(19) + (1)(-8) & (1)(3) + (2)(-2) + (1)(1) \\ (2)(8) + (6)(-5) + (7)(2) & (2)(-29) + (6)(19) + (7)(-8) & (2)(3) + (6)(-2) + (7)(1) \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.
 \end{aligned}$$

4. We have

$$[A|I_2] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array} \right] \xrightarrow{1} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array} \right] \xrightarrow{2} \left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{array} \right] = [I_2|A^{-1}].$$

Therefore,

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}.$$

1. $A_{12}(-1)$	2. $A_{21}(-2)$
-----------------	-----------------

5. We have

$$\begin{aligned}
 [A|I_2] &= \left[\begin{array}{cc|cc} 1 & 1+i & 1 & 0 \\ 1-i & 1 & 0 & 1 \end{array} \right] \xrightarrow{1} \left[\begin{array}{cc|cc} 1 & 1+i & 1 & 0 \\ 0 & -1 & -1+i & 1 \end{array} \right] \xrightarrow{2} \left[\begin{array}{cc|cc} 1 & 1+i & 1 & 0 \\ 0 & 1 & 1-i & -1 \end{array} \right] \\
 &\xrightarrow{3} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 1+i \\ 0 & 1 & 1-i & -1 \end{array} \right] = [I_2|A^{-1}].
 \end{aligned}$$

Thus,

$$A^{-1} = \begin{bmatrix} -1 & 1+i \\ 1-i & -1 \end{bmatrix}.$$

1. $A_{12}(-1+i)$	2. $M_2(-1)$	3. $A_{21}(-1-i)$
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6. We have

$$\begin{aligned}
 [A|I_2] &= \left[\begin{array}{cc|cc} 1 & -i & 1 & 0 \\ i-1 & 2 & 0 & 1 \end{array} \right] \xrightarrow{1} \left[\begin{array}{cc|cc} 1 & -i & 1 & 0 \\ 0 & 1-i & 1-i & 1 \end{array} \right] \xrightarrow{2} \left[\begin{array}{cc|cc} 1 & -i & 1 & 0 \\ 0 & 1 & 1 & \frac{1+i}{2} \end{array} \right] \\
 &\xrightarrow{3} \left[\begin{array}{cc|cc} 1 & 0 & 1+i & \frac{-1+i}{2} \\ 0 & 1 & 1 & \frac{1+i}{2} \end{array} \right] = [I_2|A^{-1}].
 \end{aligned}$$

Thus,

$$A^{-1} = \begin{bmatrix} 1+i & \frac{-1+i}{2} \\ 1 & \frac{1+i}{2} \end{bmatrix}.$$

1. $A_{12}(1-i)$	2. $M_2(1/(1-i))$	3. $A_{21}(i)$
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7. Note that $AB = 0_2$ for all 2×2 matrices B . Therefore, A is not invertible.

8. We have

$$\begin{aligned} [A|I_3] &= \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 11 & 0 & 1 & 0 \\ 4 & -3 & 10 & 0 & 0 & 1 \end{array} \right] \xrightarrow{1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 3 & 7 & -2 & 1 & 0 \\ 0 & 1 & 2 & -4 & 0 & 1 \end{array} \right] \xrightarrow{2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -4 & 0 & 1 \\ 0 & 3 & 7 & -2 & 1 & 0 \end{array} \right] \\ &\xrightarrow{3} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & -3 & 0 & 1 \\ 0 & 1 & 2 & -4 & 0 & 1 \\ 0 & 0 & 1 & 10 & 1 & -3 \end{array} \right] \xrightarrow{4} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -43 & -4 & 13 \\ 0 & 1 & 0 & -24 & -2 & 7 \\ 0 & 0 & 1 & 10 & 1 & -3 \end{array} \right] = [I_3|A^{-1}]. \end{aligned}$$

Thus,

$$A^{-1} = \begin{bmatrix} -43 & -4 & 13 \\ -24 & -2 & 7 \\ 10 & 1 & -3 \end{bmatrix}.$$

1. $A_{12}(-2), A_{13}(-4)$ 2. P_{23} 3. $A_{21}(1), A_{23}(-3)$ 4. $A_{31}(-4), A_{32}(-2)$

9. We have

$$\begin{aligned} [A|I_3] &= \left[\begin{array}{ccc|ccc} 3 & 5 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 6 & 7 & 0 & 0 & 1 \end{array} \right] \xrightarrow{1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 3 & 5 & 1 & 1 & 0 & 0 \\ 2 & 6 & 7 & 0 & 0 & 1 \end{array} \right] \xrightarrow{2} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & -2 & 1 & -3 & 0 \\ 0 & 2 & 5 & 0 & -2 & 1 \end{array} \right] \\ &\xrightarrow{3} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 3 & 0 \\ 0 & 2 & 5 & 0 & -2 & 1 \end{array} \right] \xrightarrow{4} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 2 & -5 & 0 \\ 0 & 1 & 2 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -8 & 1 \end{array} \right] \xrightarrow{5} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -29 & 3 \\ 0 & 1 & 0 & -5 & 19 & -2 \\ 0 & 0 & 1 & 2 & -8 & 1 \end{array} \right] = [I_3|A^{-1}]. \end{aligned}$$

Thus,

$$A^{-1} = \begin{bmatrix} 8 & -29 & 3 \\ -5 & 19 & -2 \\ 2 & -8 & 1 \end{bmatrix}.$$

1. P_{12} 2. $A_{12}(-3), A_{13}(-2)$ 3. $M_2(-1)$ 4. $A_{21}(-2), A_{23}(-2)$ 5. $A_{31}(3), A_{32}(-2)$

10. This matrix is not invertible, because the column of zeros guarantees that the rank of the matrix is less than three.

11. We have

$$\begin{aligned} [A|I_3] &= \left[\begin{array}{ccc|ccc} 4 & 2 & -13 & 1 & 0 & 0 \\ 2 & 1 & -7 & 0 & 1 & 0 \\ 3 & 2 & 4 & 0 & 0 & 1 \end{array} \right] \xrightarrow{1} \left[\begin{array}{ccc|ccc} 3 & 2 & 4 & 0 & 0 & 1 \\ 2 & 1 & -7 & 0 & 1 & 0 \\ 4 & 2 & -13 & 1 & 0 & 0 \end{array} \right] \xrightarrow{2} \left[\begin{array}{ccc|ccc} 1 & 1 & 11 & 0 & -1 & 1 \\ 2 & 1 & -7 & 0 & 1 & 0 \\ 4 & 2 & -13 & 1 & 0 & 0 \end{array} \right] \\ &\xrightarrow{3} \left[\begin{array}{ccc|ccc} 1 & 1 & 11 & 0 & -1 & 1 \\ 0 & -1 & -29 & 0 & 3 & -2 \\ 0 & -2 & -57 & 1 & 4 & -4 \end{array} \right] \xrightarrow{4} \left[\begin{array}{ccc|ccc} 1 & 1 & 11 & 0 & -1 & 1 \\ 0 & 1 & 29 & 0 & -3 & 2 \\ 0 & -2 & -57 & 1 & 4 & -4 \end{array} \right] \xrightarrow{5} \left[\begin{array}{ccc|ccc} 1 & 0 & -18 & 0 & 2 & -1 \\ 0 & 1 & 29 & 0 & -3 & 2 \\ 0 & 0 & 1 & 1 & -2 & 0 \end{array} \right] \\ &\xrightarrow{6} \left[\begin{array}{ccc|ccc} 1 & 0 & 18 & -34 & -1 & 1 \\ 0 & 1 & 0 & -29 & 55 & 2 \\ 0 & 0 & 1 & 1 & -2 & 0 \end{array} \right] = [I_3|A^{-1}]. \end{aligned}$$

Thus,

$$A^{-1} = \begin{bmatrix} 18 & -34 & -1 \\ -29 & 55 & 2 \\ 1 & -2 & 0 \end{bmatrix}.$$

1. P ₁₃	2. A ₂₁ (-1)	3. A ₁₂ (-2), A ₁₃ (-4)	4. M ₂ (-1)
5. A ₂₁ (-1), A ₂₃ (2) 6. A ₃₁ (18), A ₃₂ (-29)			

12. We have

$$\begin{aligned} [A|I_3] &= \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 6 & -2 & 0 & 1 & 0 \\ -1 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 2 & 4 & -2 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & \frac{1}{2} & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{array} \right] \\ &\stackrel{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -7 & 3 & -1 & 0 \\ 0 & 1 & 2 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & -5 & 4 & -\frac{3}{2} & 1 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -7 & 3 & -1 & 0 \\ 0 & 1 & 2 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{4}{5} & \frac{3}{10} & -\frac{1}{5} \end{array} \right] \\ &\stackrel{5}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{13}{5} & \frac{11}{10} & -\frac{7}{5} \\ 0 & 1 & 0 & -\frac{1}{5} & -\frac{1}{10} & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{4}{5} & \frac{3}{10} & -\frac{1}{5} \end{array} \right] = [I_3|A^{-1}]. \end{aligned}$$

Thus,

$$A^{-1} = \begin{bmatrix} -\frac{13}{5} & \frac{11}{10} & -\frac{7}{5} \\ -\frac{1}{5} & -\frac{1}{10} & \frac{2}{5} \\ -\frac{4}{5} & \frac{3}{10} & -\frac{1}{5} \end{bmatrix}.$$

1. A ₁₂ (-2), A ₁₃ (1)	2. M ₂ ($\frac{1}{2}$)	3. A ₂₁ (-2), A ₂₃ (-3)	4. M ₃ ($-\frac{1}{5}$)	5. A ₃₁ (7), A ₃₂ (-2)
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13. We have

$$\begin{aligned} [A|I_3] &= \left[\begin{array}{ccc|ccc} 1 & i & 2 & 1 & 0 & 0 \\ 1+i & -1 & 2i & 0 & 1 & 0 \\ 2 & 2i & 5 & 0 & 0 & 1 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & i & 2 & 1 & 0 & 0 \\ 0 & -i & -2 & -1-i & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & i & 2 & 1 & 0 & 0 \\ 0 & 1 & -2i & 1-i & i & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right] \\ &\stackrel{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -i & 1 & 0 \\ 0 & 1 & -2i & 1-i & i & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -i & 1 & 0 \\ 0 & 1 & 0 & 1-5i & i & 2i \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array} \right] = [I_3|A^{-1}]. \end{aligned}$$

Thus,

$$A^{-1} = \begin{bmatrix} -i & 1 & 0 \\ 1-5i & i & 2i \\ -2 & 0 & 1 \end{bmatrix}.$$

1. A ₁₂ (-1-i), A ₁₃ (-2)	2. M ₂ (i)	3. A ₂₁ (-i)	4. A ₃₂ (2i)
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14. We have

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 3 & 3 & 4 & 0 & 0 & 1 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ 3 & 3 & 4 & 0 & 0 & 1 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & -2 & 0 \\ 0 & 6 & -2 & 0 & -3 & 1 \end{array} \right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & -2 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array} \right]$$

Since $2 = \text{rank}(A) < \text{rank}(A^\#) = 3$, we know that A^{-1} does not exist (we have obtained a row of zeros in the block matrix on the left).

1. P_{12}	2. $A_{12}(-2), A_{13}(-3)$	3. $A_{23}(-2)$
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15. We have

$$\begin{aligned} [A|I_4] &= \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 2 & 0 & 3 & -4 & 0 & 1 & 0 & 0 \\ 3 & -1 & 7 & 8 & 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 5 & 0 & 0 & 0 & 1 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -10 & -2 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 & -3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 & 0 & 1 \end{array} \right] \\ &\stackrel{2}{\sim} \left[\begin{array}{cccc|cccc} 1 & -1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 & 0 & 1 \\ 0 & 2 & 1 & -1 & -3 & 0 & 1 & 0 \\ 0 & 2 & -1 & -10 & -2 & 1 & 0 & 0 \end{array} \right] \stackrel{3}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 5 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -5 & -1 & 0 & 1 & -2 \\ 0 & 0 & -3 & -14 & 0 & 1 & 0 & -2 \end{array} \right] \\ &\stackrel{4}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 3 & 5 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 & 1 & 0 & -1 & 2 \\ 0 & 0 & -3 & -14 & 0 & 1 & 0 & -2 \end{array} \right] \stackrel{5}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -10 & -3 & 0 & 3 & -5 \\ 0 & 1 & 0 & -3 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & 5 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 3 & 1 & -3 & 4 \end{array} \right] \\ &\stackrel{6}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 27 & 10 & -27 & 35 \\ 0 & 1 & 0 & 0 & 7 & 3 & -8 & 11 \\ 0 & 0 & 1 & 0 & -14 & -5 & 14 & -18 \\ 0 & 0 & 0 & 1 & 3 & 1 & -3 & 4 \end{array} \right] = [I_4|A^{-1}]. \end{aligned}$$

Thus,

$$A^{-1} = \begin{bmatrix} 27 & 10 & -27 & 35 \\ 7 & 3 & -8 & 11 \\ -14 & -5 & 14 & -18 \\ 3 & 1 & -3 & 4 \end{bmatrix}.$$

1. $A_{12}(-2), A_{13}(-3), A_{14}(-1)$	2. P_{13}	3. $A_{21}(1), A_{23}(-2), A_{24}(-2)$
4. $M_3(-1)$	5. $A_{31}(-3), A_{32}(-1), A_{34}(3)$	6. $A_{41}(10), A_{42}(3), A_{43}(5)$

16. We have

$$\begin{aligned} [A|I_4] &= \left[\begin{array}{cccc|cccc} 0 & -2 & -1 & -3 & 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 2 & 0 & 0 & 1 & 0 \\ 3 & -1 & -2 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{cccc|cccc} 1 & -2 & 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & -3 & 1 & 0 & 0 & 0 \\ 3 & -1 & -2 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \\ &\stackrel{2}{\sim} \left[\begin{array}{cccc|cccc} 1 & -2 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 4 & 2 & -3 & 0 & 1 & -2 & 0 \\ 0 & -2 & -1 & -3 & 1 & 0 & 0 & 0 \\ 0 & 5 & -2 & -6 & 0 & 0 & -3 & 1 \end{array} \right] \stackrel{3}{\sim} \left[\begin{array}{cccc|cccc} 1 & -2 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{3}{4} & 0 & \frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & -2 & -1 & -3 & 1 & 0 & 0 & 0 \\ 0 & 5 & -2 & -6 & 0 & 0 & -3 & 1 \end{array} \right] \end{aligned}$$

$$\begin{aligned}
& \zeta_4 \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & \frac{1}{3} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{3} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{9}{2} & -\frac{9}{4} & 0 & -\frac{1}{4} & -\frac{1}{2} & 1 \end{array} \right] \quad \zeta_5 \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & \frac{1}{3} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{3} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{9}{2} & -\frac{9}{4} & 0 & -\frac{1}{4} & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 \end{array} \right] \\
& \zeta_6 \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & \frac{1}{3} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{3} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & 0 & \frac{5}{18} & -\frac{2}{9} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 1 & \frac{1}{2} & -1 & 0 \end{array} \right] \quad \zeta_7 \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \\ 0 & 1 & 0 & -1 & 0 & \frac{1}{9} & -\frac{1}{9} & -\frac{1}{9} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{5}{18} & -\frac{1}{9} & -\frac{1}{9} \\ 0 & 0 & 0 & -\frac{1}{2} & 1 & \frac{1}{2} & -1 & 0 \end{array} \right] \\
& \zeta_8 \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \\ 0 & 1 & 0 & -1 & 0 & -\frac{1}{9} & -\frac{1}{9} & -\frac{1}{9} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{18} & -\frac{1}{9} & -\frac{1}{9} \\ 0 & 0 & 0 & 1 & -\frac{2}{9} & -\frac{1}{9} & 0 & 0 \end{array} \right] \quad \zeta_9 \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \\ 0 & 1 & 0 & 0 & -\frac{2}{9} & \frac{2}{9} & -\frac{1}{9} & -\frac{1}{9} \\ 0 & 0 & 1 & 0 & -\frac{1}{9} & \frac{1}{3} & 0 & -\frac{1}{9} \\ 0 & 0 & 0 & 1 & -\frac{2}{9} & -\frac{1}{9} & \frac{2}{9} & 0 \end{array} \right] = [I_4 | A^{-1}].
\end{aligned}$$

Thus,

$$A^{-1} = \begin{bmatrix} 0 & \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{2}{9} & 0 & -\frac{1}{3} & -\frac{1}{9} \\ -\frac{1}{9} & -\frac{1}{3} & 0 & -\frac{1}{9} \\ -\frac{1}{9} & -\frac{1}{9} & \frac{2}{9} & 0 \end{bmatrix}.$$

1. P_{13}	2. $A_{12}(-2), A_{14}(-3)$	3. $M_2(\frac{1}{4})$	4. $A_{21}(2), A_{23}(2), A_{24}(-5)$
5. P_{34}	6. $M_3(-\frac{2}{9})$	7. $A_{31}(-1), A_{32}(-\frac{1}{2})$	8. $M_4(-\frac{2}{9})$ 9. $A_{42}(1), A_{43}(-\frac{1}{2})$

17. To determine the second column vector of A^{-1} without determining the whole inverse, we solve the linear system $\begin{bmatrix} 2 & -1 & 4 \\ 5 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. The corresponding augmented matrix $\begin{bmatrix} 2 & -1 & 4 & 0 \\ 5 & 1 & 2 & 1 \\ 1 & -1 & 3 & 0 \end{bmatrix}$ can

be row-reduced to $\begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. Thus, back-substitution yields $z = -1$, $y = -2$, and $x = 1$. Thus,

the second column vector of A^{-1} is $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$.

18. We have $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and the Gauss-Jordan method yields $A^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$. Therefore, we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

So we have $x_1 = 4$ and $x_2 = -1$.

19. We have $A = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 2 & 4 & -3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$, and the Gauss-Jordan method yields $A^{-1} = \begin{bmatrix} 7 & 5 & -3 \\ -2 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$.

Therefore, we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 7 & 5 & -3 \\ -2 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}.$$

Hence, we have $x_1 = -2$, $x_2 = 2$, and $x_3 = 1$.

20. We have $A = \begin{bmatrix} 1 & -2i \\ 2-i & 4i \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -i \end{bmatrix}$, and the Gauss-Jordan method yields $A^{-1} = \frac{1}{2+8i} \begin{bmatrix} 4i & 2i \\ -2+i & 1 \end{bmatrix}$. Therefore, we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{2+8i} \begin{bmatrix} 4i & 2i \\ -2+i & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -i \end{bmatrix} = \frac{1}{2+8i} \begin{bmatrix} 2+8i \\ -4+i \end{bmatrix}.$$

Hence, we have $x_1 = 1$ and $x_2 = \frac{-4+i}{2+8i}$.

21. We have $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & 10 & 1 \\ 4 & 1 & 8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and the Gauss-Jordan method yields $A^{-1} = \begin{bmatrix} -79 & 27 & 46 \\ 12 & -4 & -7 \\ 38 & -13 & -22 \end{bmatrix}$.

Therefore, we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -79 & 27 & 46 \\ 12 & -4 & -7 \\ 38 & -13 & -22 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ 3 \end{bmatrix}.$$

Hence, we have $x_1 = -6$, $x_2 = 1$, and $x_3 = 3$.

22. We have $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 12 \\ 24 \\ -36 \end{bmatrix}$, and the Gauss-Jordan method yields $A^{-1} = \frac{1}{12} \begin{bmatrix} -1 & 3 & 5 \\ 3 & 3 & -3 \\ 5 & -3 & -1 \end{bmatrix}$.

Therefore, we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{12} \begin{bmatrix} -1 & 3 & 5 \\ 3 & 3 & -3 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 12 \\ 24 \\ -36 \end{bmatrix} = \begin{bmatrix} -10 \\ 18 \\ 2 \end{bmatrix}.$$

Hence, $x_1 = -10$, $x_2 = 18$, and $x_3 = 2$.

23. We have

$$AA^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (0)(0) + (1)(1) & (0)(-1) + (1)(0) \\ (-1)(0) + (0)(1) & (-1)(-1) + (0)(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2,$$

so $A^T = A^{-1}$.

24. We have

$$\begin{aligned} AA^T &= \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \\ &= \begin{bmatrix} (\sqrt{3}/2)(\sqrt{3}/2) + (1/2)(1/2) & (\sqrt{3}/2)(-1/2) + (1/2)(\sqrt{3}/2) \\ (-1/2)(\sqrt{3}/2) + (\sqrt{3}/2)(1/2) & (-1/2)(-1/2) + (\sqrt{3}/2)(\sqrt{3}/2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \end{aligned}$$

so $A^T = A^{-1}$.

25. We have

$$\begin{aligned} AA^T &= \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \alpha + \sin^2 \alpha & (\cos \alpha)(-\sin \alpha) + (\sin \alpha)(\cos \alpha) \\ (-\sin \alpha)(\cos \alpha) + (\cos \alpha)(\sin \alpha) & (-\sin \alpha)^2 + \cos^2 \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \end{aligned}$$

so $A^T = A^{-1}$.

26. We have

$$\begin{aligned} AA^T &= \left(\frac{1}{1+2x^2} \right) \begin{bmatrix} 1 & -2x & 2x^2 \\ 2x & 1-2x^2 & -2x \\ 2x^2 & 2x & 1 \end{bmatrix} \left(\frac{1}{1+2x^2} \right) \begin{bmatrix} 1 & 2x & 2x^2 \\ -2x & 1-2x^2 & 2x \\ 2x^2 & -2x & 1 \end{bmatrix} \\ &= \left(\frac{1}{1+4x^2+4x^4} \right) \begin{bmatrix} 1+4x^2+4x^4 & 0 & 0 \\ 0 & 1+4x^2+4x^4 & 0 \\ 0 & 0 & 1+4x^2+4x^4 \end{bmatrix} = I_3, \end{aligned}$$

so $A^T = A^{-1}$.

27. For part 2, we have

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n,$$

and for part 3, we have

$$(A^{-1})^T A^T = (AA^{-1})^T = I_n^T = I_n.$$

28. We prove this by induction on k , with $k = 1$ trivial and $k = 2$ proven in part 2 of Theorem 2.6.9. Assuming the statement is true for a product involving $k - 1$ matrices, we may proceed as follows:

$$\begin{aligned} (A_1A_2 \cdots A_k)^{-1} &= ((A_1A_2 \cdots A_{k-1})A_k)^{-1} = A_k^{-1}(A_1A_2 \cdots A_{k-1})^{-1} \\ &= A_k^{-1}(A_{k-1}^{-1} \cdots A_2^{-1}A_1^{-1}) = A_k^{-1}A_{k-1}^{-1} \cdots A_2^{-1}A_1^{-1}. \end{aligned}$$

In the second equality, we have applied part 2 of Theorem 2.6.9 to the two matrices $A_1A_2 \cdots A_{k-1}$ and A_k , and in the third equality, we have assumed that the desired property is true for products of $k - 1$ matrices.

29. Since A is symmetric, we know that $A^T = A$. We wish to show that $(A^{-1})^T = A^{-1}$. We have

$$(A^{-1})^T = (A^T)^{-1} = A^{-1},$$

which shows that A^{-1} is symmetric. The first equality follows from part 3 of Theorem 2.6.9, and the second equality results from the assumption that A is symmetric.

30. Since A is skew-symmetric, we know that $A^T = -A$. We wish to show that $(A^{-1})^T = -A^{-1}$. We have

$$(A^{-1})^T = (A^T)^{-1} = (-A)^{-1} = -(A^{-1}),$$

which shows that A^{-1} is skew-symmetric. The first equality follows from part 3 of Theorem 2.6.9, and the second equality results from the assumption that A^{-1} is skew-symmetric.

31. We have

$$\begin{aligned} (I_n - A)(I_n + A + A^2 + A^3) &= I_n(I_n + A + A^2 + A^3) - A(I_n + A + A^2 + A^3) \\ &= I_n + A + A^2 + A^3 - A - A^2 - A^3 - A^4 = I_n - A^4 = I_n, \end{aligned}$$

where the last equality uses the assumption that $A^4 = 0$. This calculation shows that $I_n - A$ and $I_n + A + A^2 + A^3$ are inverses of one another.

32. We have

$$B = BI_n = B(AC) = (BA)C = I_nC = C.$$

33. YES. Since $BA = I_n$, we know that $A^{-1} = B$ (see Theorem 2.6.11). Likewise, since $CA = I_n$, $A^{-1} = C$. Since the inverse of A is unique, it must follow that $B = C$.

34. We can simply compute

$$\begin{aligned} \frac{1}{\Delta} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} &= \frac{1}{\Delta} \begin{bmatrix} a_{22}a_{11} - a_{12}a_{21} & a_{22}a_{12} - a_{12}a_{22} \\ -a_{21}a_{11} + a_{11}a_{21} & -a_{21}a_{12} + a_{11}a_{22} \end{bmatrix} \\ &= \frac{1}{\Delta} \begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & 0 \\ 0 & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2. \end{aligned}$$

Therefore,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{\Delta} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

35. Assume that A is an invertible matrix and that $A\mathbf{x}_i = \mathbf{b}_i$ for $i = 1, 2, \dots, p$ (where each $\mathbf{b}_i$ is given). Use elementary row operations on the augmented matrix of the system to obtain the equivalence

$$[A|\mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3 \ \dots \ \mathbf{b}_p] \sim [I_n|\mathbf{c}_1 \ \mathbf{c}_2 \ \mathbf{c}_3 \ \dots \ \mathbf{c}_p].$$

The solutions to the system can be read from the last matrix: $\mathbf{x}_i = \mathbf{c}_i$ for each $i = 1, 2, \dots, p$.

36. We have

$$\begin{aligned} &\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & -1 & 2 \\ 2 & -1 & 4 & 1 & 2 & 3 \\ 1 & 1 & 6 & -1 & 5 & 2 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & -1 & 4 & -1 \\ 0 & 2 & 5 & -2 & 6 & 0 \end{array} \right] \\ &\stackrel{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 3 & 1 \\ 0 & 1 & 2 & -1 & 4 & -1 \\ 0 & 0 & 1 & 0 & -2 & 2 \end{array} \right] \stackrel{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 9 & -5 \\ 0 & 1 & 0 & -1 & 8 & -5 \\ 0 & 0 & 1 & 0 & -2 & 2 \end{array} \right]. \end{aligned}$$

Hence,

$$\mathbf{x}_1 = (0, -1, 0), \quad \mathbf{x}_2 = (9, 8, -2), \quad \mathbf{x}_3 = (-5, -5, 2).$$

1. $A_{12}(-2), A_{13}(-1)$ 2. $A_{21}(1), A_{23}(-2)$ 3. $A_{31}(-3), A_{32}(-2)$

37.

(a): Let $\mathbf{e}_i$ denote the i th column vector of the identity matrix I_m , and consider the m linear systems of equations

$$A\mathbf{x}_i = \mathbf{e}_i$$

for $i = 1, 2, \dots, m$. Since $\text{rank}(A) = m$ and each $\mathbf{e}_i$ is a column m -vector, it follows that

$$\text{rank}(A^\#) = m = \text{rank}(A)$$

and so each of the systems $A\mathbf{x}_i = \mathbf{e}_i$ above has a solution (Note that if $m < n$, then there will be an infinite number of solutions). If we let $B = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m]$, then

$$AB = A[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m] = [A\mathbf{x}_1, A\mathbf{x}_2, \dots, A\mathbf{x}_m] = [\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m] = I_n.$$

(b): A right inverse for A in this case is a 3×2 matrix $\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$ such that

$$\begin{bmatrix} a + 3b + c & d + 3e + f \\ 2a + 7b + 4c & 2d + 7e + 4f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, we must have

$$a + 3b + c = 1, \quad d + 3e + f = 0, \quad 2a + 7b + 4c = 0, \quad 2d + 7e + 4f = 1.$$

The first and third equation comprise a linear system with augmented matrix $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 2 & 7 & 4 & 0 \end{array} \right]$ for a , b , and

c . The row-echelon form of this augmented matrix is $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & -2 \end{array} \right]$. Setting $c = t$, we have $b = -2 - 2t$

and $a = 7 + 5t$. Next, the second and fourth equation above comprise a linear system with augmented matrix $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 2 & 7 & 4 & 1 \end{array} \right]$ for d , e , and f . The row-echelon form of this augmented matrix is $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array} \right]$. Setting $f = s$, we have $e = 1 - 2s$ and $d = -3 + 5s$. Thus, right inverses of A are precisely the matrices of the form $\begin{bmatrix} 7 + 5t & -3 + 5s \\ -2 - 2t & 1 - 2s \\ t & s \end{bmatrix}$.

Solutions to Section 2.7

True-False Review:

1. **TRUE.** Since every elementary matrix corresponds to a (reversible) elementary row operation, the reverse elementary row operation will correspond to an elementary matrix that is the inverse of the original elementary matrix.

2. **FALSE.** For instance, the matrices $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ are both elementary matrices, but their product, $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, is not.

3. **FALSE.** Every *invertible* matrix can be expressed as a product of elementary matrices. Since every elementary matrix is invertible and products of invertible matrices are invertible, any product of elementary matrices must be an invertible matrix.

4. **TRUE.** Performing an elementary row operation on a matrix does not alter its rank, and the matrix EA is obtained from A by performing the elementary row operation associated with the elementary matrix E . Therefore, A and EA have the same rank.

5. **FALSE.** If P_{ij} is a permutation matrix, then $P_{ij}^2 = I_n$, since permuting the i th and j th rows of I_n twice yields I_n . Alternatively, we can observe that $P_{ij}^2 = I_n$ from the fact that $P_{ij}^{-1} = P_{ij}$.

6. **FALSE.** For example, consider the elementary matrices $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then we have $E_1E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 7 \end{bmatrix}$ and $E_2E_1 = \begin{bmatrix} 1 & 7 \\ 0 & 7 \end{bmatrix}$.

7. **FALSE.** For example, consider the elementary matrices $E_1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

Then we have $E_1E_2 = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2E_1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

8. **FALSE.** The only matrices we perform an LU factorization for are invertible matrices for which the reduction to upper triangular form can be accomplished without permuting rows.

9. FALSE. The matrix U need not be a *unit* upper triangular matrix.

10. FALSE. As can be seen in Example 2.7.8, a 4×4 matrix with LU factorization will have 6 multipliers, not 10 multipliers.

Problems:

1.

Permutation Matrices: $P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Scaling Matrices: $M_1(k) = \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $M_2(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $M_3(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix}$.

Row Combinations:

$$A_{12}(k) = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_{13}(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}, \quad A_{23}(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix},$$

$$A_{21}(k) = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_{31}(k) = \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_{32}(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}.$$

2. We have

$$\begin{bmatrix} 3 & 5 \\ 1 & -2 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 11 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

1. P_{12}	2. $A_{12}(-3)$	3. $M_2(\frac{1}{11})$
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Elementary Matrices: $M_2(\frac{1}{11})$, $A_{12}(-3)$, P_{12} .

3. We have

$$\begin{bmatrix} 5 & 8 & 2 \\ 1 & 3 & -1 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 3 & -1 \\ 5 & 8 & 2 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -7 & 7 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

1. P_{12}	2. $A_{12}(-5)$	3. $M_2(-\frac{1}{7})$
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Elementary Matrices: $M_2(-\frac{1}{7})$, $A_{12}(-5)$, P_{12} .

4. We have

$$\begin{bmatrix} 3 & -1 & 4 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & -1 & 4 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & -1 \\ 0 & -10 & -2 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & -1 \\ 0 & 0 & 0 \end{bmatrix} \stackrel{4}{\sim} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix}.$$

1. P_{13}	2. $A_{12}(-2)$, $A_{13}(-3)$	3. $A_{23}(-2)$	4. $M_2(-\frac{1}{5})$
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Elementary Matrices: $M_2(-\frac{1}{5})$, $A_{23}(-2)$, $A_{13}(-3)$, $A_{12}(-2)$, P_{13} .

5. We have

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & -4 & -6 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\boxed{\mathbf{1. A}_{12}(-2), \mathbf{A}_{13}(-3) \quad \mathbf{2. M}_2(-1) \quad \mathbf{3. A}_{23}(2)}$$

Elementary Matrices: $A_{23}(2)$, $M_2(-1)$, $A_{13}(-3)$, $A_{12}(-2)$.

6. We reduce A to the identity matrix:

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\boxed{\mathbf{1. A}_{12}(-1) \quad \mathbf{2. A}_{21}(-2)}$$

The elementary matrices corresponding to these row operations are $E_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$.

We have $E_2 E_1 A = I_2$, so that

$$A = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

which is the desired expression since E_1^{-1} and E_2^{-1} are elementary matrices.

7. We reduce A to the identity matrix:

$$\begin{bmatrix} -2 & -3 \\ 5 & 7 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \stackrel{4}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \stackrel{5}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\boxed{\mathbf{1. A}_{12}(2) \quad \mathbf{2. P}_{12} \quad \mathbf{3. A}_{12}(2) \quad \mathbf{4. A}_{21}(1) \quad \mathbf{5. M}_2(-1)}$$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We have $E_5 E_4 E_3 E_2 E_1 A = I_2$, so

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which is the desired expression since each E_i^{-1} is an elementary matrix.

8. We reduce A to the identity matrix:

$$\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix} \stackrel{4}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \stackrel{5}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\boxed{\mathbf{1. P}_{12} \quad \mathbf{2. M}_1(-1) \quad \mathbf{3. A}_{12}(-3) \quad \mathbf{4. M}_2(\frac{1}{2}) \quad \mathbf{5. A}_{21}(2)}$$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

We have $E_5 E_4 E_3 E_2 E_1 A = I_2$, so

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix},$$

which is the desired expression since each E_i^{-1} is an elementary matrix.

9. We reduce A to the identity matrix:

$$\begin{bmatrix} 4 & -5 \\ 1 & 4 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 4 \\ 4 & -5 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 4 \\ 0 & -21 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \stackrel{4}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

1. P_{12}	2. $A_{12}(-4)$	3. $M_2(-\frac{1}{21})$	4. $A_{21}(-4)$
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The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{21} \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}.$$

We have $E_4 E_3 E_2 E_1 A = I_2$, so

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -21 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix},$$

which is the desired expression since each E_i^{-1} is an elementary matrix.

10. We reduce A to the identity matrix:

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 2 \\ 3 & 1 & 3 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 2 \\ 3 & 1 & 3 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 2 \\ 0 & 4 & 3 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\stackrel{4}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \stackrel{5}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \stackrel{6}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. $A_{12}(-2)$	2. $A_{13}(-3)$	3. $A_{23}(-1)$	4. $M_2(\frac{1}{4})$	5. $A_{32}(-\frac{1}{2})$	6. $A_{21}(1)$
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The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have $E_6 E_5 E_4 E_3 E_2 E_1 A = I_3$, so

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the desired expression since each E_i^{-1} is an elementary matrix.

11. We reduce A to the identity matrix:

$$\begin{bmatrix} 0 & -4 & -2 \\ 1 & -1 & 3 \\ -2 & 2 & 2 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -1 & 3 \\ 0 & -4 & -2 \\ -2 & 2 & 2 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -1 & 3 \\ 0 & -4 & -2 \\ 0 & 0 & 8 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & -1 & 3 \\ 0 & -4 & -2 \\ 0 & 0 & 1 \end{bmatrix} \\ \stackrel{4}{\sim} \begin{bmatrix} 1 & -1 & 3 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \stackrel{5}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \stackrel{6}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \stackrel{7}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. P_{12}	2. $A_{13}(2)$	3. $M_3(\frac{1}{8})$	4. $A_{32}(2)$	5. $A_{31}(-3)$	6. $M_2(-\frac{1}{4})$	7. $A_{21}(1)$
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The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{8} \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_5 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_7 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have $E_7 E_6 E_5 E_4 E_3 E_2 E_1 A = I_3$, so

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} E_7^{-1} \\ = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the desired expression since each E_i^{-1} is an elementary matrix.

12. We reduce A to the identity matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 8 & 0 \\ 3 & 4 & 5 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 3 & 4 & 5 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & -2 & -4 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & -2 & -4 \end{bmatrix} \\ \stackrel{4}{\sim} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \stackrel{5}{\sim} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \stackrel{6}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. $M_2(\frac{1}{8})$	2. $A_{13}(-3)$	3. $A_{21}(-2)$	4. $A_{23}(2)$	5. $M_3(-\frac{1}{4})$	6. $A_{31}(-3)$
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The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have $E_6 E_5 E_4 E_3 E_2 E_1 A = I_3$, so

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} \\ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the desired expression since each E_i^{-1} is an elementary matrix.

13. We reduce A to the identity matrix:

$$\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 3 \\ 0 & -7 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \stackrel{4}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

1. P_{12}	2. $A_{12}(-2)$	3. $M_2(-\frac{1}{7})$	4. $A_{21}(-3)$
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The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{7} \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}.$$

Direct multiplication verifies that $E_4 E_3 E_2 E_1 A = I_2$.

14. We have

$$\begin{bmatrix} 3 & -2 \\ -1 & 5 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 3 & -2 \\ 0 & \frac{13}{3} \end{bmatrix} = U.$$

1. $A_{12}(\frac{1}{3})$

Hence, $E_1 = A_{12}(\frac{1}{3})$. Then Equation (2.7.3) reads $L = E_1^{-1} = A_{12}(-\frac{1}{3}) = \begin{bmatrix} 1 & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}$. Verifying Equation (2.7.2):

$$LU = \begin{bmatrix} 1 & 0 \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 0 & \frac{13}{3} \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 5 \end{bmatrix} = A.$$

15. We have

$$\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 2 & 3 \\ 0 & -\frac{13}{2} \end{bmatrix} = U \implies m_{21} = \frac{5}{2} \implies L = \begin{bmatrix} 1 & 0 \\ \frac{5}{2} & 1 \end{bmatrix}.$$

Then

$$LU = \begin{bmatrix} 1 & 0 \\ \frac{5}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & -\frac{13}{2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = A.$$

1. $A_{12}(-\frac{5}{2})$

16. We have

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 3 & 1 \\ 0 & \frac{1}{3} \end{bmatrix} = U \implies m_{21} = \frac{5}{3} \implies L = \begin{bmatrix} 1 & 0 \\ \frac{5}{3} & 1 \end{bmatrix}.$$

Then

$$LU = \begin{bmatrix} 1 & 0 \\ \frac{5}{3} & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = A.$$

$$\boxed{1. A_{12}(-\frac{5}{3})}$$

17. We have

$$\begin{bmatrix} 3 & -1 & 2 \\ 6 & -1 & 1 \\ -3 & 5 & 2 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 4 & 4 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 16 \end{bmatrix} = U \implies m_{21} = 2, m_{31} = -1, m_{32} = 4.$$

Hence,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \quad \text{and} \quad LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 16 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 6 & -1 & 1 \\ -3 & 5 & 2 \end{bmatrix} = A.$$

$$\boxed{1. A_{12}(-2), A_{13}(1) \quad 2. A_{23}(-4)}$$

18. We have

$$\begin{bmatrix} 5 & 2 & 1 \\ -10 & -2 & 3 \\ 15 & 2 & -3 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 5 & 2 & 1 \\ 0 & 2 & 5 \\ 0 & -4 & -6 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 5 & 2 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 4 \end{bmatrix} = U \implies m_{21} = -2, m_{31} = 3, m_{32} = -2.$$

Hence,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \quad \text{and} \quad LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 1 \\ -10 & -2 & 3 \\ 15 & 2 & -3 \end{bmatrix} = A.$$

$$\boxed{1. A_{12}(2), A_{13}(-3) \quad 2. A_{23}(2)}$$

19. We have

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 0 & 3 & -4 \\ 3 & -1 & 7 & 8 \\ 1 & 3 & 4 & 5 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -1 & -10 \\ 0 & 2 & 1 & -1 \\ 0 & 4 & 2 & 2 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -1 & -10 \\ 0 & 0 & 2 & 9 \\ 0 & 0 & 4 & 22 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -1 & -10 \\ 0 & 0 & 2 & 9 \\ 0 & 0 & 0 & 4 \end{bmatrix} = U.$$

$$\boxed{1. A_{12}(-2), A_{13}(-3), A_{14}(-1) \quad 2. A_{23}(-1), A_{24}(-2) \quad 3. A_{34}(-2)}$$

Hence,

$$m_{21} = 2, \quad m_{31} = 3, m_{41} = 1, m_{32} = 1, m_{42} = 2, m_{43} = 2.$$

Hence,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \end{bmatrix} \quad \text{and} \quad LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -1 & -10 \\ 0 & 0 & 2 & 9 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 0 & 3 & -4 \\ 3 & -1 & 7 & 8 \\ 1 & 3 & 4 & 5 \end{bmatrix} = A.$$

20. We have

$$\begin{bmatrix} 2 & -3 & 1 & 2 \\ 4 & -1 & 1 & 1 \\ -8 & 2 & 2 & -5 \\ 6 & 1 & 5 & 2 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 2 & -3 & 1 & 2 \\ 0 & 5 & -1 & -3 \\ 0 & -10 & 6 & 3 \\ 0 & 10 & 2 & -4 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 2 & -3 & 1 & 2 \\ 0 & 5 & -1 & -3 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 4 & 2 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 2 & -3 & 1 & 2 \\ 0 & 5 & -1 & -3 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix} = U.$$

1. $A_{12}(-2), A_{13}(4), A_{14}(-3)$	2. $A_{23}(2), A_{24}(-2)$	3. $A_{34}(-1)$
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Hence,

$$m_{21} = 2, \quad m_{31} = -4, \quad m_{41} = 3, \quad m_{32} = -2, \quad m_{42} = 2, \quad m_{43} = 1.$$

Hence,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -4 & -2 & 1 & 0 \\ 3 & 2 & 1 & 1 \end{bmatrix} \text{ and } LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -4 & -2 & 1 & 0 \\ 3 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 & 2 \\ 0 & 5 & -1 & -3 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 & 2 \\ 4 & -1 & 1 & 1 \\ -8 & 2 & 2 & -5 \\ 6 & 1 & 5 & 2 \end{bmatrix} = A.$$

21. We have

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = U \implies m_{21} = 2 \implies L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

1. $A_{12}(-2)$

We now solve the triangular systems $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$. From $L\mathbf{y} = \mathbf{b}$, we obtain $\mathbf{y} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$. Then

$$U\mathbf{x} = \mathbf{y} \text{ yields } \mathbf{x} = \begin{bmatrix} -11 \\ 7 \end{bmatrix}.$$

22. We have

$$\begin{bmatrix} 1 & -3 & 5 \\ 3 & 2 & 2 \\ 2 & 5 & 2 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 11 & -13 \\ 0 & 11 & -8 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 11 & -13 \\ 0 & 0 & 5 \end{bmatrix} = U \implies m_{21} = 3, m_{31} = 2, m_{32} = 1.$$

1. $A_{12}(-3), A_{13}(-2)$	2. $A_{23}(-1)$
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Hence, $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$. We now solve the triangular systems $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$. From $L\mathbf{y} = \mathbf{b}$, we

$$\text{obtain } \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}. \text{ Then } U\mathbf{x} = \mathbf{y} \text{ yields } \mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}.$$

23. We have

$$\begin{bmatrix} 2 & 2 & 1 \\ 6 & 3 & -1 \\ -4 & 2 & 2 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 2 & 2 & 1 \\ 0 & -3 & -4 \\ 0 & 0 & -4 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 2 & 2 & 1 \\ 0 & -3 & -4 \\ 0 & 0 & -4 \end{bmatrix} = U \implies m_{21} = 3, m_{31} = -2, m_{32} = -2.$$

1. $A_{12}(-3), A_{13}(2)$	2. $A_{23}(2)$
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Hence, $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}$. We now solve the triangular systems $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$. From $L\mathbf{y} = \mathbf{b}$, we

$$\text{obtain } \mathbf{y} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}. \text{ Then } U\mathbf{x} = \mathbf{y} \text{ yields } \mathbf{x} = \begin{bmatrix} -1/12 \\ 1/3 \\ 1/2 \end{bmatrix}.$$

24. We have

$$\begin{bmatrix} 4 & 3 & 0 & 0 \\ 8 & 1 & 2 & 0 \\ 0 & 5 & 3 & 6 \\ 0 & 0 & -5 & 7 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 4 & 3 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 5 & 3 & 6 \\ 0 & 0 & -5 & 7 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 4 & 3 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & -5 & 7 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 4 & 3 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 13 \end{bmatrix} = U.$$

$$\boxed{\text{1. } A_{12}(-2) \quad \text{2. } A_{23}(1) \quad \text{3. } A_{34}(1)}$$

The only nonzero multipliers are $m_{21} = 2$, $m_{32} = -1$, and $m_{43} = -1$. Hence, $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$. We

now solve the triangular systems $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$. From $L\mathbf{y} = \mathbf{b}$, we obtain $\mathbf{y} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 4 \end{bmatrix}$. Then $U\mathbf{x} = \mathbf{y}$

$$\text{yields } \mathbf{x} = \begin{bmatrix} 677/1300 \\ -9/325 \\ -37/65 \\ 4/13 \end{bmatrix}.$$

25. We have

$$\begin{bmatrix} 2 & -1 \\ -8 & 3 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix} = U \implies m_{21} = -4 \implies L = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}.$$

$$\boxed{\text{1. } A_{12}(4)}$$

We now solve the triangular systems

$$L\mathbf{y}_i = \mathbf{b}_i, \quad U\mathbf{x}_i = \mathbf{y}_i$$

for $i = 1, 2, 3$. We have

$$\begin{aligned} L\mathbf{y}_1 = \mathbf{b}_1 \implies \mathbf{y}_1 &= \begin{bmatrix} 3 \\ 11 \end{bmatrix}. \text{ Then } U\mathbf{x}_1 = \mathbf{y}_1 \implies \mathbf{x}_1 = \begin{bmatrix} -4 \\ -11 \end{bmatrix}; \\ L\mathbf{y}_2 = \mathbf{b}_2 \implies \mathbf{y}_2 &= \begin{bmatrix} 2 \\ 15 \end{bmatrix}. \text{ Then } U\mathbf{x}_2 = \mathbf{y}_2 \implies \mathbf{x}_2 = \begin{bmatrix} -6.5 \\ -15 \end{bmatrix}; \\ L\mathbf{y}_3 = \mathbf{b}_3 \implies \mathbf{y}_3 &= \begin{bmatrix} 5 \\ 11 \end{bmatrix}. \text{ Then } U\mathbf{x}_3 = \mathbf{y}_3 \implies \mathbf{x}_3 = \begin{bmatrix} -3 \\ -11 \end{bmatrix}. \end{aligned}$$

26. We have

$$\begin{bmatrix} -1 & 4 & 2 \\ 3 & 1 & 4 \\ 5 & -7 & 1 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} -1 & 4 & 2 \\ 0 & 13 & 10 \\ 0 & 13 & 11 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} -1 & 4 & 2 \\ 0 & 13 & 10 \\ 0 & 0 & 1 \end{bmatrix} = U.$$

$$\boxed{\text{1. } A_{12}(3), A_{13}(5) \quad \text{2. } A_{23}(-1)}$$

Thus, $m_{21} = -3$, $m_{31} = -5$, and $m_{32} = 1$. We now solve the triangular systems

$$L\mathbf{y}_i = \mathbf{b}_i, \quad U\mathbf{x}_i = \mathbf{y}_i$$

for $i = 1, 2, 3$. We have

$$\begin{aligned}
Ly_1 = e_1 &\Rightarrow y_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}. \text{ Then } Ux_1 = y_1 \Rightarrow x_1 = \begin{bmatrix} -29/13 \\ -17/13 \\ 2 \end{bmatrix}; \\
Ly_2 = e_2 &\Rightarrow y_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}. \text{ Then } Ux_2 = y_2 \Rightarrow x_2 = \begin{bmatrix} 18/13 \\ 11/13 \\ -1 \end{bmatrix}; \\
Ly_3 = e_3 &\Rightarrow y_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \text{ Then } Ux_3 = y_3 \Rightarrow x_3 = \begin{bmatrix} -14/13 \\ -10/13 \\ 1 \end{bmatrix}.
\end{aligned}$$

27. Observe that if P_i is an elementary permutation matrix, then $P_i^{-1} = P_i = P_i^T$. Therefore, we have

$$P^{-1} = (P_1 P_2 \dots P_k)^{-1} = P_k^{-1} P_{k-1}^{-1} \dots P_2^{-1} P_1^{-1} = P_k^T P_{k-1}^T \dots P_2^T P_1^T = (P_1 P_2 \dots P_k)^T = P^T.$$

28.

(a): Let A be an invertible upper triangular matrix with inverse B . Therefore, we have $AB = I_n$. Write $A = [a_{ij}]$ and $B = [b_{ij}]$. We will show that $b_{ij} = 0$ for all $i > j$, which shows that B is upper triangular. We have

$$\sum_{k=1}^n a_{ik} b_{kj} = \delta_{ij}.$$

Since A is upper triangular, $a_{ik} = 0$ whenever $i > k$. Therefore, we can reduce the above summation to

$$\sum_{k=i}^n a_{ik} b_{kj} = \delta_{ij}.$$

Let $i = n$. Then the above summation reduces to $a_{nn} b_{nj} = \delta_{nj}$. If $j = n$, we have $a_{nn} b_{nn} = 1$, so $a_{nn} \neq 0$. For $j < n$, we have $a_{nn} b_{nj} = 0$, and therefore $b_{nj} = 0$ for all $j < n$.

Next let $i = n - 1$. Then we have

$$a_{n-1,n-1} b_{n-1,j} + a_{n-1,n} b_{nj} = \delta_{n-1,j}.$$

Setting $j = n - 1$ and using the fact that $b_{n,n-1} = 0$ by the above calculation, we obtain $a_{n-1,n-1} b_{n-1,n-1} = 1$, so $a_{n-1,n-1} \neq 0$. For $j < n - 1$, we have $a_{n-1,n-1} b_{n-1,j} = 0$ so that $b_{n-1,j} = 0$.

Next let $i = n - 2$. Then we have $a_{n-2,n-2} b_{n-2,j} + a_{n-2,n-1} b_{n-1,j} + a_{n-2,n} b_{nj} = \delta_{n-2,j}$. Setting $j = n - 2$ and using the fact that $b_{n-1,n-2} = 0$ and $b_{n,n-2} = 0$, we have $a_{n-2,n-2} b_{n-2,n-2} = 1$, so that $a_{n-2,n-2} \neq 0$. For $j < n - 2$, we have $a_{n-2,n-2} b_{n-2,j} = 0$ so that $b_{n-2,j} = 0$.

Proceeding in this way, we eventually show that $b_{ij} = 0$ for all $i > j$.

For an invertible lower triangular matrix A with inverse B , we can either modify the preceding argument, or we can proceed more briefly as follows: Note that A^T is an invertible upper triangular matrix with inverse B^T . By the preceding argument, B^T is upper triangular. Therefore, B is lower triangular, as required.

(b): Let A be an invertible unit upper triangular matrix with inverse B . Use the notations from (a). By (a), we know that B is upper triangular. We simply must show that $b_{jj} = 1$ for all j . From $a_{nn} b_{nn} = 1$ (see proof of (a)), we see that if $a_{nn} = 1$, then $b_{nn} = 1$. Moreover, from $a_{n-1,n-1} b_{n-1,n-1} = 1$, the fact that $a_{n-1,n-1} = 1$ proves that $b_{n-1,n-1} = 1$. Likewise, the fact that $a_{n-2,n-2} b_{n-2,n-2} = 1$ implies that if $a_{n-2,n-2} = 1$, then $b_{n-2,n-2} = 1$. Continuing in this fashion, we prove that $b_{jj} = 1$ for all j .

For the last part, if A is an invertible unit lower triangular matrix with inverse B , then A^T is an invertible unit upper triangular matrix with inverse B^T , and by the preceding argument, B^T is a unit upper triangular matrix. This implies that B is a unit lower triangular matrix, as desired.

29.

(a): Since A is invertible, Corollary 2.6.12 implies that both L_2 and U_1 are invertible. Since $L_1U_1 = L_2U_2$, we can left-multiply by L_2^{-1} and right-multiply by U_1^{-1} to obtain $L_2^{-1}L_1 = U_2U_1^{-1}$.

(b): By Problem 28, we know that L_2^{-1} is a unit lower triangular matrix and U_1^{-1} is an upper triangular matrix. Therefore, $L_2^{-1}L_1$ is a unit lower triangular matrix and $U_2U_1^{-1}$ is an upper triangular matrix. Since these two matrices are equal, we must have $L_2^{-1}L_1 = I_n$ and $U_2U_1^{-1} = I_n$. Therefore, $L_1 = L_2$ and $U_1 = U_2$.

30. The system $A\mathbf{x} = \mathbf{b}$ can be written as $QR\mathbf{x} = \mathbf{b}$. If we can solve $Q\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$ and then solve $R\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, then $QR\mathbf{x} = \mathbf{b}$ as desired. Multiplying $Q\mathbf{y} = \mathbf{b}$ by Q^T and using the fact that $Q^TQ = I_n$, we obtain $\mathbf{y} = Q^T\mathbf{b}$. Therefore, $R\mathbf{x} = \mathbf{y}$ can be replaced by $R\mathbf{x} = Q^T\mathbf{b}$. Therefore, to solve $A\mathbf{x} = \mathbf{b}$, we first determine $\mathbf{y} = Q^T\mathbf{b}$ and then solve the upper triangular system $R\mathbf{x} = Q^T\mathbf{b}$ by back-substitution.

Solutions to Section 2.8

True-False Review:

1. **FALSE.** According to the given information, part (c) of the Invertible Matrix Theorem fails, while part (e) holds. This is impossible.

2. **TRUE.** This holds by the equivalence of parts (d) and (f) of the Invertible Matrix Theorem.

3. **FALSE.** Part (d) of the Invertible Matrix Theorem fails according to the given information, and therefore part (b) also fails. Hence, the equation $A\mathbf{x} = \mathbf{b}$ does not have a unique solution. But it is not valid to conclude

that the equation has infinitely many solutions; it could have no solutions. For instance, if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

and $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, there are no solutions to $A\mathbf{x} = \mathbf{b}$, although $\text{rank}(A) = 2$.

4. **FALSE.** An easy counterexample is the matrix 0_n , which fails to be invertible even though it is upper triangular. Since it fails to be invertible, it cannot be row-equivalent to I_n , by the Invertible Matrix Theorem.

Problems:

1. Since A is an invertible matrix, the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. However, if we assume that $AB = AC$, then $A(B - C) = \mathbf{0}$. If $\mathbf{x}_i$ denotes the i th column of $B - C$, then $\mathbf{x}_i = \mathbf{0}$ for each i . That is, $B - C = \mathbf{0}$, or $B = C$, as required.

2. If $\text{rank}(A) = n$, then the augmented matrix $A^\#$ for the system $A\mathbf{x} = \mathbf{0}$ can be reduced to REF such that each column contains a pivot except for the right-most column of all-zeros. Solving the system by back-substitution, we find that $\mathbf{x} = \mathbf{0}$, as claimed.

3. Since $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, $\text{REF}(A)$ contains a pivot in every column. Therefore, the linear system $A\mathbf{x} = \mathbf{b}$ can be solved by back-substitution for every $\mathbf{b}$ in $\mathbb{R}^n$. Therefore, $A\mathbf{x} = \mathbf{b}$ does have a solution.

Now suppose there are two solutions $\mathbf{y}$ and $\mathbf{z}$ to the system $A\mathbf{x} = \mathbf{b}$. That is, $A\mathbf{y} = \mathbf{b}$ and $A\mathbf{z} = \mathbf{b}$. Subtracting, we find

$$A(\mathbf{y} - \mathbf{z}) = \mathbf{0},$$

and so by assumption, $\mathbf{y} - \mathbf{z} = \mathbf{0}$. That is, $\mathbf{y} = \mathbf{z}$. Therefore, there is only one solution to the linear system $A\mathbf{x} = \mathbf{b}$.

4. If A and B are each invertible matrices, then A and B can each be expressed as a product of elementary matrices, say

$$A = E_1 E_2 \dots E_k \quad \text{and} \quad B = E'_1 E'_2 \dots E'_l.$$

Then

$$AB = E_1 E_2 \dots E_k E'_1 E'_2 \dots E'_l,$$

so AB can be expressed as a product of elementary matrices. Thus, by the equivalence of (a) and (e) in the Invertible Matrix Theorem, AB is invertible.

5. We are assuming that the equations $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ each have only the trivial solution $\mathbf{x} = \mathbf{0}$. Now consider the linear system

$$(AB)\mathbf{x} = \mathbf{0}.$$

Viewing this equation as

$$A(B\mathbf{x}) = \mathbf{0},$$

we conclude that $B\mathbf{x} = \mathbf{0}$. Thus, $\mathbf{x} = \mathbf{0}$. Hence, the linear equation $(AB)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Solutions to Section 2.9

Problems:

1. We have

$$(-4)A - B^T = \begin{bmatrix} 8 & -16 & -8 & -24 \\ 4 & 4 & -20 & 0 \end{bmatrix} - \begin{bmatrix} -3 & 2 & 1 & 0 \\ 0 & 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 11 & -18 & -9 & -24 \\ 4 & 2 & -17 & -1 \end{bmatrix}.$$

2. We have

$$AB = \begin{bmatrix} -2 & 4 & 2 & 6 \\ -1 & -1 & 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 2 & 2 \\ 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 6 & -17 \end{bmatrix}.$$

Moreover,

$$\text{tr}(AB) = -1.$$

3. We have

$$(AC)(AC)^T = \begin{bmatrix} -2 \\ 26 \end{bmatrix} \begin{bmatrix} -2 & 26 \end{bmatrix} = \begin{bmatrix} 4 & -52 \\ -52 & 676 \end{bmatrix}.$$

4. We have

$$(-4B)A = \begin{bmatrix} 12 & 0 \\ -8 & -8 \\ -4 & 12 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} -2 & 4 & 2 & 6 \\ -1 & -1 & 5 & 0 \end{bmatrix} = \begin{bmatrix} -24 & 48 & 24 & 72 \\ 24 & -24 & -56 & -48 \\ -4 & -28 & 52 & -24 \\ 4 & 4 & -20 & 0 \end{bmatrix}.$$

5. Using Problem 2, we find that

$$(AB)^{-1} = \begin{bmatrix} 16 & 8 \\ 6 & -17 \end{bmatrix}^{-1} = -\frac{1}{320} \begin{bmatrix} -17 & -8 \\ -6 & 16 \end{bmatrix}.$$

6. We have

$$C^T C = \begin{bmatrix} -5 & -6 & 3 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ -6 \\ 3 \\ 1 \end{bmatrix} = [71],$$

and

$$\text{tr}(C^T C) = 71.$$

7.

(a): We have

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \end{bmatrix} \begin{bmatrix} 3 & b \\ -4 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 3a - 5 & 2a + 4b \\ 7a - 14 & 5a + 9b \end{bmatrix}.$$

In order for this product to equal I_2 , we require

$$3a - 5 = 1, \quad 2a + 4b = 0, \quad 7a - 14 = 0, \quad 5a + 9b = 1.$$

We quickly solve this for the unique solution: $a = 2$ and $b = -1$.

(b): We have

$$BA = \begin{bmatrix} 3 & -1 \\ -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{bmatrix}.$$

8. We compute the (i, j) -entry of each side of the equation. We will denote the entries of A^T by a_{ij}^T , which equals a_{ji} . On the left side, note that the (i, j) -entry of $(AB^T)^T$ is the same as the (j, i) -entry of AB^T , and

$$(j, i)\text{-entry of } AB^T = \sum_{k=0}^n a_{jk} b_{ki}^T = \sum_{k=0}^n a_{jk} b_{ik} = \sum_{k=0}^n b_{ik} a_{kj}^T,$$

and the latter expression is the (i, j) -entry of BA^T . Therefore, the (i, j) -entries of $(AB^T)^T$ and BA^T are the same, as required.

9.

(a): The (i, j) -entry of A^2 is

$$\sum_{k=1}^n a_{ik} a_{kj}.$$

(b): Assume that A is symmetric. That means that $A^T = A$. We claim that A^2 is symmetric. To see this, note that

$$(A^2)^T = (AA)^T = A^T A^T = AA = A^2.$$

Thus, $(A^2)^T = A^2$, and so A^2 is symmetric.

10. We are assuming that A is skew-symmetric, so $A^T = -A$. To show that $B^T AB$ is skew-symmetric, we observe that

$$(B^T AB)^T = B^T A^T (B^T)^T = B^T A^T B = B^T (-A) B = -(B^T AB),$$

as required.

11. We have

$$A^2 = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so A is nilpotent.

12. We have

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$A^3 = A^2 A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so A is nilpotent.

13. We have

$$A'(t) = \begin{bmatrix} -3e^{-3t} & -2\sec^2 t \tan t \\ 6t^2 & -\sin t \\ 6/t & -5 \end{bmatrix}.$$

14. We have

$$\int_0^1 B(t) dt = \begin{bmatrix} -7t & t^3/3 \\ 6t - t^2/2 & 3t^4/4 + 2t^3 \\ t + t^2/2 & \frac{2}{\pi} \sin(\pi t/2) \\ e^t & t - t^4/4 \end{bmatrix} \bigg|_0^1 = \begin{bmatrix} -7 & 1/3 \\ 11/2 & 11/4 \\ 3/2 & 2/\pi \\ e - 1 & 3/4 \end{bmatrix}.$$

15. Since $A(t)$ is 3×2 and $B(t)$ is 4×2 , it is impossible to perform the indicated subtraction.

16. Since $A(t)$ is 3×2 and $B(t)$ is 4×2 , it is impossible to perform the indicated subtraction.

17. From the last equation, we see that $x_3 = 0$. Substituting this into the middle equation, we find that $x_2 = 0.5$. Finally, putting the values of x_2 and x_3 into the first equation, we find $x_1 = -6 - 2.5 = -8.5$. Thus, there is a unique solution to the linear system, and the solution set is

$$\{(-8.5, 0.5, 0)\}.$$

18. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\begin{aligned} \left[\begin{array}{ccc|c} 5 & -1 & 2 & 7 \\ -2 & 6 & 9 & 0 \\ -7 & 5 & -3 & -7 \end{array} \right] &\stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & 11 & 20 & 7 \\ -2 & 6 & 9 & 0 \\ -7 & 5 & -3 & -7 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & 11 & 20 & 7 \\ 0 & 28 & 49 & 14 \\ 0 & 82 & 137 & 42 \end{array} \right] \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 11 & 20 & 7 \\ 0 & 1 & 7/4 & 1/2 \\ 0 & 82 & 137 & 42 \end{array} \right] \\ &\stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & 11 & 20 & 7 \\ 0 & 1 & 7/4 & 1/2 \\ 0 & 0 & -13/2 & 1 \end{array} \right] \stackrel{5}{\sim} \left[\begin{array}{ccc|c} 1 & 11 & 20 & 7 \\ 0 & 1 & 7/4 & 1/2 \\ 0 & 0 & 1 & -2/13 \end{array} \right]. \end{aligned}$$

From the last row, we conclude that $x_3 = -2/13$, and using the middle row, we can solve for x_2 : we have $x_2 + \frac{7}{4} \cdot (-\frac{2}{13}) = \frac{1}{2}$, so $x_2 = \frac{20}{26} = \frac{10}{13}$. Finally, from the first row we can get x_1 : we have $x_1 + 11 \cdot \frac{10}{13} + 20 \cdot (-\frac{2}{13}) = 7$, and so $x_1 = \frac{21}{13}$. So there is a unique solution:

$$\left\{ \left(\frac{21}{13}, \frac{10}{13}, -\frac{2}{13} \right) \right\}.$$

1. $A_{21}(2)$	2. $A_{12}(2), A_{13}(7)$	3. $M_2(1/28)$	4. $A_{23}(-82)$	5. $M_3(-2/13)$
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19. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 1 & 0 & 1 & 5 \\ 4 & 4 & 0 & 12 \end{array} \right] \xrightarrow{1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -2 & 2 & 4 \\ 0 & -4 & 4 & 8 \end{array} \right] \xrightarrow{2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & -4 & 4 & 8 \end{array} \right] \xrightarrow{3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

From this row-echelon form, we see that z is a free variable. Set $z = t$. Then from the middle row of the matrix, $y = t - 2$, and from the top row, $x + 2(t - 2) - t = 1$ or $x = -t + 5$. So the solution set is

$$\{(-t + 5, t - 2, t) : t \in \mathbb{R}\} = \{(5, -2, 0) + t(-1, 1, 1) : t \in \mathbb{R}\}.$$

1. $A_{12}(-1), A_{13}(-4)$	2. $M_2(-1/2)$	3. $A_{23}(4)$
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20. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{array} \right] \xrightarrow{1} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{array} \right] \xrightarrow{2} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{array} \right] \xrightarrow{3} \left[\begin{array}{cccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1/3 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array} \right].$$

The bottom row of this matrix shows that this system has no solutions.

1. $A_{12}(2), A_{13}(-3)$	2. $A_{23}(1)$	3. $M_2(1/3), M_3(1/3)$
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21. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{ccccc|c} 3 & 0 & -1 & 2 & -1 & 1 \\ 1 & 3 & 1 & -3 & 2 & -1 \\ 4 & -2 & -3 & 6 & -1 & 5 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array} \right] \xrightarrow{1} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 3 & 0 & -1 & 2 & -1 & 1 \\ 4 & -2 & -3 & 6 & -1 & 5 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array} \right] \xrightarrow{2} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & -9 & -4 & 11 & -7 & 4 \\ 0 & -14 & -7 & 18 & -9 & 9 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array} \right] \\ \xrightarrow{3} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & -27 & -12 & 33 & -21 & 12 \\ 0 & 28 & 14 & -36 & 18 & -18 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array} \right] \xrightarrow{4} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & -27 & -12 & 33 & -21 & 12 \\ 0 & 1 & 2 & -3 & -3 & -6 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array} \right] \\ \xrightarrow{5} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & 1 & 2 & -3 & -3 & -6 \\ 0 & -27 & -12 & 33 & -21 & 12 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array} \right] \xrightarrow{6} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & 1 & 2 & -3 & -3 & -6 \\ 0 & 0 & 42 & -48 & -102 & -150 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array} \right] \xrightarrow{7} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & 1 & 2 & -3 & -3 & -6 \\ 0 & 0 & 1 & -\frac{8}{7} & -\frac{17}{7} & -\frac{25}{7} \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array} \right].$$

We see that $x_5 = t$ is the only free variable. Back substitution yields the remaining values:

$$x_5 = t, \quad x_4 = -4t - 2, \quad x_3 = -\frac{41}{7} - \frac{15}{7}t, \quad x_2 = -\frac{2}{7} - \frac{33}{7}t, \quad x_1 = -\frac{2}{7} + \frac{16}{7}t.$$

So the solution set is

$$\left\{ \left(-\frac{2}{7} + \frac{16}{7}t, -\frac{2}{7} - \frac{33}{7}t, -\frac{41}{7} - \frac{15}{7}t, -4t - 2, t \right) : t \in \mathbb{R} \right\}$$

$$= \left\{ t \left(\frac{16}{7}, -\frac{33}{7}, -\frac{15}{7}, -4, 1 \right) + \left(-\frac{2}{7}, -\frac{2}{7}, -\frac{41}{7}, -2, 0 \right) : t \in \mathbb{R} \right\}.$$

1. P₁₂ 2. A₁₂(-3), A₁₃(-4) 3. M₂(3), M₃(-2) 4. A₂₃(1) 5. P₂₃ 6. A₂₃(27) 7. M₃(1/42)

22. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 3 & 1 & 4 & -9 & 17 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array} \right] \xrightarrow{1} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & -1 & 2 & -2 \end{array} \right] \xrightarrow{2} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\xrightarrow{3} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

From this row-echelon form, we see that $x_5 = t$ and $x_3 = s$ are free variables. Furthermore, solving this system by back-substitution, we see that

$$x_5 = t, \quad x_4 = 2t + 2, \quad x_3 = s, \quad x_2 = s - t + 1, \quad x_1 = 2t - 2s + 3.$$

So the solution set is

$$\{(2t - 2s + 3, s - t + 1, s, 2t + 2, t) : s, t \in \mathbb{R}\} = \{t(2, -1, 0, 2, 1) + s(-2, 1, 1, 0, 0) + (3, 1, 0, 2, 0) : s, t \in \mathbb{R}\}.$$

1. A₁₂(-1), A₁₃(-2), A₁₄(-2) 2. A₂₄(1) 3. P₂₃

23. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{ccc|c} 1 & -3 & 2i & 1 \\ -2i & 6 & 2 & -2 \end{array} \right] \xrightarrow{1} \left[\begin{array}{ccc|c} 1 & -3 & 2i & 1 \\ 0 & 6 - 6i & -2 & -2 + 2i \end{array} \right] \xrightarrow{2} \left[\begin{array}{ccc|c} 1 & -3 & 2i & 1 \\ 0 & 1 & -\frac{1}{6}(1 + i) & -\frac{1}{3} \end{array} \right].$$

1. A₁₂(2i) 2. M₂($\frac{1}{6-6i}$)

From the last augmented matrix above, we see that x_3 is a free variable. Let us set $x_3 = t$, where t is a complex number. Then we can solve for x_2 using the equation corresponding to the second row of the row-echelon form: $x_2 = -\frac{1}{3} + \frac{1}{6}(1 + i)t$. Finally, using the first row of the row-echelon form, we can determine that $x_1 = \frac{1}{2}t(1 - 3i)$. Therefore, the solution set for this linear system of equations is

$$\left\{ \left(\frac{1}{2}t(1 - 3i), -\frac{1}{3} + \frac{1}{6}(1 + i)t, t \right) : t \in \mathbb{C} \right\}.$$

24. We reduce the corresponding linear system as follows:

$$\left[\begin{array}{cc|c} 1 & -k & 6 \\ 2 & 3 & k \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{cc|c} 1 & -k & 6 \\ 0 & 3+2k & k-12 \end{array} \right].$$

If $k \neq -\frac{3}{2}$, then each column of the row-reduced coefficient matrix will contain a pivot, and hence, the linear system will have a unique solution. If, on the other hand, $k = -\frac{3}{2}$, then the system is inconsistent, because the last row of the row-echelon form will have a pivot in the right-most column. Under no circumstances will the linear system have infinitely many solutions.

25. First observe that if $k = 0$, then the second equation requires that $x_3 = 2$, and then the first equation requires $x_2 = 2$. However, x_1 is a free variable in this case, so there are infinitely many solutions.

Now suppose that $k \neq 0$. Then multiplying each row of the corresponding augmented matrix for the linear system by $1/k$ yields a row-echelon form with pivots in the first two columns only. Therefore, the third variable, x_3 , is free in this case. So once again, there are infinitely many solutions to the system.

We conclude that the system has infinitely many solutions for all values of k .

26. Since this linear system is homogeneous, it already has at least one solution: $(0, 0, 0)$. Therefore, it only remains to determine the values of k for which this will be the only solution. We reduce the corresponding matrix as follows:

$$\begin{aligned} & \left[\begin{array}{ccc|c} 10 & k & -1 & 0 \\ k & 1 & -1 & 0 \\ 2 & 1 & -1 & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 10k & k^2 & -k & 0 \\ 10k & 10 & -10 & 0 \\ 1 & 1/2 & -1/2 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \\ 10k & 10 & -10 & 0 \\ 10k & k^2 & -k & 0 \end{array} \right] \\ & \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \\ 0 & 10-5k & 5k-10 & 0 \\ 0 & k^2-5k & 4k & 0 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & k^2-5k & 4k & 0 \end{array} \right] \stackrel{5}{\sim} \left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & k^2-k & 0 \end{array} \right]. \end{aligned}$$

1. $M_1(k), M_2(10), M_3(1/2)$	2. P_{13}	3. $A_{12}(-10k), A_{13}(-10k)$	4. $M_2(\frac{1}{10-5k})$	5. $A_{23}(5k - k^2)$
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Note that the steps above are not valid if $k = 0$ or $k = 2$ (because Step 1 is not valid with $k = 0$ and Step 4 is not valid if $k = 2$). We will discuss those special cases individually in a moment. However if $k \neq 0, 2$, then the steps are valid, and we see from the last row of the last matrix that if $k = 1$, we have infinitely many solutions. Otherwise, if $k \neq 0, 1, 2$, then the matrix has full rank, and so there is a unique solution to the linear system.

If $k = 2$, then the last two rows of the original matrix are the same, and so the matrix of coefficients of the linear system is not invertible. Therefore, the linear system must have infinitely many solutions.

If $k = 0$, we reduce the original linear system as follows:

$$\left[\begin{array}{ccc|c} 10 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 1 & -1 & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1/10 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 1 & -1 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1/10 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -4/5 & 0 \end{array} \right] \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1/10 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1/5 & 0 \end{array} \right].$$

The last matrix has full rank, so there will be a unique solution in this case.

1. $M_1(1/10)$	2. $A_{13}(-2)$	3. $A_{23}(-1)$
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To summarize: The linear system has infinitely many solutions if and only if $k = 1$ or $k = 2$. Otherwise, the system has a unique solution.

27. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{ccc|c} 1 & -k & k^2 & 0 \\ 1 & 0 & k & 0 \\ 0 & 1 & -1 & 1 \end{array} \right] \xrightarrow{1} \left[\begin{array}{ccc|c} 1 & -k & k^2 & 0 \\ 0 & k & k-k^2 & 0 \\ 0 & 1 & -1 & 1 \end{array} \right] \xrightarrow{2} \left[\begin{array}{ccc|c} 1 & -k & k^2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & k & k-k^2 & 0 \end{array} \right] \xrightarrow{3} \left[\begin{array}{ccc|c} 1 & -k & k^2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2k-k^2 & -k \end{array} \right].$$

$$\boxed{\text{1. } A_{12}(-1) \quad \text{2. } P_{23} \quad \text{3. } A_{23}(-k)}$$

Now provided that $2k - k^2 \neq 0$, the system can be solved without free variables via back-substitution, and therefore, there is a unique solution. Consider now what happens if $2k - k^2 = 0$. Then either $k = 0$ or $k = 2$. If $k = 0$, then only the first two columns of the last augmented matrix above are pivoted, and we have a free variable corresponding to x_3 . Therefore, there are infinitely many solutions in this case. On the other hand, if $k = 2$, then the last row of the last matrix above reflects an inconsistency in the linear system, and there are no solutions.

To summarize, the system has no solutions if $k = 2$, a unique solution if $k \neq 0$ and $k \neq 2$, and infinitely many solutions if $k = 0$.

28. No, there are no common points of intersection. A common point of intersection would be indicated by a solution to the linear system consisting of the equations of the three planes. However, the corresponding augmented matrix can be row-reduced as follows:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{array} \right] \xrightarrow{1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -4 \end{array} \right] \xrightarrow{2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -5 \end{array} \right].$$

The last row of this matrix shows that the linear system is inconsistent, and so there are no points common to all three planes.

$$\boxed{\text{1. } A_{13}(-1) \quad \text{2. } A_{23}(-1)}$$

29.

(a): We have

$$\left[\begin{array}{cc} 4 & 7 \\ -2 & 5 \end{array} \right] \xrightarrow{1} \left[\begin{array}{cc} 1 & 7/4 \\ -2 & 5 \end{array} \right] \xrightarrow{2} \left[\begin{array}{cc} 1 & 7/4 \\ 0 & 17/2 \end{array} \right] \xrightarrow{3} \left[\begin{array}{cc} 1 & 7/4 \\ 0 & 1 \end{array} \right].$$

$$\boxed{\text{1. } M_1(1/4) \quad \text{2. } A_{12}(2) \quad \text{3. } M_2(2/17)}$$

(b): We have: $\text{rank}(A) = 2$, since the row-echelon form of A in (a) consists two nonzero rows.

(c): We have

$$\left[\begin{array}{cc|cc} 4 & 7 & 1 & 0 \\ -2 & 5 & 0 & 1 \end{array} \right] \xrightarrow{1} \left[\begin{array}{cc|cc} 1 & 7/4 & 1/4 & 0 \\ -2 & 5 & 0 & 1 \end{array} \right] \xrightarrow{2} \left[\begin{array}{cc|cc} 1 & 7/4 & 1/4 & 0 \\ 0 & 17/2 & 1/2 & 1 \end{array} \right] \xrightarrow{3} \left[\begin{array}{cc|cc} 1 & 7/4 & 1/4 & 0 \\ 0 & 1 & 1/17 & 2/17 \end{array} \right] \\ \xrightarrow{4} \left[\begin{array}{cc|cc} 1 & 0 & 5/34 & -7/34 \\ 0 & 1 & 1/17 & 2/17 \end{array} \right].$$

1. $M_1(1/4)$	2. $A_{12}(2)$	3. $M_2(2/17)$	4. $A_{21}(-7/4)$
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Thus,

$$A^{-1} = \begin{bmatrix} \frac{5}{34} & -\frac{7}{34} \\ \frac{1}{17} & \frac{2}{17} \end{bmatrix}.$$

30.

(a): We have

$$\begin{bmatrix} 2 & -7 \\ -4 & 14 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 2 & -7 \\ 0 & 0 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & -7/2 \\ 0 & 0 \end{bmatrix}.$$

1. $A_{12}(2)$	2. $M_1(1/2)$
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(b): We have: $\text{rank}(A) = 1$, since the row-echelon form of A in (a) has one nonzero row.

(c): Since $\text{rank}(A) < 2$, A is not invertible.

31.

(a): We have

$$\begin{bmatrix} 3 & -1 & 6 \\ 0 & 2 & 3 \\ 3 & -5 & 0 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 1 & -1/3 & 2 \\ 0 & 2 & 3 \\ 1 & -5/3 & 0 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & -1/3 & 2 \\ 0 & 2 & 3 \\ 0 & -4/3 & -2 \end{bmatrix} \xrightarrow{3} \begin{bmatrix} 1 & -1/3 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 1 & -1/3 & 2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

1. $M_1(1/3), M_3(1/3)$	2. $A_{13}(-1)$	3. $A_{23}(2/3)$	4. $M_2(1/2)$
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(b): We have: $\text{rank}(A) = 2$, since the row-echelon form of A in (a) consists of two nonzero rows.

(c): Since $\text{rank}(A) < 3$, A is not invertible.

32.

(a): We have

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{3} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

$$\xrightarrow{4} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 7 \end{bmatrix} \xrightarrow{5} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

1. P_{12}	2. $A_{12}(-2), A_{34}(-1)$	3. P_{34}	4. $M_2(-1/3), A_{34}(-3)$	5. $M_4(1/7)$
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(b): We have: $\text{rank}(A) = 4$, since the row-echelon form of A in (a) consists of four nonzero rows.

(c): We have

$$\begin{aligned}
& \left[\begin{array}{cccc|cccc} 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 3 & 0 & 0 & 0 & 1 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 3 & 0 & 0 & 0 & 1 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{array} \right] \\
& \stackrel{3}{\sim} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 7 & 0 & 0 & 4 & -3 \end{array} \right] \\
& \stackrel{5}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2/3 & -1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 4/7 & -3/7 \end{array} \right] \stackrel{6}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2/3 & -1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -3/7 & 4/7 \\ 0 & 0 & 0 & 1 & 0 & 0 & 4/7 & -3/7 \end{array} \right].
\end{aligned}$$

1. P_{12}	2. $A_{12}(-2), A_{34}(-1)$	3. P_{34}	4. $A_{34}(-3), M_2(-1/3)$	5. $M_4(1/7), A_{21}(-2)$	6. $A_{43}(1)$
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Thus,

$$A^{-1} = \begin{bmatrix} 2/3 & -1/3 & 0 & 0 \\ -1/3 & 2/3 & 0 & 0 \\ 0 & 0 & -3/7 & 4/7 \\ 0 & 0 & 4/7 & -3/7 \end{bmatrix}.$$

33.

(a): We have

$$\left[\begin{array}{ccc} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{array} \right] \stackrel{3}{\sim} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & -1 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{array} \right] \stackrel{5}{\sim} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{array} \right].$$

1. $M_1(1/3)$	2. $A_{13}(-1)$	3. P_{23}	4. $A_{23}(2)$	5. $M_2(-1), M_3(1/3)$
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(b): We have: $\text{rank}(A) = 3$, since the row-echelon form of A in (a) has 3 nonzero rows.

(c): We have

$$\begin{aligned}
& \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & -1/3 & 0 & 1 \end{array} \right] \\
& \stackrel{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & -1 & 2 & -1/3 & 0 & 1 \\ 0 & 2 & -1 & 0 & 1 & 0 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & -1 & 2 & -1/3 & 0 & 1 \\ 0 & 0 & 3 & -2/3 & 1 & 2 \end{array} \right] \\
& \stackrel{5}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & -2 & 1/3 & 0 & -1 \\ 0 & 0 & 1 & -2/9 & 1/3 & 2/3 \end{array} \right] \stackrel{6}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & -1/9 & 2/3 & 1/3 \\ 0 & 0 & 1 & -2/9 & 1/3 & 2/3 \end{array} \right].
\end{aligned}$$

1. $M_1(1/3)$ **2.** $A_{13}(-1)$ **3.** P_{23} **4.** $A_{23}(2)$ **5.** $M_2(-1), M_3(1/3)$ **6.** $A_{32}(2)$

Hence,

$$A^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ -1/9 & 2/3 & 1/3 \\ -2/9 & 1/3 & 2/3 \end{bmatrix}.$$

34.

(a): We have

$$\left[\begin{array}{ccc} -2 & -3 & 1 \\ 1 & 4 & 2 \\ 0 & 5 & 3 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc} 1 & 4 & 2 \\ -2 & -3 & 1 \\ 0 & 5 & 3 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc} 1 & 4 & 2 \\ 0 & 5 & 5 \\ 0 & 5 & 3 \end{array} \right] \stackrel{3}{\sim} \left[\begin{array}{ccc} 1 & 4 & 2 \\ 0 & 5 & 5 \\ 0 & 0 & -2 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right].$$

1. P_{12} **2.** $A_{12}(2)$ **3.** $A_{23}(-1)$ **4.** $M_2(1/5), M_3(-1/2)$

(b): We have: $\text{rank}(A) = 3$, since the row-echelon form of A in (a) consists of 3 nonzero rows.

(c): We have

$$\left[\begin{array}{ccc|ccc} -2 & -3 & 1 & 1 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 & 0 \\ 0 & 5 & 3 & 0 & 0 & 1 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 0 & 1 & 0 \\ -2 & -3 & 1 & 1 & 0 & 0 \\ 0 & 5 & 3 & 0 & 0 & 1 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 0 & 1 & 0 \\ 0 & 5 & 5 & 1 & 2 & 0 \\ 0 & 5 & 3 & 0 & 0 & 1 \end{array} \right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 0 & 1 & 0 \\ 0 & 5 & 5 & 1 & 2 & 0 \\ 0 & 0 & -2 & -1 & -2 & 1 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1/5 & 2/5 & 0 \\ 0 & 0 & 1 & 1/2 & 1 & -1/2 \end{array} \right]$$

$$\stackrel{5}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -4/5 & -3/5 & 0 \\ 0 & 1 & 1 & 1/5 & 2/5 & 0 \\ 0 & 0 & 1 & 1/2 & 1 & -1/2 \end{array} \right] \stackrel{6}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 7/5 & -1 \\ 0 & 1 & 0 & -3/10 & -3/5 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1 & -1/2 \end{array} \right].$$

1. P_{12} **2.** $A_{12}(2)$ **3.** $A_{23}(-1)$ **4.** $M_2(1/5), M_3(-1/2)$ **5.** $A_{21}(-4)$ **6.** $A_{31}(2), A_{32}(-1)$

Thus,

$$A^{-1} = \begin{bmatrix} 1/5 & 7/5 & -1 \\ -3/10 & -3/5 & 1/2 \\ 1/2 & 1 & -1/2 \end{bmatrix}.$$

35. We use the Gauss-Jordan method to find A^{-1} :

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 4 & -3 & 13 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -4 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -4 & 1 & 0 \\ 0 & 0 & -1 & 7 & -2 & 1 \end{array} \right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & -7 & 2 & -1 \end{array} \right] \stackrel{4}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & -3 & 1 & 0 \\ 0 & 1 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & -7 & 2 & -1 \end{array} \right] \stackrel{5}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 25 & -7 & 4 \\ 0 & 1 & 0 & 3 & -1 & 1 \\ 0 & 0 & 1 & -7 & 2 & -1 \end{array} \right].$$

1. $A_{12}(-4), A_{13}(-1)$ **2.** $A_{23}(-2)$ **3.** $M_3(-1)$ **4.** $A_{21}(1)$ **5.** $A_{31}(-4), A_{32}(-1)$

Thus,

$$A^{-1} = \begin{bmatrix} 25 & -7 & 4 \\ 3 & -1 & 1 \\ -7 & 2 & -1 \end{bmatrix}.$$

Now $\mathbf{x}_i = A^{-1}\mathbf{e}_i$ for each i . So

$$\mathbf{x}_1 = A^{-1}\mathbf{e}_1 = \begin{bmatrix} 25 \\ 3 \\ -7 \end{bmatrix}, \quad \mathbf{x}_2 = A^{-1}\mathbf{e}_2 = \begin{bmatrix} -7 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = A^{-1}\mathbf{e}_3 = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}.$$

36. We have $\mathbf{x}_i = A^{-1}\mathbf{b}_i$, where

$$A^{-1} = -\frac{1}{39} \begin{bmatrix} -2 & -5 \\ -7 & 2 \end{bmatrix}.$$

Therefore,

$$\mathbf{x}_1 = A^{-1}\mathbf{b}_1 = -\frac{1}{39} \begin{bmatrix} -2 & -5 \\ -7 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\frac{1}{39} \begin{bmatrix} -12 \\ -3 \end{bmatrix} = \frac{1}{39} \begin{bmatrix} 12 \\ 3 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 4 \\ 1 \end{bmatrix},$$

$$\mathbf{x}_2 = A^{-1}\mathbf{b}_2 = -\frac{1}{39} \begin{bmatrix} -2 & -5 \\ -7 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = -\frac{1}{39} \begin{bmatrix} -23 \\ -22 \end{bmatrix} = \frac{1}{39} \begin{bmatrix} 23 \\ 22 \end{bmatrix},$$

and

$$\mathbf{x}_3 = A^{-1}\mathbf{b}_3 = -\frac{1}{39} \begin{bmatrix} -2 & -5 \\ -7 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \end{bmatrix} = -\frac{1}{39} \begin{bmatrix} -21 \\ 24 \end{bmatrix} = \frac{1}{39} \begin{bmatrix} 21 \\ -24 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 7 \\ -8 \end{bmatrix}.$$

37.

(a): We have

$$(A^{-1}B)(B^{-1}A) = A^{-1}(BB^{-1})A = A^{-1}I_nA = A^{-1}A = I_n$$

and

$$(B^{-1}A)(A^{-1}B) = B^{-1}(AA^{-1})B = B^{-1}I_nB = B^{-1}B = I_n.$$

Therefore,

$$(B^{-1}A)^{-1} = A^{-1}B.$$

(b): We have

$$(A^{-1}B)^{-1} = B^{-1}(A^{-1})^{-1} = B^{-1}A,$$

as required.

38. We prove this by induction on k .

If $k = 0$, then $A^k = A^0 = I_n$ and $S^{-1}D^0S = S^{-1}I_nS = S^{-1}S = I_n$. Thus, $A^k = S^{-1}D^kS$ when $k = 0$.

Now assume that $A^{k-1} = S^{-1}D^{k-1}S$. We wish to show that $A^k = S^{-1}D^kS$. We have

$$A^k = AA^{k-1} = (S^{-1}DS)(S^{-1}D^{k-1}S) = S^{-1}D(SS^{-1})D^{k-1}S = S^{-1}DI_nD^{k-1}S = S^{-1}DD^{k-1}S = S^{-1}D^kS,$$

as required.

39.

(a): We reduce A to the identity matrix:

$$\begin{bmatrix} 4 & 7 \\ -2 & 5 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & \frac{7}{4} \\ -2 & 5 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & \frac{7}{4} \\ 0 & \frac{17}{2} \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & \frac{7}{4} \\ 0 & 1 \end{bmatrix} \stackrel{4}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

1. $M_1(\frac{1}{4})$	2. $A_{12}(2)$	3. $M_2(\frac{2}{17})$	4. $A_{21}(-\frac{7}{4})$
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The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{2}{17} \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & -\frac{7}{4} \\ 0 & 1 \end{bmatrix}.$$

We have $E_4 E_3 E_2 E_1 A = I_2$, so that

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{17}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{7}{4} \\ 0 & 1 \end{bmatrix},$$

which is the desired expression since E_i^{-1} is an elementary matrix for each i .

(b): We can reduce A to upper triangular form by the following elementary row operation:

$$\begin{bmatrix} 4 & 7 \\ -2 & 5 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 4 & 7 \\ 0 & \frac{17}{2} \end{bmatrix}.$$

1. $A_{12}(\frac{1}{2})$

Therefore we have the multiplier $m_{12} = -\frac{1}{2}$. Hence, setting

$$L = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 4 & 7 \\ 0 & \frac{17}{2} \end{bmatrix},$$

we have the LU factorization $A = LU$, which can be easily verified by direct multiplication.

40.

(a): We reduce A to the identity matrix:

$$\begin{aligned} & \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \xrightarrow{1} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \xrightarrow{3} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \xrightarrow{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \\ & \xrightarrow{5} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 4 & 3 \end{bmatrix} \xrightarrow{6} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & -\frac{7}{3} \end{bmatrix} \xrightarrow{7} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{8} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

1. P_{12}	2. $A_{12}(-2)$	3. $M_2(-\frac{1}{3})$	4. $A_{21}(-2)$	5. $M_3(\frac{1}{3})$
6. $A_{34}(-4)$	7. $M_4(-\frac{3}{7})$	8. $A_{43}(-\frac{4}{3})$		

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{bmatrix}, \quad E_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{3}{7} \end{bmatrix}, \quad E_8 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have

$$E_8 E_7 E_6 E_5 E_4 E_3 E_2 E_1 A = I_4$$

so that

$$\begin{aligned} A &= E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} E_7^{-1} E_8^{-1} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdots \\ &\quad \cdots \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{7}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

which is the desired expression since E_i^{-1} is an elementary matrix for each i .

(b): We can reduce A to upper triangular form by the following elementary row operations:

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & -\frac{7}{3} \end{bmatrix}.$$

$$\boxed{\mathbf{1.} \ A_{12}(-\frac{1}{2}) \quad \mathbf{2.} \ A_{34}(-\frac{4}{3})}$$

Therefore, the nonzero multipliers are $m_{12} = \frac{1}{2}$ and $m_{34} = \frac{4}{3}$. Hence, setting

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{4}{3} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & -\frac{7}{3} \end{bmatrix},$$

we have the LU factorization $A = LU$, which can be easily verified by direct multiplication.

41.

(a): We reduce A to the identity matrix:

$$\begin{aligned} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} &\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 3 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{9}{2} \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

1. P_{13}	2. $A_{13}(-3)$	3. $M_2(\frac{1}{2})$	4. $A_{23}(-3)$	5. $M_3(-\frac{2}{9})$
6. $A_{21}(1)$	7. $A_{31}(-\frac{3}{2})$	8. $A_{32}(\frac{1}{2})$		

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{2}{9} \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_7 = \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

We have

$$E_8 E_7 E_6 E_5 E_4 E_3 E_2 E_1 A = I_3$$

so that

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} E_7^{-1} E_8^{-1}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \dots$$

$$\dots \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{9}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix},$$

which is the desired expression since E_i^{-1} is an elementary matrix for each i .

(b): We can reduce A to upper triangular form by the following elementary row operations:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{3}{2} \end{bmatrix}.$$

1. $A_{13}(-\frac{1}{3})$	2. $A_{23}(\frac{1}{2})$
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Therefore, the nonzero multipliers are $m_{13} = \frac{1}{3}$ and $m_{23} = -\frac{1}{2}$. Hence, setting

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & -\frac{1}{2} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{3}{2} \end{bmatrix},$$

we have the LU factorization $A = LU$, which can be verified by direct multiplication.

42.

(a): We reduce A to the identity matrix:

$$\begin{bmatrix} -2 & -3 & 1 \\ 1 & 4 & 2 \\ 0 & 5 & 3 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ -2 & -3 & 1 \\ 0 & 5 & 3 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 5 & 5 \\ 0 & 5 & -3 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 5 & 5 \\ 0 & 1 & -8 \end{bmatrix} \stackrel{4}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -8 \\ 0 & 5 & 5 \end{bmatrix}$$

$$\stackrel{5}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -8 \\ 0 & 0 & 45 \end{bmatrix} \stackrel{6}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{bmatrix} \stackrel{7}{\sim} \begin{bmatrix} 1 & 0 & 34 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{bmatrix} \stackrel{8}{\sim} \begin{bmatrix} 1 & 0 & 34 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \stackrel{9}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. P_{12}	2. $A_{12}(2)$	3. $A_{23}(-1)$	4. P_{23}	5. $A_{23}(-5)$
6. $M_3(\frac{1}{45})$	7. $A_{21}(-4)$	8. $A_{32}(8)$	9. $A_{31}(-34)$	

The elementary matrices corresponding to these row operations are

$$\begin{aligned}
 E_1 &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \\
 E_4 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{45} \end{bmatrix}, \\
 E_7 &= \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_9 = \begin{bmatrix} 1 & 0 & -34 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
 \end{aligned}$$

We have

$$E_9 E_8 E_7 E_6 E_5 E_4 E_3 E_2 E_1 A = I_3$$

so that

$$\begin{aligned}
 A &= E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1} E_7^{-1} E_8^{-1} E_9^{-1} \\
 &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdots \\
 &\quad \cdots \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 45 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 34 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
 \end{aligned}$$

which is the desired expression since E_i^{-1} is an elementary matrix for each i .

(b): We can reduce A to upper triangular form by the following elementary row operations:

$$\begin{bmatrix} -2 & -3 & 1 \\ 1 & 4 & 2 \\ 0 & 5 & 3 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} -2 & -3 & 1 \\ 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & 5 & 3 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} -2 & -3 & 1 \\ 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & -2 \end{bmatrix}.$$

Therefore, the nonzero multipliers are $m_{12} = -\frac{1}{2}$ and $m_{23} = 2$. Hence, setting

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} -2 & -3 & 1 \\ 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & -2 \end{bmatrix},$$

we have the LU factorization $A = LU$, which can be verified by direct multiplication.

43.

(a): Note that

$$(A+B)^3 = (A+B)^2(A+B) = (A^2+AB+BA+B^2)(A+B) = A^3+ABA+BA^2+B^2A+A^2B+AB^2+BAB+B^3.$$

(b): We have

$$(A-B)^3 = (A-B)^2(A-B) = (A^2-AB-BA+B^2)(A-B) = A^3-ABA-BA^2+B^2A-A^2B+AB^2+BAB-B^3.$$

(c): The answer is 2^k , because each term in the expansion of $(A+B)^k$ consists of a string of k matrices, each of which is either A or B (2 possibilities for each matrix in the string). Multiplying the possibilities for each position in the string of length k , we get 2^k different strings, and hence 2^k different terms in the expansion of $(A+B)^k$.

44. We claim that

$$\begin{pmatrix} A & 0 \\ 0 & B^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & B \end{pmatrix}.$$

To see this, simply note that

$$\begin{pmatrix} A & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix} = I_{n+m}$$

and

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix} = I_{n+m}.$$

45. For a 2×4 matrix, the leading ones can occur in 6 different positions:

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For a 3×4 matrix, the leading ones can occur in 4 different positions:

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For a 4×6 matrix, the leading ones can occur in 15 different positions:

$$\begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix},$$

$$\begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For an $m \times n$ matrix with $m \leq n$, the answer is the binomial coefficient

$$C(n, m) = \binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

This represents n “choose” m , which is the number of ways to choose m columns from the n columns of the matrix in which to put the leading ones. This choice then determines the structure of the matrix.

46. The inverse of A^{10} is B^5 . To see this, we use the fact that

$$A^2B = I_n = BA^2$$

as follows:

$$\begin{aligned} A^{10}B^5 &= A^8(A^2B)B^4 = A^8I_nB^4 = A^8B^4 = A^6(A^2B)B^3 = A^6I_nB^3 \\ &= A^6B^3 = A^4(A^2B)B^2 = A^4I_nB^2 = A^4B^2 = A^2(A^2B)B = A^2I_nB = A^2B = I_n \end{aligned}$$

and

$$\begin{aligned} B^5A^{10} &= B^4(BA^2)A^8 = B^4I_nA^8 = B^4A^8 = B^3(BA^2)A^6 = B^3I_nA^6 \\ &= B^3A^6 = B^2(BA^2)A^4 = B^2I_nA^4 = B^2A^4 = B(BA^2)A^2 = BI_nA^2 = BA^2 = I_n. \end{aligned}$$

Solutions to Section 3.1

True-False Review:

1. TRUE. Let $A = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$. Then

$$\det(A) = ac - b0 = ac,$$

which is the product of the elements on the main diagonal of A .

2. TRUE. Let $A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$. Using the schematic of Figure 3.1.1, we have

$$\det(A) = adf + be0 + c00 - 0dc - 0ea - f0b = adf,$$

which is the product of the elements on the main diagonal of A .

3. FALSE. The volume of this parallelepiped is determined by the *absolute value* of $\det(A)$, since $\det(A)$ could very well be negative.

4. TRUE. There are 12 of each. The ones of even parity are $(1, 2, 3, 4)$, $(1, 3, 4, 2)$, $(1, 4, 2, 3)$, $(2, 1, 4, 3)$, $(2, 4, 3, 1)$, $(2, 3, 1, 4)$, $(3, 1, 2, 4)$, $(3, 2, 4, 1)$, $(3, 4, 1, 2)$, $(4, 1, 3, 2)$, $(4, 2, 1, 3)$, $(4, 3, 2, 1)$, and the others are all of odd parity.

5. FALSE. Many examples are possible here. If we take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $\det(A) = \det(B) = 0$, but $A + B = I_2$, and $\det(I_2) = 1 \neq 0$.

6. FALSE. Many examples are possible here. If we take $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, for example, then $\det(A) = 1 \cdot 4 - 2 \cdot 3 = -2 < 0$, even though all elements of A are positive.

7. TRUE. In the summation that defines the determinant, each term in the sum is a product consisting of one element from each row and each column of the matrix. But that means one of the factors in each term will be zero, since it must come from the row containing all zeros. Hence, each term is zero, and the summation is zero.

8. TRUE. If the determinant of the 3×3 matrix $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ is zero, then the volume of the parallelepiped determined by the three vectors is zero, and this means precisely that the three vectors all lie in the same plane.

Problems:

1. $\sigma(2, 1, 3, 4) = (-1)^{N(2,1,3,4)} = (-1)^1 = -1$, odd.

2. $\sigma(1, 3, 2, 4) = (-1)^{N(1,3,2,4)} = (-1)^1 = -1$, odd.

3. $\sigma(1, 4, 3, 5, 2) = (-1)^{N(1,4,3,5,2)} = (-1)^4 = 1$, even.

4. $\sigma(5, 4, 3, 2, 1) = (-1)^{N(5,4,3,2,1)} = (-1)^{10} = 1$, even.

5. $\sigma(1, 5, 2, 4, 3) = (-1)^{N(1,5,2,4,3)} = (-1)^4 = 1$, even.

6. $\sigma(2, 4, 6, 1, 3, 5) = (-1)^{N(2,4,6,1,3,5)} = (-1)^6 = 1$, even.

7. $\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \sigma(1, 2)a_{11}a_{22} + \sigma(2, 1)a_{12}a_{21} = a_{11}a_{22} - a_{12}a_{21}.$

8. $\det(A) = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 1 \cdot 3 - (-1)2 = 5.$

9. $\det(A) = \begin{vmatrix} 2 & -1 \\ 6 & -3 \end{vmatrix} = 2(-3) - (-1)6 = 0.$

10. $\det(A) = \begin{vmatrix} -4 & 10 \\ -1 & 8 \end{vmatrix} = -4 \cdot 8 - 10(-1) = -22.$

11. $\det(A) = \begin{vmatrix} 1 & -1 & 0 \\ 2 & 3 & 6 \\ 0 & 2 & -1 \end{vmatrix} = 1 \cdot 3(-1) + (-1)6 \cdot 0 + 0 \cdot 2 \cdot 2 - 0 \cdot 3 \cdot 0 - 6 \cdot 2 \cdot 1 - (-1)(-1)2 = -17.$

12. $\det(A) = \begin{vmatrix} 2 & 1 & 5 \\ 4 & 2 & 3 \\ 9 & 5 & 1 \end{vmatrix} = 2 \cdot 2 \cdot 1 + 1 \cdot 3 \cdot 9 + 5 \cdot 4 \cdot 5 - 5 \cdot 2 \cdot 9 - 3 \cdot 5 \cdot 2 - 1 \cdot 1 \cdot 4 = 7.$

13. $\det(A) = \begin{vmatrix} 0 & 0 & 2 \\ 0 & -4 & 1 \\ -1 & 5 & -7 \end{vmatrix} = 0(-4)(-7) + 0 \cdot 1(-1) + 2 \cdot 0 \cdot 5 - 2(-4)(-1) - 1 \cdot 5 \cdot 0 - (-7)0 \cdot 0 = -8.$

14. $\det(A) = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{vmatrix} = 400$, since of the 24 terms in the expression (3.1.3) for the determinant,

only the term $\sigma(1, 2, 3, 4)a_{11}a_{22}a_{33}a_{44} = 400$ contains all nonzero entries.

15. $\det(A) = \begin{vmatrix} 0 & 0 & 2 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 \end{vmatrix} = -60$, since of the 24 terms in the expression (3.1.3) for the determinant,

only the term $\sigma(3, 1, 4, 2)a_{13}a_{21}a_{34}a_{42}$ contains all nonzero entries, and since $\sigma(3, 1, 4, 2) = -1$, we obtain $\sigma(3, 1, 4, 2)a_{13}a_{21}a_{34}a_{42} = (-1) \cdot 2 \cdot 5 \cdot 3 \cdot 2 = -60$.

16. $\begin{vmatrix} \pi & \pi^2 \\ \sqrt{2} & 2\pi \end{vmatrix} = 2\pi^2 - \sqrt{2}\pi^2 = (2 - \sqrt{2})\pi^2$.

17. $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 4 & 1 \\ 3 & 1 & 6 \end{vmatrix} = (2)(4)(6) + (3)(1)(3) + (-1)(1)(1) - (3)(4)(-1) - (1)(1)(2) - (6)(1)(3) = 48$.

18. $\begin{vmatrix} 3 & 2 & 6 \\ 2 & 1 & -1 \\ -1 & 1 & 4 \end{vmatrix} = (3)(1)(4) + (2)(-1)(-1) + (6)(2)(1) - (-1)(1)(6) - (1)(-1)(3) - (4)(2)(2) = 19$.

19. $\begin{vmatrix} 2 & 3 & 6 \\ 0 & 1 & 2 \\ 1 & 5 & 0 \end{vmatrix} = (2)(1)(0) + (3)(2)(1) + (6)(0)(5) - (1)(1)(6) - (5)(2)(2) - (0)(0)(3) = -20$.

20. $\begin{vmatrix} \sqrt{\pi} & e^2 & e^{-1} \\ \sqrt{67} & 1/30 & 2001 \\ \pi & \pi^2 & \pi^3 \end{vmatrix} = (\sqrt{\pi})(1/30)(\pi^3) + (e^2)(2001)(\pi) + (e^{-1})(\sqrt{67})(\pi^2) - (\pi)(1/30)(e^{-1}) - (\pi^2)(2001)(\sqrt{\pi}) - (\pi^3)(\sqrt{67})(e^2) \approx 9601.882$.

21. $\begin{vmatrix} e^{2t} & e^{3t} & e^{-4t} \\ 2e^{2t} & 3e^{3t} & -4e^{-4t} \\ 4e^{2t} & 9e^{3t} & 16e^{-4t} \end{vmatrix} = (e^{2t})(3e^{3t})(16e^{-4t}) + (e^{3t})(-4e^{-4t})(4e^{2t}) + (e^{-4t})(2e^{2t})(9e^{3t}) - (4e^{2t})(3e^{3t})(e^{-4t}) - (9e^{3t})(-4e^{-4t})(e^{2t}) - (16e^{-4t})(2e^{2t})(e^{3t}) = 42e^t$.

22.

$$y_1''' - y_1'' + 4y_1' - 4y_1 = 8 \sin 2x + 4 \cos 2x - 8 \sin 2x - 4 \cos 2x = 0,$$

$$y_2''' - y_2'' + 4y_2' - 4y_2 = -8 \cos 2x + 4 \sin 2x + 8 \cos 2x - 4 \sin 2x = 0,$$

$$y_3''' - y_3'' + 4y_3' - 4y_3 = e^x - e^x + 4e^x - 4e^x = 0.$$

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x & e^x \\ -2 \sin 2x & 2 \cos 2x & e^x \\ -4 \cos 2x & -4 \sin 2x & e^x \end{vmatrix} = 2e^x \cos^2 2x - 4e^x \sin 2x \cos 2x + 8e^x \sin^2 2x + 8e^x \cos^2 2x + 4e^x \sin 2x \cos 2x + 2e^x \sin^2 2x = 10e^x.$$

23.

(a):

$$y_1''' - y_1'' - y_1' + y_1 = e^x - e^x - e^x + e^x = 0,$$

$$y_2''' - y_2'' - y_2' + y_2 = \sinh x - \cosh x - \sinh x + \cosh x = 0,$$

$$y_3''' - y_3'' - y_3' + y_3 = \cosh x - \sinh x - \cosh x + \sinh x = 0.$$

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^x & \cosh x & \sinh x \\ e^x & \sinh x & \cosh x \\ e^x & \cosh x & \sinh x \end{vmatrix}$$

$$= e^x \sinh^2 x + e^x \cosh^2 x + e^x \sinh x \cosh x - e^x \sinh^2 x - e^x \cosh^2 x - e^x \sinh x \cosh x = 0.$$

(b): The formulas we need are

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh x = \frac{e^x - e^{-x}}{2}.$$

Adding the two equations, we find that $\cosh x + \sinh x = e^x$, so that $-e^x + \cosh x + \sinh x = 0$. Therefore, we may take $d_1 = -1$, $d_2 = 1$, and $d_3 = 1$.

24.

(a): $S_4 = \{1, 2, 3, 4\}$

(1, 2, 3, 4)	(1, 2, 4, 3)
(1, 3, 2, 4)	(1, 3, 4, 2)
(1, 4, 2, 3)	(1, 4, 3, 2)
(2, 1, 3, 4)	(2, 1, 4, 3)
(2, 3, 1, 4)	(2, 3, 4, 1)
(2, 4, 1, 3)	(2, 4, 3, 1)
(3, 1, 2, 4)	(3, 1, 4, 2)
(3, 2, 1, 4)	(3, 2, 4, 1)
(3, 4, 1, 2)	(3, 4, 2, 1)
(4, 1, 2, 3)	(4, 1, 3, 2)
(4, 2, 1, 3)	(4, 2, 3, 1)
(4, 3, 1, 2)	(4, 3, 2, 1)

(b):

$N(1, 2, 3, 4) = 0$, $\sigma(1, 2, 3, 4) = 1$, even;	$N(1, 2, 4, 3) = 1$, $\sigma(1, 2, 4, 3) = -1$, odd;
$N(1, 3, 2, 4) = 1$, $\sigma(1, 3, 2, 4) = -1$, odd;	$N(1, 3, 4, 2) = 2$, $\sigma(1, 3, 4, 2) = 1$, even;
$N(1, 4, 2, 3) = 2$, $\sigma(1, 4, 2, 3) = 1$, even;	$N(1, 4, 3, 2) = 3$, $\sigma(1, 4, 3, 2) = -1$, odd;
$N(2, 1, 3, 4) = 1$, $\sigma(2, 1, 3, 4) = -1$, odd;	$N(2, 1, 4, 3) = 2$, $\sigma(2, 1, 4, 3) = 1$, even;
$N(2, 3, 1, 4) = 2$, $\sigma(2, 3, 1, 4) = 1$, even;	$N(2, 3, 4, 1) = 3$, $\sigma(2, 3, 4, 1) = -1$, odd;
$N(2, 4, 1, 3) = 3$, $\sigma(2, 4, 1, 3) = -1$, odd;	$N(2, 4, 3, 1) = 4$, $\sigma(2, 4, 3, 1) = 1$, even;
$N(3, 1, 2, 4) = 2$, $\sigma(3, 1, 2, 4) = 1$, even;	$N(3, 1, 4, 2) = 3$, $\sigma(3, 1, 4, 2) = -1$, odd;
$N(3, 2, 1, 4) = 3$, $\sigma(3, 2, 1, 4) = -1$, odd;	$N(3, 2, 4, 1) = 4$, $\sigma(3, 2, 4, 1) = 1$, even;
$N(3, 4, 1, 2) = 4$, $\sigma(3, 4, 1, 2) = 1$, even;	$N(3, 4, 2, 1) = 5$, $\sigma(3, 4, 2, 1) = -1$, odd;
$N(4, 1, 2, 3) = 3$, $\sigma(4, 1, 2, 3) = -1$, odd;	$N(4, 1, 3, 2) = 4$, $\sigma(4, 1, 3, 2) = 1$, even;
$N(4, 2, 1, 3) = 4$, $\sigma(4, 2, 1, 3) = 1$, even;	$N(4, 2, 3, 1) = 5$, $\sigma(4, 2, 3, 1) = -1$, odd;
$N(4, 3, 1, 2) = 5$, $\sigma(4, 3, 1, 2) = -1$, odd;	$N(4, 3, 2, 1) = 6$, $\sigma(4, 3, 2, 1) = 1$, even.

(c):

$$\begin{aligned} \det(A) = & a_{11}a_{22}a_{33}a_{44} - a_{11}a_{22}a_{34}a_{43} - a_{11}a_{23}a_{32}a_{44} + a_{11}a_{23}a_{34}a_{42} \\ & + a_{11}a_{24}a_{32}a_{43} - a_{11}a_{24}a_{33}a_{42} - a_{12}a_{21}a_{33}a_{44} + a_{12}a_{21}a_{34}a_{43} \\ & + a_{12}a_{23}a_{31}a_{44} - a_{12}a_{23}a_{34}a_{41} - a_{12}a_{24}a_{31}a_{43} + a_{12}a_{24}a_{33}a_{41} \\ & + a_{13}a_{21}a_{32}a_{44} - a_{13}a_{21}a_{34}a_{42} - a_{13}a_{22}a_{31}a_{44} + a_{13}a_{22}a_{34}a_{41} \\ & + a_{13}a_{24}a_{31}a_{42} - a_{13}a_{24}a_{32}a_{41} - a_{14}a_{21}a_{32}a_{43} + a_{14}a_{21}a_{33}a_{42} \\ & + a_{14}a_{22}a_{31}a_{43} - a_{14}a_{22}a_{33}a_{41} - a_{14}a_{23}a_{31}a_{42} + a_{14}a_{23}a_{32}a_{41} \end{aligned}$$

25.

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 1 & -1 & 0 & 1 \\ 3 & 0 & 2 & 5 \\ 2 & 1 & 0 & 3 \\ 9 & -1 & 2 & 1 \end{vmatrix} \\
&= 1 \cdot 0 \cdot 0 \cdot 1 - 1 \cdot 0 \cdot 2 \cdot 3 - 1 \cdot 1 \cdot 2 \cdot 1 + 1 \cdot 1 \cdot 2 \cdot 5 + 1(-1)2 \cdot 3 - 1(-1)0 \cdot 5 \\
&\quad - 3(-1)0 \cdot 1 + 3(-1)2 \cdot 3 + 3 \cdot 1 \cdot 0 \cdot 1 - 3 \cdot 1 \cdot 2 \cdot 1 - 3(-1)0 \cdot 3 + 1(-1)0 \cdot 1 \\
&\quad + 2(-1)2 \cdot 1 - 2(-1)2 \cdot 5 - 2 \cdot 0 \cdot 0 \cdot 1 + 2 \cdot 0 \cdot 2 \cdot 1 + 2(-1)0 \cdot 5 - 2(-1)2 \cdot 1 \\
&\quad - 9(-1)2 \cdot 3 + 9(-1)0 \cdot 5 + 9 \cdot 0 \cdot 0 \cdot 3 - 9 \cdot 0 \cdot 0 \cdot 1 - 9 \cdot 1 \cdot 0 \cdot 5 + 9 \cdot 1 \cdot 2 \cdot 1 \\
&= 70.
\end{aligned}$$

26.

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 1 & 1 & 0 & 1 \\ 3 & 1 & -2 & 3 \\ 2 & 3 & 1 & 2 \\ -2 & 3 & 5 & -2 \end{vmatrix} \\
&= 1 \cdot 1 \cdot 1(-2) - 1 \cdot 1 \cdot 5 \cdot 2 - 1 \cdot 3(-2)(-2) + 1 \cdot 3 \cdot 5 \cdot 3 + 1 \cdot 3(-2)2 - 1 \cdot 3 \cdot 1 \cdot 3 \\
&\quad - 3 \cdot 1 \cdot 1(-2) + 3 \cdot 1 \cdot 5 \cdot 2 + 3 \cdot 3 \cdot 0(-2) - 3 \cdot 3 \cdot 5 \cdot 1 - 3 \cdot 3 \cdot 0 \cdot 2 + 3 \cdot 3 \cdot 1 \cdot 1 \\
&\quad + 2 \cdot 1(-2)(-2) - 2 \cdot 1 \cdot 5 \cdot 3 - 2 \cdot 1 \cdot 0(-2) + 2 \cdot 1 \cdot 5 \cdot 1 + 2 \cdot 3 \cdot 0 \cdot 3 - 2 \cdot 3(-2)1 \\
&\quad - (-2)1(-2)2 + (-2)1 \cdot 1 \cdot 3 + (-2)1 \cdot 0 \cdot 2 - (-2) \cdot 1 \cdot 1 \cdot 1 - (-2)3 \cdot 0 \cdot 3 + (-2)3(-2)1 \\
&= 0.
\end{aligned}$$

27.

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 3 & 4 \\ 3 & 4 & 0 & 5 \\ 4 & 5 & 6 & 0 \end{vmatrix} \\
&= 0 \cdot 0 \cdot 0 \cdot 0 - 0 \cdot 0 \cdot 6 \cdot 5 - 0 \cdot 4 \cdot 3 \cdot 0 + 0 \cdot 4 \cdot 6 \cdot 4 + 0 \cdot 5 \cdot 3 \cdot 5 - 0 \cdot 5 \cdot 0 \cdot 4 \\
&\quad - 2 \cdot 1 \cdot 0 \cdot 0 + 2 \cdot 1 \cdot 6 \cdot 5 + 2 \cdot 4 \cdot 2 \cdot 0 - 2 \cdot 4 \cdot 6 \cdot 3 - 2 \cdot 5 \cdot 2 \cdot 5 + 2 \cdot 5 \cdot 0 \cdot 3 \\
&\quad + 3 \cdot 1 \cdot 3 \cdot 0 - 3 \cdot 1 \cdot 6 \cdot 4 - 3 \cdot 0 \cdot 2 \cdot 0 + 3 \cdot 0 \cdot 6 \cdot 3 + 3 \cdot 5 \cdot 2 \cdot 4 - 3 \cdot 5 \cdot 3 \cdot 3 \\
&\quad - 4 \cdot 1 \cdot 3 \cdot 5 + 4 \cdot 1 \cdot 0 \cdot 4 + 4 \cdot 0 \cdot 2 \cdot 5 - 4 \cdot 0 \cdot 0 \cdot 3 - 4 \cdot 4 \cdot 2 \cdot 4 + 4 \cdot 4 \cdot 3 \cdot 3 \\
&= -315.
\end{aligned}$$

28. In evaluating $\det(A)$ with the expression (3.1.3), observe that the only nonzero terms in the summation occur when $p_5 = 5$. Such terms include the factor $a_{55} = 7$, which is multiplied by each corresponding term from the 4×4 determinant calculated in Problem 27. Therefore, by factoring a_{55} out of the expression for the determinant, we are left with the determinant of the corresponding 4×4 matrix appearing in Problem 27. Therefore, the answer here is $7 \cdot (-315) = -2205$.

29.**(a):**

$$\begin{aligned}
\det(cA) &= \begin{vmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{vmatrix} = (ca_{11})(ca_{22}) - (ca_{12})(ca_{21}) \\
&= c^2 a_{11} a_{22} - c^2 a_{12} a_{21} = c^2 (a_{11} a_{22} - a_{12} a_{21}) = c^2 \det(A).
\end{aligned}$$

(b):

$$\begin{aligned}
 \det(cA) &= \sum \sigma(p_1, p_2, p_3, \dots, p_n) ca_{1p_1} ca_{2p_2} ca_{3p_3} \cdots ca_{np_n} \\
 &= c^n \sum \sigma(p_1, p_2, p_3, \dots, p_n) a_{1p_1} a_{2p_2} a_{3p_3} \cdots a_{np_n} \\
 &= c^n \det(A),
 \end{aligned}$$

where each summation above runs over all permutations σ of $\{1, 2, 3, \dots, n\}$.

30. $a_{11}a_{25}a_{33}a_{42}a_{54}$. All row and column indices are distinct, so this is a term of an order 5 determinant. Further, $N(1, 5, 3, 2, 4) = 4$, so that $\sigma(1, 5, 3, 2, 4) = (-1)^4 = +1$.

31. $a_{11}a_{23}a_{34}a_{43}a_{52}$. This is not a possible term of an order 5 determinant, since the column indices are not distinct.

32. $a_{13}a_{25}a_{31}a_{44}a_{42}$. This is not a possible term of an order 5 determinant, since the row indices are not distinct.

33. $a_{11}a_{32}a_{24}a_{43}a_{55}$. This is a possible term of an order 5 determinant. $N(1, 4, 2, 3, 5) = 2 \implies \sigma(1, 4, 2, 3, 5) = (-1)^2 = +1$.

34. $a_{13}a_{p4}a_{32}a_{2q} = a_{13}a_{2q}a_{32}a_{p4}$. We must choose $p = 4$ and $q = 1$ in order for the row and column indices to be distinct. $N(3, 1, 2, 4) = 2$ so that $\sigma(3, 1, 2, 4) = (-1)^2 = +1$.

35. $a_{21}a_{3q}a_{p2}a_{43} = a_{p2}a_{21}a_{3q}a_{43}$. We must choose $p = 1$ and $q = 4$. $N(2, 1, 4, 3) = 2$ and $\sigma(2, 1, 4, 3) = (-1)^2 = +1$.

36. $a_{3q}a_{p4}a_{13}a_{42}$. We must choose $p = 2$ and $q = 1$. $N(3, 4, 1, 2) = 4$ and $\sigma(3, 4, 1, 2) = (-1)^4 = +1$.

37. $a_{pq}a_{34}a_{13}a_{42}$. We must choose $p = 2$ and $q = 1$. $N(3, 1, 4, 2) = 3$ and $\sigma(3, 1, 4, 2) = (-1)^3 = -1$.

38.

(a): $\epsilon_{123} = 1, \epsilon_{132} = -1, \epsilon_{213} = -1, \epsilon_{231} = 1, \epsilon_{312} = 1, \epsilon_{321} = -1$.

(b): Consider $\sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \epsilon_{ijk} a_{1i} a_{2j} a_{3k}$. The only nonzero terms arise when i, j , and k are distinct. Consequently,

$$\begin{aligned}
 \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \epsilon_{ijk} a_{1i} a_{2j} a_{3k} &= \epsilon_{123} a_{11} a_{22} a_{33} + \epsilon_{132} a_{11} a_{23} a_{32} + \epsilon_{213} a_{12} a_{21} a_{33} \\
 &\quad + \epsilon_{231} a_{12} a_{23} a_{31} + \epsilon_{312} a_{13} a_{21} a_{32} + \epsilon_{321} a_{13} a_{22} a_{31} \\
 &= a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} - a_{13} a_{22} a_{31} \\
 &= \det(A).
 \end{aligned}$$

39. From the given term, we have

$$N(n, n-1, n-2, \dots, 1) = 1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2},$$

because the series of $(n-1)$ terms is just an arithmetic series which has a first term of one, common difference of one, and last term $(n-1)$. Thus, $\sigma(n, n-1, n-2, \dots, 1) = (-1)^{n(n-1)/2}$.

Solutions to Section 3.2

True-False Review:

1. FALSE. The determinant of the matrix will increase by a factor of 2^n . For instance, if $A = I_2$, then $\det(A) = 1$. However, $\det(2A) = \det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4$, so the determinant in this case increases by a factor of four.

2. TRUE. In both cases, the determinant of the matrix is multiplied by a factor of c .

3. TRUE. This follows by repeated application of Property (P8):

$$\det(A^5) = \det(AAAAA) = (\det A)(\det A)(\det A)(\det A)(\det A) = (\det A)^5.$$

4. TRUE. Since $\det(A^2) = (\det A)^2$ and since $\det(A)$ is a real number, $\det(A^2)$ must be nonnegative.

5. FALSE. The matrix is not invertible if and only if its determinant, $x^2y - xy^2 = xy(x - y)$, is zero. For example, if $x = y = 1$, the matrix is not invertible since $x = y$; however, neither x nor y is zero in this case. We conclude that the statement is false.

6. TRUE. We have

$$\det(AB) = (\det A)(\det B) = (\det B)(\det A) = \det(BA).$$

Problems:

From this point on, P2 will denote an application of the property of determinants that states that if every element in any row (column) of a matrix A is multiplied by a number c , then the determinant of the resulting matrix is $c \det(A)$.

1.

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 6 & 4 \\ 3 & -5 & 2 \end{vmatrix} \stackrel{1}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 0 & -11 & -7 \end{vmatrix} \stackrel{2}{=} 2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & -11 & -7 \end{vmatrix} \stackrel{3}{=} 2 \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & -18 \end{vmatrix} = 2(-18) = -36.$$

1. $A_{12}(-2)$, $A_{13}(-3)$ 2. P2 3. $A_{23}(11)$

2.

$$\begin{vmatrix} 2 & -1 & 4 \\ 3 & 2 & 1 \\ -2 & 1 & 4 \end{vmatrix} \stackrel{1}{=} \begin{vmatrix} 2 & -1 & 4 \\ 3 & 2 & 1 \\ 0 & 0 & 8 \end{vmatrix} \stackrel{2}{=} 2 \begin{vmatrix} 1 & -1/2 & 2 \\ 3 & 2 & 1 \\ 0 & 0 & 8 \end{vmatrix} \stackrel{3}{=} 2 \begin{vmatrix} 1 & -1/2 & 2 \\ 0 & 7/2 & -5 \\ 0 & 0 & 8 \end{vmatrix} = 2 \cdot 28 = 56.$$

1. $A_{13}(1)$ 2. P2 3. $A_{12}(-3)$

3.

$$\begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 6 \\ 4 & 1 & 12 \end{vmatrix} \stackrel{1}{=} - \begin{vmatrix} -1 & 2 & 6 \\ 2 & 1 & 3 \\ 4 & 1 & 12 \end{vmatrix} \stackrel{2}{=} - \begin{vmatrix} -1 & 2 & 6 \\ 0 & 5 & 15 \\ 0 & 9 & 36 \end{vmatrix} \stackrel{3}{=} -5 \cdot 9 \begin{vmatrix} -1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{vmatrix} \\ \stackrel{4}{=} -45 \begin{vmatrix} -1 & 2 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{vmatrix} = (-45)(-1) = 45.$$

1. P_{12}	2. $A_{12}(2), A_{13}(4)$	3. $P2$	4. $A_{23}(-1)$
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4.

$$\begin{vmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix} \stackrel{1}{=} \begin{vmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 2 & -3 & 0 \end{vmatrix} \stackrel{2}{=} \begin{vmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & -3 & 6 \end{vmatrix} \stackrel{3}{=} \begin{vmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

1. $P_{12}, P2$	2. $A_{13}(-2)$	3. $A_{23}(3)$
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5.

$$\begin{vmatrix} 3 & 7 & 1 \\ 5 & 9 & -6 \\ 2 & 1 & 3 \end{vmatrix} \stackrel{1}{=} \begin{vmatrix} 1 & 6 & -2 \\ 5 & 9 & -6 \\ 2 & 1 & 3 \end{vmatrix} \stackrel{2}{=} \begin{vmatrix} 1 & 6 & -2 \\ 0 & -21 & 4 \\ 0 & -11 & 7 \end{vmatrix} \stackrel{3}{=} \begin{vmatrix} 1 & 6 & -2 \\ 0 & 1 & -10 \\ 0 & -11 & 7 \end{vmatrix}$$

$$\stackrel{4}{=} \begin{vmatrix} 1 & 6 & -2 \\ 0 & 1 & -10 \\ 0 & 0 & -103 \end{vmatrix} = -103.$$

1. $A_{31}(-1)$	2. $A_{12}(-5), A_{13}(-2)$	3. $A_{32}(-2)$	4. $A_{24}(11)$
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6.

$$\begin{vmatrix} 1 & -1 & 2 & 4 \\ 3 & 1 & 2 & 4 \\ -1 & 1 & 3 & 2 \\ 2 & 1 & 4 & 2 \end{vmatrix} \stackrel{1}{=} \begin{vmatrix} 1 & -1 & 2 & 4 \\ 0 & 4 & -4 & -8 \\ 0 & 0 & 5 & 6 \\ 0 & 3 & 0 & -6 \end{vmatrix} \stackrel{2}{=} 3 \cdot 4 \begin{vmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 5 & 6 \\ 0 & 1 & 0 & -2 \end{vmatrix} \stackrel{3}{=} -12 \begin{vmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 5 & 6 \end{vmatrix}$$

$$\stackrel{4}{=} -12 \begin{vmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 6 \end{vmatrix} \stackrel{5}{=} -12 \begin{vmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 \end{vmatrix} = -12 \cdot 6 = -72.$$

1. $A_{12}(-3), A_{13}(1), A_{14}(-2)$	2. $P2$	3. P_{34}	4. $A_{23}(-1)$	5. $A_{34}(-5)$
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7. Note that in the first step below, we extract a factor of 13 from the second row of the matrix, and we also extract a factor of 8 from the second column of the matrix.

$$\begin{vmatrix} 2 & 32 & 1 & 4 \\ 26 & 104 & 26 & -13 \\ 2 & 56 & 2 & 7 \\ 1 & 40 & 1 & 5 \end{vmatrix} \stackrel{1}{=} 13 \cdot 8 \begin{vmatrix} 2 & 4 & 1 & 4 \\ 2 & 1 & 2 & -1 \\ 2 & 7 & 2 & 7 \\ 1 & 5 & -1 & 5 \end{vmatrix} \stackrel{2}{=} -104 \begin{vmatrix} 1 & 5 & -1 & 5 \\ 2 & 1 & 2 & -1 \\ 2 & 7 & 2 & 7 \\ 2 & 4 & 1 & 4 \end{vmatrix}$$

$$\stackrel{3}{=} -104 \begin{vmatrix} 1 & 5 & 1 & 5 \\ 0 & -9 & 0 & -11 \\ 0 & -3 & 0 & -3 \\ 0 & -6 & -1 & -6 \end{vmatrix} \stackrel{4}{=} -104(-3) \begin{vmatrix} 1 & 5 & 1 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & -9 & 0 & -11 \\ 0 & -6 & -1 & -6 \end{vmatrix}$$

$$\stackrel{5}{=} 312 \begin{vmatrix} 1 & 5 & 1 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & 0 \end{vmatrix} \stackrel{6}{=} 312 \begin{vmatrix} 1 & 5 & 1 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix}$$

$$= 312(-1)(-2) = 624.$$

1. P2	2. P ₁₄	3. A ₁₂ (-2), A ₁₃ (-2), A ₁₄ (-2)	4. P2, P ₂₃	5. A ₂₃ (9), A ₂₄ (6)	6. P ₃₄
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8.

$$\begin{vmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{vmatrix} \stackrel{1}{=} - \begin{vmatrix} 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ -1 & -1 & -1 & 0 \end{vmatrix} \stackrel{2}{=} - \begin{vmatrix} 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & -1 & 1 \end{vmatrix} \\
 \stackrel{3}{=} - \begin{vmatrix} 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -3 & -3 \end{vmatrix} \stackrel{4}{=} - \begin{vmatrix} 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 3 \end{vmatrix} = 9.$$

1. P ₁₃	2. A ₁₂ (1), A ₁₄ (1)	3. A ₂₃ (1), A ₂₄ (-2)	4. P ₃₄
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9.

$$\begin{vmatrix} 2 & 1 & 3 & 5 \\ 3 & 0 & 1 & 2 \\ 4 & 1 & 4 & 3 \\ 5 & 2 & 5 & 3 \end{vmatrix} \stackrel{1}{=} - \begin{vmatrix} 1 & 2 & 3 & 5 \\ 0 & 3 & 1 & 2 \\ 1 & 4 & 4 & 3 \\ 2 & 5 & 5 & 3 \end{vmatrix} \stackrel{2}{=} - \begin{vmatrix} 1 & 2 & 3 & 5 \\ 0 & 3 & 1 & 2 \\ 0 & 2 & 1 & -2 \\ 0 & 1 & -1 & -7 \end{vmatrix} \\
 \stackrel{3}{=} - \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & -7 \\ 0 & 2 & 1 & -2 \\ 0 & 3 & 1 & 2 \end{vmatrix} \stackrel{4}{=} \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 3 & 12 \\ 0 & 0 & 4 & 23 \end{vmatrix} \\
 \stackrel{5}{=} 3 \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 4 & 23 \end{vmatrix} \stackrel{6}{=} 3 \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 7 \end{vmatrix} \\
 = 3 \cdot 1 \cdot 1 \cdot 1 \cdot 7 = 21.$$

1. CP ₁₂	2. A ₁₃ (-1), A ₁₄ (-2)	3. P ₂₄	4. A ₂₃ (-2), A ₂₄ (-3)	5. P2	6. A ₃₄ (-4)
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10.

$$\begin{vmatrix} 2 & -1 & 3 & 4 \\ 7 & 1 & 2 & 3 \\ -2 & 4 & 8 & 6 \\ 6 & -6 & 18 & -24 \end{vmatrix} \stackrel{1}{=} -6 \begin{vmatrix} -1 & 2 & 3 & 4 \\ 1 & 7 & 2 & 3 \\ 4 & -2 & 8 & 6 \\ -1 & 1 & 3 & -4 \end{vmatrix} \stackrel{2}{=} 6 \begin{vmatrix} 1 & -2 & -3 & -4 \\ 1 & 7 & 2 & 3 \\ 4 & -2 & 8 & 6 \\ -1 & 1 & 3 & -4 \end{vmatrix} \\
 \stackrel{3}{=} 6 \begin{vmatrix} 1 & -2 & -3 & -4 \\ 0 & 9 & 5 & 7 \\ 0 & 6 & 20 & 22 \\ 0 & -1 & 0 & -8 \end{vmatrix} \stackrel{4}{=} -6 \begin{vmatrix} 1 & -2 & -3 & -4 \\ 0 & -1 & 0 & -8 \\ 0 & 6 & 20 & 22 \\ 0 & 9 & 5 & 7 \end{vmatrix} \stackrel{5}{=} -6 \begin{vmatrix} 1 & -2 & -3 & -4 \\ 0 & -1 & 0 & -8 \\ 0 & 0 & 20 & -26 \\ 0 & 0 & 5 & -65 \end{vmatrix} \\
 \stackrel{6}{=} 6 \begin{vmatrix} 1 & -2 & -3 & -4 \\ 0 & -1 & 0 & -8 \\ 0 & 0 & 5 & -65 \\ 0 & 0 & 20 & -26 \end{vmatrix} \stackrel{7}{=} 6 \begin{vmatrix} 1 & -2 & -3 & -4 \\ 0 & -1 & 0 & -8 \\ 0 & 0 & 5 & -65 \\ 0 & 0 & 0 & 234 \end{vmatrix} = 6 \cdot 1(-1) \cdot 5 \cdot 234 = -7020.$$

- | | | | |
|---|----------------------------|--|---------------------------|
| 1. CP_{12} , P_2 | 2. $\text{M}_1(-1)$ | 3. $\text{A}_{12}(-1)$, $\text{A}_{13}(-4)$, $\text{A}_{14}(1)$ | 4. P_{24} |
| 5. $\text{A}_{23}(6)$, $\text{A}_{24}(9)$ | 6. P_{34} | 7. $\text{A}_{34}(-4)$ | |

11.

$$\begin{aligned}
& \begin{vmatrix} 7 & -1 & 3 & 4 \\ 14 & 2 & 4 & 6 \\ 21 & 1 & 3 & 4 \\ -7 & 4 & 5 & 8 \end{vmatrix} \stackrel{1}{=} 7 \cdot 2 \begin{vmatrix} 1 & -1 & 3 & 2 \\ 2 & 2 & 4 & 3 \\ 3 & 1 & 3 & 2 \\ -1 & 4 & 5 & 4 \end{vmatrix} \stackrel{2}{=} 14 \begin{vmatrix} 1 & -1 & 3 & 2 \\ 0 & 4 & -2 & -1 \\ 0 & 4 & -6 & -4 \\ 0 & 3 & 8 & 6 \end{vmatrix} \\
& \stackrel{3}{=} 14 \begin{vmatrix} 1 & -1 & 3 & 2 \\ 0 & 4 & -2 & -1 \\ 0 & 0 & -4 & -3 \\ 0 & 3 & 8 & 6 \end{vmatrix} \stackrel{4}{=} 14 \begin{vmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & -10 & -7 \\ 0 & 0 & -4 & -3 \\ 0 & 3 & 8 & 6 \end{vmatrix} \stackrel{5}{=} 14 \begin{vmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & -10 & -7 \\ 0 & 0 & -4 & -3 \\ 0 & 0 & 38 & 27 \end{vmatrix} \\
& \stackrel{6}{=} -56 \begin{vmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & -10 & 7 \\ 0 & 0 & 1 & 3/4 \\ 0 & 0 & 38 & 27 \end{vmatrix} \stackrel{7}{=} -56 \begin{vmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & -10 & -7 \\ 0 & 0 & 1 & 3/4 \\ 0 & 0 & 0 & -3/2 \end{vmatrix} = -56 \cdot (-3/2) = 84.
\end{aligned}$$

- | | | | |
|-------------------------------|--|-------------------------------|--------------------------------|
| 1. P_2 | 2. $\text{A}_{12}(-2)$, $\text{A}_{13}(-3)$, $\text{A}_{14}(1)$ | 3. $\text{A}_{23}(-1)$ | 4. $\text{A}_{42}(-1)$ |
| 5. $\text{A}_{24}(-3)$ | | 6. P_2 | 7. $\text{A}_{34}(-38)$ |

12.

$$\begin{aligned}
& \begin{vmatrix} 3 & 7 & 1 & 2 & 3 \\ 1 & 1 & -1 & 0 & 1 \\ 4 & 8 & -1 & 6 & 6 \\ 3 & 7 & 0 & 9 & 4 \\ 8 & 16 & -1 & 8 & 12 \end{vmatrix} \stackrel{1}{=} - \begin{vmatrix} 1 & 1 & -1 & 0 & 1 \\ 3 & 7 & 1 & 2 & 3 \\ 4 & 8 & -1 & 6 & 6 \\ 3 & 7 & 0 & 9 & 4 \\ 8 & 16 & -1 & 8 & 12 \end{vmatrix} \stackrel{2}{=} - \begin{vmatrix} 1 & 1 & -1 & 0 & 1 \\ 0 & 4 & 4 & 2 & 0 \\ 0 & 4 & 3 & 6 & 2 \\ 0 & 4 & 3 & 9 & 1 \\ 0 & 8 & 7 & 8 & 4 \end{vmatrix} \\
& \stackrel{3}{=} - \begin{vmatrix} 1 & 1 & -1 & 0 & 1 \\ 0 & 4 & 4 & 2 & 0 \\ 0 & 0 & -1 & 4 & 2 \\ 0 & 0 & -1 & 7 & 1 \\ 0 & 0 & -1 & 4 & 4 \end{vmatrix} \stackrel{4}{=} - \begin{vmatrix} 1 & 1 & -1 & 0 & 1 \\ 0 & 4 & 4 & 2 & 0 \\ 0 & 0 & -1 & 4 & 2 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} = -(1)(4)(-1)(3)(2) = 24.
\end{aligned}$$

- | | | | |
|---------------------------|---|---|---|
| 1. P_{12} | 2. $\text{A}_{12}(-3)$, $\text{A}_{13}(-4)$, $\text{A}_{14}(-3)$, $\text{A}_{15}(-8)$ | 3. $\text{A}_{23}(-1)$, $\text{A}_{24}(-1)$, $\text{A}_{25}(-2)$ | 4. $\text{A}_{34}(-1)$, $\text{A}_{35}(-1)$ |
|---------------------------|---|---|---|

13. $\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1$; invertible.

14. $\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} = 0$; not invertible.

15. $\begin{vmatrix} 2 & 6 & -1 \\ 3 & 5 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 14$; invertible.

16. $\begin{vmatrix} -1 & 2 & 3 \\ 5 & -2 & 1 \\ 8 & -2 & 5 \end{vmatrix} = -8$; invertible.

17.

$$\begin{vmatrix} 1 & 0 & 2 & -1 \\ 3 & -2 & 1 & 4 \\ 2 & 1 & 6 & 2 \\ 1 & -3 & 4 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 & -1 \\ 0 & -2 & -5 & 7 \\ 0 & 1 & 2 & 4 \\ 0 & -3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} -2 & -5 & 7 \\ 1 & 2 & 4 \\ -3 & 2 & 1 \end{vmatrix} = 133; \text{ invertible.}$$

18.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & -2 & -2 \\ 0 & 2 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 2 \\ 0 & -2 & -2 \\ 2 & 2 & 0 \end{vmatrix} = 16; \text{ invertible.}$$

$$19. \begin{vmatrix} 1 & 2 & -3 & 5 \\ -1 & 2 & -3 & 6 \\ 2 & 3 & -1 & 4 \\ 1 & -2 & 3 & -6 \end{vmatrix} \stackrel{1}{=} - \begin{vmatrix} 1 & 2 & -3 & 5 \\ 1 & -2 & 3 & -6 \\ 2 & 3 & -1 & 4 \\ 1 & -2 & 3 & -6 \end{vmatrix} \stackrel{2}{=} 0; \text{ not invertible.} \quad \boxed{1. M_2(-1) \quad 2. P7}$$

20. The system is $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & k \\ k & 4 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. According to Corollary 3.2.5, the system has unique solution if and only if $\det(A) \neq 0$. But, $\det(A) = 4 - k^2$, so that the system has a unique solution if and only if $k \neq \pm 2$.

21. $\det(A) = \begin{vmatrix} 1 & 2 & k \\ 2 & -k & 1 \\ 3 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & k \\ 2 & -k & 1 \\ 0 & 0 & 1 - 3k \end{vmatrix} = (3k - 1)(k + 4)$. Consequently, the system has an infinite number of solutions if and only if $k = -4$ or $k = 1/3$ (Corollary 3.2.5).

22. The given system is

$$\begin{aligned} (1 - k)x_1 + 2x_2 + x_3 &= 0, \\ 2x_1 + (1 - k)x_2 + x_3 &= 0, \\ x_1 + x_2 + (2 - k)x_3 &= 0. \end{aligned}$$

The determinant of the coefficients as a function of k is given by

$$\det(A) = \begin{vmatrix} 1 - k & 2 & 1 \\ 2 & 1 - k & 1 \\ 1 & 1 & 2 - k \end{vmatrix} = -(1 + k)(k - 1)(k - 4).$$

Consequently, the system has an infinite number of solutions if and only if $k = -1$, $k = 1$, or $k = 4$.

23. $\det(A) = \begin{vmatrix} 1 & k & 0 \\ k & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1 + k - 1 - k^2 = k(1 - k)$. Consequently, the system has a unique solution if and only if $k \neq 0, 1$.

$$24. \det(A) = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \\ 0 & 1 & 3 \end{vmatrix} = 1 \cdot 1 \cdot 3 + (-1)4 \cdot 0 + 2 \cdot 3 \cdot 1 - 2 \cdot 1 \cdot 0 - 1 \cdot 4 \cdot 1 - (-1)3 \cdot 3 = 14.$$

$$\det(A^T) = \det(A) = 14.$$

$$\det(-2A) = (-2)^3 \det(A) = -8 \cdot 14 = -112.$$

25. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -2 & 4 \end{bmatrix}$. $\det(A)\det(B) = [3 \cdot 1 - (-1)2][1 \cdot 4 - 2(-2)] = 5 \cdot 8 = 40$.

$\det(AB) = \begin{vmatrix} 3 & -2 \\ -4 & 16 \end{vmatrix} = 3 \cdot 16 - (-2)(-4) = 40$. Hence, $\det(AB) = \det(A)\det(B)$.

26. $A = \begin{bmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{bmatrix}$ and $B = \begin{bmatrix} \cosh y & \sinh y \\ \sinh y & \cosh y \end{bmatrix}$.

$\det(AB) = \det(A)\det(B) = (\cosh^2 x - \sinh^2 x)(\cosh^2 y - \sinh^2 y) = 1 \cdot 1 = 1$.

27. $\begin{vmatrix} 3 & 2 & 1 \\ 6 & 4 & -1 \\ 9 & 6 & 2 \end{vmatrix} \stackrel{1}{=} 3 \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & -1 \\ 3 & 6 & 2 \end{vmatrix} \stackrel{2}{=} 3 \cdot 2 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ 3 & 3 & 2 \end{vmatrix} \stackrel{3}{=} 0$. 1. P2 2. P2 3. P7

28. $\begin{vmatrix} 1 & -3 & 1 \\ 2 & -1 & 7 \\ 3 & 1 & 13 \end{vmatrix} \stackrel{1}{=} \begin{vmatrix} 1 & -3+4 \cdot 1 & 1 \\ 2 & -1+4 \cdot 2 & 7 \\ 3 & 1+4 \cdot 3 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 7 & 7 \\ 3 & 13 & 13 \end{vmatrix} \stackrel{2}{=} 0$.

29. $\begin{vmatrix} 1+3a & 1 & 3 \\ 1+2a & 1 & 2 \\ 2 & 2 & 0 \end{vmatrix} \stackrel{1}{=} \begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & 2 \\ 2 & 2 & 0 \end{vmatrix} + \begin{vmatrix} 3a & 1 & 3 \\ 2a & 1 & 2 \\ 0 & 2 & 0 \end{vmatrix} \stackrel{2}{=} 0+a \begin{vmatrix} 3 & 1 & 3 \\ 2 & 1 & 2 \\ 0 & 2 & 0 \end{vmatrix} \stackrel{3}{=} a \cdot 0 = 0$. 1. P5 2. P2, P7 3. P7

30. B is obtained from A by the following elementary row operations: (1) $M_1(4)$, (2) $M_2(3)$, (3) P_{12} . Thus, $\det(B) = \det(A) \cdot 4 \cdot 3 \cdot (-1) = -12$.

31. B is obtained from A^T by the following elementary row operations: (1) $A_{13}(3)$, (2) $M_1(-2)$. Since $\det(A^T) = \det(A) = 1$, and (1) leaves the determinant unchanged, we have $\det(B) = -2$.

32. B is obtained from A by the following operations: (1) Interchange the two columns, (2) $M_1(-1)$, (3) $A_{12}(-4)$. Now (3) leaves the determinant unchanged, and (1) and (2) each change the sign of the determinant. Therefore, $\det(B) = \det(A) = 1$.

33. B is obtained from A by the following row operations: (1) $A_{13}(5)$, (2) $M_2(-4)$, (3) P_{12} , (4) P_{23} . Thus, $\det(B) = \det(A) \cdot (-4) \cdot (-1) \cdot (-1) = (-6) \cdot (-4) = 24$.

34. B is obtained from A by the following operations: (1) $M_1(-3)$, (2) $A_{23}(-4)$, (3) P_{12} . Thus, $\det(B) = \det(A) \cdot (-3) \cdot (-1) = (-6) \cdot (3) = -18$.

35. B is obtained from A^T by the following row operations: (1) $M_1(2)$, (2) $A_{32}(-1)$, (3) $A_{13}(-1)$. Thus, $\det(B) = \det(A^T) \cdot 2 = (-6) \cdot (2) = -12$.

36. We have $\det(AB^T) = \det(A)\det(B^T) = 5 \cdot 3 = 15$.

37. We have $\det(A^2B^5) = (\det A)^2(\det B)^5 = 5^2 \cdot 3^5 = 6075$.

38. We have $\det((A^{-1}B^2)^3) = (\det(A^{-1}B^2))^3 = [(\det A^{-1})(\det B)^2]^3 = \left(\frac{1}{5} \cdot 3^2\right)^3 = \left(\frac{9}{5}\right)^3 = \frac{729}{125} = 5.832$.

39. We have

$$\det((2B)^{-1}(AB)^T) = (\det((2B)^{-1}))(\det(AB)^T) = \left(\frac{\det(A)\det(B)}{\det(2B)}\right) = \left(\frac{5 \cdot 3}{3 \cdot 2^4}\right) = \frac{5}{16}.$$

40. We have $\det((5A)(2B)) = (5 \cdot 5^4)(3 \cdot 2^4) = 150,000$.

41.

(a): The volume of the parallelepiped is given by $|\det(A)|$. In this case, we have

$$|\det(A)| = |2 + 12k + 36 - 4k - 18 - 12| = |8 + 8k|.$$

(b): **NO.** The answer does not change because the determinants of A and A^T are the same, so the volume determined by the columns of A , which is the same as the volume determined by the rows of A^T , is $|\det(A^T)| = |\det(A)|$.

(c): The matrix A is invertible if and only if $\det(A) \neq 0$. That is, A is invertible if and only if $8 + 8k \neq 0$, if and only if $k \neq -1$.

42.

$$\begin{vmatrix} 1 & -1 & x \\ 2 & 1 & x^2 \\ 4 & -1 & x^3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & x \\ 0 & 3 & x^2 - 2x \\ 0 & 3 & x^3 - 4x \end{vmatrix} = \begin{vmatrix} 1 & -1 & x \\ 0 & 3 & x^2 - 2x \\ 0 & 0 & x^3 - x^2 - 2x \end{vmatrix} \\ = \begin{vmatrix} 1 & -1 & x \\ 0 & 3 & x(x-2) \\ 0 & 0 & x(x-2)(x+1) \end{vmatrix}.$$

We see directly that the determinant will be zero if and only if $x \in \{0, -1, 2\}$.

43.

$$\begin{vmatrix} \alpha x - \beta y & \beta x - \alpha y \\ \beta x + \alpha y & \alpha x + \beta y \end{vmatrix} = \begin{vmatrix} \alpha x & \beta x - \alpha y \\ \beta x & \alpha x + \beta y \end{vmatrix} + \begin{vmatrix} -\beta y & \beta x - \alpha y \\ \alpha y & \alpha x + \beta y \end{vmatrix} \\ = \begin{vmatrix} \alpha x & \beta x \\ \beta x & \alpha x \end{vmatrix} + \begin{vmatrix} \alpha x & -\alpha y \\ \beta x & \beta y \end{vmatrix} + \begin{vmatrix} -\beta y & \beta x \\ \alpha y & \alpha x \end{vmatrix} + \begin{vmatrix} -\beta y & -\alpha y \\ \alpha y & \beta y \end{vmatrix} \\ = x^2 \begin{vmatrix} \alpha & \beta \\ \beta & \alpha \end{vmatrix} + xy \begin{vmatrix} \alpha & -\alpha \\ \beta & \beta \end{vmatrix} - xy \begin{vmatrix} \beta & -\beta \\ \alpha & \alpha \end{vmatrix} + y^2 \begin{vmatrix} \alpha & \beta \\ \beta & \alpha \end{vmatrix} \\ = (x^2 + y^2) \begin{vmatrix} \alpha & \beta \\ \beta & \alpha \end{vmatrix} + xy \begin{vmatrix} \alpha & -\alpha \\ \beta & \beta \end{vmatrix} + xy \begin{vmatrix} \beta & \beta \\ \alpha & -\alpha \end{vmatrix} \\ = (x^2 + y^2) \begin{vmatrix} \alpha & \beta \\ \beta & \alpha \end{vmatrix} + xy \begin{vmatrix} \alpha & -\alpha \\ \beta & \beta \end{vmatrix} - xy \begin{vmatrix} \alpha & -\alpha \\ \beta & \beta \end{vmatrix} \\ = (x^2 + y^2) \begin{vmatrix} \alpha & \beta \\ \beta & \alpha \end{vmatrix}.$$

44.

$$\begin{vmatrix} a_1 + \beta b_1 & b_1 + \gamma c_1 & c_1 + \alpha a_1 \\ a_2 + \beta b_2 & b_2 + \gamma c_2 & c_2 + \alpha a_2 \\ a_3 + \beta b_3 & b_3 + \gamma c_3 & c_3 + \alpha a_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 + \gamma c_1 & c_1 + \alpha a_1 \\ a_2 & b_2 + \gamma c_2 & c_2 + \alpha a_2 \\ a_3 & b_3 + \gamma c_3 & c_3 + \alpha a_3 \end{vmatrix} + \begin{vmatrix} \beta b_1 & b_1 + \gamma c_1 & c_1 + \alpha a_1 \\ \beta b_2 & b_2 + \gamma c_2 & c_2 + \alpha a_2 \\ \beta b_3 & b_3 + \gamma c_3 & c_3 + \alpha a_3 \end{vmatrix} \\ = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \alpha \beta \gamma \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix} \\ = (1 + \alpha \beta \gamma) \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix}.$$

Now if the last expression is to be zero for all a_i, b_i , and c_i , then it must be the case that $1 + \alpha \beta \gamma = 0$; hence, $\alpha \beta \gamma = -1$.

45. Suppose A is a matrix with a row of zeros. We will use (P3) and (P7) to justify the fact that $\det(A) = 0$. If A has more than one row of zeros, then by (P7), since two rows of A are the same, $\det(A) = 0$. Assume instead that only one row of A consists entirely of zeros. Adding a nonzero row of A to the row of zeros yields a new matrix B with two equal rows. Thus, by (P7), $\det(B) = 0$. However, B was obtained from A by adding a multiple of one row to another row, and by (P3), $\det(B) = \det(A)$. Hence, $\det(A) = 0$, as required.

46. A is orthogonal, so $A^T = A^{-1}$. Using the properties of determinants, it follows that

$$1 = \det(I_n) = \det(AA^{-1}) = \det(A) \det(A^{-1}) = \det(A) \det(A^T) = \det(A) \det(A).$$

Therefore, $\det(A) = \pm 1$.

47.

(a): From the definition of determinant we have:

$$\det(A) = \sum_{n!} \sigma(p_1, p_2, p_3, \dots, p_n) a_{1p_1} a_{2p_2} a_{3p_3} \cdots a_{np_n}. \quad (47.1)$$

If A is lower triangular, then $a_{ij} = 0$ whenever $i < j$, and therefore the only nonzero terms in (47.1) are those with $p_i \leq i$ for all i . Since all the p_i must be distinct, the only possibility is $p_i = i$ for all i with $1 \leq i \leq n$, and so (47.1) reduces to the single term:

$$\det(A) = \sigma(1, 2, 3, \dots, n) a_{11} a_{22} a_{33} \cdots a_{nn}.$$

(b):

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & -1 & 3 & 5 \\ 1 & 2 & 2 & 1 \\ 3 & 0 & 1 & 4 \\ 1 & 2 & 0 & 1 \end{vmatrix} \stackrel{1}{=} \begin{vmatrix} -3 & -11 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & -8 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{vmatrix} \stackrel{2}{=} \begin{vmatrix} 0 & 13 & 0 & 0 \\ 2 & 16 & 0 & 0 \\ -1 & -8 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{vmatrix} \\ &\stackrel{3}{=} - \begin{vmatrix} 2 & 16 & 0 & 0 \\ 0 & 13 & 0 & 0 \\ -1 & -8 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{vmatrix} \stackrel{4}{=} - \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 13 & 0 & 0 \\ -1 & -8 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{vmatrix} = -26. \end{aligned}$$

1. $A_{41}(-5), A_{42}(-1), A_{43}(-4)$ 2. $A_{31}(-3), A_{32}(-2)$ 3. P_{12} 4. $A_{21}(-16/13)$

48. The problem stated in the text is wrong. It should state: “Use determinants to prove that if A is invertible and B and C are matrices with $AB = AC$, then $\det(B) = \det(C)$.” To prove this, take the determinant of both sides of $AB = AC$ to get $\det(AB) = \det(AC)$. Thus, by Property P8, we have $\det(A)\det(B) = \det(A)\det(C)$. Since A is invertible, $\det(A) \neq 0$. Thus, we can cancel $\det(A)$ from the last equality to obtain $\det(B) = \det(C)$.

49.

$$\begin{aligned} \det(S^{-1}AS) &= \det(S^{-1}) \det(A) \det(S) = \det(S^{-1}) \det(S) \det(A) = \det(S^{-1}S) \det(A) \\ &= \det(I_n) \det(A) = \det(A). \end{aligned}$$

50. No. If A were invertible, then $\det(A) \neq 0$, so that $\det(A^3) = \det(A) \det(A) \det(A) \neq 0$.

51. Let E be an elementary matrix. There are three different possibilities for E .

(a): E permutes two rows: Then E is obtained from I_n by interchanging two rows of I_n . Since $\det(I_n) = 1$ and using Property P1, we obtain $\det(E) = -1$.

(b): E adds a multiple of one row to another: Then E is obtained from I_n by adding a multiple of one row of I_n to another. Since $\det(I_n) = 1$ and using Property P3, we obtain $\det(E) = +1$.

(c): E scales a row by k : Then E is obtained from I_n by multiplying a row of I_n by k . Since $\det(I_n) = 1$ and using Property P2, we obtain $\det(E) = k$.

52. We have

$$0 = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = xy_1 + yx_2 + x_1y_2 - x_2y_1 - xy_2 - yx_1,$$

which can be rewritten as

$$x(y_1 - y_2) + y(x_2 - x_1) = x_2y_1 - x_1y_2.$$

Setting $a = y_1 - y_2$, $b = x_2 - x_1$, and $c = x_2y_1 - x_1y_2$, we can express this equation as $ax + by = c$, the equation of a line. Moreover, if we substitute x_1 for x and y_1 for y , we obtain a valid identity. Likewise, if we substitute x_2 for x and y_2 for y , we obtain a valid identity. Therefore, we have a straight line that includes the points (x_1, y_1) and (x_2, y_2) .

53.

$$\begin{aligned} \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} &\stackrel{1}{=} \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & (y-x)(y+x) \\ 0 & z-x & (z-x)(z+x) \end{vmatrix} \stackrel{2}{=} (y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix} \\ &\stackrel{3}{=} (y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 0 & z-y \end{vmatrix} = (y-x)(z-x)(z-y) = (y-z)(z-x)(x-y). \end{aligned}$$

1. $A_{12}(-1)A_{13}(-1)$	2. P2	3. $A_{23}(-1)$
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54. Since A is an $n \times n$ skew-symmetric matrix, $A^T = -A$; thus,

$$\det(A^T) = \det(-A) = (-1)^n \det(A) = -\det(A)$$

since n is given as odd. But by P4, $\det(A^T) = \det(A)$, so $\det(A) = -\det(A)$ or $\det(A) = 0$.

55. Solving $\mathbf{b} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n$ for $c_1\mathbf{a}_1$ yields $c_1\mathbf{a}_1 = \mathbf{b} - c_2\mathbf{a}_2 - \cdots - c_n\mathbf{a}_n$. Consequently, $\det(B_k)$ can be written as

$$\begin{aligned} \det(B_k) &= \det([\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{k-1}, \mathbf{b}, \mathbf{a}_{k+1}, \dots, \mathbf{a}_n]) \\ &= \det([\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{k-1}, (c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n), \mathbf{a}_{k+1}, \dots, \mathbf{a}_n]) \\ &= c_1 \det([\mathbf{a}_1, \dots, \mathbf{a}_{k-1}, \mathbf{a}_1, \mathbf{a}_{k+1}, \dots, \mathbf{a}_n]) + c_2 \det([\mathbf{a}_1, \dots, \mathbf{a}_{k-1}, \mathbf{a}_2, \mathbf{a}_{k+1}, \dots, \mathbf{a}_n]) + \cdots \\ &\quad + c_k \det([\mathbf{a}_1, \dots, \mathbf{a}_{k-1}, \mathbf{a}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_n]) + \cdots + c_n \det([\mathbf{a}_1, \dots, \mathbf{a}_{k-1}, \mathbf{a}_n, \mathbf{a}_{k+1}, \dots, \mathbf{a}_n]). \end{aligned}$$

Now by P7, all except the k^{th} determinant are zero since they have two equal columns, so that we are left with $\det(B_k) = c_k \det(A)$.

58. Using technology we find that:

$$\begin{vmatrix} 1 & 2 & 3 & 4 & a \\ 2 & 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 & 2 \\ a & 4 & 3 & 2 & 1 \end{vmatrix} = -192 + 88a - 8a^2 = -8(a-3)(a-8).$$

Consequently, the matrix is invertible provided $a \neq 3, 8$.

59. Using technology we find that:

$$\begin{vmatrix} 1-k & 4 & 1 \\ 3 & 2-k & 1 \\ 3 & 4 & -1-k \end{vmatrix} = -(k-6)(k+2)^2.$$

Consequently, the system has an infinite number of solutions if and only if $k = 6, -2$.

$k = 6$: In this case, the system is $B\mathbf{x} = \mathbf{0}$, where $B = \begin{bmatrix} -5 & 4 & 1 \\ 3 & -4 & 1 \\ 3 & 4 & -7 \end{bmatrix}$. This system has solution set $\{t(1, 1, 1) : t \in \mathbb{R}\}$.

$k = -2$: In this case, the system is $C\mathbf{x} = \mathbf{0}$, where $C = \begin{bmatrix} 3 & 4 & 1 \\ 3 & 4 & 1 \\ 3 & 4 & 1 \end{bmatrix}$. This system has solution set $\{(r, s, -3r - 4s) : r, s \in \mathbb{R}\}$.

60. Using technology we find that:

$$\det(A) = -20; A^{-1} = \begin{bmatrix} -2/5 & 1/2 & 0 & 1/10 \\ 1/2 & -1 & 1/2 & 0 \\ 0 & 1/2 & -1 & 1/2 \\ 1/10 & 0 & 1/2 & -2/5 \end{bmatrix}; \mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 19/10 \\ -5 \\ 1/2 \\ 12/5 \end{bmatrix}.$$

Solutions to Section 3.3

True-False Review:

1. **FALSE.** Because $2 + 3 = 5$ is odd, the $(2, 3)$ -cofactor is the negative of the $(2, 3)$ -minor of the matrix.

2. **TRUE.** This just requires a slight modification of the proof of Theorem 3.3.16. We compute

$$(A \cdot \text{adj}(A))_{ij} = \sum_{k=1}^n a_{ik} \text{adj}(A)_{kj} = \sum_{k=1}^n a_{ik} C_{jk} = \delta_{ij} \cdot \det(A).$$

Therefore, $A \cdot \text{adj}(A) = \det(A) \cdot I_n$.

3. **TRUE.** The Cofactor Expansion Theorem allows for expansion along any row or any column of the matrix, and in all cases, the result is the determinant of the matrix.

4. **FALSE.** For example, let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, and let $c = 2$. It is easy to see that the $(1, 1)$ -entry of $\text{adj}(A)$ is -3 . But the $(1, 1)$ -entry of $\text{adj}(2A)$ is -12 , not -6 . Therefore, the equality posed in this review item does not generally hold. Many other counterexamples could be given as well.

5. **FALSE.** For example, let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$. Then $A + B = \begin{bmatrix} 10 & 10 & 10 \\ 10 & 10 & 10 \\ 10 & 10 & 10 \end{bmatrix}$.

The $(1, 1)$ -entry of $\text{adj}(A + B)$ is therefore 0. However, the $(1, 1)$ -entry of $\text{adj}(A)$ is -3 , and the $(1, 1)$ -entry of $\text{adj}(B)$ is also -3 . But $(-3) + (-3) \neq 0$. Many other examples abound, of course.

6. **FALSE.** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and let $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Then $AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$. We compute

$$\text{adj}(AB) = \begin{bmatrix} cf + dh & -(af + bh) \\ -(ce + dg) & ae + bg \end{bmatrix} \neq \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} h & -f \\ -g & e \end{bmatrix} = \text{adj}(A)\text{adj}(B).$$

7. **TRUE.** This can be immediately deduced by substituting I_n for A in Theorem 3.3.16.

Problems:

From this point on, $\text{CET}(\text{col}\#n)$ will mean that the **Cofactor Expansion Theorem** has been applied to column n of the determinant, and $\text{CET}(\text{row}\#n)$ will mean that the **Cofactor Expansion Theorem** has been applied to row n of the determinant.

1. Minors: $M_{11} = 4$, $M_{21} = -3$, $M_{12} = 2$, $M_{22} = 1$;
Cofactors: $C_{11} = 4$, $C_{21} = 3$, $C_{12} = -2$, $C_{22} = 1$.

2. Minors: $M_{11} = -9$, $M_{21} = -7$, $M_{31} = -2$, $M_{12} = 7$, $M_{22} = 1$, $M_{32} = -2$, $M_{13} = 5$, $M_{23} = 3$, $M_{33} = 2$;
Cofactors: $C_{11} = -9$, $C_{21} = 7$, $C_{31} = -2$, $C_{12} = -7$, $C_{22} = 1$, $C_{32} = 2$, $C_{13} = 5$, $C_{23} = -3$, $C_{33} = 2$.

3. Minors: $M_{11} = -5$, $M_{21} = 47$, $M_{31} = 3$, $M_{12} = 0$, $M_{22} = -2$, $M_{32} = 0$, $M_{13} = 4$, $M_{23} = -38$, $M_{33} = -2$;
Cofactors: $C_{11} = -5$, $C_{21} = -47$, $C_{31} = 3$, $C_{12} = 0$, $C_{22} = -2$, $C_{32} = 0$, $C_{13} = 4$, $C_{23} = 38$, $C_{33} = -2$.

4.

$$\begin{aligned} M_{12} &= \begin{vmatrix} 3 & 1 & 2 \\ 7 & 4 & 6 \\ 5 & 1 & 2 \end{vmatrix} = -4, & M_{31} &= \begin{vmatrix} 3 & -1 & 2 \\ 4 & 1 & 2 \\ 0 & 1 & 2 \end{vmatrix} = 16, \\ M_{23} &= \begin{vmatrix} 1 & 3 & 2 \\ 7 & 1 & 6 \\ 5 & 0 & 2 \end{vmatrix} = 40, & M_{42} &= \begin{vmatrix} 1 & -1 & 2 \\ 3 & 1 & 2 \\ 7 & 4 & 6 \end{vmatrix} = 12, \end{aligned}$$

$$C_{12} = 4, C_{31} = 16, C_{23} = -40, C_{42} = 12.$$

5. $\begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix} = 1 \cdot |3| + 2 \cdot |1| = 5.$

6. $\begin{vmatrix} -1 & 2 & 3 \\ 1 & 4 & -2 \\ 3 & 1 & 4 \end{vmatrix} = 3 \cdot \begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix} + 2 \cdot \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} + 4 \cdot \begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix} = 3(-11) + 2(-7) + 4(-6) = -71.$

7. $\begin{vmatrix} 2 & 1 & -4 \\ 7 & 1 & 3 \\ 1 & 5 & -2 \end{vmatrix} = -7 \cdot \begin{vmatrix} 1 & -4 \\ 5 & -2 \end{vmatrix} + \begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} - 3 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} = -7 \cdot 18 + 0 - 3 \cdot 9 = -153.$

8. $\begin{vmatrix} 3 & 1 & 4 \\ 7 & 1 & 2 \\ 2 & 3 & -5 \end{vmatrix} = 3 \cdot \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} - \begin{vmatrix} 1 & 4 \\ 3 & -5 \end{vmatrix} + 2 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 2 \end{vmatrix} = 3(-11) - 7(-17) + 2(-2) = 82.$

$$9. \begin{vmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{vmatrix} = 3 \cdot 10 + 5(-6) + 0 \cdot 4 = 0.$$

$$10. \begin{vmatrix} 1 & -2 & 3 & 0 \\ 4 & 0 & 7 & -2 \\ 0 & 1 & 3 & 4 \\ 1 & 5 & -2 & 0 \end{vmatrix} \stackrel{1}{=} -2 \cdot \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 3 \\ 1 & 5 & -2 \end{vmatrix} - 4 \cdot \begin{bmatrix} 1 & -2 & 3 \\ 4 & 0 & 7 \\ 1 & 5 & -2 \end{bmatrix} = -2 \cdot (-26) - 4 \cdot (-5) = 72.$$

1. CET(col#4)

$$11. \begin{vmatrix} 1 & 0 & -2 \\ 3 & 1 & -1 \\ 7 & 2 & 5 \end{vmatrix} \stackrel{1}{=} \begin{vmatrix} 1 & -1 \\ 2 & 5 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 1 \\ 7 & 2 \end{vmatrix} = 7 - 2(-1) = 9. \quad \boxed{1. \text{ CET(row\#1)}}$$

$$12. \begin{vmatrix} -1 & 2 & 3 \\ 0 & 1 & 4 \\ 2 & -1 & 3 \end{vmatrix} \stackrel{1}{=} -1 \cdot \begin{vmatrix} 1 & 4 \\ -1 & 3 \end{vmatrix} + 2 \cdot \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = -7 + 10 = 3. \quad \boxed{1. \text{ CET(col\#1)}}$$

$$13. \begin{vmatrix} 2 & -1 & 3 \\ 5 & 2 & 1 \\ 3 & -3 & 7 \end{vmatrix} \stackrel{1}{=} 2 \cdot \begin{vmatrix} 2 & 1 \\ -3 & 7 \end{vmatrix} - 5 \cdot \begin{vmatrix} -1 & 3 \\ -3 & 7 \end{vmatrix} + 3 \cdot \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} = 2 \cdot 17 - 5 \cdot 2 + 3(-7) = 3.$$

1. CET(col#1)

$$14. \begin{vmatrix} 0 & -2 & 1 \\ 2 & 0 & -3 \\ -1 & 3 & 0 \end{vmatrix} \stackrel{1}{=} 0 \cdot \begin{vmatrix} 0 & -3 \\ 3 & 0 \end{vmatrix} - 2 \cdot \begin{vmatrix} -2 & 1 \\ 3 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} -2 & 1 \\ 0 & -3 \end{vmatrix} = 0 + 6 - 6 = 0.$$

1. CET(col#1)

$$15. \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix} \stackrel{1}{=} 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{vmatrix} = -2 - 2 = -4.$$

1. CET(col#1)

$$16. \begin{vmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 3 \\ 0 & 2 & -1 & 0 \\ 1 & 3 & -2 & 4 \end{vmatrix} \stackrel{1}{=} \begin{vmatrix} 2 & 5 & 3 & 1 \\ 1 & 0 & -2 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & -2 & 4 \end{vmatrix} \stackrel{2}{=} - \begin{vmatrix} 2 & 5 & 1 \\ 1 & 0 & 3 \\ 1 & -1 & 4 \end{vmatrix} \\ = \begin{vmatrix} 5 & 1 \\ -1 & 4 \end{vmatrix} + 3 \cdot \begin{vmatrix} 2 & 5 \\ 1 & -1 \end{vmatrix} = 21 - 21 = 0.$$

1. CA₃₂(2) 2. CET(row#3)

17.

$$\begin{vmatrix} 3 & 5 & 2 & 6 \\ 2 & 3 & 5 & -5 \\ 7 & 5 & -3 & -16 \\ 9 & -6 & 27 & -12 \end{vmatrix} \stackrel{1}{=} \begin{vmatrix} 1 & 2 & -3 & 11 \\ 2 & 3 & 5 & -5 \\ 7 & 5 & -3 & -16 \\ 9 & -6 & 27 & -12 \end{vmatrix} \stackrel{2}{=} \begin{vmatrix} 1 & 2 & -3 & 11 \\ 0 & -1 & 11 & -27 \\ 0 & -9 & 18 & -93 \\ 0 & -24 & 54 & -111 \end{vmatrix} \\ \stackrel{3}{=} \begin{vmatrix} -1 & 11 & -27 \\ -9 & 18 & -93 \\ -24 & 54 & -111 \end{vmatrix} \stackrel{4}{=} \begin{vmatrix} -1 & 11 & -27 \\ 0 & -81 & 150 \\ 0 & -210 & 537 \end{vmatrix} \stackrel{5}{=} - \begin{vmatrix} -81 & 150 \\ -210 & 537 \end{vmatrix} = 11997.$$

- | | | |
|------------------------------|---|---------------|
| 1. $A_{21}(-1)$ | 2. $A_{12}(-2), A_{13}(-7), A_{14}(-9)$ | 3. CET(col#1) |
| 4. $A_{12}(-9), A_{13}(-24)$ | 5. CET(col#1) | |

18.

$$\begin{vmatrix} 2 & -7 & 4 & 3 \\ 5 & 5 & -3 & 7 \\ 6 & 2 & 6 & 3 \\ 4 & 2 & -4 & 5 \end{vmatrix} \stackrel{1}{=} \begin{vmatrix} 2 & -7 & 4 & 3 \\ 1 & 3 & 1 & 2 \\ 6 & 2 & 6 & 3 \\ 4 & 2 & -4 & 5 \end{vmatrix} \stackrel{2}{=} \begin{vmatrix} 0 & -13 & 2 & -1 \\ 1 & 3 & 1 & 2 \\ 0 & -16 & 0 & -9 \\ 0 & -10 & -8 & -3 \end{vmatrix} \stackrel{3}{=} - \begin{vmatrix} -13 & 2 & -1 \\ -16 & 0 & -9 \\ -10 & -8 & -3 \end{vmatrix} \\ \stackrel{4}{=} \begin{vmatrix} 13 & -2 & 1 \\ -16 & 0 & -9 \\ -10 & -8 & -3 \end{vmatrix} \stackrel{5}{=} \begin{vmatrix} 13 & -2 & 1 \\ 101 & -18 & 0 \\ 29 & -14 & 0 \end{vmatrix} \stackrel{6}{=} \begin{vmatrix} 101 & -18 \\ 29 & 14 \end{vmatrix} = -892.$$

- | | | | |
|---------------------------|---|---------------|-------|
| 1. $A_{42}(-1)$ | 2. $A_{21}(-2), A_{23}(-6), A_{24}(-4)$ | 3. CET(col#1) | 4. P2 |
| 5. $A_{12}(9), A_{13}(3)$ | 6. CET(col#3) | | |

19.

$$\begin{vmatrix} 2 & 0 & -1 & 3 & 0 \\ 0 & 3 & 0 & 1 & 2 \\ 0 & 1 & 3 & 0 & 4 \\ 1 & 0 & 1 & -1 & 0 \\ 3 & 0 & 2 & 0 & 5 \end{vmatrix} \stackrel{1}{=} \begin{vmatrix} 0 & 0 & -3 & 5 & 0 \\ 0 & 3 & 0 & 1 & 2 \\ 0 & 1 & 3 & 0 & 4 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 3 & 5 \end{vmatrix} \stackrel{2}{=} - \begin{vmatrix} 0 & -3 & 5 & 0 \\ 3 & 0 & 1 & 2 \\ 1 & 3 & 0 & 4 \\ 0 & -1 & 3 & 5 \end{vmatrix} \\ \stackrel{3}{=} - \begin{vmatrix} 0 & -3 & 5 & 0 \\ 0 & -9 & 1 & -10 \\ 1 & 3 & 0 & 4 \\ 0 & -1 & 3 & 5 \end{vmatrix} \stackrel{4}{=} - \begin{vmatrix} -3 & 5 & 0 \\ -9 & 1 & -10 \\ -1 & 3 & 5 \end{vmatrix} \stackrel{5}{=} \begin{vmatrix} -3 & 5 & 0 \\ -9 & 1 & -10 \\ 1 & -3 & -5 \end{vmatrix} \\ \stackrel{6}{=} \begin{vmatrix} 42 & 0 & 50 \\ -9 & 1 & -10 \\ -26 & 0 & -35 \end{vmatrix} \stackrel{7}{=} \begin{vmatrix} 42 & 50 \\ -26 & -35 \end{vmatrix} = -170.$$

- | | | | |
|-----------------------------|----------------------------|-----------------|---------------|
| 1. $A_{41}(-2), A_{45}(-3)$ | 2. CET(col#1) | 3. $A_{12}(-3)$ | 4. CET(col#1) |
| 5. P2 | 6. $A_{21}(-5), A_{23}(3)$ | 7. CET(col#2) | |

20.

$$\begin{vmatrix} 0 & x & y & z \\ -x & 0 & 1 & -1 \\ -y & -1 & 0 & 1 \\ -z & 1 & -1 & 0 \end{vmatrix} \stackrel{1}{=} \begin{vmatrix} xz & 0 & x+y & z \\ -x & 0 & 1 & -1 \\ -y-z & 0 & -1 & 1 \\ -z & 1 & -1 & 0 \end{vmatrix} \stackrel{2}{=} \begin{vmatrix} xz & x+y & z \\ -x & 1 & -1 \\ -y-z & -1 & 1 \end{vmatrix}$$

$$\stackrel{3}{=} \begin{vmatrix} xz - yz + z^2 & x + y + z & 0 \\ -x + y - z & 0 & 0 \\ -y - z & -1 & 1 \end{vmatrix} \stackrel{4}{=} \begin{vmatrix} xz - yz + z^2 & x + y + z \\ -x - y - z & 0 \end{vmatrix} = (x + y + z)^2.$$

1. $A_{41}(-x), A_{43}(1)$ 2. CET(col#2) 3. $A_{32}(1), A_{31}(-z)$ 4. CET(col#3)

21.

(a):

$$\begin{aligned} V(r_1, r_2, r_3) &= \begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{vmatrix} \stackrel{1}{=} \begin{vmatrix} 1 & 0 & 0 \\ r_1 & r_2 - r_1 & r_3 - r_1 \\ r_1^2 & r_2^2 - r_1^2 & r_3^2 - r_1^2 \end{vmatrix} \\ &\stackrel{2}{=} \begin{vmatrix} r_2 - r_1 & r_3 - r_1 \\ r_2^2 - r_1^2 & r_3^2 - r_1^2 \end{vmatrix} = (r_2 - r_1)(r_3^2 - r_1^2) - (r_3 - r_1)(r_2^2 - r_1^2) \\ &= (r_3 - r_1)(r_2 - r_1)[(r_3 + r_1) - (r_2 + r_1)] = (r_2 - r_1)(r_3 - r_1)(r_3 - r_2). \end{aligned}$$

1. $CA_{12}(-1), CA_{13}(-1)$ 2. CET(row#1)

(b): We use mathematical induction. The result is vacuously true when $n = 1$ and quickly verified when $n = 2$. Suppose that the result is true when $n = k - 1$, for $k \geq 3$, and consider

$$V(r_1, r_2, \dots, r_k) = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ r_1 & r_2 & r_3 & \dots & r_k \\ r_1^2 & r_2^2 & r_3^2 & \dots & r_k^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_1^{k-1} & r_2^{k-1} & r_3^{k-1} & \dots & r_k^{k-1} \end{vmatrix}.$$

The determinant vanishes when $r_k = r_1, r_2, \dots, r_{k-1}$, so we can write

$$V(r_1, r_2, \dots, r_k) = a(r_1, r_2, \dots, r_k) \prod_{i=1}^{n-1} (r_k - r_i),$$

where $a(r_1, r_2, \dots, r_k)$ is the coefficient of r_k^{k-1} in the expansion of $V(r_1, r_2, \dots, r_k)$. However, using the Cofactor Expansion Theorem along column k , we see that this coefficient is just $V(r_1, r_2, \dots, r_{k-1})$, so by hypothesis,

$$a(r_1, r_2, \dots, r_k) = V(r_1, r_2, \dots, r_{k-1}) = \prod_{1 \leq i < m \leq n-1} (r_m - r_i).$$

Thus,

$$V(r_1, r_2, \dots, r_k) = \prod_{1 \leq i < m \leq n-1} (r_m - r_i) \prod_{i=1}^{n-1} (r_k - r_i) = \prod_{1 \leq i < m \leq n} (r_m - r_i).$$

Hence the result is true for $n = k$, so, by induction, is true for all non-negative integers n .

22.

(a): $\det(A) = 11$;

$$(b): M_C = \begin{bmatrix} 5 & -4 \\ -1 & 3 \end{bmatrix};$$

$$(c): \text{adj}(A) = \begin{bmatrix} 5 & -1 \\ -4 & 3 \end{bmatrix};$$

$$(d): A^{-1} = \frac{1}{11} \begin{bmatrix} 5 & -1 \\ -4 & 3 \end{bmatrix}.$$

23.

$$(a): \det(A) = 7;$$

$$(b): M_C = \begin{bmatrix} 1 & -4 \\ 2 & -1 \end{bmatrix};$$

$$(c): \text{adj}(A) = \begin{bmatrix} 1 & 2 \\ -4 & -1 \end{bmatrix};$$

$$(d): A^{-1} = \frac{1}{7} \begin{bmatrix} 1 & 2 \\ -4 & -1 \end{bmatrix}.$$

24.

$$(a): \det(A) = 0;$$

$$(b): M_C = \begin{bmatrix} -6 & 15 \\ -2 & 5 \end{bmatrix};$$

$$(c): \text{adj}(A) = \begin{bmatrix} -6 & -2 \\ 15 & 5 \end{bmatrix};$$

$$(d): A^{-1} \text{ does not exist because } \det(A) = 0.$$

25.

$$(a): \det(A) = \begin{vmatrix} 2 & -3 & 0 \\ 2 & 1 & 5 \\ 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -3 & 0 \\ 0 & 4 & 5 \\ 0 & -1 & 2 \end{vmatrix} = 2 \cdot 13 = 26;$$

$$(b): M_C = \begin{bmatrix} 7 & -4 & -2 \\ 6 & 4 & 2 \\ -15 & -10 & 8 \end{bmatrix};$$

$$(c): \text{adj}(A) = \begin{bmatrix} 7 & 6 & -15 \\ -4 & 4 & -10 \\ -2 & 2 & 8 \end{bmatrix};$$

$$(d): A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{26} \begin{bmatrix} 7 & 6 & -15 \\ -4 & 4 & -10 \\ -2 & 2 & 8 \end{bmatrix}.$$

26.

$$(a): \det(A) = \begin{vmatrix} -2 & 3 & -1 \\ 2 & 1 & 5 \\ 0 & 2 & 3 \end{vmatrix} = \begin{vmatrix} -2 & 3 & -1 \\ 0 & 4 & 4 \\ 0 & 2 & 3 \end{vmatrix} = -2 \cdot 4 = -8;$$

$$(b): M_C = \begin{bmatrix} -7 & -6 & 4 \\ -11 & -6 & 4 \\ 16 & 8 & -8 \end{bmatrix};$$

$$(c): \text{adj}(A) = \begin{bmatrix} -7 & -11 & 16 \\ -6 & -6 & 8 \\ 4 & 4 & -8 \end{bmatrix};$$

$$(d): A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = -\frac{1}{8} \begin{bmatrix} -7 & -11 & 16 \\ -6 & -6 & 8 \\ 4 & 4 & -8 \end{bmatrix}.$$

27.

$$(a): \det(A) = \begin{vmatrix} 1 & -1 & 2 \\ 3 & -1 & 4 \\ 5 & 1 & 7 \end{vmatrix} = \begin{vmatrix} 6 & 0 & 9 \\ 8 & 0 & 11 \\ 5 & 1 & 7 \end{vmatrix} = 6;$$

$$(b): M_C = \begin{bmatrix} -11 & -1 & 8 \\ 9 & -3 & -6 \\ -2 & 2 & 2 \end{bmatrix};$$

$$(c): \text{adj}(A) = \begin{bmatrix} -11 & 9 & -2 \\ -1 & -3 & 2 \\ 8 & -6 & 2 \end{bmatrix};$$

$$(d): A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{6} \begin{bmatrix} -11 & 9 & -2 \\ -1 & -3 & 2 \\ 8 & -6 & 2 \end{bmatrix} = \begin{bmatrix} -11/6 & 3/2 & -1/3 \\ -1/6 & -1/2 & 1/3 \\ 4/3 & -1 & 1/3 \end{bmatrix}.$$

28.

$$(a): \det(A) = \begin{vmatrix} 0 & 1 & 2 \\ -1 & -1 & 3 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 \\ 0 & -3 & 4 \\ 1 & -2 & 1 \end{vmatrix} = 10;$$

$$(b): M_C = \begin{bmatrix} 5 & 4 & 3 \\ -5 & -2 & 1 \\ 5 & -2 & 1 \end{bmatrix};$$

$$(c): \text{adj}(A) = \begin{bmatrix} 5 & -5 & 5 \\ 4 & -2 & -2 \\ 3 & 1 & 1 \end{bmatrix};$$

$$(d): A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{10} \begin{bmatrix} 5 & -5 & 5 \\ 4 & -2 & -2 \\ 3 & 1 & 1 \end{bmatrix}.$$

29.

$$(a): \det(A) = \begin{vmatrix} 2 & -3 & 5 \\ 1 & 2 & 1 \\ 0 & 7 & -1 \end{vmatrix} = \begin{vmatrix} 0 & -7 & 3 \\ 1 & 2 & 1 \\ 0 & 7 & -1 \end{vmatrix} = 14;$$

$$(b): M_C = \begin{bmatrix} -9 & 1 & 7 \\ 32 & -2 & -14 \\ -13 & 3 & 7 \end{bmatrix};$$

$$(c): \operatorname{adj}(A) = \begin{bmatrix} -9 & 32 & -13 \\ 1 & -2 & 3 \\ 7 & -14 & 7 \end{bmatrix};$$

$$(d): A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{14} \begin{bmatrix} -9 & 32 & -13 \\ 1 & -2 & 3 \\ 7 & -14 & 7 \end{bmatrix} = \begin{bmatrix} -9/14 & 16/7 & -13/14 \\ 1/14 & -1/7 & 3/14 \\ 1/2 & -1 & 1/2 \end{bmatrix}.$$

30.

$$(a): \det(A) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & -2 & 0 \\ 0 & 2 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{vmatrix} = 16;$$

$$(b): M_C = \begin{bmatrix} 4 & 4 & 4 & 4 \\ -4 & 4 & -4 & 4 \\ 4 & 4 & -4 & -4 \\ -4 & 4 & 4 & -4 \end{bmatrix};$$

$$(c): \operatorname{adj}(A) = \begin{bmatrix} 4 & -4 & 4 & -4 \\ 4 & 4 & 4 & 4 \\ 4 & -4 & -4 & 4 \\ 4 & 4 & -4 & -4 \end{bmatrix};$$

$$(d): A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{16} \begin{bmatrix} 4 & -4 & 4 & -4 \\ 4 & 4 & 4 & 4 \\ 4 & -4 & -4 & 4 \\ 4 & 4 & -4 & -4 \end{bmatrix}.$$

31.

$$(a): \det(A) = \begin{vmatrix} 1 & 0 & 3 & 5 \\ -2 & 1 & 1 & 3 \\ 3 & 9 & 0 & 2 \\ 2 & 0 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 11 & 0 & 18 & 5 \\ 4 & 1 & 10 & 3 \\ 7 & 9 & 6 & 2 \\ 0 & 0 & 0 & -1 \end{vmatrix} = - \begin{vmatrix} 11 & 0 & 18 \\ 4 & 1 & 10 \\ 7 & 9 & 6 \end{vmatrix} = - \begin{vmatrix} 11 & 0 & 18 \\ 4 & 1 & 10 \\ -29 & 0 & -84 \end{vmatrix} = 402;$$

$$(b): M_C = \begin{bmatrix} 84 & -46 & -29 & 81 \\ -162 & 60 & 99 & -27 \\ 18 & 38 & -11 & 3 \\ -30 & 26 & 130 & -72 \end{bmatrix};$$

$$(c): \operatorname{adj}(A) = \begin{bmatrix} 84 & -162 & 18 & -30 \\ -46 & 60 & 38 & 26 \\ -29 & 99 & -11 & 130 \\ 81 & -27 & 3 & -72 \end{bmatrix};$$

(d):

$$\begin{aligned}
 A^{-1} &= \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{402} \begin{bmatrix} 84 & -162 & 18 & -30 \\ -46 & 60 & 38 & 26 \\ -29 & 99 & -11 & 130 \\ 81 & -27 & 3 & -72 \end{bmatrix} \\
 &= \begin{bmatrix} 14/67 & -27/67 & 3/67 & -5/67 \\ -23/201 & 10/67 & 19/201 & 13/201 \\ -29/402 & 33/134 & -11/402 & 65/201 \\ 27/134 & -9/134 & 1/134 & -12/67 \end{bmatrix}.
 \end{aligned}$$

32.

(a): Using the Cofactor Expansion Theorem along row 1 yields

$$\begin{aligned}
 \det(A) &= (1 + 2x^2) + 2x(2x + 4x^3) + 2x^2(2x^2 + 4x^4) \\
 &= 8x^6 + 12x^4 + 6x^2 + 1 = (1 + 2x^2)^3.
 \end{aligned}$$

(b):

$$\begin{aligned}
 M_C &= \begin{bmatrix} 1 + 2x^2 & -(2x + 4x^3) & 2x^2 + 4x^4 \\ 2x + 4x^3 & 1 - 4x^4 & -(2x + 4x^3) \\ 2x^2 + 4x^4 & 2x + 4x^3 & 1 + 2x^2 \end{bmatrix} \\
 &= \begin{bmatrix} 1 + 2x^2 & -2x(1 + 2x^2) & 2x^2(1 + 2x^2) \\ 2x(1 + 2x^2) & (1 + 2x^2)(1 - 2x^2) & -2x(1 + 2x^2) \\ 2x^2(1 + 2x^2) & 2x(1 + 2x^2) & 1 + 2x^2 \end{bmatrix},
 \end{aligned}$$

so that

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{(1 + 2x^2)^2} \begin{bmatrix} 1 & 2x & 2x^2 \\ -2x & 1 - 2x^2 & 2x \\ 2x^2 & -2x & 1 \end{bmatrix}.$$

$$\mathbf{33.} \quad \det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1,$$

and

$$C_{23} = - \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -(2 - 1) = -1.$$

Thus,

$$(A^{-1})_{32} = \frac{(\operatorname{adj}(A))_{32}}{\det(A)} = \frac{C_{23}}{\det(A)} = -1.$$

$$\mathbf{34.} \quad \det(A) = \begin{vmatrix} 2 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 4 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{vmatrix} = - \begin{vmatrix} 4 & 1 \\ 3 & -1 \end{vmatrix} = -(-4 - 3) = 7,$$

and

$$C_{13} = \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = -2 - 3 = -5.$$

Thus,

$$(A^{-1})_{31} = \frac{(\text{adj}(A))_{31}}{\det(A)} = \frac{C_{13}}{\det(A)} = -\frac{5}{7}.$$

35.

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 3 \\ 0 & 1 & -1 & 2 \\ -1 & 1 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & -1 & 3 \\ 0 & 1 & -1 & 2 \\ -1 & 1 & 3 & 0 \end{vmatrix} = \begin{vmatrix} -1 & -1 & 3 \\ 1 & -1 & 2 \\ 1 & 3 & 0 \end{vmatrix} \\ &= \begin{vmatrix} -1 & 2 & 3 \\ 1 & -4 & 2 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ -4 & 2 \end{vmatrix} = 4 + 12 = 16, \end{aligned}$$

and

$$C_{32} = - \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ -1 & 2 & 0 \end{vmatrix} = -(-3) \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = 3(2 + 1) = 9.$$

Thus,

$$(A^{-1})_{23} = \frac{(\text{adj}(A))_{23}}{\det(A)} = \frac{C_{32}}{\det(A)} = \frac{9}{16}.$$

36. $M_C = \begin{bmatrix} 2e^{2t} & -2e^t \\ -e^{2t} & 3e^t \end{bmatrix}$, $\text{adj}(A) = \begin{bmatrix} 2e^{2t} & -e^{2t} \\ -2e^t & 3e^t \end{bmatrix}$, and $\det(A) = 4e^{3t}$, so

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{4e^{3t}} \begin{bmatrix} 2e^{2t} & -e^{2t} \\ -2e^t & 3e^t \end{bmatrix}.$$

37. $M_C = \begin{bmatrix} e^{-t} \sin(2t) & -e^t \cos(2t) \\ -e^t \cos(2t) & e^t \sin(2t) \end{bmatrix}$, $\text{adj}(A) = \begin{bmatrix} e^{-t} \sin(2t) & e^t \cos(2t) \\ -e^t \cos(2t) & e^t \sin(2t) \end{bmatrix}$, and

$$\det(A) = \sin^2(2t) + \cos^2(2t) = 1,$$

so

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \begin{bmatrix} e^{-t} \sin(2t) & e^t \cos(2t) \\ -e^t \cos(2t) & e^t \sin(2t) \end{bmatrix}.$$

38. $M_C = \begin{bmatrix} 3te^{-t} & -e^{-t} & -te^{2t} \\ -te^{-t} & e^{-t} & 0 \\ -te^{-t} & 0 & te^{2t} \end{bmatrix}$, $\text{adj}(A) = \begin{bmatrix} 3te^{-t} & -te^{-t} & -te^{-t} \\ -e^{-t} & e^{-t} & 0 \\ -te^{2t} & 0 & te^{2t} \end{bmatrix}$, and

$$\det(A) = \begin{vmatrix} e^t & 2te^t & e^{-2t} \\ e^t & 2te^t & e^{-2t} \\ e^t & te^t & 2e^{-2t} \end{vmatrix} = \begin{vmatrix} e^t & te^t & e^{-2t} \\ 0 & te^t & 0 \\ 0 & 0 & e^{-2t} \end{vmatrix} = e^t(te^{-t}) = t,$$

so

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{t} \begin{bmatrix} 3te^{-t} & -te^{-t} & -te^{-t} \\ -e^{-t} & e^{-t} & 0 \\ -te^{2t} & 0 & te^{2t} \end{bmatrix}.$$

39. $M_C = \begin{bmatrix} -1 & 2 & -1 \\ 3 & -6 & 3 \\ -2 & 4 & -2 \end{bmatrix}$, $\text{adj}(A) = \begin{bmatrix} -1 & 3 & -2 \\ 2 & -6 & 4 \\ -1 & 3 & -2 \end{bmatrix}$. Hence,

$$A \cdot \text{adj}(A) = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 3 & -2 \\ 2 & -6 & 4 \\ -1 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0_3.$$

From Equation (3.3.4) of the text we have that, in general, $A \cdot \text{adj}(A) = \det(A) \cdot I_n$. Since, for the given matrix, $A \cdot \text{adj}(A) = 0_3$, we must have $\det(A) = 0$.

40. $\det(A) = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 7$, $\det(B_1) = \begin{vmatrix} 2 & -3 \\ 4 & 2 \end{vmatrix} = 16$, and $\det(B_2) = \begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix} = 6$. Thus,

$$x_1 = \frac{\det(B_1)}{\det(A)} = \frac{16}{7} \quad \text{and} \quad x_2 = \frac{\det(B_2)}{\det(A)} = \frac{6}{7}.$$

Solution: $(16/7, 6/7)$.

41.

$$\begin{aligned} \det(A) &= \begin{vmatrix} 3 & -2 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 6, \\ \det(B_1) &= \begin{vmatrix} 4 & -2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 9, \\ \det(B_2) &= \begin{vmatrix} 3 & 4 & 1 \\ 1 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 0, \\ \det(B_3) &= \begin{vmatrix} 3 & -2 & 4 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = -3. \end{aligned}$$

Thus, $x_1 = \frac{\det(B_1)}{\det(A)} = \frac{9}{6}$, $x_2 = \frac{\det(B_2)}{\det(A)} = 0$, and $x_3 = \frac{\det(B_3)}{\det(A)} = -\frac{3}{6}$.

Solution: $(3/2, 0, -1/2)$.

42. $\det(A) = \begin{vmatrix} 1 & -3 & 1 \\ 1 & 4 & -1 \\ 2 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 1 \\ 0 & 7 & -2 \\ 0 & 7 & -5 \end{vmatrix} = \begin{vmatrix} 7 & -2 \\ 7 & -5 \end{vmatrix} = -35 + 14 = -21 \neq 0$.

Thus, the system has only the trivial solution.

43.

$$\begin{aligned}
\det(A) &= \begin{vmatrix} 1 & -2 & 3 & -1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} -5 & -2 & 3 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \\ 4 & 1 & -2 & 1 \end{vmatrix} \\
&= - \begin{vmatrix} -5 & -2 & -1 \\ 1 & 1 & -1 \\ 4 & 1 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & -1 & 0 \\ 5 & 2 & 0 \\ 4 & 1 & 1 \end{vmatrix} = -3, \\
\det(B_1) &= \begin{vmatrix} 1 & -2 & 3 & -1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 3 & 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 3 & -1 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 3 & 2 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -3 & 3 \\ 2 & 0 & 1 \\ 3 & 2 & -2 \end{vmatrix} = \begin{vmatrix} -5 & -3 & 3 \\ 0 & 0 & 1 \\ 7 & 2 & -2 \end{vmatrix} = -11, \\
\det(B_2) &= \begin{vmatrix} 1 & 1 & 3 & -1 \\ 2 & 2 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 3 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 3 & 0 \\ 2 & 2 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 1 & 2 \\ 3 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 \\ -4 & 5 & 0 \\ 3 & -2 & 1 \end{vmatrix} = 17, \\
\det(B_3) &= \begin{vmatrix} 1 & -2 & 1 & -1 \\ 2 & 0 & 2 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 0 & -2 & 1 & -1 \\ 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & -1 \\ -3 & 1 & 3 & 1 \end{vmatrix} \\
&= -2 \begin{vmatrix} 0 & -2 & -1 \\ 1 & 1 & -1 \\ -3 & 1 & 1 \end{vmatrix} = -2 \begin{vmatrix} -3 & -1 & 0 \\ -2 & 2 & 0 \\ -3 & 1 & 1 \end{vmatrix} = -2(-8) = 16, \\
\det(B_4) &= \begin{vmatrix} 1 & -2 & 3 & 1 \\ 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -2 & 3 & 1 \\ 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & -2 & 3 \end{vmatrix} = - \begin{vmatrix} 3 & 3 & 1 \\ 2 & 1 & 2 \\ -1 & -2 & 3 \end{vmatrix} \\
&= - \begin{vmatrix} -3 & 3 & -5 \\ 0 & 1 & 0 \\ 3 & 2 & 7 \end{vmatrix} = - \begin{vmatrix} -3 & -5 \\ 3 & 7 \end{vmatrix} = 6.
\end{aligned}$$

Therefore,

$$x_1 = \frac{11}{3}, \quad x_2 = -\frac{17}{3}, \quad x_3 = -\frac{16}{3}, \quad \text{and} \quad x_4 = -2.$$

Solution: $(11/3, -17/3, -16/3, -2)$.**44.**

$$\begin{aligned}
\det(A) &= \begin{vmatrix} e^t & e^{-2t} \\ e^t & -2e^{-2t} \end{vmatrix} = e^t e^{-2t} \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -3e^{-t}, \\
\det(B_1) &= \begin{vmatrix} 3 \sin t & e^{-2t} \\ 4 \cos t & -2e^{-2t} \end{vmatrix} = e^{-2t} \begin{vmatrix} 3 \sin t & 1 \\ 4 \cos t & -2 \end{vmatrix} = -2e^{-2t}(3 \sin t + 2 \cos t), \\
\det(B_2) &= \begin{vmatrix} e^t & 3 \sin t \\ e^t & 4 \cos t \end{vmatrix} = e^t \begin{vmatrix} 1 & 3 \sin t \\ 1 & 4 \cos t \end{vmatrix} = e^t(4 \cos t - 3 \sin t).
\end{aligned}$$

Thus, $x_1 = \frac{\det(B_1)}{\det(A)} = \frac{2(3 \sin t + 2 \cos t)}{3e^t}$ and $x_2 = \frac{\det(B_2)}{\det(A)} = \frac{e^{2t}(3 \sin t - 4 \cos t)}{3}$.

Solution: $\left(\frac{2}{3}e^{-t}(3 \sin t + 2 \cos t), \frac{1}{3}e^{2t}(3 \sin t - 4 \cos t) \right)$.

45.

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 & 4 & -2 & 1 \\ 2 & 9 & -3 & -2 \\ 1 & 5 & 0 & -1 \\ 3 & 14 & 7 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -2 & 2 \\ 2 & -1 & -3 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & -1 & 7 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -2 & 2 \\ -1 & -3 & 0 \\ -1 & 7 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -16 & 0 \\ -1 & -3 & 0 \\ -1 & 7 & 1 \end{vmatrix} = -19, \\ \det(B_2) &= \begin{vmatrix} 1 & 2 & -2 & 1 \\ 2 & 5 & -3 & -2 \\ 1 & 3 & 0 & -1 \\ 3 & 6 & 7 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -2 & 2 \\ 2 & -1 & -3 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & -3 & 7 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -2 & 2 \\ -1 & -3 & 0 \\ -3 & 7 & 1 \end{vmatrix} = \begin{vmatrix} 5 & -16 & 0 \\ -1 & -3 & 0 \\ -3 & 7 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 5 & -16 \\ -1 & -3 \end{vmatrix} = -31. \text{ Therefore } x_2 = \frac{\det(B_2)}{\det(A)} = \frac{31}{19}. \end{aligned}$$

46.

$$\begin{aligned} \det(A) &= \begin{vmatrix} b+c & a & a \\ b & a+c & b \\ c & c & a+b \end{vmatrix} = \begin{vmatrix} -a+b+c & a & a \\ -a+b-c & a+c & b \\ 0 & c & a+b \end{vmatrix} \\ &= -c \begin{vmatrix} -a+b+c & a \\ -a+b-c & b \end{vmatrix} + (a+b) \begin{vmatrix} -a+b+c & a \\ -a+b-c & a+c \end{vmatrix} = 4abc, \\ \det(B_1) &= \begin{vmatrix} a & a & a \\ b & a+c & b \\ c & c & a+b \end{vmatrix} = \begin{vmatrix} a & 0 & a \\ b & a-b+c & b \\ c & 0 & a+b \end{vmatrix} \\ &= (a-b+c) \begin{vmatrix} a & a \\ c & a+b \end{vmatrix} = a(a-b+c)(a+b-c), \\ \det(B_2) &= \begin{vmatrix} b+c & a & a \\ b & b & b \\ c & c & a+b \end{vmatrix} = \begin{vmatrix} b+c & a-b-c & a \\ b & 0 & b \\ c & 0 & a+b \end{vmatrix} \\ &= -(a-b-c) \begin{vmatrix} b & b \\ c & a+b \end{vmatrix} = -b(a-b-c)(a+b-c), \\ \det(B_3) &= \begin{vmatrix} b+c & a & a \\ b & a+c & b \\ c & c & c \end{vmatrix} = \begin{vmatrix} -a+b+c & a & a \\ 0 & a+c & b \\ 0 & c & c \end{vmatrix} \\ &= (-a+b+c) \begin{vmatrix} a+c & b \\ c & c \end{vmatrix} = -c(a-b-c)(a-b+c). \end{aligned}$$

Case 1: If $a \neq 0$, $b \neq 0$, and $c \neq 0$, then $\det(A) \neq 0$, so it follows by Theorem 3.2.4 that the given system of equations has a unique solution. The solution in this case is given by:

$$\begin{aligned}
x_1 &= \frac{\det(B_1)}{\det(A)} = \frac{(a-b+c)(a+b-c)}{4bc}, \\
x_2 &= \frac{\det(B_2)}{\det(A)} = \frac{-(a-b-c)(a+b-c)}{4ac}, \\
x_3 &= \frac{\det(B_3)}{\det(A)} = \frac{-(a-b-c)(a-b+c)}{4ab}.
\end{aligned}$$

Case 2: If $a = b = 0$ and $c \neq 0$, then it is easy to see from the system that the system is inconsistent. By the symmetry of the equations, the same will be true if $a = c = 0$ with $b \neq 0$, or $b = c = 0$ with $a \neq 0$.

Case 3: Suppose $a = 0$ where $b \neq 0$ and $c \neq 0$, and consider the reduced row echelon form of the augmented matrix:

$$\left[\begin{array}{ccc|c} b+c & 0 & 0 & 0 \\ c & b & b & b \\ b & c & c & c \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & b-c & b-c & b-c \\ 0 & c-b & c-b & c-b \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

From the last matrix we see that the system has an infinite number of solutions of the form $\{(0, 1-r, r) : r \in \mathbb{R}\}$. By the symmetry of the three equations, it follows that the other two cases: $b = 0$ with $a \neq 0$ and $c \neq 0$, and $c = 0$ with $a \neq 0$ and $b \neq 0$ have similar forms of solutions.

47. Let B be the matrix obtained from A by adding column i to column j ($i \neq j$) in the matrix A . By the property for columns corresponding to Property P3, we have $\det(B) = \det(A)$. Cofactor expansion of B along column j gives

$$\det(A) = \det(B) = \sum_{k=1}^n (a_{kj} + a_{ki})C_{kj} = \sum_{k=1}^n a_{kj}C_{kj} + \sum_{k=1}^n a_{ki}C_{kj}.$$

That is,

$$\det(A) = \det(A) + \sum_{k=1}^n a_{ki}C_{kj},$$

since by the Cofactor Expansion Theorem the first summation on the right-hand side is simply $\det(A)$. It follows immediately that

$$\sum_{k=1}^n a_{ki}C_{kj} = 0, \quad i \neq j.$$

51.

$$\begin{aligned}
A &= \begin{bmatrix} 1.21 & 3.42 & 2.15 \\ 5.41 & 2.32 & 7.15 \\ 21.63 & 3.51 & 9.22 \end{bmatrix}, B_1 = \begin{bmatrix} 3.25 & 3.42 & 2.15 \\ 4.61 & 2.32 & 7.15 \\ 9.93 & 3.51 & 9.22 \end{bmatrix}, \\
B_2 &= \begin{bmatrix} 1.21 & 3.25 & 2.15 \\ 5.41 & 4.61 & 7.15 \\ 21.63 & 9.93 & 9.22 \end{bmatrix}, B_3 = \begin{bmatrix} 1.21 & 3.42 & 3.25 \\ 5.41 & 2.32 & 4.61 \\ 21.63 & 3.51 & 9.93 \end{bmatrix}.
\end{aligned}$$

From Cramer's Rule,

$$x_1 = \frac{\det(B_1)}{\det(A)} \approx 0.25, \quad x_2 = \frac{\det(B_2)}{\det(A)} \approx 0.72, \quad x_3 = \frac{\det(B_3)}{\det(A)} \approx 0.22.$$

52. $\det(A) = 32$, $\det(B_1) = -3218$, $\det(B_2) = 3207$, $\det(B_3) = 2896$, $\det(B_4) = -9682$, $\det(B_5) = 2414$.
So,

$$x_1 = -\frac{1609}{16}, x_2 = \frac{3207}{32}, x_3 = \frac{181}{2}, x_4 = -\frac{4841}{32}, x_5 = \frac{1207}{16}.$$

53. We have

$$(BA)_{ji} = \sum_{k=1}^n b_{jk} a_{ki} = \sum_{k=1}^n \frac{1}{\det(A)} \cdot \operatorname{adj}(A)_{jk} \cdot a_{ki} = \frac{1}{\det(A)} \sum_{k=1}^n C_{kj} a_{ki} = \delta_{ij},$$

where we have used Equation (3.3.4) in the last step.

Solutions to Section 3.4

Problems:

1. $\begin{vmatrix} 5 & -1 \\ 3 & 7 \end{vmatrix} = 5 \cdot 7 - 3(-1) = 38.$

2. $\begin{vmatrix} 3 & 5 & 7 \\ -1 & 2 & 4 \\ 6 & 3 & -2 \end{vmatrix} = -43.$

3. $\begin{vmatrix} 5 & 1 & 4 \\ 6 & 1 & 3 \\ 14 & 2 & 7 \end{vmatrix} = -3.$

4.

$$\begin{vmatrix} 2.3 & 1.5 & 7.9 \\ 4.2 & 3.3 & 5.1 \\ 6.8 & 3.6 & 5.7 \end{vmatrix} = 2.3 \begin{vmatrix} 3.3 & 5.1 \\ 3.6 & 5.7 \end{vmatrix} - 1.5 \begin{vmatrix} 4.2 & 5.1 \\ 6.8 & 5.7 \end{vmatrix} + 7.9 \begin{vmatrix} 4.2 & 3.3 \\ 6.8 & 3.6 \end{vmatrix} \\ = 1.035 + 16.11 - 57.828 = -40.683.$$

5.

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a \begin{vmatrix} c & a \\ a & b \end{vmatrix} - b \begin{vmatrix} b & a \\ c & b \end{vmatrix} + c \begin{vmatrix} b & c \\ c & a \end{vmatrix} \\ = a(bc - a^2) - b(b^2 - ac) + c(ab - c^2) = 3abc - a^3 - b^3 - c^3.$$

6.

$$\begin{vmatrix} 3 & 5 & -1 & 2 \\ 2 & 1 & 5 & 2 \\ 3 & 2 & 5 & 7 \\ 1 & -1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 8 & -7 & -1 \\ 0 & 3 & 1 & 0 \\ 0 & 5 & -1 & 4 \\ 1 & -1 & 2 & 1 \end{vmatrix} = -1 \begin{vmatrix} 8 & -7 & -1 \\ 3 & 1 & 0 \\ 5 & -1 & 4 \end{vmatrix} \\ = -1 \begin{vmatrix} 29 & -7 & -1 \\ 0 & 1 & 0 \\ 8 & -1 & 4 \end{vmatrix} = -1 \begin{vmatrix} 29 & -1 \\ 8 & 4 \end{vmatrix} = -124.$$

7.

$$\begin{aligned}
\begin{vmatrix} 7 & 1 & 2 & 3 \\ 2 & -2 & 4 & 6 \\ 3 & -1 & 5 & 4 \\ 18 & 9 & 27 & 54 \end{vmatrix} &= 9 \begin{vmatrix} 7 & 1 & 2 & 3 \\ 2 & -2 & 4 & 6 \\ 3 & -1 & 5 & 4 \\ 2 & 1 & 3 & 6 \end{vmatrix} = 9 \begin{vmatrix} 7 & 1 & 2 & 3 \\ 16 & 0 & 8 & 12 \\ 10 & 0 & 7 & 7 \\ -5 & 0 & 1 & 3 \end{vmatrix} = -9 \begin{vmatrix} 16 & 8 & 12 \\ 10 & 7 & 7 \\ -5 & 1 & 3 \end{vmatrix} \\
&= -36 \begin{vmatrix} 4 & 2 & 3 \\ 10 & 7 & 7 \\ -5 & 1 & 3 \end{vmatrix} = -36 \begin{vmatrix} 14 & 0 & -3 \\ 45 & 0 & -14 \\ -5 & 1 & 3 \end{vmatrix} \\
&= 36 \begin{vmatrix} 14 & -3 \\ 45 & -14 \end{vmatrix} = -2196.
\end{aligned}$$

$$8. \det(A) = 11. \quad M_C = \begin{bmatrix} 7 & -2 \\ -5 & 3 \end{bmatrix} \Rightarrow \operatorname{adj}(A) = \begin{bmatrix} 7 & -5 \\ -2 & 3 \end{bmatrix}, \text{ so that}$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \begin{bmatrix} 7/11 & -5/11 \\ -2/11 & 3/11 \end{bmatrix}.$$

$$9. \det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & -5 & -7 \end{vmatrix} = -18.$$

$$M_C = \begin{bmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{bmatrix} \Rightarrow \operatorname{adj}(A) = \begin{bmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{bmatrix}, \text{ so that}$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \begin{bmatrix} -5/18 & 1/18 & 7/18 \\ 1/18 & 7/18 & -5/18 \\ 7/18 & -5/18 & 1/18 \end{bmatrix}.$$

$$10. \det(A) = \begin{vmatrix} 3 & 4 & 7 \\ 2 & 6 & 1 \\ 3 & 14 & -1 \end{vmatrix} = \begin{vmatrix} -11 & -38 & 7 \\ 0 & 0 & 1 \\ 5 & 20 & -1 \end{vmatrix} = - \begin{vmatrix} -11 & -38 \\ 5 & 20 \end{vmatrix} = 30.$$

$$M_C = \begin{bmatrix} -20 & 5 & 10 \\ 102 & -24 & -30 \\ -38 & 11 & 10 \end{bmatrix} \Rightarrow \operatorname{adj}(A) = \begin{bmatrix} -20 & 102 & -38 \\ 5 & -24 & 11 \\ 10 & -30 & 10 \end{bmatrix}, \text{ so that}$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \begin{bmatrix} -2/3 & 17/5 & -19/15 \\ 1/6 & -4/5 & 11/30 \\ 1/3 & -1 & 1/3 \end{bmatrix}.$$

$$11. \det(A) = \begin{vmatrix} 2 & 5 & 7 \\ 4 & -3 & 2 \\ 6 & 9 & 11 \end{vmatrix} = 116.$$

$$M_C = \begin{bmatrix} -51 & -32 & 54 \\ 8 & -20 & 12 \\ 31 & 24 & -26 \end{bmatrix} \Rightarrow \operatorname{adj}(A) = \begin{bmatrix} -51 & 8 & 31 \\ -32 & -20 & 24 \\ 54 & 12 & -26 \end{bmatrix}, \text{ so that}$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \begin{bmatrix} -51/116 & 2/29 & 31/116 \\ -8/29 & -5/29 & 6/29 \\ 27/58 & 3/29 & -13/58 \end{bmatrix}.$$

$$12. \det(A) = \begin{vmatrix} 5 & -1 & 2 & 5 \\ 3 & -1 & 4 & 5 \\ 1 & -1 & 2 & 1 \\ 5 & 9 & -3 & 2 \end{vmatrix} = -152.$$

$$M_C = \begin{bmatrix} -38 & 32 & 34 & 2 \\ 0 & 28 & 44 & -60 \\ 38 & -124 & -222 & 130 \\ 0 & -24 & -16 & 8 \end{bmatrix} \Rightarrow \operatorname{adj}(A) = \begin{bmatrix} -38 & 0 & 38 & 0 \\ 32 & 28 & -124 & -24 \\ 34 & 44 & -222 & -16 \\ 2 & -60 & 130 & 8 \end{bmatrix}, \text{ so that}$$

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \begin{bmatrix} 1/4 & 0 & -1/4 & 0 \\ -4/19 & -7/38 & 31/38 & 3/19 \\ -17/76 & -11/38 & 111/76 & 2/19 \\ -1/76 & 15/38 & -65/76 & -1/19 \end{bmatrix}.$$

$$13. A = \begin{bmatrix} 3 & 5 \\ 6 & 2 \end{bmatrix}, B_1 = \begin{bmatrix} 4 & 5 \\ 9 & 2 \end{bmatrix}, B_2 = \begin{bmatrix} 3 & 4 \\ 6 & 9 \end{bmatrix}, \text{ so that}$$

$$x_1 = \frac{\det(B_1)}{\det(A)} = \frac{37}{24}, \quad x_2 = \frac{\det(B_2)}{\det(A)} = -\frac{1}{8}.$$

$$14. A = \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix}, B_1 = \begin{bmatrix} e^{-t} & \sin t \\ 3e^{-t} & -\cos t \end{bmatrix}, B_2 = \begin{bmatrix} \cos t & e^{-t} \\ \sin t & 3e^{-t} \end{bmatrix}, \text{ so that}$$

$$x_1 = \frac{\det(B_1)}{\det(A)} = e^{-t}[\cos t + 3 \sin t], \quad x_2 = \frac{\det(B_2)}{\det(A)} = e^{-t}[\sin t - 3 \cos t].$$

$$15. A = \begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 5 \\ 2 & 3 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 5 & 1 & 3 \\ 7 & -1 & 5 \\ 2 & 3 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 4 & 5 & 3 \\ 2 & 7 & 5 \\ 2 & 2 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 4 & 1 & 5 \\ 2 & -1 & 7 \\ 2 & 3 & 2 \end{bmatrix}, \text{ so that}$$

$$x_1 = \frac{\det(B_1)}{\det(A)} = \frac{1}{4}, \quad x_2 = \frac{\det(B_2)}{\det(A)} = \frac{1}{16}, \quad x_3 = \frac{\det(B_3)}{\det(A)} = \frac{21}{16}.$$

$$16. A = \begin{bmatrix} 5 & 3 & 6 \\ 2 & 4 & -7 \\ 2 & 5 & 9 \end{bmatrix}, B_1 = \begin{bmatrix} 3 & 3 & 6 \\ -1 & 4 & -7 \\ 4 & 5 & 9 \end{bmatrix}, B_2 = \begin{bmatrix} 5 & 3 & 6 \\ 2 & -1 & -7 \\ 2 & 4 & 9 \end{bmatrix}, B_3 = \begin{bmatrix} 5 & 3 & 3 \\ 2 & 4 & -1 \\ 2 & 5 & 4 \end{bmatrix}, \text{ so that}$$

$$x_1 = \frac{\det(B_1)}{\det(A)} = \frac{30}{271}, \quad x_2 = \frac{\det(B_2)}{\det(A)} = \frac{59}{271}, \quad x_3 = \frac{\det(B_3)}{\det(A)} = \frac{81}{271}.$$

$$17. A = \begin{bmatrix} 3.1 & 3.5 & 7.1 \\ 2.2 & 5.2 & 6.3 \\ 1.4 & 8.1 & 0.9 \end{bmatrix}, B_1 = \begin{bmatrix} 3.6 & 3.5 & 7.1 \\ 2.5 & 5.2 & 6.3 \\ 9.3 & 8.1 & 0.9 \end{bmatrix}, B_2 = \begin{bmatrix} 3.1 & 3.6 & 7.1 \\ 2.2 & 2.5 & 6.3 \\ 1.4 & 9.3 & 0.9 \end{bmatrix}, B_3 = \begin{bmatrix} 3.1 & 3.5 & 3.6 \\ 2.2 & 5.2 & 2.5 \\ 1.4 & 8.1 & 9.3 \end{bmatrix},$$

so that

$$x_1 = \frac{\det(B_1)}{\det(A)} = 3.77, \quad x_2 = \frac{\det(B_2)}{\det(A)} = 0.66, \quad x_3 = \frac{\det(B_3)}{\det(A)} = -1.46.$$

18. Since A is invertible, A^{-1} exists. Then,

$$AA^{-1} = I_n \Rightarrow \det(AA^{-1}) = \det(I_n) = 1 \Rightarrow \det(A) \det(A^{-1}) = 1, \text{ so that } \det(A^{-1}) = \frac{1}{\det(A)}.$$

19. $\det(2A) = 2^3 \det(A) = 24.$

From the result of the preceding problem, $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{3}.$

$\det(A^T B) = \det(A^T) \det(B) = \det(A) \det(B) = -12.$

$\det(B^5) = [\det(B)]^5 = -1024.$

$\det(B^{-1}AB) = \det(B^{-1}) \det(A) \det(B) = \frac{1}{\det(B)} \det(A) \det(B) = \det(A) = 3.$

Solutions to Section 3.5

Additional Problems:

1.

(a): We have $\det(A) = (-7)(-5) - (1)(-2) = 37.$

(b): We have

$$A \stackrel{1}{\sim} \begin{bmatrix} 1 & -5 \\ -7 & -2 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -5 \\ 0 & -37 \end{bmatrix} = B.$$

1. P_{12} 2. $A_{12}(7)$
--

Now, $\det(A) = -\det(B) = -(-37) = 37.$

(c): Let us use the Cofactor Expansion Theorem along the first row:

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} = (-7)(-5) + (-2)(-1) = 37.$$

2.

(a): We have $\det(A) = (6)(1) - (-2)(6) = 18.$

(b): We have

$$A \stackrel{1}{\sim} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = B.$$

1. $M_1(1/6)$ 2. $A_{12}(2)$
--

Now, $\det(A) = 6 \det(B) = 6 \cdot 3 = 18.$

(c): Let us use the Cofactor Expansion Theorem along the first row:

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} = (6)(1) + (6)(2) = 18.$$

3.

(a): Using Equation (3.1.2), we have

$$\det(A) = (-1)(2)(-3) + (4)(2)(2) + (1)(0)(2) - (-1)(2)(2) - (4)(0)(-3) - (1)(2)(2) = 6 + 16 + 0 - (-4) - 0 - 4 = 22.$$

(b): We have

$$A \stackrel{1}{\sim} \begin{bmatrix} -1 & 4 & 1 \\ 0 & 2 & 2 \\ 0 & 10 & -1 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} -1 & 4 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & -11 \end{bmatrix} = B.$$

1. $A_{13}(2)$ 2. $A_{23}(-5)$

Now, $\det(A) = \det(B) = (-1)(2)(-11) = 22$.

(c): Let us use the Cofactor Expansion Theorem along the first column:

$$\det(A) = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} = (-1)(-10) + (0)(14) + (2)(6) = 22.$$

4.

(a): Using Equation (3.1.2), we have

$$\det(A) = (2)(0)(3) + (3)(2)(6) + (-5)(-4)(-3) - (-6)(0)(-5) - (-3)(2)(2) - (3)(-4)(3) = 0 + 36 - 60 + 0 + 12 + 36 = 24.$$

(b): We have

$$A \stackrel{1}{\sim} \begin{bmatrix} 2 & 3 & -5 \\ 0 & 6 & -8 \\ 0 & -12 & 18 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 2 & 3 & -5 \\ 0 & 6 & -8 \\ 0 & 0 & 2 \end{bmatrix} = B.$$

1. $A_{12}(2), A_{13}(-3)$ 2. $A_{23}(2)$

Now, $\det(A) = \det(B) = (2)(6)(2) = 24$.

(c): Let us use the Cofactor Expansion Theorem along the second row:

$$\det(A) = (-4)(6) + (0)(36) + (2)(24) = -24 + 0 + 48 = 24.$$

5.

(a): Of the 24 terms appearing in the determinant expression (3.1.3), only terms containing the factors a_{11} and a_{44} will be nonzero (all other entries in the first column and fourth row of A are zero). Looking at entries in the second and third rows and columns of A , we see that only the product $a_{23}a_{32}$ is nonzero. Therefore, the only nonzero term in the summation (3.1.3) is $a_{11}a_{23}a_{32}a_{44} = (3)(1)(2)(-4) = -24$. The permutation associated with this term is $(1, 3, 2, 4)$ which contains one inversion. Therefore, $\sigma(1, 3, 2, 4) = -1$, and so the determinant is $(-24)(-1) = 24$.

(b): We have

$$A \stackrel{1}{\sim} \begin{bmatrix} 3 & -1 & -2 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix} = B.$$

1. P_{23}

Now, $\det(A) = -\det(B) = -(3)(2)(1)(-4) = 24$.

(c): Cofactor expansion along the first column yields: $\det(A) = 3 \cdot \begin{vmatrix} 0 & 1 & 4 \\ 2 & 1 & -1 \\ 0 & 0 & -4 \end{vmatrix}$. This latter determinant can be found by cofactor expansion along the last column: $(-4)[(0)(1) - (2)(1)] = 8$. Thus, $\det(A) = 3 \cdot 8 = 24$.

6.

(a): Of the 24 terms in the determinant expression (3.1.3), the only nonzero term is $a_{41}a_{32}a_{23}a_{14}$, which has a positive sign: $\sigma(4, 3, 2, 1) = +1$. Therefore, $\det(A) = (-3)(1)(-5)(-2) = -30$.

(b): By permuting the first and last rows, and by permuting the middle two rows, we bring A to upper triangular form. The two permutations introduce two sign changes, so the resulting upper triangular matrix has the same determinant as A , and it is $(-3)(1)(-5)(-2) = -30$.

(c): Cofactor expansion along the first column yields: $\det(A) = -(-3) \cdot \begin{vmatrix} 0 & 0 & -2 \\ 0 & -5 & 1 \\ 1 & -4 & 1 \end{vmatrix}$. This latter determinant can be found by cofactor expansion along the first column: $(1)[(0)(1) - (-5)(-2)] = -10$. Hence, $\det(A) = 3 \cdot (-10) = -30$.

7. To obtain the given matrix from A , we perform two row permutations, multiply a row through by -4 , and multiply a row through by 2 . The combined effect of these operations is to multiply the determinant of A by $(-1)^2 \cdot (-4) \cdot (2) = -8$. Hence, the given matrix has determinant $\det(A) \cdot (-8) = 4 \cdot (-8) = -32$.

8. To obtain the given matrix from A , we add 5 times the middle row to the top row, we multiply the last row by 3 , we multiply the middle row by -1 , we add a multiple of the last row to the middle row, and we perform a row permutation. The combined effect of these operations is to multiply the determinant of A by $(1) \cdot (3) \cdot (-1) \cdot (-1) = 3$. Hence, the given matrix has determinant $\det(A) \cdot (3) = 4 \cdot (3) = 12$.

9. To obtain the given matrix from A^T , we do two row permutations, multiply a row by -1 , multiply a row by 3 , and add 2 times one row to another. The combined effect of these operations is to multiply the determinant of A by $(-1)(-1)(-1)(3)(1) = -3$. Hence, the given matrix has determinant $\det(A) \cdot (-3) = (4) \cdot (-3) = -12$.

10. To obtain the given matrix from A , we permute the bottom two rows, multiply a row by 2 , multiply a row by -1 , add -1 times one row to another, and then multiply each row of the matrix by 3 . The combined effect of these operations is to multiply the determinant of A by $(-1)(2)(-1)(1)(3)^3 = 54$. Hence, the given matrix has determinant $\det(A) \cdot 54 = (4)(54) = 216$.

11. We have $\det(AB) = \det(A)\det(B) = (-2) \cdot 3 = -6$.

12. We have $\det(B^2A^{-1}) = (\det(B))^2\det(A^{-1}) = \frac{3^2}{-2} = -4.5$.

13. We have $\det((A^{-1}B)^T(2B^{-1})) = \det(A^{-1}B) \cdot \det(2B^{-1}) = -\frac{1}{2} \cdot 3 \cdot 2^4 \cdot \frac{1}{3} = -8$.

14. We have $\det((-A)^3(2B^2)) = \det(-A)^3\det(2B^2) = (-1)^3(\det A)^3 \cdot 2^4 \cdot (\det B)^2 = (-1)^3(-2)^3 \cdot 2^4 \cdot 3^2 = 1152$.

15. Since A and B are not square matrices, $\det(A)$ and $\det(B)$ are not possible to compute. We have $\det(C) = -18$, and therefore, $\det(C^T) = -18$. Now, $AB = \begin{bmatrix} 8 & -10 \\ 25 & 28 \end{bmatrix}$, and so $\det(AB) = 474$. Since

$BA = \begin{bmatrix} 4 & 5 & 2 \\ 1 & 8 & -13 \\ 18 & 15 & 24 \end{bmatrix}$, we have $\det(BA) = 0$. Next, $\det(B^T A^T) = \det((AB)^T) = \det(AB) = 474$. Next,

$\det(BAC) = \det(BA)\det(C) = 0 \cdot (-18) = 0$. Finally, we have $\det(ACB) = \det \begin{bmatrix} 38 & 54 \\ 133 & 297 \end{bmatrix} = 4104$.

16. The matrix of cofactors of B is $M_C = \begin{bmatrix} 1 & -1 \\ -4 & 5 \end{bmatrix}$, and so $\text{adj}(B) = \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$. Since $\det(B) = 1$,

we have $B^{-1} = \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$. Therefore,

$$(A^{-1}B^T)^{-1} = (B^T)^{-1}A = (B^{-1})^T A = \begin{bmatrix} 1 & -1 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 11 & 12 \end{bmatrix}.$$

17.

$$M_C = \begin{bmatrix} 16 & -1 & -5 \\ 4 & 5 & -3 \\ -4 & 2 & 10 \end{bmatrix};$$

$$\text{adj}(A) = \begin{bmatrix} 16 & 4 & -4 \\ -1 & 5 & 2 \\ -5 & -3 & 10 \end{bmatrix};$$

$$\det(A) = 28;$$

$$A^{-1} = \begin{bmatrix} 4/7 & 1/7 & -1/7 \\ -1/28 & 5/28 & 1/14 \\ -5/28 & -3/28 & 5/14 \end{bmatrix}.$$

18.

$$M_C = \begin{bmatrix} 5 & 12 & -11 & -1 \\ 65 & -24 & -23 & -13 \\ -25 & 0 & -5 & 5 \\ -35 & 0 & 5 & -5 \end{bmatrix};$$

$$\text{adj}(A) = \begin{bmatrix} 5 & 65 & -25 & -35 \\ 12 & -24 & 0 & 0 \\ -11 & -23 & -5 & 5 \\ -1 & -13 & 5 & -5 \end{bmatrix};$$

$$\det(A) = -60;$$

$$A^{-1} = -\frac{1}{60} \begin{bmatrix} 5 & 65 & -25 & -35 \\ 12 & -24 & 0 & 0 \\ -11 & -23 & -5 & 5 \\ -1 & -13 & 5 & -5 \end{bmatrix}.$$

19.

$$M_C = \begin{bmatrix} 88 & -24 & -40 & -48 \\ 32 & 12 & -20 & 0 \\ 16 & 12 & -4 & 0 \\ -4 & 6 & -2 & 0 \end{bmatrix};$$

$$\text{adj}(A) = \begin{bmatrix} 88 & 32 & 16 & -4 \\ -24 & 12 & 12 & 6 \\ -40 & -20 & -4 & -2 \\ -48 & 0 & 0 & 0 \end{bmatrix};$$

$$\det(A) = -48;$$

$$A^{-1} = -\frac{1}{48} \begin{bmatrix} 88 & 32 & 16 & -4 \\ -24 & 12 & 12 & 6 \\ -40 & -20 & -4 & -2 \\ -48 & 0 & 0 & 0 \end{bmatrix}.$$

20.

$$\begin{aligned}
M_C &= \begin{bmatrix} 21 & 12 & -12 \\ 24 & 9 & -12 \\ 48 & 24 & -27 \end{bmatrix}; \\
\text{adj}(A) &= \begin{bmatrix} 21 & 24 & 48 \\ 12 & 9 & 24 \\ -12 & -12 & -27 \end{bmatrix}; \\
\det(A) &= 9; \\
A^{-1} &= \begin{bmatrix} 7/3 & 8/3 & 16/3 \\ 4/3 & 1 & 8/3 \\ -4/3 & -4/3 & -3 \end{bmatrix}.
\end{aligned}$$

21.

$$\begin{aligned}
M_C &= \begin{bmatrix} 7 & -2 & 0 \\ 0 & 2 & -2 \\ -6 & 0 & 2 \end{bmatrix}; \\
\text{adj}(A) &= \begin{bmatrix} 7 & 0 & -6 \\ -2 & 2 & 0 \\ 0 & -2 & 2 \end{bmatrix}; \\
\det(A) &= 2; \\
A^{-1} &= \begin{bmatrix} 3.5 & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.
\end{aligned}$$

22. There are many ways to do this. One choice is to let $B = \begin{bmatrix} 4 & -1 & 0 \\ 5 & 1 & 4 \\ 0 & 0 & \frac{10}{9} \end{bmatrix}$.

23. **FALSE.** For instance, if one entry of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ is changed, two of the rows of A will still be identical, and therefore, the determinant of the resulting matrix must be zero. It is not possible to force the determinant to equal r .

24. Note that

$$\det(A) = 2 + 12k + 36 - 4k - 18 - 12 = 8k + 8 = 8(k + 1).$$

(a): Based on the calculation above, we see that A fails to be invertible if and only if $k = -1$.

(b): The volume of the parallelepiped determined by the row vectors of A is precisely $|\det(A)| = |8k + 8|$. The volume of the parallelepiped determined by the column vectors of A is the same as the volume of the parallelepiped determined by the row vectors of A^T , which is $|\det(A^T)| = |\det(A)| = |8k + 8|$. Thus, the volume is the same.

25. Note that

$$\det(A) = 3(k + 1) + 2k + 0 - 3 - k(k + 1) - 0 = -k^2 + 4k = k(4 - k).$$

(a): Based on the calculation above, we see that A fails to be invertible if and only if $k = 0$ or $k = 4$.

(b): The volume of the parallelepiped determined by the row vectors of A is precisely $|\det(A)| = |-k^2 + 4k|$. The volume of the parallelepiped determined by the column vectors of A is the same as the volume of the parallelepiped determined by the row vectors of A^T , which is $|\det(A^T)| = |\det(A)| = |-k^2 + 4k|$. Thus, the volume is the same.

26. Note that

$$\det(A) = 0 + 4(k-3) + 2k^3 - k^2 - 8k - 0 = 2k^3 - k^2 - 4k - 12.$$

(a): Based on the calculation above, we see that A fails to be invertible if and only if $2k^3 - k^2 - 4k - 12 = 0$, and the only real solution to this equation for k is $k \approx 2.39$ (from calculator).

(b): The volume of the parallelepiped determined by the row vectors of A is precisely $|\det(A)| = |2k^3 - k^2 - 4k - 12|$. The volume of the parallelepiped determined by the column vectors of A is the same as the volume of the parallelepiped determined by the row vectors of A^T , which is $|\det(A^T)| = |\det(A)| = |2k^3 - k^2 - 4k - 12|$. Thus, the volume is the same.

27. From the assumption that $AB = -BA$, we can take the determinant of each side: $\det(AB) = \det(-BA)$. Hence, $\det(A)\det(B) = (-1)^n \det(B)\det(A) = -\det(A)\det(B)$. From this, it follows that $\det(A)\det(B) = 0$, and therefore, either $\det(A) = 0$ or $\det(B) = 0$. Thus, either A or B (or both) fails to be invertible.

28. Since $AA^T = I_n$, we can take the determinant of both sides to get $\det(AA^T) = \det(I_n) = 1$. Hence, $\det(A)\det(A^T) = 1$. Therefore, we have $(\det(A))^2 = 1$. We conclude that $\det(A) = \pm 1$.

29. The coefficient matrix of this linear system is $A = \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix}$. We have

$$\det(A) = \begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix} = -7, \quad \det(B_1) = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5, \quad \text{and} \quad \det(B_2) = \begin{vmatrix} -3 & 3 \\ 1 & 1 \end{vmatrix} = -6.$$

Thus,

$$x_1 = \frac{\det(B_1)}{\det(A)} = -\frac{5}{7} \quad \text{and} \quad x_2 = \frac{\det(B_2)}{\det(A)} = \frac{6}{7}.$$

Solution: $(-5/7, 6/7)$.

30. The coefficient matrix of this linear system is $A = \begin{bmatrix} 2 & -1 & 1 \\ 4 & 5 & 3 \\ 4 & -3 & 3 \end{bmatrix}$. We have

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & -1 & 1 \\ 4 & 5 & 3 \\ 4 & -3 & 3 \end{vmatrix} = 16, \quad \det(B_1) = \begin{vmatrix} 2 & -1 & 1 \\ 0 & 5 & 3 \\ 2 & -3 & 3 \end{vmatrix} = 32, \\ \det(B_2) &= \begin{vmatrix} 2 & 2 & 1 \\ 4 & 0 & 3 \\ 4 & 2 & 3 \end{vmatrix} = -4, \quad \det(B_3) = \begin{vmatrix} 2 & -1 & 2 \\ 4 & 5 & 0 \\ 4 & -3 & 2 \end{vmatrix} = -36. \end{aligned}$$

Thus,

$$x_1 = \frac{\det(B_1)}{\det(A)} = 2, \quad x_2 = \frac{\det(B_2)}{\det(A)} = -\frac{1}{4}, \quad \text{and} \quad x_3 = \frac{\det(B_3)}{\det(A)} = -\frac{9}{4}.$$

Solution: $(2, -1/4, -9/4)$.

31. The coefficient matrix of this linear system is $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -1 & 1 \\ 0 & 5 & 5 \end{bmatrix}$. We have

$$\det(A) = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -1 & 1 \\ 0 & 5 & 5 \end{vmatrix} = -20, \quad \det(B_1) = \begin{vmatrix} -1 & 1 & 2 \\ -1 & -1 & 1 \\ -5 & 5 & 5 \end{vmatrix} = -10,$$

$$\det(B_2) = \begin{vmatrix} 3 & -1 & 2 \\ 2 & -1 & 1 \\ 0 & -5 & 5 \end{vmatrix} = -10, \quad \det(B_3) = \begin{vmatrix} 3 & 1 & -1 \\ 2 & -1 & -1 \\ 0 & 5 & -5 \end{vmatrix} = 30.$$

Thus,

$$x_1 = \frac{\det(B_1)}{\det(A)} = \frac{1}{2}, \quad x_2 = \frac{\det(B_2)}{\det(A)} = \frac{1}{2}, \quad \text{and} \quad x_3 = \frac{\det(B_3)}{\det(A)} = -\frac{3}{2}.$$

Solution: $(1/2, 1/2, -3/2)$.

Solutions to Section 4.1

True-False Review:

1. **FALSE.** The vectors (x, y) and $(x, y, 0)$ do not belong to the same set, so they are not even comparable, let alone equal to one another.
2. **TRUE.** The *unique* additive inverse of (x, y, z) is $(-x, -y, -z)$.
3. **TRUE.** The solution set refers to collections of the *unknowns* that solve the linear system. Since this system has 6 unknowns, the solution set will consist of vectors belonging to $\mathbb{R}^6$.
4. **TRUE.** The vector $(-1) \cdot (x_1, x_2, \dots, x_n)$ is precisely $(-x_1, -x_2, \dots, -x_n)$, and this is the additive inverse of $(x_1, x_2, \dots, x_n)$.
5. **FALSE.** There is no such name for a vector whose components are all positive.
6. **FALSE.** The correct result is

$$(s+t)(\mathbf{x} + \mathbf{y}) = (s+t)\mathbf{x} + (s+t)\mathbf{y} = s\mathbf{x} + t\mathbf{x} + s\mathbf{y} + t\mathbf{y},$$

and in this item, only the first and last of these four terms appear.

7. **TRUE.** When the vector $\mathbf{x}$ is scalar multiplied by zero, each component becomes zero: $0\mathbf{x} = \mathbf{0}$. This is the zero vector in $\mathbb{R}^n$.
8. **TRUE.** This is seen geometrically from addition and subtraction of geometric vectors.
9. **FALSE.** If $k < 0$, then $k\mathbf{x}$ is a vector in the *third* quadrant. For instance, $(1, 1)$ lies in the first quadrant, but $(-2)(1, 1) = (-2, -2)$ lies in the third quadrant.
10. **TRUE.** Recalling that $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$, we have

$$5\mathbf{i} - 6\mathbf{j} + \sqrt{2}\mathbf{k} = 5(1, 0, 0) - 6(0, 1, 0) + \sqrt{2}(0, 0, 1) = (5, -6, \sqrt{2}),$$

as stated.

11. **FALSE.** If the three vectors lie on the same line or the same plane, the resulting object may determine a one-dimensional segment or two-dimensional area. For instance, if $\mathbf{x} = \mathbf{y} = \mathbf{z} = (1, 0, 0)$, then the vectors $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ rest on the segment from $(0, 0, 0)$ to $(1, 0, 0)$, and do not determine a three-dimensional solid region.
12. **FALSE.** The components of $k\mathbf{x}$ only remain even integers if k is an integer. But, for example, if $k = \pi$, then the components of $k\mathbf{x}$ are not even integers at all, let alone even integers.

Problems:

1. $\mathbf{v}_1 = (6, 2)$, $\mathbf{v}_2 = (-3, 6)$, $\mathbf{v}_3 = (6, 2) + (-3, 6) = (3, 8)$.

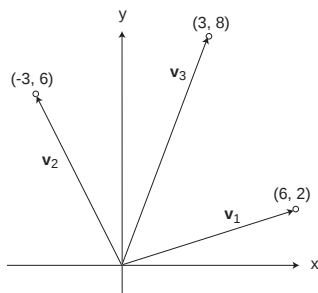


Figure 61: Figure for Problem 1

2. $\mathbf{v}_1 = (-3, -12)$, $\mathbf{v}_2 = (20, -4)$, $\mathbf{v}_3 = (-3, -12) + (20, -4) = (17, -16)$.

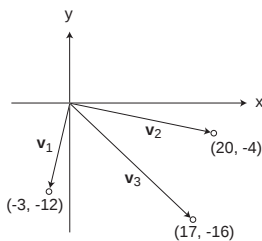


Figure 62: Figure for Problem 2

3. $\mathbf{v} = 5(3, -1, 2, 5) - 7(-1, 2, 9, -2) = (15, -5, 10, 25) - (-7, 14, 63, -14) = (22, -19, -53, 39)$. Additive inverse: $-\mathbf{v} = (-1)\mathbf{v} = (-22, 19, 53, -39)$.

4. We have

$$\mathbf{y} = \frac{2}{3}\mathbf{x} + \frac{1}{3}\mathbf{z} = \frac{2}{3}(1, 2, 3, 4, 5) + \frac{1}{3}(-1, 0, -4, 1, 2) = \left(\frac{1}{3}, \frac{4}{3}, \frac{2}{3}, 3, 4\right).$$

5. Let $\mathbf{x} = (x_1, x_2, x_3, x_4)$, $\mathbf{y} = (y_1, y_2, y_3, y_4)$ be arbitrary vectors in $\mathbb{R}^4$. Then

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) \\ &= (y_1 + x_1, y_2 + x_2, y_3 + x_3, y_4 + x_4) = (y_1, y_2, y_3, y_4) + (x_1, x_2, x_3, x_4) \\ &= \mathbf{y} + \mathbf{x}. \end{aligned}$$

6. Let $\mathbf{x} = (x_1, x_2, x_3, x_4)$, $\mathbf{y} = (y_1, y_2, y_3, y_4)$, $\mathbf{z} = (z_1, z_2, z_3, z_4)$ be arbitrary vectors in $\mathbb{R}^4$. Then

$$\begin{aligned}
 \mathbf{x} + (\mathbf{y} + \mathbf{z}) &= (x_1, x_2, x_3, x_4) + (y_1 + z_1, y_2 + z_2, y_3 + z_3, y_4 + z_4) \\
 &= (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), x_3 + (y_3 + z_3), x_4 + (y_4 + z_4)) \\
 &= ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2, (x_3 + y_3) + z_3, (x_4 + y_4) + z_4) \\
 &= (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) + (z_1, z_2, z_3, z_4) \\
 &= (\mathbf{x} + \mathbf{y}) + \mathbf{z}.
 \end{aligned}$$

7. Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ be arbitrary vectors in $\mathbb{R}^3$, and let r, s, t be arbitrary real numbers. Then:

$$\begin{aligned}
 1\mathbf{x} &= 1(x_1, x_2, x_3) = (1x_1, 1x_2, 1x_3) = (x_1, x_2, x_3) = \mathbf{x}. \\
 (st)\mathbf{x} &= (st)(x_1, x_2, x_3) = ((st)x_1, (st)x_2, (st)x_3) = (s(tx_1), s(tx_2), s(tx_3)) = s(tx_1, tx_2, tx_3) = s(t\mathbf{x}). \\
 r(\mathbf{x} + \mathbf{y}) &= r(x_1 + y_1, x_2 + y_2, x_3 + y_3) = (r(x_1 + y_1), r(x_2 + y_2), r(x_3 + y_3)) \\
 &= (rx_1 + ry_1, rx_2 + ry_2, rx_3 + ry_3) = (rx_1, rx_2, rx_3) + (ry_1, ry_2, ry_3) \\
 &= r\mathbf{x} + r\mathbf{y}. \\
 (s + t)\mathbf{x} &= (s + t)(x_1, x_2, x_3) = ((s + t)x_1, (s + t)x_2, (s + t)x_3) \\
 &= (sx_1 + tx_1, sx_2 + tx_2, sx_3 + tx_3) = (sx_1, sx_2, sx_3) + (tx_1, tx_2, tx_3) \\
 &= s\mathbf{x} + t\mathbf{x}.
 \end{aligned}$$

8. For example, if $\mathbf{x} = (2, 2)$ and $\mathbf{y} = (-1, -1)$, then $\mathbf{x} + \mathbf{y} = (1, 1)$ lies in the first quadrant. If $\mathbf{x} = (1, 2)$ and $\mathbf{y} = (-5, 1)$, then $\mathbf{x} + \mathbf{y} = (-4, 3)$ lies in the second quadrant. If $\mathbf{x} = (1, 1)$ and $\mathbf{y} = (-2, -2)$, then $\mathbf{x} + \mathbf{y} = (-1, -1)$ lies in the third quadrant. If $\mathbf{x} = (2, 1)$ and $\mathbf{y} = (-1, -5)$, then $\mathbf{x} + \mathbf{y} = (1, -4)$ lies in the fourth quadrant.

Solutions to Section 4.2

True-False Review:

1. **TRUE.** This is part 1 of Theorem 4.2.6.
2. **FALSE.** The statement would be true if it was required that $\mathbf{v}$ be nonzero. However, if $\mathbf{v} = \mathbf{0}$, then $r\mathbf{v} = s\mathbf{v} = \mathbf{0}$ for all values of r and s , and r and s need not be equal. We conclude that the statement is not true.
3. **FALSE.** This set is not closed under scalar multiplication. In particular, if k is an irrational number such as $k = \pi$ and $\mathbf{v}$ is an integer, then $k\mathbf{v}$ is not an integer.
4. **TRUE.** We have

$$(\mathbf{x} + \mathbf{y}) + ((-\mathbf{x}) + (-\mathbf{y})) = (\mathbf{x} + (-\mathbf{x})) + (\mathbf{y} + (-\mathbf{y})) = \mathbf{0} + \mathbf{0} = \mathbf{0},$$

where we have used the vector space axioms in these steps. Therefore, the additive inverse of $\mathbf{x} + \mathbf{y}$ is $(-\mathbf{x}) + (-\mathbf{y})$.

5. **TRUE.** This is part 1 of Theorem 4.2.6.
6. **TRUE.** This is called the *trivial* vector space. Since $0 + 0 = 0$ and $k \cdot 0 = 0$, it is closed under addition and scalar multiplication. Both sides of the remaining axioms yield 0, and 0 is the zero vector, and it is its own additive inverse.

7. FALSE. This set is not closed under addition, since $1 + 1 \notin \{0, 1\}$. Therefore, (A1) fails, and hence, this set does not form a vector space. (It is worth noting that the set is also not closed under scalar multiplication.)

8. FALSE. This set is not closed under scalar multiplication. If $k < 0$ and x is a positive real number, the result kx is a negative real number, and therefore no longer belongs to the set of positive real numbers.

Problems:

1. If $x = p/q$ and $y = r/s$, where p, q, r, s are integers ($q \neq 0, s \neq 0$), then $x + y = (ps + qr)/(qs)$, which is a rational number. Consequently, the set of all rational numbers is closed under addition. The set is not closed under scalar multiplication since, if we multiply a rational number by an irrational number, the result is an irrational number.

2. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be upper triangular matrices, and let k be an arbitrary real number. Then whenever $i > j$, it follows that $a_{ij} = 0$ and $b_{ij} = 0$. Consequently, $a_{ij} + b_{ij} = 0$ whenever $i > j$, and $ka_{ij} = 0$ whenever $i > j$. Therefore, $A + B$ and kA are upper triangular matrices, so the set of all upper triangular matrices with real elements is closed under both addition and scalar multiplication.

3. $V = \{y : y' + 9y = 4x^2\}$ is not a vector space because it is not closed under vector addition. Let $u, v \in V$. Then $u' + 9u = 4x^2$ and $v' + 9v = 4x^2$. It follows that $(u + v)' + 9(u + v) = (u' + v') + 9(u + v) = u' + 9u + v' + 9v = 4x^2 + 4x^2 = 8x^2 \neq 4x^2$. Thus, $u + v \notin V$. Likewise, V is not closed under scalar multiplication.

4. $V = \{y : y' + 9y = 0 \text{ for all } x \in I\}$ is closed under addition and scalar multiplication, as we now show:

A1: Addition: For $u, v \in V$, $u' + 9u = 0$ and $v' + 9v = 0$, so
 $(u + v)' + 9(u + v) = u' + v' + 9u + 9v = u' + 9u + v' + 9v = 0 + 0 = 0$
 $\implies u + v \in V$, therefore we have closure under addition.

A2: Scalar Multiplication: If $\alpha \in \mathbb{R}$ and $u \in V$, then
 $(\alpha u)' + 9(\alpha u) = \alpha u' + 9\alpha u = \alpha(u' + 9u) = \alpha \cdot 0 = 0$, so we also have closure under multiplication.

5. $V = \{x \in \mathbb{R}^n : Ax = \mathbf{0}, \text{ where } A \text{ is a fixed matrix}\}$ is closed under addition and scalar multiplication, as we now show:

Let $\mathbf{u}, \mathbf{v} \in V$ and $k \in \mathbb{R}$.

A1: Addition: $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$, so $\mathbf{u} + \mathbf{v} \in V$.

A2: Scalar Multiplication: $A(k\mathbf{u}) = kA\mathbf{u} = k\mathbf{0} = \mathbf{0}$, thus $k\mathbf{u} \in V$.

6. (a): The zero vector in $M_2(\mathbb{R})$ is $0_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Since $\det(0_2) = 0$, 0_2 is an element of S .

(b): Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\det(A) = 0 = \det(B)$, so that $A, B \in S$. However,
 $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies \det(A + B) = 1$, so that $A + B \notin S$. Consequently, S is not closed under addition.

(c): YES. Note that $\det(cA) = c^2 \det(A)$, so if $\det(A) = 0$, then $\det(cA) = 0$.

7. (1) $\mathbb{N}$ is not closed under scalar multiplication, since multiplication of a positive integer by a real number does not, in general, result in a positive integer.

- (2) There is no zero vector in $\mathbb{N}$.
 (3) No element of $\mathbb{N}$ has an additive inverse in $\mathbb{N}$.

8. Let V be $\mathbb{R}^2$, i.e. $\{(x, y) : x, y \in \mathbb{R}\}$. With addition and scalar multiplication as defined in the text, this set is clearly closed under both operations.

A3: $\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2) = (v_1 + u_1, v_2 + u_2) = \mathbf{v} + \mathbf{u}$.

A4: $[\mathbf{u} + \mathbf{v}] + \mathbf{w} = [(u_1 + v_1, u_2 + v_2)] + (w_1, w_2) = ([u_1 + v_1] + w_1, [u_2 + v_2] + w_2) = (u_1 + [v_1 + w_1], u_2 + [v_2 + w_2]) = (u_1, u_2) + [(v_1 + w_1, v_2 + w_2)] = \mathbf{u} + [\mathbf{v} + \mathbf{w}]$.

A5: $\mathbf{0} = (0, 0)$ since $(x, y) + (0, 0) = (x + 0, y + 0) = (x, y)$.

A6: If $\mathbf{u} = (a, b)$, then $-\mathbf{u} = (-a, -b)$, $a, b \in \mathbb{R}$, since $(a, b) + (-a, -b) = (a - a, b - b) = (0, 0)$.

Now, let $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$, where $u_1, u_2, v_1, v_2 \in \mathbb{R}$, and let $r, s, t \in \mathbb{R}$.

A7: $1 \cdot \mathbf{v} = 1(v_1, v_2) = (1 \cdot v_1, 1 \cdot v_2) = (v_1, v_2) = \mathbf{v}$.

A8: $(rs)\mathbf{v} = (rs)(v_1, v_2) = ((rs)v_1, (rs)v_2) = (rsv_1, rsv_2) = (r(sv_1), r(sv_2)) = r(sv_1, sv_2) = r(s\mathbf{v})$.

A9: $r(\mathbf{u} + \mathbf{v}) = r((u_1, u_2) + (v_1, v_2)) = r(u_1 + v_1, u_2 + v_2) = (r(u_1 + v_1), r(u_2 + v_2)) = (ru_1 + rv_1, ru_2 + rv_2) = (ru_1, ru_2) + (rv_1, rv_2) = r(u_1, u_2) + r(v_1, v_2) = r\mathbf{u} + r\mathbf{v}$.

A10: $(r + s)\mathbf{u} = (r + s)(u_1, u_2) = ((r + s)u_1, (r + s)u_2) = (ru_1 + su_1, ru_2 + su_2) = (ru_1, ru_2) + (su_1, su_2) = r(u_1, u_2) + s(u_1, u_2) = r\mathbf{u} + s\mathbf{u}$.

Thus, $\mathbb{R}^2$ is a vector space.

9. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \in M_{2 \times 3}(\mathbb{R})$. Then we see that the zero vector is $0_{2 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, since

$A + 0_{2 \times 3} = A$. Further, A has additive inverse $-A = \begin{bmatrix} -a & -b & -c \\ -d & -e & -f \end{bmatrix}$ since $A + (-A) = 0_{2 \times 3}$.

10. The zero vector of $M_{m \times n}(\mathbb{R})$ is $0_{m \times n}$, the $m \times n$ zero matrix. The additive inverse of the $m \times n$ matrix A with (i, j) -element a_{ij} is the $m \times n$ matrix $-A$ with (i, j) -element $-a_{ij}$.

11. $V = \{p : p \text{ is a polynomial in } x \text{ of degree } 2\}$. V is not a vector space because it is not closed under addition. For example, $x^2 \in V$ and $-x^2 \in V$, yet $x^2 + (-x^2) = 0 \notin V$.

12. YES. We verify the ten axioms of a vector space.

A1: The product of two positive real numbers is a positive real number, so the set is closed under addition.

A2: Any power of a positive real number is a positive real number, so the set is closed under scalar multiplication.

A3: We have

$$x + y = xy = yx = y + x$$

for all $x, y \in \mathbb{R}^+$, so commutativity under addition holds.

A4: We have

$$(x + y) + z = (xy) + z = (xy)z = x(yz) = x(y + z) = x + (y + z)$$

for all $x, y, z \in \mathbb{R}^+$, so that associativity under addition holds.

A5: We claim that the zero vector in this set is the real number 1. To see this, note that $1 + x = 1x = x = x1 = x + 1$ for all $x \in \mathbb{R}^+$.

A6: We claim that the additive inverse of the vector $x \in \mathbb{R}^+$ is $\frac{1}{x} \in \mathbb{R}^+$. To see this, note that $x + \frac{1}{x} = x \frac{1}{x} = 1 = \frac{1}{x} x = \frac{1}{x} + x$.

A7: Note that $1 \cdot x = x^1 = x$ for all $x \in \mathbb{R}^+$, so that the unit property holds.

A8: For all $r, s \in \mathbb{R}$ and $x \in \mathbb{R}^+$, we have

$$(rs) \cdot x = x^{rs} = x^{sr} = (x^s)^r = r \cdot x^s = r \cdot (s \cdot x),$$

as required for associativity of scalar multiplication.

A9: For all $r \in \mathbb{R}$ and $x, y \in \mathbb{R}^+$, we have

$$r \cdot (x + y) = r \cdot (xy) = (xy)^r = x^r y^r = x^r + y^r = r \cdot x + r \cdot y,$$

as required for distributivity of scalar multiplication over vector addition.

A10: For all $r, s \in \mathbb{R}$ and $x \in \mathbb{R}^+$, we have

$$(r + s) \cdot x = x^{r+s} = x^r x^s = x^r + x^s = r \cdot x + s \cdot x,$$

as required for distributivity of scalar multiplication over scalar addition.

The above verification of axioms A1-A10 shows that we have a vector space structure here.

13. Axioms A1 and A2 clearly hold under the given operations.

A3: $\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 - v_1, u_2 - v_2) = -(v_1 - u_1, -(v_2 - u_2)) \neq (v_1 - u_1, v_2 - u_2) = \mathbf{v} + \mathbf{u}$. Consequently, A3 does not hold.

A4: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = (u_1 - v_1, u_2 - v_2) + (w_1, w_2) = ((u_1 - v_1) - w_1, (u_2 - v_2) - w_2) = (u_1 - (v_1 + w_1), u_2 - (v_2 + w_2)) = (u_1, u_2) + (v_1 + w_1, v_2 + w_2) \neq (u_1, u_2) + (v_1 - w_1, v_2 - w_2) = \mathbf{u} + (\mathbf{v} + \mathbf{w})$. Consequently, A4 does not hold.

A5: $\mathbf{0} = (0, 0)$ since $\mathbf{u} + \mathbf{0} = (u_1, u_2) + (0, 0) = (u_1 - 0, u_2 - 0) = (u_1, u_2) = \mathbf{u}$.

A6: If $\mathbf{u} = (u_1, u_2)$, then $-\mathbf{u} = (u_1, u_2)$ since $\mathbf{u} + (-\mathbf{u}) = (u_1, u_2) + (u_1, u_2) = (u_1 - u_1, u_2 - u_2) = (0, 0) = \mathbf{0}$. Each of the remaining axioms do not hold.

14. Axiom A5: The zero vector on $\mathbb{R}^2$ with the defined operation of addition is given by $(1, 1)$, for if (u_1, u_2) is any element in $\mathbb{R}^2$, then $(u_1, u_2) + (1, 1) = (u_1 \cdot 1, u_2 \cdot 1) = (u_1, u_2)$. Thus, Axiom A5 holds.

Axiom A6: Suppose that (u_1, v_1) is any element in $\mathbb{R}^2$ with additive inverse (a, b) . From the first part of the problem, we know that $(1, 1)$ is the zero element, so it must be the case that $(u_1, v_1) + (a, b) = (1, 1)$ so that $(u_1 a, v_1 b) = (1, 1)$; hence, it follows that $u_1 a = 1$ and $v_1 b = 1$, but this system is not satisfied for all $(u_1, v_1) \in \mathbb{R}$, namely, $(0, 0)$. Thus, Axiom A6 is not satisfied.

15. Let $A, B, C \in M_2(\mathbb{R})$ and $r, s, t \in \mathbb{R}$.

A3: The addition operation is not commutative since

$$A + B = AB \neq BA = B + A.$$

A4: Addition is associative since

$$(A + B) + C = AB + C = (AB)C = A(BC) = A(B + C) = A + (B + C).$$

A5: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the zero vector in $M_2(\mathbb{R})$ because $A + I_2 = AI_2 = A$ for all $A \in M_2(\mathbb{R})$.

A6: We wish to determine whether for each matrix $A \in M_2(\mathbb{R})$ we can find a matrix $B \in M_2(\mathbb{R})$ such that $A + B = I_2$ (remember that we have shown in A5 that the zero vector is I_2), equivalently, such that $AB = I_2$. However, this equation can be satisfied only if A is nonsingular, therefore the axiom fails.

A7: $1 \cdot A = A$ is true for all $A \in M_2(\mathbb{R})$.

A8: $(st)A = s(tA)$ is true for all $A \in M_2(\mathbb{R})$ and $s, t \in \mathbb{R}$.

A9: $sA + tA = (sA)(tA) = st(AA) \neq (s + t)A$ for all $s, t \in \mathbb{R}$ and $A \in M_2(\mathbb{R})$. Consequently, the axiom fails.

A10: $rA + rB = (rA)(rB) = r^2 AB = r^2(A + B)$. Thus, $rA + rB \neq rA + rB$ for all $r \in \mathbb{R}$, so the axiom fails.

16. $M_2(\mathbb{R}) = \{A : A \text{ is a } 2 \times 2 \text{ real matrix}\}$. Let $A, B \in M_2(\mathbb{R})$ and $k \in \mathbb{R}$.

A1: $A \oplus B = -(A + B)$.

A2: $k \cdot A = -kA$.

A3 and A4: $A \oplus B = -(A + B) = -(B + A) = B \oplus A$. Hence, the operation $\oplus$ is commutative.

$(A \oplus B) \oplus C = -((A \oplus B) + C) = -(-(A + B) + C) = A + B - C$, but

$A \oplus (B \oplus C) = -(A + (B \oplus C)) = -(A + (-(B + C))) = -A + B + C$. Thus the operation $\oplus$ is not associative.

A5: An element B is needed such that $A \oplus B = A$ for all $A \in M_2(\mathbb{R})$, but $-(A + B) = A \implies B = -2A$. Since this depends on A , there is no zero vector.

A6: Since there is no zero vector, we cannot define the additive inverse.

Let $r, s, t \in \mathbb{R}$.

A7: $1 \cdot A = -1A = -A \neq A$.

A8: $(st) \cdot A = -[(st)A]$, but $s \cdot (t \cdot A) = s \cdot (-tA) = -[s(-tA)] = s(tA) = (st)A$, so it follows that $(st) \cdot A \neq s \cdot (t \cdot A)$.

A9: $r \cdot (A \oplus B) = -r(A \oplus B) = -r(-(A + B)) = r(A + B) = rA + rB = -[(-rA) + (-rB)] = -rA \oplus (-rB) = r \cdot A + r \cdot B$, whereas $rA \oplus rB = -(rA + rB) = -rA - rB$, so this axiom fails to hold.

A10: $(s + t) \cdot A = -(s + t)A = -sA + (-tA)$, but $s \cdot A \oplus t \cdot A = -(sA) \oplus (-tA) = -[-(sA) + (-tA)] = sA + tA$, hence $(s + t) \cdot A \neq s \cdot A \oplus t \cdot A$. We conclude that only the axioms (A1)-(A3) hold.

17. Let $\mathbb{C}^2 = \{(z_1, z_2) : z_i \in \mathbb{C}\}$ under the usual operations of addition and scalar multiplication.

A3 and A4: Follow from the properties of addition in $\mathbb{C}^2$.

A5: $(0, 0)$ is the zero vector in $\mathbb{C}^2$ since $(z_1, z_2) + (0, 0) = (z_1 + 0, z_2 + 0) = (z_1, z_2)$ for all $(z_1, z_2) \in \mathbb{C}^2$.

A6: The additive inverse of the vector $(z_1, z_2) \in \mathbb{C}^2$ is the vector $(-z_1, -z_2)$ for all $(z_1, z_2) \in \mathbb{C}^2$.

A7-A10: Follows from properties in $\mathbb{C}^2$.

Thus, $\mathbb{C}^2$ together with its defined operations, is a complex vector space.

18. Let $M_2(\mathbb{C}) = \{A : A \text{ is a } 2 \times 2 \text{ matrix with complex entries}\}$ under the usual operations of matrix addition and multiplication.

A3 and A4: Follows from properties of matrix addition.

A5: The zero vector, $\mathbf{0}$, for $M_2(\mathbb{C})$ is the 2×2 zero matrix, 0_2 .

A6: For each vector $A = [a_{ij}] \in M_2(\mathbb{C})$, the vector $-A = [-a_{ij}] \in M_2(\mathbb{C})$ satisfies $A + (-A) = 0_2$.

A7-A10: Follow from properties of matrix algebra.

Hence, $M_2(\mathbb{C})$ together with its defined operations, is a complex vector space.

19. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors in $\mathbb{C}^3$, and let $k \in \mathbb{R}$.

A1: $\mathbf{u} + \mathbf{v} = (u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \in \mathbb{C}^3$.

A2: $k\mathbf{u} = k(u_1, u_2, u_3) = (ku_1, ku_2, ku_3) \in \mathbb{C}^3$.

A3 and A4: Satisfied by the properties of addition in $\mathbb{C}^3$.

A5: $(0, 0, 0)$ is the zero vector in $\mathbb{C}^3$ since $(0, 0, 0) + (z_1, z_2, z_3) = (0 + z_1, 0 + z_2, 0 + z_3) = (z_1, z_2, z_3)$ for all $(z_1, z_2, z_3) \in \mathbb{C}^3$.

A6: $(-z_1, -z_2, -z_3)$ is the additive inverse of (z_1, z_2, z_3) because $(z_1, z_2, z_3) + (-z_1, -z_2, -z_3) = (0, 0, 0)$ for all $(z_1, z_2, z_3) \in \mathbb{C}^3$.

Let $r, s, t \in \mathbb{R}$.

A7: $1 \cdot \mathbf{u} = 1 \cdot (u_1, u_2, u_3) = (1u_1, 1u_2, 1u_3) = (u_1, u_2, u_3) = \mathbf{u}$.

A8: $(st)\mathbf{u} = (st)(u_1, u_2, u_3) = ((st)u_1, (st)u_2, (st)u_3) = (s(tu_1), s(tu_2), s(tu_3)) = s(tu_1, tu_2, tu_3) = s(t(u_1, u_2, u_3)) = s(t\mathbf{u})$.

A9: $r(\mathbf{u} + \mathbf{v}) = r(u_1 + v_1, u_2 + v_2, u_3 + v_3) = (r(u_1 + v_1), r(u_2 + v_2), r(u_3 + v_3)) = (ru_1 + rv_1, ru_2 + rv_2, ru_3 + rv_3) = (ru_1, ru_2, ru_3) + (rv_1, rv_2, rv_3) = r(u_1, u_2, u_3) + r(v_1, v_2, v_3) = r\mathbf{u} + r\mathbf{v}$.

A10: $(s + t)\mathbf{u} = (s + t)(u_1, u_2, u_3) = ((s + t)u_1, (s + t)u_2, (s + t)u_3) = (su_1 + tu_1, su_2 + tu_2, su_3 + tu_3) = (su_1, su_2, su_3) + (tu_1, tu_2, tu_3) = s(u_1, u_2, u_3) + t(u_1, u_2, u_3) = s\mathbf{u} + t\mathbf{u}$.

Thus, $\mathbb{C}^3$ is a real vector space.

20. NO. If we scalar multiply a vector $(x, y, z) \in \mathbb{R}^3$ by a non-real (complex) scalar r , we will obtain the vector $(rx, ry, rz) \notin \mathbb{R}^3$, since $rx, ry, rz \notin \mathbb{R}$.

21. Let k be an arbitrary scalar, and let $\mathbf{u}$ be an arbitrary vector in V . Then, using property 2 of Theorem

4.2.6, we have

$$k\mathbf{0} = k(0\mathbf{u}) = (k0)\mathbf{u} = 0\mathbf{u} = \mathbf{0}.$$

22. Assume that k is a scalar and $\mathbf{u} \in V$ such that $k\mathbf{u} = \mathbf{0}$. If $k = 0$, the desired conclusion is already reached. If, on the other hand, $k \neq 0$, then we have $\frac{1}{k} \in \mathbb{R}$ and

$$\frac{1}{k} \cdot (k\mathbf{u}) = \frac{1}{k} \cdot \mathbf{0} = \mathbf{0},$$

or

$$\left(\frac{1}{k} \cdot k\right) \mathbf{u} = \mathbf{0}.$$

Hence, $1 \cdot \mathbf{u} = \mathbf{0}$, and the unit property A7 now shows that $\mathbf{u} = \mathbf{0}$, and again, we reach the desired conclusion.

23. We verify the axioms A1-A10 for a vector space.

A1: If $a_0 + a_1x + \cdots + a_nx^n$ and $b_0 + b_1x + \cdots + b_nx^n$ belong to P_n , then

$$(a_0 + a_1x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_nx^n) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n,$$

which again belongs to P_n . Therefore, P_n is closed under addition.

A2: If $a_0 + a_1x + \cdots + a_nx^n$ and r is a scalar, then

$$r \cdot (a_0 + a_1x + \cdots + a_nx^n) = (ra_0) + (ra_1)x + \cdots + (ra_n)x^n,$$

which again belongs to P_n . Therefore, P_n is closed under scalar multiplication.

A3: Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$ and $q(x) = b_0 + b_1x + \cdots + b_nx^n$ belong to P_n . Then

$$\begin{aligned} p(x) + q(x) &= (a_0 + a_1x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_nx^n) \\ &= (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n \\ &= (b_0 + a_0) + (b_1 + a_1)x + \cdots + (b_n + a_n)x^n \\ &= (b_0 + b_1x + \cdots + b_nx^n) + (a_0 + a_1x + \cdots + a_nx^n) \\ &= q(x) + p(x), \end{aligned}$$

so P_n satisfies commutativity under addition.

A4: Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$, $q(x) = b_0 + b_1x + \cdots + b_nx^n$, and $r(x) = c_0 + c_1x + \cdots + c_nx^n$ belong to P_n . Then

$$\begin{aligned} [p(x) + q(x)] + r(x) &= [(a_0 + a_1x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_nx^n)] + (c_0 + c_1x + \cdots + c_nx^n) \\ &= [(a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n] + (c_0 + c_1x + \cdots + c_nx^n) \\ &= [(a_0 + b_0) + c_0] + [(a_1 + b_1) + c_1]x + \cdots + [(a_n + b_n) + c_n]x^n \\ &= [a_0 + (b_0 + c_0)] + [a_1 + (b_1 + c_1)]x + \cdots + [a_n + (b_n + c_n)]x^n \\ &= (a_0 + a_1x + \cdots + a_nx^n) + [(b_0 + c_0) + (b_1 + c_1)x + \cdots + (b_n + c_n)x^n] \\ &= (a_0 + a_1x + \cdots + a_nx^n) + [(b_0 + b_1x + \cdots + b_nx^n) + (c_0 + c_1x + \cdots + c_nx^n)] \\ &= p(x) + [q(x) + r(x)], \end{aligned}$$

so P_n satisfies associativity under addition.

A5: The zero vector is the zero polynomial $z(x) = 0 + 0 \cdot x + \cdots + 0 \cdot x^n$, and it is readily verified that this polynomial satisfies $z(x) + p(x) = p(x) = p(x) + z(x)$ for all $p(x) \in P_n$.

A6: The additive inverse of $p(x) = a_0 + a_1x + \cdots + a_nx^n$ is

$$-p(x) = (-a_0) + (-a_1)x + \cdots + (-a_n)x^n.$$

It is readily verified that $p(x) + (-p(x)) = z(x)$, where $z(x)$ is defined in A5.

A7: We have

$$1 \cdot (a_0 + a_1x + \cdots + a_nx^n) = a_0 + a_1x + \cdots + a_nx^n,$$

which demonstrates the unit property in P_n .

A8: Let $r, s \in \mathbb{R}$, and $p(x) = a_0 + a_1x + \cdots + a_nx^n \in P_n$. Then

$$\begin{aligned} (rs) \cdot p(x) &= (rs) \cdot (a_0 + a_1x + \cdots + a_nx^n) \\ &= [(rs)a_0] + [(rs)a_1]x + \cdots + [(rs)a_n]x^n \\ &= r[(sa_0) + (sa_1)x + \cdots + (sa_n)x^n] \\ &= r[s(a_0 + a_1x + \cdots + a_nx^n)] \\ &= r \cdot (s \cdot p(x)), \end{aligned}$$

which verifies the associativity of scalar multiplication.

A9: Let $r \in \mathbb{R}$, let $p(x) = a_0 + a_1x + \cdots + a_nx^n \in P_n$, and let $q(x) = b_0 + b_1x + \cdots + b_nx^n \in P_n$. Then

$$\begin{aligned} r \cdot (p(x) + q(x)) &= r \cdot ((a_0 + a_1x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_nx^n)) \\ &= r \cdot [(a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n] \\ &= [r(a_0 + b_0)] + [r(a_1 + b_1)]x + \cdots + [r(a_n + b_n)]x^n \\ &= [(ra_0) + (ra_1)x + \cdots + (ra_n)x^n] + [(rb_0) + (rb_1)x + \cdots + (rb_n)x^n] \\ &= [r(a_0 + a_1x + \cdots + a_nx^n)] + [r(b_0 + b_1x + \cdots + b_nx^n)] \\ &= r \cdot p(x) + r \cdot q(x), \end{aligned}$$

which verifies the distributivity of scalar multiplication over vector addition.

A10: Let $r, s \in \mathbb{R}$ and let $p(x) = a_0 + a_1x + \cdots + a_nx^n \in P_n$. Then

$$\begin{aligned} (r + s) \cdot p(x) &= (r + s) \cdot (a_0 + a_1x + \cdots + a_nx^n) \\ &= [(r + s)a_0] + [(r + s)a_1]x + \cdots + [(r + s)a_n]x^n \\ &= [ra_0 + sa_0] + [ra_1x + sa_1x] + \cdots + [ra_nx^n + sa_nx^n] \\ &= r(a_0 + a_1x + \cdots + a_nx^n) + s(a_0 + a_1x + \cdots + a_nx^n) \\ &= r \cdot p(x) + s \cdot p(x), \end{aligned}$$

which verifies the distributivity of scalar multiplication over scalar addition.

The above verification of axioms A1-A10 shows that P_n is a vector space.

Solutions to Section 4.3

True-False Review:

1. **FALSE.** The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$, not $\mathbb{R}^m$.
2. **FALSE.** It is not necessarily the case that $\mathbf{0}$ belongs to the solution set of the linear system. In fact, $\mathbf{0}$ belongs to the solution set of the linear system if and only if the system is homogeneous.
3. **TRUE.** If $b = 0$, then the line is $y = mx$, which is a line through the origin of $\mathbb{R}^2$, a one-dimensional subspace of $\mathbb{R}^2$. On the other hand, if $b \neq 0$, then the origin does not lie on the given line, and therefore since the line does not contain the zero vector, it cannot form a subspace of $\mathbb{R}^2$ in this case.

4. FALSE. The spaces $\mathbb{R}^m$ and $\mathbb{R}^n$, with $m < n$, are not comparable. Neither of them is a subset of the other, and therefore, neither of them can form a subspace of the other.

5. TRUE. Choosing any vector $\mathbf{v}$ in S , the scalar multiple $0\mathbf{v} = \mathbf{0}$ still belongs to S .

6. FALSE. In order for a subset of V to form a subspace, the same operations of addition and scalar multiplication must be used in the subset as used in V .

7. FALSE. This set is not closed under addition. For instance, the point $(1, 1, 0)$ lies in the xy -plane, the point $(0, 1, 1)$ lies in the yz -plane, but

$$(1, 1, 0) + (0, 1, 1) = (1, 2, 1)$$

does not belong to S . Therefore, S is not a subspace of V .

8. FALSE. For instance, if we consider $V = \mathbb{R}^3$, then the xy -plane forms a subspace of V , and the x -axis forms a subspace of V . Both of these subspaces contain in common all points along the x -axis. Other examples abound as well.

Problems:

1. $S = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = (2k, -3k), k \in \mathbb{R}\}$.

(a) S is certainly nonempty. Let $\mathbf{x}, \mathbf{y} \in S$. Then for some $r, s \in \mathbb{R}$,

$$\mathbf{x} = (2r, -3r) \quad \text{and} \quad \mathbf{y} = (2s, -3s).$$

Hence,

$$\mathbf{x} + \mathbf{y} = (2r, -3r) + (2s, -3s) = (2(r + s), -3(r + s)) = (2k, -3k),$$

where $k = r + s$. Consequently, S is closed under addition. Further, if $c \in \mathbb{R}$, then

$$c\mathbf{x} = c(2r, -3r) = (2cr, -3cr) = (2t, -3t),$$

where $t = cr$. Therefore S is also closed under scalar multiplication. It follows from Theorem 4.3.2 that S is a subspace of $\mathbb{R}^2$.

(b) The subspace S consists of all points lying along the line in the accompanying figure.

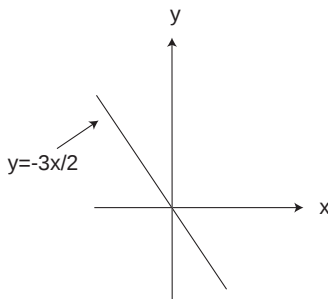


Figure 63: Figure for Problem 1(b)

2. $S = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = (r - 2s, 3r + s, s), r, s \in \mathbb{R}\}$.

(a) S is certainly nonempty. Let $\mathbf{x}, \mathbf{y} \in S$. Then for some $r, s, u, v \in \mathbb{R}$,

$$\mathbf{x} = (r - 2s, 3r + s, s) \quad \text{and} \quad \mathbf{y} = (u - 2v, 3u + v, v).$$

Hence,

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (r - 2s, 3r + s, s) + (u - 2v, 3u + v, v) \\ &= ((r + u) - 2(s + v), 3(r + u) + (s + v), s + v) \\ &= (a - 2b, 3a + b, b), \end{aligned}$$

where $a = r + u$, and $b = s + v$. Consequently, S is closed under addition. Further, if $c \in \mathbb{R}$, then

$$c\mathbf{x} = c(r - 2s, 3r + s, s) = (cr - 2cs, 3cr + cs, cs) = (k - 2l, 3k + l, l),$$

where $k = cr$ and $l = cs$. Therefore S is also closed under scalar multiplication. It follows from Theorem 4.3.2 that S is a subspace of $\mathbb{R}^3$.

(b) The coordinates of the points in S are (x, y, z) where

$$x = r - 2s, y = 3r + s, z = s.$$

Eliminating r and s between these three equations yields $3x - y + 7z = 0$.

3. $S = \{(x, y) \in \mathbb{R}^2 : 3x + 2y = 0\}$. $S \neq \emptyset$ since $(0, 0) \in S$.

Closure under Addition: Let $(x_1, x_2), (y_1, y_2) \in S$. Then $3x_1 + 2x_2 = 0$ and $3y_1 + 2y_2 = 0$, so $3(x_1 + y_1) + 2(x_2 + y_2) = 0$, which implies that $(x_1 + y_1, x_2 + y_2) \in S$.

Closure under Scalar Multiplication: Let $a \in \mathbb{R}$ and $(x_1, x_2) \in S$. Then $3x_1 + 2x_2 = 0 \implies a(3x_1 + 2x_2) = a \cdot 0 \implies 3(ax_1) + 2(ax_2) = 0$, which shows that $(ax_1, ax_2) \in S$. Thus, S is a subspace of $\mathbb{R}^2$ by Theorem 4.3.2.

4. $S = \{(x_1, 0, x_3, 2) : x_1, x_3 \in \mathbb{R}\}$ is not a subspace of $\mathbb{R}^4$ because it is not closed under addition. To see this, let $(a, 0, b, 2), (c, 0, d, 2) \in S$. Then $(a, 0, b, 2) + (c, 0, d, 2) = (a + c, 0, b + d, 4) \notin S$.

5. $S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\}$ is not a subspace of $\mathbb{R}^3$ because $(0, 0, 0) \notin S$ since $0 + 0 + 0 \neq 1$.

6. $S = \{\mathbf{u} \in \mathbb{R}^2 : A\mathbf{u} = \mathbf{b}, A \text{ is a fixed } m \times n \text{ matrix}\}$ is not a subspace of $\mathbb{R}^n$ since $\mathbf{0} \notin S$.

7. $S = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 0\}$ is not a subspace of $\mathbb{R}^2$, since it is not closed under addition, as we now observe: If $(x_1, y_1), (x_2, y_2) \in S$, then $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$.

$$\begin{aligned} (x_1 + x_2)^2 - (y_1 + y_2)^2 &= x_1^2 + 2x_1x_2 + x_2^2 - (y_1^2 + 2y_1y_2 + y_2^2) \\ &= (x_1^2 - y_1^2) + (x_2^2 - y_2^2) + 2(x_1x_2 - y_1y_2) \\ &= 0 + 0 + 2(x_1x_2 - y_1y_2) \neq 0, \text{ in general.} \end{aligned}$$

Thus, $(x_1, y_1) + (x_2, y_2) \notin S$.

8. $S = \{A \in M_2(\mathbb{R}) : \det(A) = 1\}$ is not a subspace of $M_2(\mathbb{R})$. To see this, let $k \in \mathbb{R}$ be a scalar and let $A \in S$. Then $\det(kA) = k^2 \det(A) = k^2 \cdot 1 = k^2 \neq 1$, unless $k = \pm 1$. Note also that $\det(A) = 1$ and $\det(B) = 1$ does not imply that $\det(A + B) = 1$.

9. $S = \{A = [a_{ij}] \in M_n(\mathbb{R}) : a_{ij} = 0 \text{ whenever } i < j\}$. Note that $S \neq \emptyset$ since $0_n \in S$. Now let $A = [a_{ij}]$ and $B = [b_{ij}]$ be lower triangular matrices. Then $a_{ij} = 0$ and $b_{ij} = 0$ whenever $i < j$. Then $a_{ij} + b_{ij} = 0$

and $ca_{ij} = 0$ whenever $i < j$. Hence $A + B = [a_{ij} + b_{ij}]$ and $cA = [ca_{ij}]$ are also lower triangular matrices. Therefore S is closed under addition and scalar multiplication. Consequently, S is a subspace of $M_2(\mathbb{R})$ by Theorem 4.3.2.

10. $S = \{A \in M_n(\mathbb{R}) : A \text{ is invertible}\}$ is not a subspace of $M_n(\mathbb{R})$ because $0_n \notin S$.

11. $S = \{A \in M_2(\mathbb{R}) : A^T = A\}$. $S \neq \emptyset$ since $0_2 \in S$.

Closure under Addition: If $A, B \in S$, then $(A + B)^T = A^T + B^T = A + B$, which shows that $A + B \in S$.

Closure under Scalar Multiplication: If $r \in \mathbb{R}$ and $A \in S$, then $(rA)^T = rA^T = rA$, which shows that $rA \in S$. Consequently, S is a subspace of $M_2(\mathbb{R})$ by Theorem 4.3.2.

12. $S = \{A \in M_2(\mathbb{R}) : A^T = -A\}$. $S \neq \emptyset$ because $0_2 \in S$.

Closure under Addition: If $A, B \in S$, then $(A + B)^T = A^T + B^T = -A + (-B) = -(A + B)$, which shows that $A + B \in S$.

Closure under Scalar Multiplication: If $k \in \mathbb{R}$ and $A \in S$, then $(kA)^T = kA^T = k(-A) = -(kA)$, which shows that $kA \in S$.

Thus, S is a subspace of $M_2(\mathbb{R})$ by Theorem 4.3.2.

13. $S = \{f \in V : f(a) = f(b)\}$, where V is the vector space of all real-valued functions defined on $[a, b]$.

Note that $S \neq \emptyset$ since the zero function $O(x) = 0$ for all x belongs to S .

Closure under Addition: If $f, g \in S$, then $(f + g)(a) = f(a) + g(a) = f(b) + g(b) = (f + g)(b)$, which shows that $f + g \in S$.

Closure under Scalar Multiplication: If $k \in \mathbb{R}$ and $f \in S$, then $(kf)(a) = kf(a) = kf(b) = (kf)(b)$, which shows that $kf \in S$.

Therefore S is a subspace of V by Theorem 4.3.2.

14. $S = \{f \in V : f(a) = 1\}$, where V is the vector space of all real-valued functions defined on $[a, b]$. We claim that S is not a subspace of V .

Not Closed under Addition: If $f, g \in S$, then $(f + g)(a) = f(a) + g(a) = 1 + 1 = 2 \neq 1$, which shows that $f + g \notin S$.

It can also be shown that S is not closed under scalar multiplication.

15. $S = \{f \in V : f(-x) = f(x) \text{ for all } x \in \mathbb{R}\}$. Note that $S \neq \emptyset$ since the zero function $O(x) = 0$ for all x belongs to S . Let $f, g \in S$. Then

$$(f + g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)(x)$$

and if $c \in \mathbb{R}$, then

$$(cf)(-x) = cf(-x) = cf(x) = (cf)(x),$$

so $f + g$ and $c \cdot f$ belong to S . Therefore, S is closed under addition and scalar multiplication. Therefore, S is a subspace of V by Theorem 4.3.2.

16. $S = \{p \in P_2 : p(x) = ax^2 + b, a, b \in \mathbb{R}\}$. Note that $S \neq \emptyset$ since $p(x) = 0$ belongs to S .

Closure under Addition: Let $p, q \in S$. Then for some $a_1, a_2, b_1, b_2 \in \mathbb{R}$,

$$p(x) = a_1x^2 + b_1 \quad \text{and} \quad q(x) = a_2x^2 + b_2.$$

Hence,

$$(p+q)(x) = p(x) + q(x) = (a_1 + a_2)x^2 + b_1 + b_2 = ax^2 + b,$$

where $a = a_1 + a_2$ and $b = b_1 + b_2$, so that S is closed under addition.

Closure under Scalar Multiplication: If $k \in \mathbb{R}$, then

$$(kp)(x) = kp(x) = ka_1x^2 + kb_1 = cx^2 + d,$$

where $c = ka_1$ and $d = kb_1$, so that S is also closed under scalar multiplication.

It follows from Theorem 4.3.2 that S is a subspace of P_2 .

17. $S = \{p \in P_2 : p(x) = ax^2 + 1, a \in \mathbb{R}\}$. We claim that S is not closed under addition:

Not Closed under Addition: Let $p, q \in S$. Then for some $a_1, a_2 \in \mathbb{R}$,

$$p(x) = a_1x^2 + 1 \quad \text{and} \quad q(x) = a_2x^2 + 1.$$

Hence,

$$(p+q)(x) = p(x) + q(x) = (a_1 + a_2)x^2 + 2 = ax^2 + 2$$

where $a = a_1 + a_2$. Consequently, $p+q \notin S$, and therefore S is not closed under addition. It follows that S is not a subspace of P_2 .

18. $S = \{y \in C^2(I) : y'' + 2y' - y = 0\}$. Note that S is nonempty since the function $y = 0$ belongs to S .

Closure under Addition: Let $y_1, y_2 \in S$.

$$\begin{aligned} (y_1 + y_2)'' + 2(y_1 + y_2)' - (y_1 + y_2) &= y_1'' + y_2'' + 2(y_1' + y_2') - y_1 - y_2 \\ &= y_1'' + 2y_1' - y_1 + y_2'' + 2y_2' - y_2 \\ &= (y_1'' + 2y_1' - y_1) + (y_2'' + 2y_2' - y_2) \quad \text{Thus, } y_1 + y_2 \in S. \\ &= 0 + 0 = 0. \end{aligned}$$

Closure under Scalar Multiplication: Let $k \in \mathbb{R}$ and $y_1 \in S$.

$$(ky_1)'' + 2(ky_1)' - (ky_1) = ky_1'' + 2ky_1' - ky_1 = k(y_1'' + 2y_1' - y_1) = k \cdot 0 = 0, \text{ which shows that } ky_1 \in S.$$

Hence, S is a subspace of V by Theorem 4.3.2.

19. $S = \{y \in C^2(I) : y'' + 2y' - y = 1\}$. S is not a subspace of V .

We show that S fails to be closed under addition (one can also verify that it is not closed under scalar multiplication, but this is unnecessary if one shows the failure of closure under addition):

Not Closed under Addition: Let $y_1, y_2 \in S$.

$$\begin{aligned} (y_1 + y_2)'' + 2(y_1 + y_2)' - (y_1 + y_2) &= y_1'' + y_2'' + 2(y_1' + y_2') - y_1 - y_2 \\ &= (y_1'' + 2y_1' - y_1) + (y_2'' + 2y_2' - y_2) \quad \text{Thus, } y_1 + y_2 \notin S. \\ &= 1 + 1 = 2 \neq 1. \end{aligned}$$

Or alternatively:

Not Closed under Scalar Multiplication: Let $k \in \mathbb{R}$ and $y_1 \in S$.

$$((ky_1)'' + 2(ky_1)' - (ky_1)) = ky_1'' + 2ky_1' - ky_1 = k(y_1'' + 2y_1' - y_1) = k \cdot 1 = k \neq 1, \text{ unless } k = 1. \text{ Therefore, } ky_1 \notin S \text{ unless } k = 1.$$

20. $A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & -2 \\ -1 & 3 & 4 \end{bmatrix}$. $\text{nullspace}(A) = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{0}\}$. The reduced row echelon form of the

augmented matrix of the system $A\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$. Consequently, the linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $(0, 0, 0)$, so $\text{nullspace}(A) = \{(0, 0, 0)\}$.

21. $A = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 3 & 10 & -4 & 6 \\ 2 & 5 & -6 & -1 \end{bmatrix}$. $\text{nullspace}(A) = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{0}\}$. The REDUCED ROW ECHELON

FORM of the augmented matrix of the system $A\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{cccc|c} 1 & 0 & -8 & -8 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$, so that $\text{nullspace}(A) = \{(8r + 8s, -2r - 3s, r, s) : r, s \in \mathbb{R}\}$.

22. $A = \begin{bmatrix} 1 & i & -2 \\ 3 & 4i & -5 \\ -1 & -3i & i \end{bmatrix}$. $\text{nullspace}(A) = \{\mathbf{x} \in \mathbb{C}^3 : A\mathbf{x} = \mathbf{0}\}$. The REDUCED ROW ECHELON

FORM of the augmented matrix of the system $A\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$. Consequently, the linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $(0, 0, 0)$, so $\text{nullspace}(A) = \{(0, 0, 0)\}$.

23. Since the zero function $y(x) = 0$ for all $x \in I$ is not a solution to the differential equation, the set of all solutions does not contain the zero vector from $C^2(I)$, hence it is not a vector space at all and cannot be a subspace.

24.

(a) As an example, we can let $V = \mathbb{R}^2$. If we take S_1 to be the set of points lying on the x -axis and S_2 to be the set of points lying on the y -axis, then it is readily seen that S_1 and S_2 are both subspaces of V . However, the union of these subspaces is not closed under addition. For instance, the points $(1, 0)$ and $(0, 1)$ lie in $S_1 \cup S_2$, but $(1, 0) + (0, 1) = (1, 1) \notin S_1 \cup S_2$. Therefore, the union $S_1 \cup S_2$ does not form a subspace of V .

(b) Since S_1 and S_2 are both subspaces of V , they both contain the zero vector. It follows that the zero vector is an element of $S_1 \cap S_2$, hence this subset is nonempty. Now let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $S_1 \cap S_2$, and let c be a scalar. Then $\mathbf{u}$ and $\mathbf{v}$ are both in S_1 and both in S_2 . Since S_1 and S_2 are each subspaces of V , $\mathbf{u} + \mathbf{v}$ and $c\mathbf{u}$ are vectors in both S_1 and S_2 , hence they are in $S_1 \cap S_2$. This implies that $S_1 \cap S_2$ is a nonempty subset of V , which is closed under both addition and scalar multiplication. Therefore, $S_1 \cap S_2$ is a subspace of V .

(c) Note that S_1 is a subset of $S_1 + S_2$ (every vector $\mathbf{v} \in S_1$ can be written as $\mathbf{v} + \mathbf{0} \in S_1 + S_2$), so $S_1 + S_2$ is nonempty.

Closure under Addition: Now let $\mathbf{u}$ and $\mathbf{v}$ belong to $S_1 + S_2$. We may write $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ and $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{u}_1, \mathbf{v}_1 \in S_1$ and $\mathbf{u}_2, \mathbf{v}_2 \in S_2$. Then

$$\mathbf{u} + \mathbf{v} = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{v}_1) + (\mathbf{u}_2 + \mathbf{v}_2).$$

Since S_1 and S_2 are closed under addition, we have that $\mathbf{u}_1 + \mathbf{v}_1 \in S_1$ and $\mathbf{u}_2 + \mathbf{v}_2 \in S_2$. Therefore,

$$\mathbf{u} + \mathbf{v} = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{v}_1) + (\mathbf{u}_2 + \mathbf{v}_2) \in S_1 + S_2.$$

Hence, $S_1 + S_2$ is closed under addition.

Closure under Scalar Multiplication: Next, let $\mathbf{u} \in S_1 + S_2$ and let c be a scalar. We may write $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where $\mathbf{u}_1 \in S_1$ and $\mathbf{u}_2 \in S_2$. Thus,

$$c \cdot \mathbf{u} = c \cdot \mathbf{u}_1 + c \cdot \mathbf{u}_2,$$

and since S_1 and S_2 are closed under scalar multiplication, $c \cdot \mathbf{u}_1 \in S_1$ and $c \cdot \mathbf{u}_2 \in S_2$. Therefore,

$$c \cdot \mathbf{u} = c \cdot \mathbf{u}_1 + c \cdot \mathbf{u}_2 \in S_1 + S_2.$$

Hence, $S_1 + S_2$ is closed under scalar multiplication.

Solutions to Section 4.4

True-False Review:

1. **TRUE.** By its very definition, when a linear span of a set of vectors is formed, that span becomes closed under addition and under scalar multiplication. Therefore, it is a subspace of V .
2. **FALSE.** In order to say that S spans V , it must be true that *all* vectors in V can be expressed as a linear combination of the vectors in S , not simply “some” vector.
3. **TRUE.** Every vector in V can be expressed as a linear combination of the vectors in S , and therefore, it is also true that every vector in W can be expressed as a linear combination of the vectors in S . Therefore, S spans W , and S is a spanning set for W .
4. **FALSE.** To illustrate this, consider $V = \mathbb{R}^2$, and consider the spanning set $\{(1, 0), (0, 1), (1, 1)\}$. Then the vector $\mathbf{v} = (2, 2)$ can be expressed as a linear combination of the vectors in S in more than one way:

$$\mathbf{v} = 2(1, 1) \quad \text{and} \quad \mathbf{v} = 2(1, 0) + 2(0, 1).$$

Many other illustrations, using a variety of different vector spaces, are also possible.

5. **TRUE.** To say that a set S of vectors in V spans V is to say that every vector in V belongs to $\text{span}(S)$. So V is a subset of $\text{span}(S)$. But of course, every vector in $\text{span}(S)$ belongs to the vector space V , and so $\text{span}(S)$ is a subset of V . Therefore, $\text{span}(S) = V$.
6. **FALSE.** This is not necessarily the case. For example, the linear span of the vectors $(1, 1, 1)$ and $(2, 2, 2)$ is simply a *line* through the origin, not a plane.
7. **FALSE.** There are vector spaces that do not contain finite spanning sets. For instance, if V is the vector space consisting of all polynomials with coefficients in $\mathbb{R}$, then since a finite spanning set could not contain polynomials of arbitrarily large degree, no finite spanning set is possible for V .
8. **FALSE.** To illustrate this, consider $V = \mathbb{R}^2$, and consider the spanning set $S = \{(1, 0), (0, 1), (1, 1)\}$. The proper subset $S' = \{(1, 0), (0, 1)\}$ is still a spanning set for V . Therefore, it is possible for a proper subset of a spanning set for V to still be a spanning set for V .
9. **TRUE.** The general matrix $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ in this vector space can be written as $aE_{11} + bE_{12} + cE_{13} + dE_{22} + eE_{23} + fE_{33}$, and therefore the matrices in the set $\{E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33}\}$ span the vector space.
10. **FALSE.** For instance, it is easily verified that $\{x^2, x^2 + x, x^2 + 1\}$ is a spanning set for P_2 , and yet, it contains only polynomials of degree 2.

11. FALSE. For instance, consider $m = 2$ and $n = 3$. Then one spanning set for $\mathbb{R}^2$ is $\{(1, 0), (0, 1), (1, 1), (2, 2)\}$, which consists of four vectors. On the other hand, one spanning set for $\mathbb{R}^3$ is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, which consists of only three vectors.

12. TRUE. This is explained in True-False Review Question 7 above.

Problems:

1. $\{(1, -1), (2, -2), (2, 3)\}$. Since $\mathbf{v}_1 = (1, -1)$, and $\mathbf{v}_2 = (2, 3)$ are noncolinear, the given set of vectors does span $\mathbb{R}^2$. (See the comment preceding Example 4.4.3 in the text.)

2. The given set of vectors does not span $\mathbb{R}^2$ since it does not contain two nonzero and non-collinear vectors.

3. The three vectors in the given set are all collinear. Consequently, the set of vectors does not span $\mathbb{R}^2$.

4. Since $\begin{vmatrix} 1 & 2 & 4 \\ -1 & 5 & -2 \\ 1 & 3 & 1 \end{vmatrix} = -23 \neq 0$, the given vectors are not coplanar, and therefore span $\mathbb{R}^3$.

5. Since $\begin{vmatrix} 1 & 2 & 4 \\ -2 & 3 & -1 \\ 1 & 1 & 2 \end{vmatrix} = -7 \neq 0$, the given vectors are not coplanar, and therefore span $\mathbb{R}^3$. Note that we can simply ignore the zero vector $(0, 0, 0)$.

6. Since $\begin{vmatrix} 2 & 3 & 1 \\ -1 & -3 & 1 \\ 4 & 5 & 3 \end{vmatrix} = 0$, the vectors are coplanar, and therefore the given set does not span $\mathbb{R}^3$.

7. Since $\begin{vmatrix} 1 & 3 & 4 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix} = 0$, the vectors are coplanar, and therefore the given set does not span $\mathbb{R}^3$. The linear span of the vectors is those points (x, y, z) for which the system

$$c_1(1, 2, 3) + c_2(3, 4, 5) + c_3(4, 5, 6) = (x, y, z)$$

is consistent. Reducing the augmented matrix $\left[\begin{array}{ccc|c} 1 & 3 & 4 & x \\ 2 & 4 & 5 & y \\ 3 & 5 & 6 & z \end{array} \right]$ of this system yields $\left[\begin{array}{ccc|c} 1 & 3 & 4 & x \\ 0 & 2 & 3 & 2x - y \\ 0 & 0 & 0 & x - 2y + z \end{array} \right]$.

This system is consistent if and only if $x - 2y + z = 0$. Consequently, the linear span of the given set of vectors consists of all points lying on the plane with the equation $x - 2y + z = 0$.

8. Let $(x_1, x_2) \in \mathbb{R}^2$ and $a, b \in \mathbb{R}$.

$(x_1, x_2) = a\mathbf{v}_1 + b\mathbf{v}_2 = a(2, -1) + b(3, 2) = (2a, -a) + (3b, 2b) = (2a + 3b, -a + 2b)$. It follows that $2a + 3b = x_1$ and $-a + 2b = x_2$, which implies that $a = \frac{2x_1 - 3x_2}{7}$ and $b = \frac{x_1 + 2x_2}{7}$ so $(x_1, x_2) = \left(\frac{2x_1 - 3x_2}{7}\right)\mathbf{v}_1 + \left(\frac{x_1 + 2x_2}{7}\right)\mathbf{v}_2$. $\{\mathbf{v}_1, \mathbf{v}_2\}$ spans $\mathbb{R}^2$, and in particular, $(5, -7) = \frac{31}{7}\mathbf{v}_1 - \frac{9}{7}\mathbf{v}_2$.

9. Let $(x, y, z) \in \mathbb{R}^3$ and $a, b, c \in \mathbb{R}$.

$$\begin{aligned} (x, y, z) &= a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = a(-1, 3, 2) + b(1, -2, 1) + c(2, 1, 1) \\ &= (-a, 3a, 2a) + (b, -2b, b) + (2c, c, c) \\ &= (-a + b + 2c, 3a - 2b + c, 2a + b + c). \end{aligned}$$

These equalities result in the system:
$$\begin{cases} a + b + 2c = x \\ 3a - 2b + c = y \\ 2a + b + c = z \end{cases}$$

Upon solving for a, b , and c we obtain

$$a = \frac{-3x + y + 5z}{16}, b = \frac{-x - 5y + 7z}{16}, \text{ and } c = \frac{7x + 3y - z}{16}.$$

Consequently, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$, and

$$(x, y, z) = \left(\frac{-3x + y + 5z}{16} \right) \mathbf{v}_1 + \left(\frac{-x - 5y + 7z}{16} \right) \mathbf{v}_2 + \left(\frac{7x + 3y - z}{16} \right) \mathbf{v}_3.$$

10. Let $(x, y) \in \mathbb{R}^2$ and $a, b, c \in \mathbb{R}$.

$$\begin{aligned} (x, y) &= a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = a(1, 1) + b(-1, 2) + c(1, 4) \\ &= (a, a) + (-b, 2b) + (c, 4c) \\ &= (a - b + c, a + 2b + 4c). \end{aligned}$$

These equalities result in the system:
$$\begin{cases} a - b + c = x \\ a + 2b + 4c = y \end{cases}$$

Solving the system we find that

$$a = \frac{2x + y - 6c}{3}, \text{ and } b = \frac{y - x - 3c}{3}$$

where c is a free real variable. It follows that

$$(x, y) = \left(\frac{2x + y - 6c}{3} \right) \mathbf{v}_1 + \left(\frac{y - x - 3c}{3} \right) \mathbf{v}_2 + c\mathbf{v}_3,$$

which implies that the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ span $\mathbb{R}^2$. Moreover, if $c = 0$, then

$$(x, y) = \left(\frac{2x + y}{3} \right) \mathbf{v}_1 + \left(\frac{y - x}{3} \right) \mathbf{v}_2,$$

so $\mathbb{R}^2 = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ also.

11. $\mathbf{x} = (c_1, c_2, c_2 - 2c_1) = (c_1, 0, -2c_1) + (0, c_2, c_2) = c_1(1, 0, -2) + c_2(0, 1, 1) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Thus, $\{(1, 0, -2), (0, 1, 1)\}$ spans S .

12. $\mathbf{v} = (c_1, c_2, c_2 - 2c_1, c_1 - 2c_2) = (c_1, 0, -c_1, c_1) + (0, c_2, c_2, -2c_2) = c_1(1, 0, -1, 1) + c_2(0, 1, 1, -2)$. Thus, $\{(1, 0, -1, 1), (0, 1, 1, -2)\}$ spans S .

13. $x - 2y - z = 0 \implies x = 2y + z$, so $\mathbf{v} \in \mathbb{R}^3$.

$$\implies \mathbf{v} = (2y + z, y, z) = (2y, y, 0) + (z, 0, z) = y(2, 1, 0) + z(1, 0, 1).$$

Therefore $S = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = a(2, 1, 0) + b(1, 0, 1), a, b \in \mathbb{R}\}$, hence $\{(2, 1, 0), (1, 0, 1)\}$ spans S .

14. $\text{nullspace}(A) = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{0}\}$. $A\mathbf{x} = \mathbf{0} \implies \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

Performing Gauss-Jordan elimination on the augmented matrix of the system we obtain:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 3 & 4 & 0 \\ 2 & 4 & 6 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

From the last matrix we find that $x_1 = -r$ and $x_2 = -r$ where $x_3 = r$, with $r \in \mathbb{R}$. Consequently,

$$\mathbf{x} = (x_1, x_2, x_3) = (-r, -r, r) = r(-1, -1, 1).$$

Thus,

$$\text{nullspace}(A) = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = r(-1, -1, 1), r \in \mathbb{R}\} = \text{span}\{(-1, -1, 1)\}.$$

15. $A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 1 & 3 & 4 & 2 \\ 2 & 4 & 6 & -1 \end{bmatrix}$. $\text{nullspace}(A) = \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{0}\}$. The REDUCED ROW ECHELON FORM

of the augmented matrix of this system is $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$. Consequently, $\text{nullspace}(A) = \{r(1, 1, -1, 0) : r \in \mathbb{R}\} = \text{span}\{(1, 1, -1, 0)\}$.

16. $A \in S \implies A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ where $a, b, c \in \mathbb{R}$. Thus,

$$A = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = aA_1 + bA_2 + cA_3.$$

Therefore $S = \text{span}\{A_1, A_2, A_3\}$.

17. $S = \left\{ A \in M_2(\mathbb{R}) : A = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}, \alpha \in \mathbb{R} \right\} = \left\{ A \in M_2(\mathbb{R}) : A = \alpha \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$.

18.

(a) $S \neq \emptyset$ since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$.

Closure under Addition: Let $\mathbf{x}, \mathbf{y} \in S$. Then,

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{bmatrix} + \begin{bmatrix} y_{11} & y_{12} \\ 0 & y_{22} \end{bmatrix} = \begin{bmatrix} x_{11} + y_{11} & x_{12} + y_{12} \\ 0 & x_{22} + y_{22} \end{bmatrix},$$

which implies that $\mathbf{x} + \mathbf{y} \in S$.

Closure under Scalar Multiplication: Let $r \in \mathbb{R}$ and $\mathbf{x} \in S$. Then

$$r\mathbf{x} = r \begin{bmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{bmatrix} = \begin{bmatrix} rx_{11} & rx_{12} \\ 0 & rx_{22} \end{bmatrix},$$

which implies that $r\mathbf{x} \in S$.

Consequently, S is a subspace of $M_2(\mathbb{R})$ by Theorem 4.3.2.

(b) $A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} = a_{11} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Therefore, $S = \text{span} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

19. Let $\mathbf{v} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $a, b \in \mathbb{R}$.

$$\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2 = a(1, -1, 2) + b(2, -1, 3) = (a, -a, 2a) + (2b, -b, 3b) = (a + 2b, -a - b, 2a + 3b).$$

Thus, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = (a + 2b, -a - b, 2a + 3b), a, b \in \mathbb{R}\}$.

Geometrically, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane through the origin determined by the two given vectors. The plane has parametric equations $x = a + 2b$, $y = -a - b$, and $z = 2a + 3b$. If a, b , and c are eliminated from the equations, then the resulting Cartesian equation is given by $x - y - z = 0$.

20. Let $\mathbf{v} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $a, b \in \mathbb{R}$.

$$\begin{aligned}\mathbf{v} &= a\mathbf{v}_1 + b\mathbf{v}_2 = a(1, 2, -1) + b(-2, -4, 2) = (a, 2a, -a) + (-2b, -4b, 2b) = (a - 2b, 2a - 4b, -a + 2b) \\ &= (a - 2b)(1, 2, -1) = k(1, 2, -1), \text{ where } k = a - 2b.\end{aligned}$$

Thus, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = k(1, 2, -1), k \in \mathbb{R}\}$. Geometrically, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is the line through the origin determined by the vector $(1, 2, -1)$.

21. Let $\mathbf{v} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $a, b, c \in \mathbb{R}$.

$$\begin{aligned}\mathbf{v} &= a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = a(1, 1, -1) + b(2, 1, 3) + c(-2, -2, 2) = (a, a, -a) + (2b, b, 3b) + (-2c, -2c, 2c) \\ &= (a + 2b - 2c, a + b - 2c, -a + 3b + 2c).\end{aligned}$$

Assuming that $\mathbf{v} = (x, y, z)$ and using the last ordered triple, we obtain the system:
$$\begin{cases} a + 2b - 2c = x \\ a + b - 2c = y \\ -a + 3b + 2c = z \end{cases}$$

Performing Gauss-Jordan elimination on the augmented matrix of the system, we obtain:

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & x \\ 1 & 1 & -2 & y \\ -1 & 3 & 2 & z \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -2 & x \\ 0 & -1 & 0 & y - x \\ 0 & 5 & 0 & x + z \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 2y - x \\ 0 & 1 & 0 & x - y \\ 0 & 0 & 0 & 5y - 4x + z \end{array} \right].$$

It is clear from the last matrix that the subspace, S , of $\mathbb{R}^3$ is a plane through $(0, 0, 0)$ with Cartesian equation $4x - 5y - z = 0$. Moreover, $\{\mathbf{v}_1, \mathbf{v}_2\}$ also spans the subspace S since

$$\begin{aligned}\mathbf{v} &= a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = a(1, 1, -1) + b(2, 1, 3) + c(-2, -2, 2) = a(1, 1, -1) + b(2, 1, 3) - 2c(1, 1, -1) \\ &= (a - 2c)(1, 1, -1) + b(2, 1, 3) = d\mathbf{v}_1 + b\mathbf{v}_2 \text{ where } d = a - 2c \in \mathbb{R}.\end{aligned}$$

22. If $\mathbf{v} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ then there exist $a, b \in \mathbb{R}$ such that

$$\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2 = a(1, -1, 2) + b(2, 1, 3) = (a, -a, 2a) + (2b, b, 3b) = (a + 2b, -a + b, 2a + 3b).$$

Hence, $\mathbf{v} = (3, 3, 4)$ is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ provided there exists $a, b \in \mathbb{R}$ satisfying the system:
$$\begin{cases} a + 2b = 3 \\ -a + b = 3 \\ 2a + 3b = 4 \end{cases}$$

Solving this system we find that $a = -1$ and $b = 2$.

Consequently, $\mathbf{v} = -\mathbf{v}_1 + 2\mathbf{v}_2$ so that $(3, 3, 4) \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

23. If $\mathbf{v} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ then there exist $a, b \in \mathbb{R}$ such that

$$\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2 = a(-1, 1, 2) + b(3, 1, -4) = (-a, a, 2a) + (3b, b, -4b) = (-a + 3b, a + b, 2a - 4b).$$

Hence $\mathbf{v} = (5, 3, -6)$ is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ provided there exists $a, b \in \mathbb{R}$ satisfying the system:
$$\begin{cases} -a + 3b = 5 \\ a + b = 3 \\ 2a - 4b = -6 \end{cases}$$

Solving this system we find that $a = 1$ and $b = 2$.

Consequently, $\mathbf{v} = \mathbf{v}_1 + 2\mathbf{v}_2$ so that $(5, 3, -6) \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

24. If $\mathbf{v} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ then there exist $a, b \in \mathbb{R}$ such that

$$\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2 = (3a, a, 2a) + (-2b, -b, b) = (3a - 2b, a - b, 2a + b).$$

Hence $\mathbf{v} = (1, 1, -2)$ is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ provided there exists $a, b \in \mathbb{R}$ satisfying the system:
$$\begin{cases} 3a - 2b = 1 \\ a - b = 1 \\ 2a + b = -2 \end{cases}$$

Since $\left[\begin{array}{cc|c} 3 & -2 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -2 \end{array} \right]$ reduces to $\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{array} \right]$, it follows that the system has no solution. Hence it must be the case that $(1, 1, -2) \notin \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

25. If $p \in \text{span}\{p_1, p_2\}$ then there exist $a, b \in \mathbb{R}$ such that $p(x) = ap_1(x) + bp_2(x)$, so $p(x) = 2x^2 - x + 2$ is in $\text{span}\{p_1, p_2\}$ provided there exist $a, b \in \mathbb{R}$ such that

$$\begin{aligned} 2x^2 - x + 2 &= a(x - 4) + b(x^2 - x + 3) \\ &= ax - 4a + bx^2 - bx + 3b \\ &= bx^2 + (a - b)x + (3b - 4a). \end{aligned}$$

Equating like coefficients and solving, we find that $a = 1$ and $b = 2$.

Thus, $2x^2 - x + 2 = 1 \cdot (x - 4) + 2 \cdot (x^2 - x + 3) = p_1(x) + 2p_2(x)$ so $p \in \text{span}\{p_1, p_2\}$.

26. Let $A \in \text{span}\{A_1, A_2, A_3\}$ and $c_1, c_2, c_3 \in \mathbb{R}$.

$$\begin{aligned} A &= c_1 A_1 + c_2 A_2 + c_3 A_3 = c_1 \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} c_1 & -c_1 \\ 2c_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c_2 \\ -2c_2 & c_2 \end{bmatrix} + \begin{bmatrix} 3c_3 & 0 \\ c_3 & 2c_3 \end{bmatrix} = \begin{bmatrix} c_1 + 3c_3 & -c_1 + c_2 \\ 2c_1 - 2c_2 + c_3 & c_2 + 2c_3 \end{bmatrix}. \end{aligned}$$

$$\text{Therefore } \text{span}\{A_1, A_2, A_3\} = \left\{ A \in M_2(\mathbb{R}) : A = \begin{bmatrix} c_1 + 3c_3 & -c_1 + c_2 \\ 2c_1 - 2c_2 + c_3 & c_2 + 2c_3 \end{bmatrix} \right\}.$$

27. Let $A \in \text{span}\{A_1, A_2\}$ and $a, b \in \mathbb{R}$.

$$A = aA_1 + bA_2 = a \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} + b \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} a & 2a \\ -a & 3a \end{bmatrix} + \begin{bmatrix} -2b & b \\ b & -b \end{bmatrix} = \begin{bmatrix} a - 2b & 2a + b \\ -a + b & 3a - b \end{bmatrix}.$$

So $\text{span}\{A_1, A_2\} = \left\{ A \in M_2(\mathbb{R}) : A = \begin{bmatrix} a - 2b & 2a + b \\ -a + b & 3a - b \end{bmatrix} \right\}$. Now, to determine whether $B \in \text{span}\{A_1, A_2\}$,

let $\begin{bmatrix} a - 2b & 2a + b \\ -a + b & 3a - b \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$. This implies that $a = 1$ and $b = -1$, thus $B \in \text{span}\{A_1, A_2\}$.

28. (a) The general vector in $\text{span}\{f, g\}$ is of the form $h(x) = c_1 \cosh x + c_2 \sinh x$ where $c_1, c_2 \in \mathbb{R}$.

(b) Let $h \in S$ and $c_1, c_2 \in \mathbb{R}$. Then

$$\begin{aligned} h(x) &= c_1 f(x) + c_2 g(x) = c_1 \cosh x + c_2 \sinh x \\ &= c_1 \frac{e^x + e^{-x}}{2} + c_2 \frac{e^x - e^{-x}}{2} = \frac{c_1 e^x}{2} + \frac{c_1 e^{-x}}{2} + \frac{c_2 e^x}{2} - \frac{c_2 e^{-x}}{2} \\ &= \frac{c_1 + c_2}{2} e^x + \frac{c_1 - c_2}{2} e^{-x} = d_1 e^x + d_2 e^{-x} \end{aligned}$$

where $d_1 = \frac{c_1 + c_2}{2}$ and $d_2 = \frac{c_1 - c_2}{2}$. Therefore $S = \text{span}\{e^x, e^{-x}\}$.

29. The origin in $\mathbb{R}^3$.

30. All points lying on the line through the origin with direction $\mathbf{v}_1$.

31. All points lying on the plane through the origin containing $\mathbf{v}_1$ and $\mathbf{v}_2$.

32. If $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{0}$, then the subspace is the origin in $\mathbb{R}^3$. If at least one of the vectors is nonzero, then the subspace consists of all points lying on the line through the origin in the direction of the nonzero vector.

33. Suppose that S is a subset of S' . We must show that every vector in $\text{span}(S)$ also belongs to $\text{span}(S')$. Every vector $\mathbf{v}$ that lies in $\text{span}(S)$ can be expressed as $\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_k \mathbf{v}_k$, where $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$

belong to S . However, since S is a subset of S' , $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ also belong to S' , and therefore, $\mathbf{v}$ belongs to $\text{span}(S')$. Thus, we have shown that every vector in $\text{span}(S)$ also lies in $\text{span}(S')$.

34.

Proof of $\implies$: We begin by supposing that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Since $\mathbf{v}_3 \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, our supposition implies that $\mathbf{v}_3 \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, which means that $\mathbf{v}_3$ can be expressed as a linear combination of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

Proof of $\impliedby$: Now suppose that $\mathbf{v}_3$ can be expressed as a linear combination of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. We must show that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, and we do this by showing that each of these subsets is a subset of the other. Since it is clear that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a subset of $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, we focus our attention on proving that every vector in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ belongs to $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. To see this, suppose that $\mathbf{v}$ belongs to $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, so that we may write $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. By assumption, $\mathbf{v}_3$ can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, so that we may write $\mathbf{v}_3 = d_1\mathbf{v}_1 + d_2\mathbf{v}_2$. Hence, we obtain

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3(d_1\mathbf{v}_1 + d_2\mathbf{v}_2) = (c_1 + c_3d_1)\mathbf{v}_1 + (c_2 + c_3d_2)\mathbf{v}_2 \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2\},$$

as required.

Solutions to Section 4.5

1. FALSE. For instance, consider the vector space $V = \mathbb{R}^2$. Here are two different minimal spanning sets for V :

$$\{(1, 0), (0, 1)\} \quad \text{and} \quad \{(1, 0), (1, 1)\}.$$

Many other examples of this abound.

2. TRUE. We have seven column vectors, and each of them belongs to $\mathbb{R}^5$. Therefore, the number of vectors present exceeds the number of components in those vectors, and hence they must be linearly dependent.

3. FALSE. For instance, the 7×5 zero matrix, $0_{7 \times 5}$, does not have linearly independent columns.

4. TRUE. Any linear dependencies within the subset also represent linear dependencies within the original, larger set of vectors. Therefore, if the nonempty subset were linearly dependent, then this would require that the original set is also linearly dependent. In other words, if the original set is linearly independent, then so is the nonempty subset.

5. TRUE. This is stated in Theorem 4.5.21.

6. TRUE. If we can write $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k$, then $\{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a linearly dependent set.

7. TRUE. This is a rephrasing of the statement in True-False Review Question 5 above.

8. FALSE. None of the vectors $(1, 0)$, $(0, 1)$, and $(1, 1)$ in $\mathbb{R}^2$ are proportional to each other, and yet, they form a linearly dependent set of vectors.

9. FALSE. The illustration given in part (c) of Example 4.5.22 gives an excellent case-in-point here.

Problems:

1. $\{(1, -1), (1, 1)\}$. These vectors are elements of $\mathbb{R}^2$. Since there are two vectors, and the dimension of $\mathbb{R}^2$ is two, Corollary 4.5.15 states that the vectors will be linearly dependent if and only if $\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 0$. Now

$\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2 \neq 0$. Consequently, the given vectors are linearly independent.

2. $\{(2, -1), (3, 2), (0, 1)\}$. These vectors are elements of $\mathbb{R}^2$, but since there are three vectors, the vectors are linearly dependent by Corollary 4.5.15. Let $\mathbf{v}_1 = (2, -1)$, $\mathbf{v}_2 = (3, 2)$, $\mathbf{v}_3 = (0, 1)$. We now determine a dependency relationship. The condition

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

requires $2c_1 + 3c_2 = 0$ and $-c_1 + 2c_2 + c_3 = 0$.

The REDUCED ROW ECHELON FORM of the augmented matrix of this system is $\left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{7} & 0 \\ 0 & 1 & \frac{2}{7} & 0 \end{array} \right]$.

Hence the system has solution $c_1 = 3r$, $c_2 = -2r$, $c_3 = 7r$. Consequently, $3\mathbf{v}_1 - 2\mathbf{v}_2 + 7\mathbf{v}_3 = \mathbf{0}$.

3. $\{(1, -1, 0), (0, 1, -1), (1, 1, 1)\}$. These vectors are elements of $\mathbb{R}^3$.

$\begin{vmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix} = 3 \neq 0$, so by Corollary 4.5.15, the vectors are linearly independent.

4. $\{(1, 2, 3), (1, -1, 2), (1, -4, 1)\}$. These vectors are elements of $\mathbb{R}^3$.

$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -4 \\ 3 & 2 & 1 \end{vmatrix} = 0$, so by Corollary 4.5.15, the vectors are linearly dependent. Let $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (1, -1, 2)$, $\mathbf{v}_3 = (1, -4, 1)$. We determine a dependency relation. The condition $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ requires

$$c_1 + c_2 + c_3 = 0, \quad 2c_1 - c_2 - 4c_3 = 0, \quad 3c_1 + 2c_2 + c_3 = 0.$$

The REDUCED ROW ECHELON FORM of the augmented matrix of this system is $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$.

Hence $c_1 = r$, $c_2 = -2r$, $c_3 = r$, and so $\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$.

5. Given $\{(-2, 4, -6), (3, -6, 9)\}$. The vectors are linearly dependent because $3(-2, 4, -6) + 2(3, -6, 9) = (0, 0, 0)$, which gives a linear dependency relation.

Alternatively, let $a, b \in \mathbb{R}$ and observe that

$$a(-2, 4, -6) + b(3, -6, 9) = (0, 0, 0) \implies (-2a, 4a, -6a) + (3b, -6b, 9b) = (0, 0, 0).$$

The last equality results in the system:
$$\begin{cases} -2a + 3b = 0 \\ 4a - 6b = 0 \\ -6a + 9b = 0 \end{cases}$$

The REDUCED ROW ECHELON FORM of the augmented matrix of this system is $\left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right]$, which

implies that $a = \frac{3}{2}b$. Thus, the given set of vectors is linearly dependent.

6. $\{(1, -1, 2), (2, 1, 0)\}$. Let $a, b \in \mathbb{R}$.

$a(1, -1, 2) + b(2, 1, 0) = (0, 0, 0) \implies (a, -a, 2a) + (2b, b, 0) = (0, 0, 0) \implies (a + 2b, -a + b, 2a) = (0, 0, 0)$. The

last equality results in the system:
$$\begin{cases} a + 2b = 0 \\ -a + b = 0 \\ 2a = 0 \end{cases}$$
 Since the only solution of the system is $a = b = 0$, it

follows that the vectors are linearly independent.

7. $\{(-1, 1, 2), (0, 2, -1), (3, 1, 2), (-1, -1, 1)\}$. These vectors are elements of $\mathbb{R}^3$.

Since there are four vectors, it follows from Corollary 4.5.15 that the vectors are linearly dependent. Let $\mathbf{v}_1 = (-1, 1, 2)$, $\mathbf{v}_2 = (0, 2, -1)$, $\mathbf{v}_3 = (3, 1, 2)$, $\mathbf{v}_4 = (-1, -1, 1)$. Then

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$$

requires

$$-c_1 + 3c_3 - c_4 = 0, \quad c_1 + 2c_2 + c_3 - c_4 = 0, \quad 2c_1 - c_2 + 2c_3 + c_4 = 0.$$

The REDUCED ROW ECHELON FORM of the augmented matrix of this system is
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{2}{5} & 0 \\ 0 & 1 & 0 & -\frac{3}{5} & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & 0 \end{array} \right]$$

so that

$$c_1 = -2r, c_2 = 3r, c_3 = r, c_4 = 5r. \text{ Hence, } -2\mathbf{v}_1 + 3\mathbf{v}_2 + \mathbf{v}_3 + 5\mathbf{v}_4 = \mathbf{0}.$$

8. $\{(1, -1, 2, 3), (2, -1, 1, -1), (-1, 1, 1, 1)\}$. Let $a, b, c \in \mathbb{R}$.

$$a(1, -1, 2, 3) + b(2, -1, 1, -1) + c(-1, 1, 1, 1) = (0, 0, 0, 0)$$

$$\implies (a, -a, 2a, 3a) + (2b, -b, b, -b) + (-c, c, c, c) = (0, 0, 0, 0)$$

$$\implies (a + 2b - c, -a - b + c, 2a + b + c, 3a - b + c) = (0, 0, 0, 0).$$

The last equality results in the system:
$$\begin{cases} a + 2b - c = 0 \\ -a - b + c = 0 \\ 2a + b + c = 0 \\ 3a - b + c = 0 \end{cases}$$
 The REDUCED ROW ECHELON FORM of the

augmented matrix of this system is
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$
 Consequently, $a = b = c = 0$. Thus, the given set of

vectors is linearly independent.

9. $\{(2, -1, 0, 1), (1, 0, -1, 2), (0, 3, 1, 2), (-1, 1, 2, 1)\}$. These vectors are elements in $\mathbb{R}^4$.

By CET (row 1), we obtain:

$$\begin{vmatrix} 2 & 1 & 0 & -1 \\ -1 & 0 & 3 & 1 \\ 0 & -1 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{vmatrix} = 2 \begin{vmatrix} 0 & 3 & 1 \\ -1 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 3 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 0 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 21 \neq 0.$$

Thus, it follows from Corollary 4.5.15 that the vectors are linearly independent.

10. Since $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 0$, the given set of vectors is linearly dependent in $\mathbb{R}^3$. Further, since the vectors

are not collinear, it follows that $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a plane 3-space.

11. (a) Since $\begin{vmatrix} 2 & 1 & -3 \\ -1 & 3 & -9 \\ 5 & -4 & 12 \end{vmatrix} = 0$, the given set of vectors is linearly dependent.

(b) By inspection, we see that $\mathbf{v}_3 = -3\mathbf{v}_2$. Hence $\mathbf{v}_2$ and $\mathbf{v}_3$ are collinear and therefore $\text{span}\{\mathbf{v}_2, \mathbf{v}_3\}$ is the line through the origin that has the direction of $\mathbf{v}_2$. Further, since $\mathbf{v}_1$ is not proportional to either of these vectors, it does not lie along the same line, hence $\mathbf{v}_1$ is not in $\text{span}\{\mathbf{v}_2, \mathbf{v}_3\}$.

12. $\begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & k \\ k & k & 6 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & k-1 \\ 0 & k & 6-k \end{vmatrix} = \begin{vmatrix} 2 & k-1 \\ k & 6-k \end{vmatrix} = (3-k)(k+4)$. Now, the determinant is zero when $k = 3$ or $k = -4$. Consequently, by Corollary 4.5.15, the vectors are linearly dependent if and only if $k = 3$ or $k = -4$.

13. $\{(1, 0, 1, k), (-1, 0, k, 1), (2, 0, 1, 3)\}$. These vectors are elements in $\mathbb{R}^4$.

Let $a, b, c \in \mathbb{R}$.

$$a(1, 0, 1, k) + b(-1, 0, k, 1) + c(2, 0, 1, 3) = (0, 0, 0, 0)$$

$$\implies (a, 0, a, ka) + (-b, 0, kb, b) + (2c, 0, c, 3c) = (0, 0, 0, 0)$$

$$\implies (a - b + 2c, 0, a + kb + c, ka + b + 3c) = (0, 0, 0, 0).$$

The last equality results in the system:
$$\begin{cases} a - b + 2c = 0 \\ a + kb + c = 0 \\ ka + b + 3c = 0 \end{cases}$$
 Evaluating the determinant of the coefficient matrix, we obtain

$$\begin{vmatrix} 1 & -1 & 2 \\ 1 & k & 1 \\ k & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & k+1 & -1 \\ k & k+1 & 3-2k \end{vmatrix} = 2(k+1)(2-k).$$

Consequently, the system has only the trivial solution, hence the given set of vectors are linearly independent if and only if $k \neq 2, -1$.

14. $\{(1, 1, 0, -1), (1, k, 1, 1), (2, 1, k, 1), (-1, 1, 1, k)\}$. These vectors are elements in $\mathbb{R}^4$.

Let $a, b, c, d \in \mathbb{R}$.

$$a(1, 1, 0, -1) + b(1, k, 1, 1) + c(2, 1, k, 1) + d(-1, 1, 1, k) = (0, 0, 0, 0)$$

$$\implies (a, a, 0, -a) + (b, kb, b, b) + (2c, c, kc, c) + (-d, d, d, kd) = (0, 0, 0, 0)$$

$$\implies (a + b + 2c - d, a + kb + c + d, b + kc + d, -a + b + c + kd) = (0, 0, 0, 0).$$

The last equality results in the system:
$$\begin{cases} a + b + 2c - d = 0 \\ a + kb + c + d = 0 \\ b + kc + d = 0 \\ -a + b + c + kd = 0 \end{cases}$$
 By Corollary 3.2.5, this system has the

trivial solution if and only if the determinant of the coefficient matrix is zero. Evaluating the determinant of the coefficient matrix, we obtain:

$$\begin{vmatrix} 1 & 1 & 2 & -1 \\ 1 & k & 1 & 1 \\ 0 & 1 & k & 1 \\ -1 & 1 & 1 & k \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 & -1 \\ 0 & k-1 & -1 & 2 \\ 0 & 1 & k & 1 \\ 0 & 2 & 3 & k-1 \end{vmatrix} = \begin{vmatrix} k-1 & -1 & 2 \\ 1 & k & 1 \\ 2 & 3 & k-1 \end{vmatrix} = \begin{vmatrix} 0 & -k^2+k-1 & 3-k \\ 1 & k & 1 \\ 0 & 3-2k & k-3 \end{vmatrix} \\ = \begin{vmatrix} k^2-k+1 & k-3 \\ 3-2k & k-3 \end{vmatrix} = (k-3) \begin{vmatrix} k^2-k+1 & 1 \\ 3-2k & 1 \end{vmatrix} = (k-3)(k-1)(k+2).$$

For the original set of vectors to be linearly independent, we need $a = b = c = 0$. We see that this condition will be true provided that $k \in \mathbb{R}$ such that $k \neq 3, 1, -2$.

15. Let $a, b, c \in \mathbb{R}$. $a \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} + c \begin{bmatrix} 3 & 6 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\implies \begin{bmatrix} a+2b+3c & a-b+6c \\ 0 & a+b+4c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \text{ The last equation results in the system: } \begin{cases} a+2b+3c=0 \\ a-b+6c=0 \\ a+b+4c=0 \end{cases}$$

The REDUCED ROW ECHELON FORM of the augmented matrix of this system is $\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$, which implies that the system has an infinite number of solutions. Consequently, the given matrices are linearly dependent.

16. Let $a, b \in \mathbb{R}$. $a \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} + b \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2a - b & -a + 2b \\ 3a + b & 4a + 3b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \text{ The last equation results in the system: } \begin{cases} 2a - b = 0 \\ 3a + b = 0 \\ -a + 2b = 0 \\ 4a + 3b = 0 \end{cases}. \text{ This system}$$

has only the trivial solution $a = b = 0$, thus the given matrices are linearly independent in $M_2(\mathbb{R})$.

17. Let $a, b, c \in \mathbb{R}$. $a \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + b \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} + c \begin{bmatrix} 2 & 1 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a - b + 2c & b + c \\ a + 2b + 5c & 2a + b + 7c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \text{ The last equation results in the system: } \begin{cases} a - b + 2c = 0 \\ a + 2b + 5c = 0 \\ 2a + b + 7c = 0 \\ b + c = 0 \end{cases}.$$

The REDUCED ROW ECHELON FORM of the augmented matrix of this system is $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$, which

implies that the system has an infinite number of solutions. Consequently, the given matrices are linearly dependent.

18. Let $a, b \in \mathbb{R}$. $ap_1(x) + bp_2(x) = 0 \Rightarrow a(1 - x) + b(1 + x) = 0 \Rightarrow (a + b) + (-a + b)x = 0$.

Equating like coefficients, we obtain the system: $\begin{cases} a + b = 0 \\ -a + b = 0 \end{cases}$. Since the only solution to this system is $a = b = 0$, it follows that the given vectors are linearly independent.

19. Let $a, b \in \mathbb{R}$. $ap_1(x) + bp_2(x) = 0 \Rightarrow a(2 + 3x) + b(4 + 6x) = 0 \Rightarrow (2a + 4b) + (3a + 6b)x = 0$.

Equating like coefficients, we obtain the system: $\begin{cases} 2a + 4b = 0 \\ 3a + 6b = 0 \end{cases}$. The REDUCED ROW ECHELON FORM

of the augmented matrix of this system is $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right]$, which implies that the system has an infinite number of solutions. Thus, the given vectors are linearly dependent.

20. Let $c_1, c_2 \in \mathbb{R}$. $c_1p_1(x) + c_2p_2(x) = 0 \Rightarrow c_1(a + bx) + c_2(c + dx) = 0 \Rightarrow (ac_1 + cc_2) + (bc_1 + dc_2)x = 0$.

Equating like coefficients, we obtain the system: $\begin{cases} ac_1 + cc_2 = 0 \\ bc_1 + dc_2 = 0 \end{cases}$. The determinant of the matrix of coef-

ficients is $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$. Consequently, the system has just the trivial solution, and hence $p_1(x)$ and $p_2(x)$ are linearly independent if and only if $ad - bc \neq 0$.

21. Since $\cos 2x = \cos^2 x - \sin^2 x$, $f_1(x) = f_3(x) - f_2(x)$ so it follows that f_1, f_2 , and f_3 are linearly dependent in $C^\infty(-\infty, \infty)$.

22. Let $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (-3, 4, 5)$, $\mathbf{v}_3 = (1, -\frac{4}{3}, -\frac{5}{3})$. By inspection, we see that $\mathbf{v}_2 = -3\mathbf{v}_3$. Further, since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not proportional they are linearly independent. Consequently, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set of vectors and $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

23. Let $\mathbf{v}_1 = (3, 1, 5)$, $\mathbf{v}_2 = (0, 0, 0)$, $\mathbf{v}_3 = (1, 2, -1)$, $\mathbf{v}_4 = (-1, 2, 3)$. Since $\mathbf{v}_2 = \mathbf{0}$, it is certainly true that $\text{span}\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. Further, since $\det[\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4] = 42 \neq 0$, $\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$ is a linearly independent set.

24. Since we have four vectors in $\mathbb{R}^3$, the given set is linearly dependent. We could determine the specific linear dependency between the vectors to find a linearly independent subset, but in this case, if we just take any

three of the vectors, say $(1, -1, 1)$, $(1, -3, 1)$, $(3, 1, 2)$, then $\begin{vmatrix} 1 & 1 & 3 \\ -1 & -3 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 2 \neq 0$, so that these vectors are linearly independent. Consequently, $\text{span}\{(1, -1, 1), (1, -3, 1), (3, 1, 2)\} = \text{span}\{(1, 1, 1), (1, -1, 1), (1, -3, 1), (3, 1, 2)\}$.

25. Let $\mathbf{v}_1 = (1, 1, -1, 1)$, $\mathbf{v}_2 = (2, -1, 3, 1)$, $\mathbf{v}_3 = (1, 1, 2, 1)$, $\mathbf{v}_4 = (2, -1, 2, 1)$.

Since $\begin{vmatrix} 1 & 2 & 1 & 2 \\ 1 & -1 & 1 & -1 \\ -1 & 3 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent. We now determine the linearly dependent relationship. The REDUCED ROW ECHELON FORM of the augmented matrix corresponding to the system

$$c_1(1, 1, -1, 1) + c_2(2, -1, 3, 1) + c_3(1, 1, 2, 1) + c_4(2, -1, 2, 1) = (0, 0, 0, 0)$$

is $\begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, so that $c_1 = -r$, $c_2 = -3r$, $c_3 = r$, $c_4 = 3r$, where r is a free variable. It follows that a linearly dependent relationship between the given set of vectors is

$$-\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3 + 3\mathbf{v}_4 = \mathbf{0}$$

so that

$$\mathbf{v}_1 = -3\mathbf{v}_2 + \mathbf{v}_3 + 3\mathbf{v}_4.$$

Consequently, $\text{span}\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, and $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a linearly independent set.

26. Let $A_1 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 2 \\ 5 & 7 \end{bmatrix}$, $A_3 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$. Then

$$c_1A_1 + c_2A_2 + c_3A_3 = \mathbf{0}_2$$

requires that

$$c_1 - c_2 + 3c_3 = 0, \quad 2c_1 + 2c_2 + 2c_3 = 0, \quad 3c_1 + 5c_2 + c_3 = 0, \quad 4c_1 + 7c_2 + c_3 = 0.$$

The REDUCED ROW ECHELON FORM of the augmented matrix of this system is $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Consequently, the matrices are linearly dependent. Solving the system gives $c_1 = -2r$, $c_2 = c_3 = r$. Hence, a linearly dependent relationship is $-2A_1 + A_2 + A_3 = \mathbf{0}_2$.

27. We first determine whether the given set of polynomials is linearly dependent. Let

$$p_1(x) = 2 - 5x, \quad p_2(x) = 3 + 7x, \quad p_3(x) = 4 - x.$$

Then

$$c_1(2 - 5x) + c_2(3 + 7x) + c_3(4 - x) = 0$$

requires

$$2c_1 + 3c_2 + 4c_3 = 0 \quad \text{and} \quad -5c_1 + 7c_2 - c_3 = 0.$$

This system has solution $(-31r, -18r, 29r)$, where r is a free variable. Consequently, the given set of polynomials is linearly dependent, and a linearly dependent relationship is

$$-31p_1(x) - 18p_2(x) + 29p_3(x) = 0,$$

or equivalently,

$$p_3(x) = \frac{1}{29}[31p_1(x) + 18p_2(x)].$$

Hence, the linearly independent set of vectors $\{2 - 5x, 3 + 7x\}$ spans the same subspace of P_1 as that spanned by $\{2 - 5x, 3 + 7x, 4 - x\}$.

28. We first determine whether the given set of polynomials is linearly dependent. Let

$$p_1(x) = 2 + x^2, \quad p_2(x) = 4 - 2x + 3x^2, \quad p_3(x) = 1 + x.$$

Then

$$c_1(2 + x^2) + c_2(4 - 2x + 3x^2) + c_3(1 + x) = 0$$

leads to the system

$$2c_1 + 4c_2 + c_3 = 0, \quad -2c_2 + c_3 = 0, \quad c_1 + 3c_2 = 0.$$

This system has solution $(-3r, r, 2r)$ where r is a free variable. Consequently, the given set of vectors is linearly dependent, and a specific linear relationship is

$$-3p_1(x) + p_2(x) + 2p_3(x) = 0,$$

or equivalently,

$$p_2(x) = 3p_1(x) - 2p_3(x).$$

Hence, the linearly independent set of vectors $\{2 + x^2, 1 + x\}$ spans the same subspace of P_2 as that spanned by the given set of vectors.

29. $W[f_1, f_2, f_3](x) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2$. Since $W[f_1, f_2, f_3](x) \neq 0$ on I , it follows that the functions are linearly independent on I .

30.

$$\begin{aligned} W[f_1, f_2, f_3](x) &= \begin{vmatrix} \sin x & \cos x & \tan x \\ \cos x & -\sin x & \sec^2 x \\ -\sin x & -\cos x & 2 \tan x \sec^2 x \end{vmatrix} = \begin{vmatrix} \sin x & \cos x & \tan x \\ \cos x & -\sin x & \sec^2 x \\ 0 & 0 & \tan x(2 \sec^2 x + 1) \end{vmatrix} \\ &= -\tan x(2 \sec^2 x + 1). \end{aligned}$$

$W[f_1, f_2, f_3](x)$ is not always zero over I , so the vectors are linearly independent by Theorem 4.5.21

31. $W[f_1, f_2, f_3](x) = \begin{vmatrix} 1 & 3x & x^2 - 1 \\ 0 & 3 & 2x \\ 0 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 2x \\ 0 & 2 \end{vmatrix} = 6 \neq 0$ on I . Consequently, $\{f_1, f_2, f_3\}$ is a linearly independent set on I by Theorem 4.5.21.

32. $W[f_1, f_2, f_3](x) = \begin{vmatrix} e^{2x} & e^{3x} & e^{-x} \\ 2e^{2x} & 3e^{3x} & -e^{-x} \\ 4e^{2x} & 9e^{3x} & e^{-x} \end{vmatrix} = e^{4x} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 4 & 9 & 1 \end{vmatrix} = e^{4x} \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -3 \\ 4 & 5 & -3 \end{vmatrix} = 12e^{4x}$. Since the Wronskian is never zero, the functions are linearly independent on $(-\infty, \infty)$.

33. On $[0, \infty)$, $f_2 = 7f_1$, so that the functions are linearly dependent on this interval. Therefore $W[f_1, f_2](x) = 0$ for $x \in [0, \infty)$. However, on $(-\infty, 0)$, $W[f_1, f_2](x) = \begin{vmatrix} 3x^3 & 7x^2 \\ 9x^2 & 14x \end{vmatrix} = -21x^4 \neq 0$. Since the Wronskian is not zero for all $x \in (-\infty, \infty)$, the functions are linearly independent on that interval.

34. $W[f_1, f_2, f_3](x) = \begin{vmatrix} 1 & x & 2x - 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix} = 0$. By inspection, we see that $f_3 = 2f_2 - f_1$, so that the functions are linearly dependent on $(-\infty, \infty)$.

35. We show that the Wronskian (the determinant can be computed by cofactor expansion along the first row) is identically zero:

$$\begin{vmatrix} e^x & e^{-x} & \cosh x \\ e^x & -e^{-x} & \sinh x \\ e^x & e^{-x} & \cosh x \end{vmatrix} = -(\cosh x + \sinh x) - (\cosh x - \sinh x) + 2 \cosh x = 0.$$

Thus, the Wronskian is identically zero on $(-\infty, \infty)$. Furthermore, $\{f_1, f_2, f_3\}$ is a linearly dependent set because

$$-\frac{1}{2}f_1(x) - \frac{1}{2}f_2(x) + f_3(x) = -\frac{1}{2}e^x - \frac{1}{2}e^{-x} + \cosh x = -\frac{1}{2}e^x - \frac{1}{2}e^{-x} + \frac{e^x + e^{-x}}{2} = 0 \text{ for all } x \in I.$$

36. We show that the Wronskian is identically zero for $f_1(x) = ax^3$ and $f_2(x) = bx^3$, which covers the functions in this problem as a special case:

$$\begin{vmatrix} ax^3 & bx^3 \\ 3ax^2 & 3bx^2 \end{vmatrix} = 3abx^5 - 3abx^5 = 0.$$

Next, let $a, b \in \mathbb{R}$.

If $x \geq 0$, then

$$af_1(x) + bf_2(x) = 0 \implies 2ax^3 + 5bx^3 = 0 \implies (2a + 5b)x^3 = 0 \implies 2a + 5b = 0.$$

If $x \leq 0$, then

$$af_1(x) + bf_2(x) = 0 \implies 2ax^3 - 3bx^3 = 0 \implies (2a - 3b)x^3 = 0 \implies 2a - 3b = 0.$$

Solving the resulting system, we obtain $a = b = 0$; therefore, $\{f_1, f_2\}$ is a linearly independent set of vectors on $(-\infty, \infty)$.

37. (a) When $x > 0$, $f'_2(x) = 1$ and when $x < 0$, $f'_2(x) = -1$; thus $f'_2(0)$ does not exist, which implies that $f_2 \notin C^1(-\infty, \infty)$.

(b) Let $a, b \in \mathbb{R}$. On the interval $(-\infty, 0)$, $ax + b(-x) = 0$, which has more than the trivial solution for a and b . Thus, $\{f_1, f_2\}$ is a linearly dependent set of vectors on $(-\infty, 0)$.

On the interval $[0, \infty)$, $ax + bx = 0 \implies a + b = 0$, which has more than the trivial solution for a and b . Therefore $\{f_1, f_2\}$ is a linearly dependent set of vectors on $[0, \infty)$.

On the interval $(-\infty, \infty)$, a and b must satisfy: $ax + b(-x) = 0$ and $ax + bx = 0$, that is, $\begin{cases} a - b = 0 \\ a + b = 0 \end{cases}$. Since this system has only $a = b = 0$ as its solution, $\{f_1, f_2\}$ is a linearly independent set of vectors on $(-\infty, \infty)$.

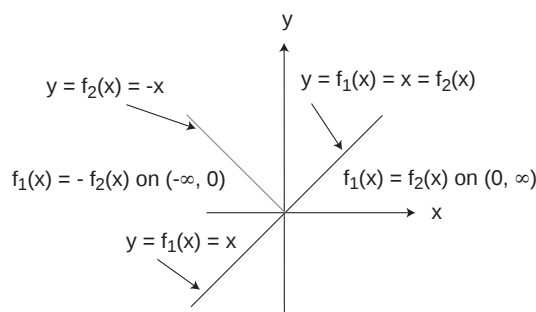


Figure 64: Figure for Problem 37

38. Let $a, b \in \mathbb{R}$.

$$af_1(x) + bf_2(x) = 0 \implies \begin{cases} ax + bx = 0 \text{ if } x \neq 0 \\ a(0) + b(1) = 0 \text{ if } x = 0 \end{cases} \implies \begin{cases} (a+b)x = 0 \\ b = 0 \end{cases} \\ \implies a = 0 \text{ and } b = 0 \text{ so } \{f_1, f_2\} \text{ is a linearly independent set on } I.$$

39. Let $a, b, c \in \mathbb{R}$ and $x \in (-\infty, \infty)$.

$$af_1(x) + bf_2(x) + cf_3(x) = 0 \implies \begin{cases} a(x-1) + b(2x) + c(3) = 0 \text{ for } x \geq 1 \\ 2a(x-1) + b(2x) + c(3) = 0 \text{ for } x < 1 \end{cases} \\ \implies \begin{cases} (a+2b)x + (-a+3c) = 0 \\ (2a+2b)x + (-2a+3c) = 0 \end{cases} \implies \begin{cases} a+2b = 0 \text{ and } -a+3c = 0 \\ 2a+2b = 0 \text{ and } -2a+3c = 0. \end{cases}$$

Since the only solution to this system of equations is $a = b = c = 0$, it follows that the given functions are linearly independent on $(-\infty, \infty)$.

The domain space may be divided into three types of intervals: (1) interval subsets of $(-\infty, 1)$, (2) interval subset of $[1, \infty)$, (3) intervals containing 1 where 1 is not an endpoint of the intervals.

For intervals of type (3):

Intervals such as type (3) are treated as above [with domain space of $(-\infty, \infty)$]: vectors are linearly independent.

For intervals of type (1):

$$a(2(x-1)) + b(2x) + c(3) = 0 \implies (2a+2b)x + (-2a+3c) = 0 \implies 2a+2b = 0, \text{ and } -2a+3c = 0.$$

Since this system has three variables with only two equations, the solution to the system is not unique, hence intervals of type (1) result in linearly dependent vectors.

For intervals of type (2):

$a(x-1)+b(2x)+c(3)=0 \implies a+2b=0$ and $-a+3c=0$. As in the last case, this system has three variables with only two equations, so it must be the case that intervals of type (2) result in linearly dependent vectors.

40. (a) Let $f_0(x)=1, f_1(x)=x, f_2(x)=x^2, f_3(x)=x^3$.

Then $W[f_1, f_2, f_3, f_4](x) = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} = 12 \neq 0$. Hence, $\{f_1, f_2, f_3, f_4\}$ is linearly independent on any interval.

$$(b) \quad W[f_0, f_1, f_2, \dots, f_n](x) = \begin{vmatrix} 1 & x & x^2 & \dots & x^n \\ 0 & 1 & 2x & \dots & nx^{n-1} \\ 0 & 0 & 2 & \dots & n(n-1)x^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & n! \end{vmatrix}.$$

The matrix corresponding to this determinant is upper triangular, so the value of the determinant is given by the product of all of the diagonal entries. $W[f_0, f_1, f_2, \dots, f_n](x) = 1 \cdot 1 \cdot 2 \cdot 6 \cdot 24 \cdots n!$, which is not zero regardless of the actual domain of x . Consequently, the functions are linearly independent on any interval.

41. (a) Let $f_1(x)=e^{r_1x}, f_2(x)=e^{r_2x}$ and $f_3(x)=e^{r_3x}$. Then

$$W[f_1, f_2, f_3](x) = \begin{vmatrix} e^{r_1x} & e^{r_2x} & e^{r_3x} \\ r_1e^{r_1x} & r_2e^{r_2x} & r_3e^{r_3x} \\ r_1^2e^{r_1x} & r_2^2e^{r_2x} & r_3^2e^{r_3x} \end{vmatrix} = e^{r_1x}e^{r_2x}e^{r_3x} \begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{vmatrix} \\ = e^{(r_1+r_2+r_3)x} (r_3-r_1)(r_3-r_2)(r_2-r_1).$$

If $r_i \neq r_j$ for $i \neq j$, then $W[f_1, f_2, f_3](x)$ is never zero, and hence the functions are linearly independent on any interval. If, on the other hand, $r_i = r_j$ with $i \neq j$, then $f_i - f_j = 0$, so that the functions are linearly dependent. Thus, r_1, r_2, r_3 must all be different in order that f_1, f_2, f_3 are linearly independent.

(b)

$$W[f_1, f_2, f_3, \dots, f_n](x) = \begin{vmatrix} e^{r_1x} & e^{r_2x} & \dots & e^{r_nx} \\ r_1e^{r_1x} & r_2e^{r_2x} & \dots & r_ne^{r_nx} \\ r_1^2e^{r_1x} & r_2^2e^{r_2x} & \dots & r_n^2e^{r_nx} \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1}e^{r_1x} & r_2^{n-1}e^{r_2x} & \dots & r_n^{n-1}e^{r_nx} \end{vmatrix} \\ = e^{r_1x}e^{r_2x} \dots e^{r_nx} \begin{vmatrix} 1 & 1 & \dots & 1 \\ r_1 & r_2 & \dots & r_n \\ r_1^2 & r_2^2 & \dots & r_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \dots & r_n^{n-1} \end{vmatrix} \\ \stackrel{*}{=} e^{r_1x}e^{r_2x} \dots e^{r_nx} V(r_1, r_2, \dots, r_n) \\ = e^{r_1x}e^{r_2x} \dots e^{r_nx} \prod_{1 \leq i < m \leq n} (r_m - r_i).$$

Now from the last equality, we see that if $r_i \neq r_j$ for $i \neq j$, where $i, j \in \{1, 2, 3, \dots, n\}$, then $W[f_1, f_2, \dots, f_n](x)$ is never zero, and hence the functions are linearly independent on any interval. If, on the other hand, $r_i = r_j$ with $i \neq j$, then $f_i - f_j = 0$, so that the functions are linearly dependent. Thus $\{f_1, f_2, \dots, f_n\}$ is a linearly independent set if and only if all the r_i are distinct for $i \in \{1, 2, 3, \dots, n\}$.

(*Note: $V(r_1, r_2, \dots, r_n)$ is the $n \times n$ Vandermonde determinant. See Section 3.3 Problem 21).

42. Let $a, b \in \mathbb{R}$. Assume that $a\mathbf{v} + b\mathbf{w} = \mathbf{0}$. Then $a(\alpha\mathbf{v}_1 + \mathbf{v}_2) + b(\mathbf{v}_1 + \alpha\mathbf{v}_2) = \mathbf{0}$ which implies that $(\alpha a + b)\mathbf{v}_1 + (a + b\alpha)\mathbf{v}_2 = \mathbf{0}$. Now since it is given that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, $\alpha a + b = 0$ and $a + b\alpha = 0$. This system has only the trivial solution for a and b if and only if $\begin{vmatrix} \alpha & 1 \\ 1 & \alpha \end{vmatrix} \neq 0$. That is, if and only if $\alpha^2 - 1 \neq 0$, or $\alpha \neq \pm 1$. Therefore, the vectors are linearly independent provided that $\alpha \neq \pm 1$.

43. It is given that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. Let $\mathbf{u}_1 = a_1\mathbf{v}_1 + b_1\mathbf{v}_2$, $\mathbf{u}_2 = a_2\mathbf{v}_1 + b_2\mathbf{v}_2$, and $\mathbf{u}_3 = a_3\mathbf{v}_1 + b_3\mathbf{v}_2$ where $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$. Let $c_1, c_2, c_3 \in \mathbb{R}$. Then
 $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{0}$
 $\implies c_1(a_1\mathbf{v}_1 + b_1\mathbf{v}_2) + c_2(a_2\mathbf{v}_1 + b_2\mathbf{v}_2) + c_3(a_3\mathbf{v}_1 + b_3\mathbf{v}_2) = \mathbf{0}$
 $\implies (c_1a_1 + c_2a_2 + c_3a_3)\mathbf{v}_1 + (c_1b_1 + c_2b_2 + c_3b_3)\mathbf{v}_2 = \mathbf{0}$.

Now since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent: $\begin{cases} c_1a_1 + c_2a_2 + c_3a_3 = 0 \\ c_1b_1 + c_2b_2 + c_3b_3 = 0. \end{cases}$ There are an infinite number of solutions to this homogeneous system since there are three unknowns but only two equations. Hence, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a linearly dependent set of vectors.

44. Given that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is a linearly independent set of vectors in a vector space V , and

$$\mathbf{u}_k = \sum_{i=1}^m a_{ik}\mathbf{v}_i, k \in \{1, 2, \dots, n\}. \quad (44.1)$$

(a) Let $c_k \in \mathbb{R}, k \in \{1, 2, \dots, n\}$. Using system (44.1) and $\sum_{k=1}^n c_k\mathbf{u}_k = \mathbf{0}$, we obtain:

$$\sum_{k=1}^n c_k \left(\sum_{i=1}^m a_{ik}\mathbf{v}_i \right) = \mathbf{0} \iff \sum_{i=1}^m \left(\sum_{k=1}^n a_{ik}c_k \right) \mathbf{v}_i = \mathbf{0}.$$

Since the $\mathbf{v}_i$ for each $i \in \{1, 2, \dots, m\}$ are linearly independent,

$$\sum_{k=1}^n a_{ik}c_k = 0, 1 \leq i \leq m. \quad (44.2)$$

But this is a system of m equations with n unknowns $c_1, c_2, \dots, c_n$. Since $n > m$, the system has more unknowns than equations, and so has nontrivial solutions. Thus, $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is a linearly dependent set.

(b) If $m = n$, then the system (44.2) has a trivial solution $\iff$ the coefficient matrix of the system is invertible $\iff \det[a_{ij}] \neq 0$.

(c) If $n < m$, then the homogeneous system (44.2) has just the trivial solution if and only if $\text{rank}(A) = n$. Recall that for a homogeneous system, $\text{rank}(A^\#) = \text{rank}(A)$.

(d) Corollary 4.5.15.

45. We assume that

$$c_1(A\mathbf{v}_1) + c_2(A\mathbf{v}_2) + \dots + c_n(A\mathbf{v}_n) = \mathbf{0}.$$

Our aim is to show that $c_1 = c_2 = \dots = c_n = 0$. We manipulate the left side of the above equation as

follows:

$$\begin{aligned} c_1(A\mathbf{v}_1) + c_2(A\mathbf{v}_2) + \cdots + c_n(A\mathbf{v}_n) &= \mathbf{0} \\ A(c_1\mathbf{v}_1) + A(c_2\mathbf{v}_2) + \cdots + A(c_n\mathbf{v}_n) &= \mathbf{0} \\ A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) &= \mathbf{0}. \end{aligned}$$

Since A is invertible, we can left multiply the last equation by A^{-1} to obtain

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is linearly independent, we can now conclude directly that $c_1 = c_2 = \cdots = c_n = 0$, as required.

46. Assume that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$. We must show that $c_1 = c_2 = c_3 = 0$. Let us suppose for the moment that $c_3 \neq 0$. In that case, we can solve the above equation for $\mathbf{v}_3$:

$$\mathbf{v}_3 = -\frac{c_1}{c_3}\mathbf{v}_1 - \frac{c_2}{c_3}\mathbf{v}_2.$$

However, this contradicts the assumption that $\mathbf{v}_3$ does not belong to $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Therefore, we conclude that $c_3 = 0$. Our starting equation therefore reduces to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$. Now the assumption that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent shows that $c_1 = c_2 = 0$. Therefore, $c_1 = c_2 = c_3 = 0$, as required.

47. Assume that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k + c_{k+1}\mathbf{v}_{k+1} = \mathbf{0}$. We must show that $c_1 = c_2 = \cdots = c_{k+1} = 0$. Let us suppose for the moment that $c_{k+1} \neq 0$. In that case, we can solve the above equation for $\mathbf{v}_{k+1}$:

$$\mathbf{v}_{k+1} = -\frac{c_1}{c_{k+1}}\mathbf{v}_1 - \frac{c_2}{c_{k+1}}\mathbf{v}_2 - \cdots - \frac{c_k}{c_{k+1}}\mathbf{v}_k.$$

However, this contradicts the assumption that $\mathbf{v}_{k+1}$ does not belong to $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$. Therefore, we conclude that $c_{k+1} = 0$. Our starting equation therefore reduces to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$. Now the assumption that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is linearly independent shows that $c_1 = c_2 = \cdots = c_k = 0$. Therefore, $c_1 = c_2 = \cdots = c_k = c_{k+1} = 0$, as required.

48. Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a set of vectors with $k \geq 2$. Suppose that $\mathbf{v}_k$ can be expressed as a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k-1}\}$. We claim that

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k-1}\}.$$

Since every vector belonging to $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k-1}\}$ evidently belongs to $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$, we focus on showing that every vector in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ also belongs to $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k-1}\}$:

Let $\mathbf{v} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$. We therefore may write

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k.$$

By assumption, we may write $\mathbf{v}_k = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_{k-1}\mathbf{v}_{k-1}$. Therefore, we obtain

$$\begin{aligned} \mathbf{v} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \\ &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{k-1}\mathbf{v}_{k-1} + c_k(d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_{k-1}\mathbf{v}_{k-1}) \\ &= (c_1 + c_k d_1)\mathbf{v}_1 + (c_2 + c_k d_2)\mathbf{v}_2 + \cdots + (c_{k-1} + c_k d_{k-1})\mathbf{v}_{k-1} \\ &\in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k-1}\}. \end{aligned}$$

This shows that every vector belonging to $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ also belongs to $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{k-1}\}$, as needed.

49. We first prove part 1 of Proposition 4.5.7. Suppose that we have a set $\{\mathbf{u}, \mathbf{v}\}$ of two vectors in a vector space V . If $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent, then we have

$$c\mathbf{u} + d\mathbf{v} = \mathbf{0},$$

where c and d are not both zero. Without loss of generality, suppose that $c \neq 0$. Then we have

$$\mathbf{u} = -\frac{d}{c}\mathbf{v},$$

so that $\mathbf{u}$ and $\mathbf{v}$ are proportional. Conversely, if $\mathbf{u}$ and $\mathbf{v}$ are proportional, then $\mathbf{v} = c\mathbf{u}$ for some constant c . Thus, $c\mathbf{u} - \mathbf{v} = \mathbf{0}$, which shows that $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent.

For part 2 of Proposition 4.5.7, suppose the zero vector $\mathbf{0}$ belongs to a set S of vectors in a vector space V . Then $1 \cdot \mathbf{0}$ is a linear dependency among the vectors in S , and therefore S is linearly dependent.

50. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ spans V and let $\mathbf{v}$ be any vector in V . By assumption, we can write

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k,$$

for some constants $c_1, c_2, \dots, c_k$. Therefore,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k - \mathbf{v} = \mathbf{0}$$

is a linear dependency among the vectors in $\{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$. Thus, $\{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is linearly dependent.

51. Let $S = \{p_1, p_2, \dots, p_k\}$ and assume without loss of generality that the polynomials are listed in decreasing order by degree:

$$\deg(p_1) > \deg(p_2) > \dots > \deg(p_k).$$

To show that S is linearly independent, assume that

$$c_1p_1 + c_2p_2 + \dots + c_kp_k = 0.$$

We wish to show that $c_1 = c_2 = \dots = c_k = 0$. We require that each coefficient on the left side of the above equation is zero, since we have 0 on the right-hand side. Since p_1 has the highest degree, none of the terms $c_2p_2, c_3p_3, \dots, c_kp_k$ can cancel the leading coefficient of p_1 . Therefore, we conclude that $c_1 = 0$. Thus, we now have

$$c_2p_2 + c_3p_3 + \dots + c_kp_k = 0,$$

and we can repeat this argument again now to show successively that $c_2 = c_3 = \dots = c_k = 0$.

Solutions to Section 4.6

- 1. FALSE.** It is not enough that S spans V . It must also be the case that S is *linearly independent*.
- 2. FALSE.** For example, $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$, since $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$.
- 3. TRUE.** Any set of two non-proportional vectors in $\mathbb{R}^2$ will form a basis for $\mathbb{R}^2$.
- 4. FALSE.** We have $\dim[P_n] = n + 1$ and $\dim[\mathbb{R}^n] = n$.
- 5. FALSE.** For example, if $V = \mathbb{R}^2$, then the set $S = \{(1, 0), (2, 0), (3, 0)\}$, consisting of $3 > 2$ vectors, fails to span V , a 2-dimensional vector space.

6. TRUE. We have $\dim[P_3] = 4$, and so any set of more than four vectors in P_3 must be linearly dependent (a maximal linearly independent set in a 4-dimensional vector space consists of four vectors).

7. FALSE. For instance, the two vectors $1 + x$ and $2 + 2x$ in P_3 are linearly *dependent*.

8. TRUE. Since $M_3(\mathbb{R})$ is 9-dimensional, any set of 10 vectors in this vector space must be linearly dependent by Theorem 4.6.4.

9. FALSE. Only *linearly independent* sets with fewer than n vectors can be extended to a basis for V .

10. TRUE. We can build such a subset by choosing vectors from the set as follows. Choose $\mathbf{v}_1$ to be any vector in the set. Now choose $\mathbf{v}_2$ in the set such that $\mathbf{v}_2 \notin \text{span}\{\mathbf{v}_1\}$. Next, choose $\mathbf{v}_3$ in the set such that $\mathbf{v}_3 \notin \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Proceed in this manner until it is no longer possible to find a vector in the set that is not spanned by the collection of previously chosen vectors. This point will occur eventually, since V is finite-dimensional. Moreover, the chosen vectors will form a linearly independent set, since each $\mathbf{v}_i$ is chosen from outside $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{i-1}\}$. Thus, the set we obtain in this way is a linearly independent set of vectors that spans V , hence forms a basis for V .

11. FALSE. The set of all 3×3 upper triangular matrices forms a 6-dimensional subspace of $M_3(\mathbb{R})$, not a 3-dimensional subspace. One basis is given by $\{E_{11}, E_{12}, E_{13}, E_{22}, E_{23}, E_{33}\}$.

Problems:

1. $\dim[\mathbb{R}^2] = 2$. There are two vectors, so if they are to form a basis for $\mathbb{R}^2$, they need to be linearly independent: $\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \neq 0$. This implies that the vectors are linearly independent, hence they form a basis for $\mathbb{R}^2$.

2. $\dim[\mathbb{R}^3] = 3$. There are three vectors, so if they are to form a basis for $\mathbb{R}^3$, they need to be linearly independent: $\begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 13 \neq 0$. This implies that the vectors are linearly independent, hence they form a basis for $\mathbb{R}^3$.

3. $\dim[\mathbb{R}^3] = 3$. There are three vectors, so if they are to form a basis for $\mathbb{R}^3$, they need to be linearly independent: $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 5 & 11 \\ 1 & -2 & -5 \end{vmatrix} = 0$. This implies that the vectors are linearly dependent, hence they do not form a basis for $\mathbb{R}^3$.

4. $\dim[\mathbb{R}^4] = 4$. We need 4 linearly independent vectors in order to span $\mathbb{R}^4$. However, there are only 3 vectors in this set. Thus, the vectors cannot be a basis for $\mathbb{R}^4$.

5. $\dim[\mathbb{R}^4] = 4$. There are four vectors, so if they are to form a basis for $\mathbb{R}^3$, they need to be linearly independent: $\begin{vmatrix} 1 & 2 & -1 & 2 \\ 1 & 1 & 1 & -1 \\ 0 & 3 & 1 & 1 \\ 2 & -1 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 & 2 \\ 0 & -1 & 2 & -3 \\ 0 & 3 & 1 & 1 \\ 0 & -5 & 0 & -2 \end{vmatrix} = \begin{vmatrix} -1 & 2 & -3 \\ 3 & 1 & 1 \\ -5 & 0 & -2 \end{vmatrix} = \begin{vmatrix} -7 & 0 & -5 \\ 3 & 1 & 1 \\ -5 & 0 & -2 \end{vmatrix} = -11$.

Since this determinant is nonzero, the given vectors are linearly independent. Consequently, they form a basis for $\mathbb{R}^4$.

6. $\dim[\mathbb{R}^4] = 4$. There are four vectors, so if they are to form a basis for $\mathbb{R}^3$, they need to be linearly

independent: $\begin{vmatrix} 0 & 1 & 0 & k \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ k & 0 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} -1 & 1 & 0 \\ 0 & 1 & 2 \\ k & 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 0 & k \\ -1 & 1 & 0 \\ k & 0 & 1 \end{vmatrix} = -(-1 + 2k) - (-k^2) = k^2 - 2k + 1 = (k - 1)^2 \neq 0$ when $k \neq 1$. Thus, the vectors will form a basis for $\mathbb{R}^4$ provided $k \neq 1$.

7. The general vector $p(x) \in P_3$ can be represented as $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. Thus $P_3 = \text{span}\{1, x, x^2, x^3\}$. Further, $\{1, x, x^2, x^3\}$ is a linearly independent set since

$$W[1, x, x^2, x^3] = \begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{vmatrix} = 12 \neq 0.$$

Consequently, $S = \{1, x, x^2, x^3\}$ is a basis for P_3 and $\dim[P_3] = 4$. Of course, S is not the only basis for P_3 .

8. Many acceptable bases are possible here. One example is

$$S = \{x^3, x^3 + 1, x^3 + x, x^3 + x^2\}.$$

All of the polynomials in this set have degree 3. We verify that S is a basis: Assume that

$$c_1(x^3) + c_2(x^3 + 1) + c_3(x^3 + x) + c_4(x^3 + x^2) = 0.$$

Thus,

$$(c_1 + c_2)x^3 + c_4x^2 + c_3x + c_2 = 0,$$

from which we quickly see that $c_1 = c_2 = c_3 = c_4 = 0$. Thus, S is linearly independent. Since P_3 is 4-dimensional, we can now conclude from Corollary 4.6.13 that S is a basis for P_3 .

9. $A\mathbf{x} = \mathbf{0} \implies \begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The augmented matrix for this system is $\left[\begin{array}{cc|c} 1 & 3 & 0 \\ -2 & -6 & 0 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array} \right]$; thus, $x_1 + 3x_2 = 0$, or $x_1 = -3x_2$. Let $x_2 = r$ so that $x_1 = -3r$ where $r \in \mathbb{R}$. Consequently, $S = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = r(-3, 1), r \in \mathbb{R}\} = \text{span}\{(-3, 1)\}$. It follows that $\{(-3, 1)\}$ is a basis for S and $\dim[S] = 1$.

10. $A\mathbf{x} = \mathbf{0} \implies \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The REDUCED ROW ECHELON FORM of the augmented matrix for this system is $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$. We see that $x_2 = 0$, and x_1 and x_3 are free variables:

$x_1 = r$ and $x_2 = s$. Hence, $(x_1, x_2, x_3) = (r, 0, s) = r(1, 0, 0) + s(0, 0, 1)$, so that the solution set of the system is $S = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = r(1, 0, 0) + s(0, 0, 1), r, s \in \mathbb{R}\}$. Therefore we see that $\{(1, 0, 0), (0, 0, 1)\}$ is a basis for S and $\dim[S] = 2$.

11. $A\mathbf{x} = \mathbf{0} \implies \begin{bmatrix} 1 & -1 & 4 \\ 2 & 3 & -2 \\ 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The REDUCED ROW ECHELON FORM of the augmented matrix for this system is $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$. If we let $x_3 = r$ then $(x_1, x_2, x_3) = (-2r, 2r, r) =$

$r(-2, 2, 1)$, so that the solution set of the system is $S = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = r(-2, 2, 1), r \in \mathbb{R}\}$. Therefore we see that $\{(-2, 2, 1)\}$ is a basis for S and $\dim[S] = 1$.

12. $A\mathbf{x} = \mathbf{0} \implies \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -1 & 3 & 4 \\ 1 & 0 & 1 & 1 \\ 3 & -1 & 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. The REDUCED ROW ECHELON FORM of the augmented matrix for this system is $\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$. If we let $x_3 = r, x_4 = s$ then $x_2 = r + 2s$ and

$x_1 = -r - s$. Hence, the solution set of the system is $S = \{\mathbf{x} \in \mathbb{R}^4 : \mathbf{x} = r(-1, 1, 1, 0) + s(-1, 2, 0, 1), r, s \in \mathbb{R}\} = \text{span}\{(-1, 1, 1, 0), (-1, 2, 0, 1)\}$. Further, the vectors $\mathbf{v}_1 = (-1, 1, 1, 0), \mathbf{v}_2 = (-1, 2, 0, 1)$ are linearly independent since $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0} \implies c_1(-1, 1, 1, 0) + c_2(-1, 2, 0, 1) = (0, 0, 0, 0) \implies c_1 = c_2 = 0$. Consequently, $\{(-1, 1, 1, 0), (-1, 2, 0, 1)\}$ is a basis for S and $\dim[S] = 2$.

13. If we let $y = r$ and $z = s$ where $r, s \in \mathbb{R}$, then $x = 3r - s$. It follows that any ordered triple in S can be written in the form:

$(x, y, z) = (3r - s, r, s) = (3r, r, 0) + (-s, 0, s) = r(3, 1, 0) + s(-1, 0, 1)$, where $r, s \in \mathbb{R}$. If we let $\mathbf{v}_1 = (3, 1, 0)$ and $\mathbf{v}_2 = (-1, 0, 1)$, then $S = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = r(3, 1, 0) + s(-1, 0, 1), r, s \in \mathbb{R}\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$; moreover, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent for if $a, b \in \mathbb{R}$ and $a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{0}$, then

$$a(3, 1, 0) + b(-1, 0, 1) = (0, 0, 0),$$

which implies that $(3a, a, 0) + (-b, 0, b) = (0, 0, 0)$, or $(3a - b, a, b) = (0, 0, 0)$. In other words, $a = b = 0$. Since $\{\mathbf{v}_1, \mathbf{v}_2\}$ spans S and is linearly independent, it is a basis for S . Also, $\dim[S] = 2$.

14.

$$S = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = (r, r - 2s, 3s - 5r), r, s \in \mathbb{R}\}$$

$$= \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = (r, r, -5r) + (0, -2s, 3s), r, s \in \mathbb{R}\}$$

$$= \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = r(1, 1, -5) + s(0, -2, 3), r, s \in \mathbb{R}\}.$$

Thus, $S = \text{span}\{(1, 1, -5), (0, -2, 3)\}$. The vectors $\mathbf{v}_1 = (1, 1, -5)$ and $\mathbf{v}_2 = (0, -2, 3)$ are linearly independent for if $a, b \in \mathbb{R}$ and $a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{0}$, then

$$a(1, 1, -5) + b(0, -2, 3) = (0, 0, 0) \implies (a, a, -5a) + (0, -2b, 3b) = (0, 0, 0) \implies (a, a - 2b, 3b - 5a) = (0, 0, 0) \implies a = b = 0.$$

It follows that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for S and $\dim[S] = 2$.

15. $S = \{A \in M_2(\mathbb{R}) : A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, a, b, c \in \mathbb{R}\}$. Each vector in S can be written as

$$A = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

so that $S = \text{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$. Since the vectors in this set are clearly linearly independent, it follows that a basis for S is $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$, and therefore $\dim[S] = 3$.

16. $S = \{A \in M_2(\mathbb{R}) : A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}, a, b, c \in \mathbb{R}\}$. Each vector in S can be written as

$$A = a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

so that $S = \text{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$. Since this set of vectors is clearly linearly independent, it follows that a basis for S is $\left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}$, and therefore $\dim[S] = 3$.

17. We see directly that $\mathbf{v}_3 = 2\mathbf{v}_1$. Let $\mathbf{v}$ be an arbitrary vector in S . Then

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = (c_1 + 2c_3)\mathbf{v}_1 + c_2\mathbf{v}_2 = d_1\mathbf{v}_1 + d_2\mathbf{v}_2,$$

where $d_1 = (c_1 + 2c_3)$ and $d_2 = c_2$. Hence $S = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Further, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent for if $a, b \in \mathbb{R}$ and $a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{0}$, then

$$a(1, 0, 1) + b(0, 1, 1) = (0, 0, 0) \implies (a, 0, a) + (0, b, b) = (0, 0, 0) \implies (a, b, a + b) = (0, 0, 0) \implies a = b = 0.$$

Consequently, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for S and $\dim[S] = 2$.

18. f_3 depends on f_1 and f_2 since $\sinh x = \frac{e^x - e^{-x}}{2}$. Thus, $f_3(x) = \frac{f_1(x) - f_2(x)}{2}$.

$W[f_1, f_2](x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 \neq 0$ for all $x \in \mathbb{R}$, so $\{f_1, f_2\}$ is a linearly independent set. Thus, $\{f_1, f_2\}$ is a basis for S and $\dim[S] = 2$.

19. The given set of matrices is linearly dependent because it contains the zero vector. Consequently, the matrices $A_1 = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 5 & -6 \\ -5 & 1 \end{bmatrix}$ span the same subspace of $M_2(\mathbb{R})$ as that spanned by the original set. We now determine whether $\{A_1, A_2, A_3\}$ is linearly independent. The vector equation:

$$c_1A_1 + c_2A_2 + c_3A_3 = 0_2$$

leads to the linear system

$$c_1 - c_2 + 5c_3 = 0, 3c_1 + 4c_2 - 6c_3 = 0, -c_1 + c_2 - 5c_3 = 0, 2c_1 + c_2 + c_3 = 0.$$

This system has solution $(-2r, 3r, r)$, where r is a free variable. Consequently, $\{A_1, A_2, A_3\}$ is linearly dependent with linear dependency relation

$$-2A_1 + 3A_2 + A_3 = 0,$$

or equivalently,

$$A_3 = 2A_1 - 3A_2.$$

It follows that the set of matrices $\{A_1, A_2\}$ spans the same subspace of $M_2(\mathbb{R})$ as that spanned by the original set of matrices. Further $\{A_1, A_2\}$ is linearly independent by inspection, and therefore it is a basis for the subspace.

20. (a) We must show that every vector $(x, y) \in \mathbb{R}^2$ can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Mathematically, we express this as

$$c_1(1, 1) + c_2(-1, 1) = (x, y)$$

which implies that

$$c_1 - c_2 = x \quad \text{and} \quad c_1 + c_2 = y.$$

Adding the equations here, we obtain $2c_1 = x + y$ or

$$c_1 = \frac{1}{2}(x + y).$$

Now we can solve for c_2 :

$$c_2 = y - c_1 = \frac{1}{2}(y - x).$$

Therefore, since we were able to solve for c_1 and c_2 in terms of x and y , we see that the system of equations is consistent for all x and y . Therefore, $\{\mathbf{v}_1, \mathbf{v}_2\}$ spans $\mathbb{R}^2$.

(b) $\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \neq 0$, so the vectors are linearly independent.

(c) We can draw this conclusion from part (a) alone by using Theorem 4.6.12 or from part (b) alone by using Theorem 4.6.10.

21. (a) We must show that every vector $(x, y) \in \mathbb{R}^2$ can be expressed as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Mathematically, we express this as

$$c_1(2, 1) + c_2(3, 1) = (x, y)$$

which implies that

$$2c_1 + 3c_2 = x \quad \text{and} \quad c_1 + c_2 = y.$$

From this, we can solve for c_1 and c_2 in terms of x and y :

$$c_1 = 3y - x \quad \text{and} \quad c_2 = x - 2y.$$

Therefore, since we were able to solve for c_1 and c_2 in terms of x and y , we see that the system of equations is consistent for all x and y . Therefore, $\{\mathbf{v}_1, \mathbf{v}_2\}$ spans $\mathbb{R}^2$.

(b) $\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1 \neq 0$, so the vectors are linearly independent.

(c) We can draw this conclusion from part (a) alone by using Theorem 4.6.12 or from part (b) alone by using Theorem 4.6.10.

22. $\dim[\mathbb{R}^3] = 3$. There are 3 vectors, so if they are linearly independent, then they are a basis of $\mathbb{R}^3$ by

Theorem 4.6.1. Since $\begin{vmatrix} 0 & 3 & 6 \\ 6 & 0 & -3 \\ 3 & 3 & 0 \end{vmatrix} = 81 \neq 0$, the given vectors are linearly independent, and so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

is a basis for $\mathbb{R}^3$.

23. $\dim[P_2] = 3$. There are 3 vectors, so $\{p_1, p_2, p_3\}$ may be a basis for P_2 depending on α . To be a basis, the set of vectors must be linearly independent. Let $a, b, c \in \mathbb{R}$. Then $ap_1(x) + bp_2(x) + cp_3(x) = 0$

$$\implies a(1 + \alpha x^2) + b(1 + x + x^2) + c(2 + x) = 0$$

$$\implies (a + b + 2c) + (b + c)x + (a\alpha + b)x^2 = 0. \text{ Equating like coefficients in the last equality, we obtain the}$$

$$\text{system: } \begin{cases} a + b + 2c = 0 \\ b + c = 0 \\ a\alpha + b = 0. \end{cases} \text{ Reduce the augmented matrix of this system.}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ \alpha & 1 & 0 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & \alpha - 1 & 2\alpha & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \alpha + 1 & 0 \end{array} \right].$$

For this system to have only the trivial solution, the last row of the matrix must not be a zero row. This means that $\alpha \neq -1$. Therefore, for the given set of vectors to be linearly independent (and thus a basis), α can be any value except -1 .

24. $\dim[P_2] = 3$. There are 3 vectors, so in order to form a basis for P_2 , they must be linearly independent. Since we are dealing with functions, we will use the Wronskian.

$$W[p_1, p_2, p_3](x) = \begin{vmatrix} 1+x & x(x-1) & 1+2x^2 \\ 1 & 2x-1 & 4x \\ 0 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1+x & x(x-1) & 1+2x \\ 1 & 2x-1 & 2 \\ 0 & 2 & 0 \end{vmatrix} = -2.$$

Since the Wronskian is nonzero, $\{p_1, p_2, p_3\}$ is linearly independent, and hence forms a basis for P_2 .

25. $W[p_0, p_1, p_2](x) = \begin{vmatrix} 1 & x & \frac{3x^2-1}{2} \\ 0 & 1 & 3x \\ 0 & 0 & 3 \end{vmatrix} = 3 \neq 0$, so $\{p_0, p_1, p_2\}$ is a linearly independent set. Since $\dim[P_2] = 3$, it follows that $\{p_0, p_1, p_2\}$ is a basis for P_2 .

26. (a) Suppose that $a, b, c, d \in \mathbb{R}$.

$$a \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} + c \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + d \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} = 0_2$$

$$\Rightarrow \begin{bmatrix} -a+b+c & a+3b-c \\ -b+c+2d & a+2c+3d \end{bmatrix} = 0_2. \quad \text{The last matrix results in the system: } \begin{cases} -a+b+c=0 \\ a+3b-d=0 \\ -b+c+2d=0 \\ a+2c+3d=0. \end{cases} \quad \text{The}$$

REDUCED ROW ECHELON FORM of the augmented matrix of this system is $\left[\begin{array}{cccc|c} 1 & -1 & -1 & 0 & 0 \\ 0 & 4 & 1 & -1 & 0 \\ 0 & 0 & 5 & 7 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$,

which implies that $a = b = c = d = 0$. Hence, the given set of vectors is linearly independent. Since $\dim[M_2(\mathbb{R})] = 4$, it follows that $\{A_1, A_2, A_3, A_4\}$ is a basis for $M_2(\mathbb{R})$.

(b) We wish to express the vector $\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ in the form

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = a \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} + c \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + d \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}.$$

Matching entries on each side of this equation (upper left, upper right, lower left, and lower right), we obtain a linear system with augmented matrix

$$\left[\begin{array}{cccc|c} -1 & 1 & 1 & 0 & 5 \\ 1 & 3 & 0 & -1 & 6 \\ 0 & -1 & 1 & 2 & 7 \\ 1 & 0 & 2 & 3 & 8 \end{array} \right].$$

Solving this system by Gaussian elimination, we find that $a = -\frac{34}{3}$, $b = 12$, $c = -\frac{55}{3}$, and $d = \frac{56}{3}$. Thus, we have

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = -\frac{34}{3} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} + 12 \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} - \frac{55}{3} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + \frac{56}{3} \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}.$$

27.

(a) $A\mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & -3 & 5 & -6 \\ 5 & 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The augmented matrix for this linear system is

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 2 & -3 & 5 & -6 & 0 \\ 5 & 0 & 2 & -3 & 0 \end{array} \right] \stackrel{1}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & -5 & 7 & -8 & 0 \\ 0 & -5 & 7 & -8 & 0 \end{array} \right] \stackrel{2}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & -5 & 7 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \stackrel{3}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1.4 & 1.6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

$$\boxed{\text{1. } A_{12}(-2), A_{13}(-5) \quad \text{2. } A_{23}(-1) \quad \text{3. } M_2(-\frac{1}{5})}$$

We see that $x_3 = r$ and $x_4 = s$ are free variables, and therefore $\text{nullspace}(A)$ is 2-dimensional. Now we can check directly that $A\mathbf{v}_1 = \mathbf{0}$ and $A\mathbf{v}_2 = \mathbf{0}$, and since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent (they are non-proportional), they therefore form a basis for $\text{nullspace}(A)$ by Corollary 4.6.13.

(b) An arbitrary vector (x, y, z, w) in $\text{nullspace}(A)$ can be expressed as a linear combination of the basis vectors:

$$(x, y, z, w) = c_1(-2, 7, 5, 0) + c_2(3, -8, 0, 5),$$

where $c_1, c_2 \in \mathbb{R}$.

28.

(a) An arbitrary matrix in S takes the form $\begin{bmatrix} a & b & -a-b \\ c & d & -c-d \\ -a-c & -b-d & a+b+c+d \end{bmatrix}$

$$= a \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Therefore, we have the following basis for S :

$$\left\{ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \right\}.$$

From this, we see that $\dim[S] = 4$.

(b) We see that each of the matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

has a different row or column that does *not* sum to zero, and thus none of these matrices belong to S , and they are linearly independent from one another. Therefore, supplementing the basis for S in part (a) with the five matrices here extends the basis in (a) to a basis for $M_3(\mathbb{R})$.

29.

(a) An arbitrary matrix in S takes the form $\begin{bmatrix} a & b & c \\ d & e & a+b+c-d-e \\ b+c-d & a+c-e & d+e-c \end{bmatrix}$

$$= a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} + e \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Therefore, we have the following basis for S :

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \right\}.$$

From this, we see that $\dim[S] = 5$.

(b) We must include four additional matrices that are linearly independent and outside of S . The matrices E_{11} , E_{12} , $E_{11} + E_{13}$, and E_{22} will suffice in this case.

30. Let $A \in \text{Sym}_2(\mathbb{R})$ so $A = A^T$. $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \text{Sym}_2(\mathbb{R})$; $a, b, c \in \mathbb{R}$. This vector can be represented as

a linear combination of $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$:

$A = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Since $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a linearly independent set that also spans $\text{Sym}_2(\mathbb{R})$, it is a basis for $\text{Sym}_2(\mathbb{R})$. Thus, $\dim[\text{Sym}_2(\mathbb{R})] = 3$. Let $B \in \text{Skew}_2(\mathbb{R})$ so $B = -B^T$. Then $B = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, where $b \in \mathbb{R}$. The set $\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ is linearly independent

and spans $\text{Skew}_2(\mathbb{R})$ so that a basis for $\text{Skew}_2(\mathbb{R})$ is $\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$. Consequently, $\dim[\text{Skew}_2(\mathbb{R})] = 1$. Now, $\dim[\text{Sym}_2(\mathbb{R})] = 4$, and hence $\dim[\text{Sym}_2(\mathbb{R})] + \dim[\text{Skew}_2(\mathbb{R})] = 3 + 1 = 4 = \dim[M_2(\mathbb{R})]$.

31. We know that $\dim[M_n(\mathbb{R})] = n^2$. Let $S \in \text{Sym}_n(\mathbb{R})$ and let $[S_{ij}]$ be the matrix with ones in the (i, j) and (j, i) positions and zeroes elsewhere. Then the general $n \times n$ symmetric matrix can be expressed as:

$$\begin{aligned} S = & a_{11}S_{11} + a_{12}S_{12} + a_{13}S_{13} + \cdots + a_{1n}S_{1n} \\ & + a_{22}S_{22} + a_{23}S_{23} + \cdots + a_{2n}S_{2n} \\ & + \cdots \\ & + a_{n-1,n-1}S_{n-1,n-1} + a_{n-1,n}S_{n-1,n} + a_{nn}S_{nn}. \end{aligned}$$

We see that S has been resolved into a linear combination of

$n + (n-1) + (n-2) + \cdots + 1 = \frac{n(n+1)}{2}$ linearly independent matrices, which therefore form a basis for $\text{Sym}_n(\mathbb{R})$; hence $\dim[\text{Sym}_n(\mathbb{R})] = \frac{n(n+1)}{2}$.

Now let $T \in \text{Skew}_n(\mathbb{R})$ and let $[T_{ij}]$ be the matrix with one in the (i, j) -position, negative one in the (j, i) -position, and zeroes elsewhere, including the main diagonal. Then the general $n \times n$ skew-symmetric matrix can be expressed as:

$$\begin{aligned} T = & a_{12}T_{12} + a_{13}T_{13} + a_{14}T_{14} + \cdots + a_{1n}T_{1n} \\ & + a_{23}T_{23} + a_{24}T_{24} + \cdots + a_{2n}T_{2n} \\ & + \cdots \\ & + a_{n-1,n}T_{n-1,n} \end{aligned}$$

We see that T has been resolved into a linear combination of $(n-1) + (n-2) + (n-3) + \cdots + 2 + 1 = \frac{(n-1)n}{2}$ linearly independent vectors, which therefore form a basis for $\text{Skew}_n(\mathbb{R})$; hence $\dim[\text{Skew}_n(\mathbb{R})] = \frac{(n-1)n}{2}$.

Consequently, using these results, we have

$$\dim[\text{Sym}_n(\mathbb{R})] + \dim[\text{Skew}_n(\mathbb{R})] = \frac{n(n+1)}{2} + \frac{(n-1)n}{2} = n^2 = \dim[M_n(\mathbb{R})].$$

32. (a) S is a two-dimensional subspace of $\mathbb{R}^3$. Consequently, any two linearly independent vectors lying

in this subspace determine a basis for S . By inspection we see, for example, that $\mathbf{v}_1 = (4, -1, 0)$ and $\mathbf{v}_2 = (3, 0, 1)$ both lie in the plane. Further, since they are not proportional, these vectors are linearly independent. Consequently, a basis for S is $\{(4, -1, 0), (3, 0, 1)\}$.

(b) To extend the basis obtained in Part (a) to obtain a basis for $\mathbb{R}^3$, we require one more vector that does not lie in S . For example, since $\mathbf{v}_3 = (1, 0, 0)$ does not lie on the plane it is an appropriate vector. Consequently, a basis for $\mathbb{R}^3$ is $\{(4, -1, 0), (3, 0, 1), (1, 0, 0)\}$.

33. Each vector in S can be written as

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Consequently, a basis for S is given by the linearly independent set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$. To extend this basis to $M_2(\mathbb{R})$, we can choose, for example, the two vectors $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then the linearly independent set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ is a basis for $M_2(\mathbb{R})$.

34. The vectors in S can be expressed as

$$(2a_1 + a_2)x^2 + (a_1 + a_2)x + (3a_1 - a_2) = a_1(2x^2 + x + 3) + a_2(x^2 + x - 1),$$

and since $\{2x^2 + x + 3, x^2 + x - 1\}$ is linearly independent (these polynomials are non-proportional) and certainly span S , they form a basis for S . To extend this basis to $V = P_2$, we must include one additional vector (since P_2 is 3-dimensional). Any polynomial that is not in S will suffice. For example, $x \notin S$, since x cannot be expressed in the form $(2a_1 + a_2)x^2 + (a_1 + a_2)x + (3a_1 - a_2)$, since the equations

$$2a_1 + a_2 = 0, \quad a_1 + a_2 = 1, \quad 3a_1 - a_2 = 0$$

are inconsistent. Thus, the extension we use as a basis for V here is $\{2x^2 + x + 3, x^2 + x - 1, x\}$. Many other correct answers can also be given here.

35. Since S is a basis for P_{n-1} , S contains n vectors. Therefore, $S \cup \{x^n\}$ is a set of $n + 1$ vectors, which is precisely the dimension of P_n . Moreover, x^n does not lie in $P_{n-1} = \text{span}(S)$, and therefore, $S \cup \{x^n\}$ is linearly independent by Problem 47 in Section 4.5. By Corollary 4.6.13, we conclude that $S \cup \{x^n\}$ is a basis for P_n .

36. Since S is a basis for P_{n-1} , S contains n vectors. Therefore, $S \cup \{p\}$ is a set of $n + 1$ vectors, which is precisely the dimension of P_n . Moreover, p does not lie in $P_{n-1} = \text{span}(S)$, and therefore, $S \cup \{p\}$ is linearly independent by Problem 47 in Section 4.5. By Corollary 4.6.13, we conclude that $S \cup \{p\}$ is a basis for P_n .

37.

(a) Let $\mathbf{e}_k$ denote the k th standard basis vector. Then a basis for $\mathbb{C}^n$ with scalars in $\mathbb{R}$ is given by

$$\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n, i\mathbf{e}_1, i\mathbf{e}_2, \dots, i\mathbf{e}_n\},$$

and the dimension is $2n$.

(b) Using the notation in part (a), a basis for $\mathbb{C}^n$ with scalars in $\mathbb{C}$ is given by

$$\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\},$$

and the dimension is n .

Solutions to Section 4.7

True-False Review:

1. TRUE. This is the content of Theorem 4.7.1. The existence of such a linear combination comes from the fact that a basis for V must span V , and the uniqueness of such a linear combination follows from the linear independence of the vectors comprising a basis.

2. TRUE. This follows from the Equation $[\mathbf{v}]_B = P_{B \leftarrow C}[\mathbf{v}]_C$, which is just Equation (4.7.6) with the roles of B and C reversed.

3. TRUE. The number of columns in the change-of-basis matrix $P_{C \leftarrow B}$ is the number of vectors in B , while the number of rows of $P_{C \leftarrow B}$ is the number of vectors in C . Since all bases for the vector space V contain the same number of vectors, this implies that $P_{C \leftarrow B}$ contains the same number of rows and columns.

4. TRUE. If V is an n -dimensional vector space, then

$$P_{C \leftarrow B} P_{B \leftarrow C} = I_n = P_{B \leftarrow C} P_{C \leftarrow B},$$

which implies that $P_{C \leftarrow B}$ is invertible.

5. TRUE. This follows from the linearity properties:

$$[\mathbf{v} - \mathbf{w}]_B = [\mathbf{v} + (-1)\mathbf{w}]_B = [\mathbf{v}]_B + [(-1)\mathbf{w}]_B = [\mathbf{v}]_B + (-1)[\mathbf{w}]_B = [\mathbf{v}]_B - [\mathbf{w}]_B.$$

6. FALSE. It depends on the order in which the vectors in the bases B and C are listed. For instance, if we consider the bases $B = \{(1, 0), (0, 1)\}$ and $C = \{(0, 1), (1, 0)\}$ for $\mathbb{R}^2$, then although B and C contain the same vectors, if we let $\mathbf{v} = (1, 0)$, then $[\mathbf{v}]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ while $[\mathbf{v}]_C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

7. FALSE. For instance, if we consider the bases $B = \{(1, 0), (0, 1)\}$ and $C = \{(0, 1), (1, 0)\}$ for $\mathbb{R}^2$, and if we let $\mathbf{v} = (1, 0)$ and $\mathbf{w} = (0, 1)$, then $\mathbf{v} \neq \mathbf{w}$, but $[\mathbf{v}]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = [\mathbf{w}]_C$.

8. TRUE. If $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, then the column vector $[\mathbf{v}_i]_B$ is the i th standard basis vector (1 in the i th position and zeroes elsewhere). Thus, for each i , the i th column of $P_{B \leftarrow B}$ consists of a 1 in the i th position and zeroes elsewhere. This describes precisely the identity matrix.

Problems:

1. Write

$$(5, -10) = c_1(2, -2) + c_2(1, 4).$$

Then

$$2c_1 + c_2 = 5 \quad \text{and} \quad -2c_1 + 4c_2 = -10.$$

Solving this system of equations gives $c_1 = 3$ and $c_2 = -1$. Thus,

$$[\mathbf{v}]_B = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

2. Write

$$(8, -2) = c_1(-1, 3) + c_2(3, 2).$$

Then

$$-c_1 + 3c_2 = 8 \quad \text{and} \quad 3c_1 + 2c_2 = -2.$$

Solving this system of equations gives $c_1 = -2$ and $c_2 = 2$. Thus,

$$[\mathbf{v}]_B = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

3. Write

$$(-9, 1, -8) = c_1(1, 0, 1) + c_2(1, 1, -1) + c_3(2, 0, 1).$$

Then

$$c_1 + c_2 + 2c_3 = -9 \quad \text{and} \quad c_2 = 1 \quad \text{and} \quad c_1 - c_2 + c_3 = -8.$$

Solving this system of equations gives $c_1 = -4$, $c_2 = 1$, and $c_3 = -3$. Thus,

$$[\mathbf{v}]_B = \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix}.$$

4. Write

$$(1, 7, 7) = c_1(1, -6, 3) + c_2(0, 5, -1) + c_3(3, -1, -1).$$

Then

$$c_1 + 3c_3 = 1 \quad \text{and} \quad -6c_1 + 5c_2 - c_3 = 7 \quad \text{and} \quad 3c_1 - c_2 - c_3 = 7.$$

Using Gaussian elimination to solve this system of equations gives $c_1 = 4$, $c_2 = 6$, and $c_3 = -1$. Thus,

$$[\mathbf{v}]_B = \begin{bmatrix} 4 \\ 6 \\ -1 \end{bmatrix}.$$

5. Write

$$(1, 7, 7) = c_1(3, -1, -1) + c_2(1, -6, 3) + c_3(0, 5, -1).$$

Then

$$3c_1 + c_2 = 1 \quad \text{and} \quad -c_1 - 6c_2 + 5c_3 = 7 \quad \text{and} \quad -c_1 + 3c_2 - c_3 = 7.$$

Using Gaussian elimination to solve this system of equations gives $c_1 = -1$, $c_2 = 4$, and $c_3 = 6$. Thus,

$$[\mathbf{v}]_B = \begin{bmatrix} -1 \\ 4 \\ 6 \end{bmatrix}.$$

6. Write

$$(5, 5, 5) = c_1(-1, 0, 0) + c_2(0, 0, -3) + c_3(0, -2, 0).$$

Then

$$-c_1 = 5 \quad \text{and} \quad -2c_3 = 5 \quad \text{and} \quad -3c_2 = 5.$$

Therefore, $c_1 = -5$, $c_2 = -\frac{5}{3}$, and $c_3 = -\frac{5}{2}$. Thus,

$$[\mathbf{v}]_B = \begin{bmatrix} -5 \\ -5/3 \\ -5/2 \end{bmatrix}.$$

7. Write

$$-4x^2 + 2x + 6 = c_1(x^2 + x) + c_2(2 + 2x) + c_3(1).$$

Equating the powers of x on each side, we have

$$c_1 = -4 \quad \text{and} \quad c_1 + 2c_2 = 2 \quad \text{and} \quad 2c_2 + c_3 = 6.$$

Solving this system of equations, we find that $c_1 = -4$, $c_2 = 3$, and $c_3 = 0$. Hence,

$$[p(x)]_B = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}.$$

8. Write

$$15 - 18x - 30x^2 = c_1(5 - 3x) + c_2(1) + c_3(1 + 2x^2).$$

Equating the powers of x on each side, we have

$$5c_1 + c_2 + c_3 = 15 \quad \text{and} \quad -3c_1 = -18 \quad \text{and} \quad 2c_3 = -30.$$

Solving this system of equations, we find that $c_1 = 6$, $c_2 = 0$, and $c_3 = -15$. Hence,

$$[p(x)]_B = \begin{bmatrix} 6 \\ 0 \\ -15 \end{bmatrix}.$$

9. Write

$$4 - x + x^2 - 2x^3 = c_1(1) + c_2(1 + x) + c_3(1 + x + x^2) + c_4(1 + x + x^2 + x^3).$$

Equating the powers of x on each side, we have

$$c_1 + c_2 + c_3 + c_4 = 4 \quad \text{and} \quad c_2 + c_3 + c_4 = -1 \quad \text{and} \quad c_3 + c_4 = 1 \quad \text{and} \quad c_4 = -2.$$

Solving this system of equations, we find that $c_1 = 5$, $c_2 = -2$, $c_3 = 3$, and $c_4 = -2$. Hence,

$$[p(x)]_B = \begin{bmatrix} 5 \\ -2 \\ 3 \\ -2 \end{bmatrix}.$$

10. Write

$$8 + x + 6x^2 + 9x^3 = c_1(x^3 + x^2) + c_2(x^3 - 1) + c_3(x^3 + 1) + c_4(x^3 + x).$$

Equating the powers of x on each side, we have

$$-c_2 + c_3 = 8 \quad \text{and} \quad c_4 = 1 \quad \text{and} \quad c_1 = 6 \quad \text{and} \quad c_1 + c_2 + c_3 + c_4 = 9.$$

Solving this system of equations, we find that $c_1 = 6$, $c_2 = -3$, $c_3 = 5$, and $c_4 = 1$. Hence

$$[p(x)]_B = \begin{bmatrix} 6 \\ -3 \\ 5 \\ 1 \end{bmatrix}.$$

11. Write

$$\begin{bmatrix} -3 & -2 \\ -1 & 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Equating the individual entries of the matrices on each side of this equation (upper left, upper right, lower left, and lower right, respectively) gives

$$c_1 + c_2 + c_3 + c_4 = -3 \quad \text{and} \quad c_1 + c_2 + c_3 = -2 \quad \text{and} \quad c_1 + c_2 = -1 \quad \text{and} \quad c_1 = 2.$$

Solving this system of equations, we find that $c_1 = 2$, $c_2 = -3$, $c_3 = -1$, and $c_4 = -1$. Thus,

$$[A]_B = \begin{bmatrix} 2 \\ -3 \\ -1 \\ -1 \end{bmatrix}.$$

12. Write

$$\begin{bmatrix} -10 & 16 \\ -15 & -14 \end{bmatrix} = c_1 \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 4 \\ -1 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}.$$

Equating the individual entries of the matrices on each side of this equation (upper left, upper right, lower left, and lower right, respectively) gives

$$2c_1 + c_3 + 3c_4 = -10, \quad -c_1 + 4c_2 + c_3 - c_4 = 16, \quad 3c_1 - c_2 + c_3 + 2c_4 = -15, \quad 5c_1 + c_2 + c_3 + 5c_4 = -14.$$

Solving this system of equations, we find that $c_1 = -2$, $c_2 = 4$, $c_3 = -3$, and $c_4 = -1$. Thus,

$$[A]_B = \begin{bmatrix} -2 \\ 4 \\ -3 \\ -1 \end{bmatrix}.$$

13. Write

$$\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = c_1 \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + c_4 \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}.$$

Equating the individual entries of the matrices on each side of this equation (upper left, upper right, lower left, and lower right, respectively) gives

$$-c_1 + c_2 + c_3 = 5 \quad \text{and} \quad c_1 + 3c_2 - c_4 = 6 \quad \text{and} \quad -c_2 + c_3 + 2c_4 = 7 \quad \text{and} \quad c_1 + 2c_3 + 3c_4 = 8.$$

Solving this system of equations, we find that $c_1 = -34/3$, $c_2 = 12$, $c_3 = -55/3$, and $c_4 = 56/3$. Thus,

$$[A]_B = \begin{bmatrix} -34/3 \\ 12 \\ -55/3 \\ 56/3 \end{bmatrix}.$$

14. Write

$$(x, y, z) = c_1(0, 6, 3) + c_2(3, 0, 3) + c_3(6, -3, 0).$$

Then

$$3c_2 + 6c_3 = x \quad \text{and} \quad 6c_1 - 3c_3 = y \quad \text{and} \quad 3c_1 + 3c_2 = z.$$

The augmented matrix for this linear system is $\left[\begin{array}{ccc|c} 3 & 3 & 0 & x \\ 6 & 0 & -3 & y \\ 0 & 3 & 6 & z \end{array} \right]$. We can reduce this to row-echelon form

as $\left[\begin{array}{ccc|c} 1 & 1 & 0 & x/3 \\ 0 & 1 & 2 & y/3 \\ 0 & 0 & 9 & y + 2x - 2z \end{array} \right]$. Solving this system by back-substitution gives

$$c_1 = \frac{1}{9}x + \frac{2}{9}y - \frac{1}{9}z \quad \text{and} \quad c_2 = -\frac{1}{9}x - \frac{2}{9}y + \frac{4}{9}z \quad \text{and} \quad c_3 = \frac{2}{9}x + \frac{1}{9}y - \frac{2}{9}z.$$

Hence, denote the ordered basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ by B , we have

$$[\mathbf{v}]_B = \begin{bmatrix} \frac{1}{9}x + \frac{2}{9}y - \frac{1}{9}z \\ -\frac{1}{9}x - \frac{2}{9}y + \frac{4}{9}z \\ \frac{2}{9}x + \frac{1}{9}y - \frac{2}{9}z \end{bmatrix}.$$

15. Write

$$a_0 + a_1x + a_2x^2 = c_1(1+x) + c_2x(x-1) + c_3(1+2x^2).$$

Equating powers of x on both sides of this equation, we have

$$c_1 + c_3 = a_0 \quad \text{and} \quad c_1 - c_2 = a_1 \quad \text{and} \quad c_2 + 2c_3 = a_2.$$

The augmented matrix corresponding to this system of equations is $\left[\begin{array}{ccc|c} 1 & 0 & 1 & a_0 \\ 1 & -1 & 0 & a_1 \\ 0 & 1 & 2 & a_2 \end{array} \right]$. We can reduce this

to row-echelon form as $\left[\begin{array}{ccc|c} 1 & 0 & 1 & a_0 \\ 0 & 1 & 2 & a_2 \\ 0 & 0 & 1 & -a_0 + a_1 + a_2 \end{array} \right]$. Thus, solving by back-substitution, we have

$$c_1 = 2a_0 - a_1 - a_2 \quad \text{and} \quad c_2 = 2a_0 - 2a_1 - a_2 \quad \text{and} \quad c_3 = -a_0 + a_1 + a_2.$$

Hence, relative to the ordered basis $B = \{p_1, p_2, p_3\}$, we have

$$[p(x)]_B = \begin{bmatrix} 2a_0 - a_1 - a_2 \\ 2a_0 - 2a_1 - a_2 \\ -a_0 + a_1 + a_2 \end{bmatrix}.$$

16. Let $\mathbf{v}_1 = (9, 2)$ and $\mathbf{v}_2 = (4, -3)$. Setting

$$(9, 2) = c_1(2, 1) + c_2(-3, 1)$$

and solving, we find $c_1 = 3$ and $c_2 = -1$. Thus, $[\mathbf{v}_1]_C = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Next, setting

$$(4, -3) = c_1(2, 1) + c_2(-3, 1)$$

and solving, we find $c_1 = -1$ and $c_2 = -2$. Thus, $[\mathbf{v}_2]_C = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$. Therefore,

$$P_{C \leftarrow B} = \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix}.$$

17. Let $\mathbf{v}_1 = (-5, -3)$ and $\mathbf{v}_2 = (4, 28)$. Setting

$$(-5, -3) = c_1(6, 2) + c_2(1, -1)$$

and solving, we find $c_1 = -1$ and $c_2 = 1$. Thus, $[\mathbf{v}_1]_C = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Next, setting

$$(4, 28) = c_1(6, 2) + c_2(1, -1)$$

and solving, we find $c_1 = 4$ and $c_2 = -20$. Thus, $[\mathbf{v}_2]_C = \begin{bmatrix} 4 \\ -20 \end{bmatrix}$. Therefore,

$$P_{C \leftarrow B} = \begin{bmatrix} -1 & 4 \\ 1 & -20 \end{bmatrix}.$$

18. Let $\mathbf{v}_1 = (2, -5, 0)$, $\mathbf{v}_2 = (3, 0, 5)$, and $\mathbf{v}_3 = (8, -2, -9)$. Setting

$$(2, -5, 0) = c_1(1, -1, 1) + c_2(2, 0, 1) + c_3(0, 1, 3)$$

and solving, we find $c_1 = 4$, $c_2 = -1$, and $c_3 = -1$. Thus, $[\mathbf{v}_1]_C = \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$. Next, setting

$$(3, 0, 5) = c_1(1, -1, 1) + c_2(2, 0, 1) + c_3(0, 1, 3)$$

and solving, we find $c_1 = 1$, $c_2 = 1$, and $c_3 = 1$. Thus, $[\mathbf{v}_2]_C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Finally, setting

$$(8, -2, -9) = c_1(1, -1, 1) + c_2(2, 0, 1) + c_3(0, 1, 3)$$

and solving, we find $c_1 = -2$, $c_2 = 5$, and $c_3 = -4$. Thus, $[\mathbf{v}_3]_C = \begin{bmatrix} -2 \\ 5 \\ -4 \end{bmatrix}$. Therefore,

$$P_{C \leftarrow B} = \begin{bmatrix} 4 & 1 & -2 \\ -1 & 1 & 5 \\ -1 & 1 & -4 \end{bmatrix}.$$

19. Let $\mathbf{v}_1 = (-7, 4, 4)$, $\mathbf{v}_2 = (4, 2, -1)$, and $\mathbf{v}_3 = (-7, 5, 0)$. Setting

$$(-7, 4, 4) = c_1(1, 1, 0) + c_2(0, 1, 1) + c_3(3, -1, -1)$$

and solving, we find $c_1 = 0$, $c_2 = 5/3$ and $c_3 = -7/3$. Thus, $[\mathbf{v}_1]_C = \begin{bmatrix} 0 \\ 5/3 \\ -7/3 \end{bmatrix}$. Next, setting

$$(4, 2, -1) = c_1(1, 1, 0) + c_2(0, 1, 1) + c_3(3, -1, -1)$$

and solving, we find $c_1 = 0$, $c_2 = 19/3$ and $c_3 = -7/3$. Setting

$$(4, 2, -1) = c_1(1, 1, 0) + c_2(0, 1, 1) + c_3(3, -1, -1)$$

and solving, we find $c_1 = 3$, $c_2 = -2/3$, and $c_3 = 1/3$. Thus, $[\mathbf{v}_2]_C = \begin{bmatrix} 3 \\ -2/3 \\ 1/3 \end{bmatrix}$. Finally, setting

$$(-7, 5, 0) = c_1(1, 1, 0) + c_2(0, 1, 1) + c_3(3, -1, -1)$$

and solving, we find $c_1 = 5$, $c_2 = -4$, and $c_3 = -4$. Hence, $[\mathbf{v}_3]_C = \begin{bmatrix} 5 \\ -4 \\ -4 \end{bmatrix}$. Therefore,

$$P_{C \leftarrow B} = \begin{bmatrix} 0 & 3 & 5 \\ \frac{5}{3} & -\frac{2}{3} & -4 \\ -\frac{7}{3} & \frac{1}{3} & -4 \end{bmatrix}.$$

20. Let $\mathbf{v}_1 = 7 - 4x$ and $\mathbf{v}_2 = 5x$. Setting

$$7 - 4x = c_1(1 - 2x) + c_2(2 + x)$$

and solving, we find $c_1 = 3$ and $c_2 = 2$. Thus, $[\mathbf{v}_1]_C = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Next, setting

$$5x = c_1(1 - 2x) + c_2(2 + x)$$

and solving, we find $c_1 = -2$ and $c_2 = 1$. Hence, $[\mathbf{v}_2]_C = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Therefore,

$$P_{C \leftarrow B} = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}.$$

21. Let $\mathbf{v}_1 = -4 + x - 6x^2$, $\mathbf{v}_2 = 6 + 2x^2$, and $\mathbf{v}_3 = -6 - 2x + 4x^2$. Setting

$$-4 + x - 6x^2 = c_1(1 - x + 3x^2) + c_2(2) + c_3(3 + x^2)$$

and solving, we find $c_1 = -1$, $c_2 = 3$, and $c_3 = -3$. Thus, $[\mathbf{v}_1]_C = \begin{bmatrix} -1 \\ 3 \\ -3 \end{bmatrix}$. Next, setting

$$6 + 2x^2 = c_1(1 - x + 3x^2) + c_2(2) + c_3(3 + x^2)$$

and solving, we find $c_1 = 0$, $c_2 = 0$, and $c_3 = 2$. Thus, $[\mathbf{v}_2]_C = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$. Finally, setting

$$-6 - 2x + 4x^2 = c_1(1 - x + 3x^2) + c_2(2) + c_3(3 + x^2)$$

and solving, we find $c_1 = 2$, $c_2 = -1$, and $c_3 = -2$. Thus, $[\mathbf{v}_3]_C = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$. Therefore,

$$P_{C \leftarrow B} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 0 & -1 \\ -3 & 2 & -2 \end{bmatrix}.$$

22. Let $\mathbf{v}_1 = -2 + 3x + 4x^2 - x^3$, $\mathbf{v}_2 = 3x + 5x^2 + 2x^3$, $\mathbf{v}_3 = -5x^2 - 5x^3$, and $\mathbf{v}_4 = 4 + 4x + 4x^2$. Setting

$$-2 + 3x + 4x^2 - x^3 = c_1(1 - x^3) + c_2(1 + x) + c_3(x + x^2) + c_4(x^2 + x^3)$$

and solving, we find $c_1 = 0$, $c_2 = -2$, $c_3 = 5$, and $c_4 = -1$. Thus, $[\mathbf{v}_1]_C = \begin{bmatrix} 0 \\ -2 \\ 5 \\ -1 \end{bmatrix}$. Next, setting

$$3x + 5x^2 + 2x^3 = c_1(1 - x^3) + c_2(1 + x) + c_3(x + x^2) + c_4(x^2 + x^3)$$

and solving, we find $c_1 = 0$, $c_2 = 0$, $c_3 = 3$, and $c_4 = 2$. Thus, $[\mathbf{v}_2]_C = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 2 \end{bmatrix}$. Next, solving

$$-5x^2 - 5x^3 = c_1(1 - x^3) + c_2(1 + x) + c_3(x + x^2) + c_4(x^2 + x^3)$$

and solving, we find $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, and $c_4 = -5$. Thus, $[\mathbf{v}_3]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -5 \end{bmatrix}$. Finally, setting

$$4 + 4x + 4x^2 = c_1(1 - x^3) + c_2(1 + x) + c_3(x + x^2) + c_4(x^2 + x^3)$$

and solving, we find $c_1 = 2$, $c_2 = 2$, $c_3 = 2$, and $c_4 = 2$. Thus, $[\mathbf{v}_4]_C = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$. Therefore,

$$P_{C \leftarrow B} = \begin{bmatrix} 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 2 \\ 5 & 3 & 0 & 2 \\ -1 & 2 & -5 & 2 \end{bmatrix}.$$

23. Let $\mathbf{v}_1 = 2 + x^2$, $\mathbf{v}_2 = -1 - 6x + 8x^2$, and $\mathbf{v}_3 = -7 - 3x - 9x^2$. Setting

$$2 + x^2 = c_1(1 + x) + c_2(-x + x^2) + c_3(1 + 2x^2)$$

and solving, we find $c_1 = 3$, $c_2 = 3$, and $c_3 = -1$. Thus, $[\mathbf{v}_1]_C = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}$. Next, solving

$$-1 - 6x + 8x^2 = c_1(1 + x) + c_2(-x + x^2) + c_3(1 + 2x^2)$$

and solving, we find $c_1 = -4$, $c_2 = 2$, and $c_3 = 3$. Thus, $[\mathbf{v}_2]_C = \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix}$. Finally, solving

$$-7 - 3x - 9x^2 = c_1(1 + x) + c_2(-x + x^2) + c_3(1 + 2x^2)$$

and solving, we find $c_1 = -2$, $c_2 = 1$, and $c_3 = -5$. Thus, $[\mathbf{v}_3]_C = \begin{bmatrix} -2 \\ 1 \\ -5 \end{bmatrix}$. Therefore,

$$P_{C \leftarrow B} = \begin{bmatrix} 3 & -4 & -2 \\ 3 & 2 & 1 \\ -1 & 3 & -5 \end{bmatrix}.$$

24. Let $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} -2 & -4 \\ 0 & 0 \end{bmatrix}$. Setting

$$\begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and solving, we find $c_1 = -2$, $c_2 = c_3 = c_4 = 1$. Thus, $[\mathbf{v}_1]_C = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Next, setting

$$\begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix} c_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and solving, we find $c_1 = 0$, $c_2 = 3$, $c_3 = -4$, and $c_4 = 1$. Thus, $[\mathbf{v}_2]_C = \begin{bmatrix} 0 \\ 3 \\ -4 \\ 1 \end{bmatrix}$. Next, setting

$$\begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} c_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and solving, we find $c_1 = 0$, $c_2 = 0$, $c_3 = 5$, and $c_4 = -2$. Thus, $[\mathbf{v}_3]_C = \begin{bmatrix} 0 \\ 0 \\ 5 \\ -2 \end{bmatrix}$. Finally, setting

$$\begin{bmatrix} -2 & -4 \\ 0 & 0 \end{bmatrix} c_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and solving, we find $c_1 = 0$, $c_2 = 0$, $c_3 = -4$, and $c_4 = 2$. Thus, $[\mathbf{v}_4]_C = \begin{bmatrix} 0 \\ 0 \\ -4 \\ 2 \end{bmatrix}$. Therefore, we have

$$P_{C \leftarrow B} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & -4 & 5 & -4 \\ 1 & 1 & -2 & 2 \end{bmatrix}.$$

25. Let $\mathbf{v}_1 = E_{12}$, $\mathbf{v}_2 = E_{22}$, $\mathbf{v}_3 = E_{21}$, and $\mathbf{v}_4 = E_{11}$. We see by inspection that

$$[\mathbf{v}_1]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad [\mathbf{v}_2]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [\mathbf{v}_3]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad [\mathbf{v}_4]_C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore,

$$P_{C \leftarrow B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

26. We could simply compute the inverse of the matrix obtained in Problem 16. For instructive purposes, however, we proceed directly. Let $\mathbf{w}_1 = (2, 1)$ and $\mathbf{w}_2 = (-3, 1)$. Setting

$$(2, 1) = c_1(9, 2) + c_2(4, -3)$$

and solving, we obtain $c_1 = 2/7$ and $c_2 = -1/7$. Thus, $[\mathbf{w}_1]_B = \begin{bmatrix} 2/7 \\ -1/7 \end{bmatrix}$. Next, setting

$$(-3, 1) = c_1(9, 2) + c_2(4, -3)$$

and solving, we obtain $c_1 = -1/7$ and $c_2 = -3/7$. Thus, $[\mathbf{w}_2]_B = \begin{bmatrix} -1/7 \\ -3/7 \end{bmatrix}$. Therefore,

$$P_{B \leftarrow C} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{1}{7} & -\frac{3}{7} \end{bmatrix}.$$

27. We could simply compute the inverse of the matrix obtained in Problem 17. For instructive purposes, however, we proceed directly. Let $\mathbf{w}_1 = (6, 2)$ and $\mathbf{w}_2 = (1, -1)$. Setting

$$(6, 2) = c_1(-5, -3) + c_2(4, 28)$$

and solving, we obtain $c_1 = -5/4$ and $c_2 = -1/16$. Thus, $[\mathbf{w}_1]_B = \begin{bmatrix} -5/4 \\ -1/16 \end{bmatrix}$. Next, setting

$$(1, -1) = c_1(-5, -3) + c_2(4, 28)$$

and solving, we obtain $c_1 = -1/4$ and $c_2 = -1/16$. Thus, $[\mathbf{w}_2]_B = \begin{bmatrix} -1/4 \\ -1/16 \end{bmatrix}$. Therefore,

$$P_{B \leftarrow C} = \begin{bmatrix} -\frac{5}{4} & -\frac{1}{4} \\ -\frac{1}{16} & -\frac{1}{16} \end{bmatrix}.$$

28. We could simply compute the inverse of the matrix obtained in Problem 18. For instructive purposes, however, we proceed directly. Let $\mathbf{w}_1 = (1, -1, 1)$, $\mathbf{w}_2 = (2, 0, 1)$, and $\mathbf{w}_3 = (0, 1, 3)$. Setting

$$(1, -1, 1) = c_1(2, -5, 0) + c_2(3, 0, 5) + c_3(8, -2, -9)$$

and solving, we find $c_1 = 1/5$, $c_2 = 1/5$, and $c_3 = 0$. Thus, $[\mathbf{w}_1]_B = \begin{bmatrix} 1/5 \\ 1/5 \\ 0 \end{bmatrix}$. Next, setting

$$(2, 0, 1) = c_1(2, -5, 0) + c_2(3, 0, 5) + c_3(8, -2, -9)$$

and solving, we find $c_1 = -2/45$, $c_2 = 2/5$, and $c_3 = 1/9$. Thus, $[\mathbf{w}_2]_B = \begin{bmatrix} -2/45 \\ 2/5 \\ 1/9 \end{bmatrix}$. Finally, setting

$$(0, 1, 3) = c_1(2, -5, 0) + c_2(3, 0, 5) + c_3(8, -2, -9)$$

and solving, we find $c_1 = -7/45$, $c_2 = 2/5$, and $c_3 = -1/9$. Thus, $[\mathbf{w}_3]_B = \begin{bmatrix} -7/45 \\ 2/5 \\ -1/9 \end{bmatrix}$. Therefore,

$$P_{B \leftarrow C} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{45} & -\frac{7}{45} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ 0 & \frac{1}{9} & -\frac{1}{9} \end{bmatrix}.$$

29. We could simply compute the inverse of the matrix obtained in Problem 20. For instructive purposes, however, we proceed directly. Let $\mathbf{w}_1 = 1 - 2x$ and $\mathbf{w}_2 = 2 + x$. Setting

$$1 - 2x = c_1(7 - 4x) + c_2(5x)$$

and solving, we find $c_1 = 1/7$ and $c_2 = -2/7$. Thus, $[\mathbf{w}_1]_B = \begin{bmatrix} 1/7 \\ -2/7 \end{bmatrix}$. Setting

$$2 + x = c_1(7 - 4x) + c_2(5x)$$

and solving, we find $c_1 = 2/7$ and $c_2 = 3/7$. Thus, $[\mathbf{w}_2]_B = \begin{bmatrix} 2/7 \\ 3/7 \end{bmatrix}$. Therefore,

$$P_{B \leftarrow C} = \begin{bmatrix} \frac{1}{7} & \frac{2}{7} \\ -\frac{2}{7} & \frac{3}{7} \end{bmatrix}.$$

30. Referring to Problem 22, we have

$$P_{B \leftarrow C} = (P_{C \leftarrow B})^{-1} = \begin{bmatrix} 0 & 0 & 0 & 2 \\ -2 & 0 & 0 & 2 \\ 5 & 3 & 0 & 2 \\ -1 & 2 & -5 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & -1/2 & 0 & 0 \\ -7/6 & 5/6 & 1/3 & 0 \\ -11/30 & 13/30 & 2/15 & -1/5 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}.$$

31. We could simply compute the inverse of the matrix obtained in Problem 25. For instructive purposes, however, we proceed directly. Let $\mathbf{w}_1 = E_{22}$, $\mathbf{w}_2 = E_{11}$, $\mathbf{w}_3 = E_{21}$, and $\mathbf{w}_4 = E_{12}$. We see by inspection that

$$[\mathbf{w}_1]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad [\mathbf{w}_2]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad [\mathbf{w}_3]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad [\mathbf{w}_4]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore

$$P_{B \leftarrow C} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

32. We first compute $[\mathbf{v}]_B$ and $[\mathbf{v}]_C$ directly. Setting

$$(-5, 3) = c_1(9, 2) + c_2(4, -3)$$

and solving, we obtain $c_1 = -3/35$ and $c_2 = -37/35$. Thus, $[\mathbf{v}]_B = \begin{bmatrix} -\frac{3}{35} \\ -\frac{37}{35} \end{bmatrix}$. Setting

$$(-5, 3) = c_1(2, 1) + c_2(-3, 1)$$

and solving, we obtain $c_1 = 4/5$ and $c_2 = 11/5$. Thus, $[\mathbf{v}]_C = \begin{bmatrix} \frac{4}{5} \\ \frac{11}{5} \end{bmatrix}$. Now, according to Problem 16,

$$P_{C \leftarrow B} = \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix}, \text{ so}$$

$$P_{C \leftarrow B}[\mathbf{v}]_B = \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -\frac{3}{35} \\ -\frac{37}{35} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{11}{5} \end{bmatrix} = [\mathbf{v}]_C,$$

which confirms Equation (4.7.6).

33. We first compute $[\mathbf{v}]_B$ and $[\mathbf{v}]_C$ directly. Setting

$$(-1, 2, 0) = c_1(-7, 4, 4) + c_2(4, 2, -1) + c_3(-7, 5, 0)$$

and solving, we obtain $c_1 = 3/43$, $c_2 = 12/43$, and $c_3 = 10/43$. Thus, $[\mathbf{v}]_B = \begin{bmatrix} \frac{3}{43} \\ \frac{12}{43} \\ \frac{10}{43} \end{bmatrix}$. Setting

$$(-1, 2, 0) = c_1(1, 1, 0) + c_2(0, 1, 1) + c_3(3, -1, -1)$$

and solving, we obtain $c_1 = 2$, $c_2 = -1$, and $c_3 = -1$. Thus, $[\mathbf{v}]_C = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$. Now, according to Problem

$$19, P_{C \leftarrow B} = \begin{bmatrix} 0 & 3 & 5 \\ \frac{5}{3} & -\frac{2}{3} & -4 \\ -\frac{7}{3} & \frac{1}{3} & -4 \end{bmatrix}, \text{ so } P_{C \leftarrow B}[\mathbf{v}]_B = \begin{bmatrix} 0 & 3 & 5 \\ \frac{5}{3} & -\frac{2}{3} & -4 \\ -\frac{7}{3} & \frac{1}{3} & -4 \end{bmatrix} \begin{bmatrix} \frac{3}{43} \\ \frac{12}{43} \\ \frac{10}{43} \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} = [\mathbf{v}]_C, \text{ which}$$

confirms Equation (4.7.6).

34. We first compute $[\mathbf{v}]_B$ and $[\mathbf{v}]_C$ directly. Setting

$$6 - 4x = c_1(7 - 4x) + c_2(5x)$$

and solving, we obtain $c_1 = 6/7$ and $c_2 = -4/35$. Thus, $[\mathbf{v}]_B = \begin{bmatrix} \frac{6}{7} \\ -\frac{4}{35} \end{bmatrix}$. Next, setting

$$6 - 4x = c_1(1 - 2x) + c_2(2 + x)$$

and solving, we obtain $c_1 = 14/5$ and $c_2 = 8/5$. Thus, $[\mathbf{v}]_C = \begin{bmatrix} \frac{14}{5} \\ \frac{8}{5} \end{bmatrix}$. Now, according to Problem 20,

$$P_{C \leftarrow B} = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}, \text{ so}$$

$$P_{C \leftarrow B}[\mathbf{v}]_B = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{6}{7} \\ -\frac{4}{35} \end{bmatrix} = \begin{bmatrix} \frac{14}{5} \\ \frac{8}{5} \end{bmatrix} = [\mathbf{v}]_C,$$

which confirms Equation (4.7.6).

35. We first compute $[\mathbf{v}]_B$ and $[\mathbf{v}]_C$ directly. Setting

$$5 - x + 3x^2 = c_1(-4 + x - 6x^2) + c_2(6 + 2x^2) + c_3(-6 - 2x + 4x^2)$$

and solving, we obtain $c_1 = 1$, $c_2 = 5/2$, and $c_3 = 1$. Thus, $[\mathbf{v}]_B = \begin{bmatrix} 1 \\ \frac{5}{2} \\ 1 \end{bmatrix}$. Next, setting

$$5 - x + 3x^2 = c_1(1 - x + 3x^2) + c_2(2) + c_3(3 + x^2)$$

and solving, we obtain $c_1 = 1$, $c_2 = 2$, and $c_3 = 0$. Thus, $[\mathbf{v}]_C = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Now, according to Problem 21,

$$P_{C \leftarrow B} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 0 & -1 \\ -3 & 2 & -2 \end{bmatrix}, \text{ so}$$

$$P_{C \leftarrow B}[\mathbf{v}]_B = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 0 & -1 \\ -3 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{5}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = [\mathbf{v}]_C,$$

which confirms Equation (4.7.6).

36. We first compute $[\mathbf{v}]_B$ and $[\mathbf{v}]_C$ directly. Setting

$$\begin{bmatrix} -1 & -1 \\ -4 & 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix} + c_2 \begin{bmatrix} 0 & -1 \\ 3 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} + c_4 \begin{bmatrix} -2 & -4 \\ 0 & 0 \end{bmatrix}$$

and solving, we obtain $c_1 = -5/2$, $c_2 = -13/6$, $c_3 = 37/6$, and $c_4 = 17/2$. Thus, $[\mathbf{v}]_B = \begin{bmatrix} -5/2 \\ -13/6 \\ 37/6 \\ 17/2 \end{bmatrix}$. Next,

setting

$$\begin{bmatrix} -1 & -1 \\ -4 & 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and solving, we find $c_1 = 5$, $c_2 = -9$, $c_3 = 3$, and $c_4 = 0$. Thus, $[\mathbf{v}]_C = \begin{bmatrix} 5 \\ -9 \\ 3 \\ 0 \end{bmatrix}$. Now, according to Problem

$$24, P_{C \leftarrow B} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & -4 & 5 & -4 \\ 1 & 1 & -2 & 2 \end{bmatrix}, \text{ and so}$$

$$P_{C \leftarrow B}[\mathbf{v}]_B = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & -4 & 5 & -4 \\ 1 & 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} -5/2 \\ -13/6 \\ 37/6 \\ 17/2 \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \\ 3 \\ 0 \end{bmatrix} = [\mathbf{v}]_C,$$

which confirms Equation (4.7.6).

37. Write $\mathbf{x} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n$. We have

$$c\mathbf{x} = c(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) = (ca_1)\mathbf{v}_1 + (ca_2)\mathbf{v}_2 + \cdots + (ca_n)\mathbf{v}_n.$$

Hence,

$$[c\mathbf{x}]_B = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix} = c \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = c[\mathbf{x}]_B.$$

38. We must show that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is linearly independent and spans V .

Check linear independence: Assume that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

We wish to show that $c_1 = c_2 = \cdots = c_n = 0$. Now by assumption, the zero vector $\mathbf{0}$ can be uniquely written as a linear combination of the vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. Since

$$\mathbf{0} = 0 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2 + \cdots + 0 \cdot \mathbf{v}_n,$$

we therefore conclude that

$$c_1 = c_2 = \cdots = c_n = 0,$$

as needed.

Check spanning property: Let $\mathbf{v}$ be an arbitrary vector in V . By assumption, it is possible to express $\mathbf{v}$ (uniquely) as a linear combination of the vectors in $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$; say

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n.$$

Therefore, $\mathbf{v}$ lies in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$. Since $\mathbf{v}$ is an arbitrary member of V , we conclude that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ spans V .

39. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and let $C = \{\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(n)}\}$, where σ is a permutation of the set $\{1, 2, \dots, n\}$. We will show that $P_{C \leftarrow B}$ contains exactly one 1 in each row and column, and zeroes elsewhere (the argument for $P_{B \leftarrow C}$ is essentially identical, or can be deduced from the fact that $P_{B \leftarrow C}$ is the inverse of $P_{C \leftarrow B}$).

Let i be in $\{1, 2, \dots, n\}$. The i th column of $P_{C \leftarrow B}$ is $[\mathbf{v}_i]_C$. Suppose that $k_i \in \{1, 2, \dots, n\}$ is such that $\sigma(k_i) = i$. Then $[\mathbf{v}_i]_C$ is a column vector with a 1 in the k_i th position and zeroes elsewhere. Since the values $k_1, k_2, \dots, k_n$ are distinct, we see that each column of $P_{C \leftarrow B}$ contains a single 1 (with zeroes elsewhere) in a different position from any other column. Hence, when we consider all n columns as a whole, each position in the column vector must have a 1 occurring exactly once in one of the columns. Therefore, $P_{C \leftarrow B}$ contains exactly one 1 in each row and column, and zeroes elsewhere.

Solutions to Section 4.8

True-False Review:

1. **TRUE.** Note that $\text{rowspace}(A)$ is a subspace of $\mathbb{R}^n$ and $\text{colspace}(A)$ is a subspace of $\mathbb{R}^m$, so certainly if $\text{rowspace}(A) = \text{colspace}(A)$, then $\mathbb{R}^n$ and $\mathbb{R}^m$ must be the same. That is, $m = n$.
2. **FALSE.** A basis for the row space of A consists of the *nonzero* row vectors of any row-echelon form of A .
3. **FALSE.** The nonzero column vectors of the original matrix A that correspond to the nonzero column vectors of a row-echelon form of A form a basis for $\text{colspace}(A)$.
4. **TRUE.** Both $\text{rowspace}(A)$ and $\text{colspace}(A)$ have dimension equal to $\text{rank}(A)$, the number of nonzero rows in a row-echelon form of A . Equivalently, their dimensions are both equal to the number of pivots occurring in a row-echelon form of A .
5. **TRUE.** For an invertible $n \times n$ matrix, $\text{rank}(A) = n$. That means there are n nonzero rows in a row-echelon form of A , and so $\text{rowspace}(A)$ is n -dimensional. Therefore, we conclude that $\text{rowspace}(A) = \mathbb{R}^n$.
6. **TRUE.** This follows immediately from the (true) statements in True-False Review Questions 4-5 above.

Problems:

1. A row-echelon form of A is $\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$. Consequently, a basis for $\text{rowspace}(A)$ is $\{(1, -2)\}$, whereas a basis for $\text{colspace}(A)$ is $\{(1, -3)\}$.
2. A row-echelon form of A is $\begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$. Consequently, a basis for $\text{rowspace}(A)$ is $\{(1, 1, -3, 2), (0, 1, -2, 1)\}$, whereas a basis for $\text{colspace}(A)$ is $\{(1, 3), (1, 4)\}$.
3. A row-echelon form of A is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Consequently, a basis for $\text{rowspace}(A)$ is $\{(1, 2, 3), (0, 1, 2)\}$, whereas a basis for $\text{colspace}(A)$ is $\{(1, 5, 9), (2, 6, 10)\}$.
4. A row-echelon form of A is $\begin{bmatrix} 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Consequently, a basis for $\text{rowspace}(A)$ is $\{(0, 1, \frac{1}{3})\}$, whereas a basis for $\text{colspace}(A)$ is $\{(3, -6, 12)\}$.
5. A row-echelon form of A is $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Consequently, a basis for $\text{rowspace}(A)$ is $\{(1, 2, -1, 3), (0, 0, 0, 1)\}$, whereas a basis for $\text{colspace}(A)$ is $\{(1, 3, 1, 5), (3, 5, -1, 7)\}$.
6. We can reduce A to $\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -4 & 3 \\ 0 & 0 & 6 & -13 \end{bmatrix}$ (Note: This is not row-echelon form, but it is not necessary to bring the leading nonzero element in each row to 1.). Consequently, a basis for $\text{rowspace}(A)$ is $\{(1, -1, 2, 3), (0, 2, -4, 3), (0, 0, 6, -13)\}$, whereas a basis for $\text{colspace}(A)$ is $\{(1, 1, 3), (-1, 1, 1), (2, -2, 4)\}$.
7. We determine a basis for the row space of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 5 & -4 & 1 \\ 7 & -5 & -4 \end{bmatrix}$. A row-echelon form of this matrix is

$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{bmatrix}$. Consequently, a basis for the subspace spanned by the given vectors is $\{(1, -1, 2), (0, 1, -9)\}$.

8. We determine a basis for the row space of the matrix $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 5 & -1 \\ 2 & 7 & 4 \\ 1 & 4 & 1 \end{bmatrix}$. A row-echelon form of this matrix is

$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Consequently, a basis for the subspace spanned by the given vectors is $\{(1, 3, 3), (0, 1, -2)\}$.

9. We determine a basis for the row space of the matrix $\begin{bmatrix} 1 & 1 & -1 & 2 \\ 2 & 1 & 3 & -4 \\ 1 & 2 & -6 & 10 \end{bmatrix}$. A row-echelon form of

this matrix is $\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -5 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Consequently, a basis for the subspace spanned by the given vectors is $\{(1, 1, -1, 2), (0, 1, -5, 8)\}$.

10. We determine a basis for the row space of the matrix $\begin{bmatrix} 1 & 4 & 1 & 3 \\ 2 & 8 & 3 & 5 \\ 1 & 4 & 0 & 4 \\ 2 & 8 & 2 & 6 \end{bmatrix}$. A row-echelon form of this

matrix is $\begin{bmatrix} 1 & 4 & 1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Consequently, a basis for the subspace spanned by the given vectors is $\{(1, 4, 1, 3), (0, 0, 1, -1)\}$.

11. A row-echelon form of A is $\begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$. Consequently, a basis for $\text{row space}(A)$ is $\{(1, -3)\}$, whereas a basis for $\text{col space}(A)$ is $\{(-3, 1)\}$. Both of these subspaces are lines in the xy -plane.

12. (a) A row-echelon form of A is $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Consequently, a basis for $\text{row space}(A)$ is $\{(1, 2, 4), (0, 1, 1)\}$, whereas a basis for $\text{col space}(A)$ is $\{(1, 5, 3), (2, 11, 7)\}$.

(b) We see that both of these subspaces are 2-dimensional, and therefore each corresponds geometrically to a plane. By inspection, we see that the two basis vectors for $\text{row space}(A)$ satisfy the equations $2x + y - z = 0$, and therefore $\text{row space}(A)$ corresponds to the plane with this equation. Similarly, we see that the two basis vectors for $\text{col space}(A)$ satisfy the equations $2x - y + z = 0$, and therefore $\text{col space}(A)$ corresponds to the plane with this equations.

13. If $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $\text{col space}(A)$ is spanned by $(1, 2)$, but if we permute the two rows of A , we obtain a new matrix whose column space is spanned by $(2, 1)$. On the other hand, if we multiply the first row by 2, then we obtain a new matrix whose column space is spanned by $(2, 2)$. And if we add -2 times the first row to the second row, we obtain a new matrix whose column space is spanned by $(1, 0)$. Therefore, in all cases, $\text{col space}(A)$ is altered by the row operations performed.

14. Many examples are possible here, but an easy 2×2 example is the matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. We have $\text{rowspace}(A) = \{(r, r) : r \in \mathbb{R}\}$, while $\text{colspace}(A) = \{(r, -r) : r \in \mathbb{R}\}$. Thus, $\text{rowspace}(A)$ and $\text{colspace}(A)$ have no nonzero vectors in common.

Solutions to Section 4.9

True-False Review:

1. FALSE. For example, consider the 7×3 zero matrix, $0_{7 \times 3}$. We have $\text{rank}(0_{7 \times 3}) = 0$, and therefore by the Rank-Nullity Theorem, $\text{nullity}(0_{7 \times 3}) = 3$. But $|m - n| = |7 - 3| = 4$. Many other examples can be given. In particular, provided that $m > 2n$, the $m \times n$ zero matrix will show the falsity of the statement.

2. FALSE. In this case, $\text{rowspace}(A)$ is a subspace of $\mathbb{R}^9$, hence it cannot possibly be equal to $\mathbb{R}^7$. The correct conclusion here should have been that $\text{rowspace}(A)$ is a 7-dimensional subspace of $\mathbb{R}^9$.

3. TRUE. By the Rank-Nullity Theorem, $\text{rank}(A) = 7$, and therefore, $\text{rowspace}(A)$ is a 7-dimensional subspace of $\mathbb{R}^7$. Hence, $\text{rowspace}(A) = \mathbb{R}^7$.

4. FALSE. For instance, the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is an upper triangular matrix with two zeros appearing on the main diagonal. However, since $\text{rank}(A) = 1$, we also have $\text{nullity}(A) = 1$.

5. TRUE. An invertible matrix A must have $\text{nullspace}(A) = \{\mathbf{0}\}$, but if $\text{colspace}(A)$ is also $\{\mathbf{0}\}$, then A would be the zero matrix, which is certainly not invertible.

6. FALSE. For instance, if we take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $\text{nullity}(A) + \text{nullity}(B) = 1 + 1 = 2$, but $A + B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, and $\text{nullity}(A + B) = 1$.

7. FALSE. For instance, if we take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $\text{nullity}(A) \cdot \text{nullity}(B) = 1 \cdot 1 = 1$, but $AB = 0_2$, and $\text{nullity}(AB) = 2$.

8. TRUE. If $\mathbf{x}$ belongs to the nullspace of B , then $B\mathbf{x} = \mathbf{0}$. Therefore, $(AB)\mathbf{x} = A(B\mathbf{x}) = A\mathbf{0} = \mathbf{0}$, so that $\mathbf{x}$ also belongs to the nullspace of AB . Thus, $\text{nullspace}(B)$ is a subspace of $\text{nullspace}(AB)$. Hence, $\text{nullity}(B) \leq \text{nullity}(AB)$, as claimed.

9. TRUE. If $\mathbf{y}$ belongs to $\text{nullspace}(A)$, then $A\mathbf{y} = \mathbf{0}$. Hence, if $A\mathbf{x}_p = \mathbf{b}$, then

$$A(\mathbf{y} + \mathbf{x}_p) = A\mathbf{y} + A\mathbf{x}_p = \mathbf{0} + \mathbf{b} = \mathbf{b},$$

which demonstrates that $\mathbf{y} + \mathbf{x}_p$ is also a solution to the linear system $A\mathbf{x} = \mathbf{b}$.

Problems:

1. The matrix is already in row-echelon form. A vector (x, y, z, w) in $\text{nullspace}(A)$ must satisfy $x - 6z - w = 0$. We see that y, z , and w are free variables, and

$$\text{nullspace}(A) = \left\{ \begin{bmatrix} 6z + w \\ y \\ z \\ w \end{bmatrix} : y, z, w \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Therefore, $\text{nullity}(A) = 3$. Moreover, since this row-echelon form contains one nonzero row, $\text{rank}(A) = 1$. Since the number of columns of A is $4 = 1 + 3$, the Rank-Nullity Theorem is verified.

2. We bring A to row-echelon form:

$$\text{REF}(A) = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}.$$

A vector (x, y) in $\text{nullspace}(A)$ must satisfy $x - \frac{1}{2}y = 0$. Setting $y = 2t$, we get $x = t$. Therefore,

$$\text{nullspace}(A) = \left\{ \begin{bmatrix} t \\ 2t \end{bmatrix} : t \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}.$$

Therefore, $\text{nullity}(A) = 1$. Since $\text{REF}(A)$ contains one nonzero row, $\text{rank}(A) = 1$. Since the number of columns of A is $2 = 1 + 1$, the Rank-Nullity Theorem is verified.

3. We bring A to row-echelon form:

$$\text{REF}(A) = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since there are no unpivoted columns, there are no free variables in the associated homogeneous linear system, and so

$$\text{nullspace}(A) = \{\mathbf{0}\}.$$

Therefore, $\text{nullity}(A) = 0$. Since $\text{REF}(A)$ contains three nonzero rows, $\text{rank}(A) = 3$. Since the number of columns of A is $3 = 3 + 0$, the Rank-Nullity Theorem is verified.

4. We bring A to row-echelon form:

$$\text{REF}(A) = \begin{bmatrix} 1 & 4 & -1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

A vector (x, y, z, w) in $\text{nullspace}(A)$ must satisfy $x + 4y - z + 3w = 0$ and $y + z + w = 0$. We see that z and w are free variables. Set $z = t$ and $w = s$. Then $y = -z - w = -t - s$ and $x = z - 4y - 3w = t - 4(-t - s) - 3s = 5t + s$. Therefore,

$$\text{nullspace}(A) = \left\{ \begin{bmatrix} 5t + s \\ -t - s \\ t \\ s \end{bmatrix} : s, t \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 5 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Therefore, $\text{nullity}(A) = 2$. Moreover, since $\text{REF}(A)$ contains two nonzero rows, $\text{rank}(A) = 2$. Since the number of columns of A is $4 = 2 + 2$, the Rank-Nullity Theorem is verified.

5. Since all rows (or columns) of this matrix are proportional to the first one, $\text{rank}(A) = 1$. Since A has two columns, we conclude from the Rank-Nullity Theorem that

$$\text{nullity}(A) = 2 - \text{rank}(A) = 2 - 1 = 1.$$

6. The first and last rows of A are not proportional, but the middle rows are proportional to the first row. Therefore, $\text{rank}(A) = 2$. Since A has five columns, we conclude from the Rank-Nullity Theorem that

$$\text{nullity}(A) = 5 - \text{rank}(A) = 5 - 2 = 3.$$

7. Since the second and third columns are not proportional and the first column is all zeros, we have $\text{rank}(A) = 2$. Since A has three columns, we conclude from the Rank-Nullity Theorem that

$$\text{nullity}(A) = 3 - \text{rank}(A) = 3 - 2 = 1.$$

8. This matrix (already in row-echelon form) has one nonzero row, so $\text{rank}(A) = 1$. Since it has four columns, we conclude from the Rank-Nullity Theorem that

$$\text{nullity}(A) = 4 - \text{rank}(A) = 4 - 1 = 3.$$

9. The augmented matrix for this linear system is $\left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 2 & 7 & 9 & 11 \\ 1 & 5 & 21 & 10 \end{array} \right]$. We quickly reduce this augmented matrix to row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & 1 & 11 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

A solution (x, y, z) to the system will have a free variable corresponding to the third column: $z = t$. Then $y + 11t = 3$, so $y = 3 - 11t$. Finally, $x + 3y - z = 4$, so $x = 4 + t - 3(3 - 11t) = -5 + 34t$. Thus, the solution set is

$$\left\{ \begin{bmatrix} -5 + 34t \\ 3 - 11t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} 34 \\ -11 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Observe that $\mathbf{x}_p = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$ is a particular solution to $A\mathbf{x} = \mathbf{b}$, and that $\begin{bmatrix} 34 \\ -11 \\ 1 \end{bmatrix}$ forms a basis for $\text{nullspace}(A)$. Therefore, the set of solution vectors obtained does indeed take the form (4.9.3).

10. The augmented matrix for this linear system is $\left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 6 \\ 1 & -2 & 5 & 5 & 13 \\ 2 & -1 & 1 & 4 & 5 \end{array} \right]$. We quickly reduce this augmented matrix to row-echelon form:

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 & 6 \\ 0 & 1 & -3 & -2 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

A solution (x, y, z, w) to the system will have free variables corresponding to the third and fourth columns: $z = t$ and $w = s$. Then $y - 3z - 2w = -7$ requires that $y = -7 + 3t + 2s$, and $x - y + 2z + 3w = 6$ requires that $x = 6 + (-7 + 3t + 2s) - 2t - 3s = -1 + t - s$. Thus, the solution set is

$$\left\{ \begin{bmatrix} -1 + t - s \\ -7 + 3t + 2s \\ t \\ s \end{bmatrix} : s, t \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -7 \\ 0 \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

Observe that $\mathbf{x}_p = \begin{bmatrix} -1 \\ -7 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution to $A\mathbf{x} = \mathbf{b}$, and that

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\text{nullspace}(A)$. Therefore, the set of solution vectors obtained does indeed take the form (4.9.3).

11. The augmented matrix for this linear system is $\left[\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 3 & -1 & -7 & 2 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & -4 & -6 \end{array} \right]$. We quickly reduce this augmented matrix to row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 0 & 1 & \frac{1}{4} & -\frac{11}{4} \\ 0 & 0 & 1 & 1 \end{array} \right].$$

There are no free variables in the solution set (since none of the first three columns is unpivoted), and we find the solution set by back-substitution: $\left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \right\}$. It is easy to see that this is indeed a particular solution:

$\mathbf{x}_p = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$. Since the row-echelon form of A has three nonzero rows, $\text{rank}(A) = 3$. Thus, $\text{nullity}(A) = 0$.

Hence, $\text{nullspace}(A) = \{\mathbf{0}\}$. Thus, the only term in the expression (4.9.3) that appears in the solution is $\mathbf{x}_p$, and this is precisely the unique solution we obtained in the calculations above.

12. By inspection, we see that a particular solution to this (homogeneous) linear system is $\mathbf{x}_p = \mathbf{0}$. We quickly reduce this augmented matrix to row-echelon form:

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 5 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{7}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

A solution (x, y, z, w) to the system will have free variables corresponding to the third and fourth columns: $z = t$ and $w = s$. Then $y - \frac{1}{2}z + \frac{7}{2}w = 0$ requires that $y = \frac{1}{2}t - \frac{7}{2}s$, and $x + y - z + 5w = 0$ requires that $x = t - (\frac{1}{2}t - \frac{7}{2}s) - 5s = \frac{1}{2}t - \frac{3}{2}s$. Thus, the solution set is

$$\left\{ \begin{bmatrix} \frac{1}{2}t - \frac{3}{2}s \\ \frac{1}{2}t - \frac{7}{2}s \\ t \\ s \end{bmatrix} : s, t \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -\frac{3}{2} \\ -\frac{7}{2} \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

Since the vectors $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -\frac{3}{2} \\ -\frac{7}{2} \\ 0 \\ 1 \end{bmatrix}$ form a basis for $\text{nullspace}(A)$, our solutions take the proper form given in (4.9.3).

13. By the Rank-Nullity Theorem, $\text{rank}(A) = 7 - \text{nullity}(A) = 7 - 4 = 3$, and hence, $\text{colspace}(A)$ is 3-dimensional. But since A has three rows, $\text{colspace}(A)$ is a subspace of $\mathbb{R}^3$. Therefore, since the only 3-dimensional subspace of $\mathbb{R}^3$ is $\mathbb{R}^3$ itself, we conclude that $\text{colspace}(A) = \mathbb{R}^3$. Now $\text{rowspace}(A)$ is also 3-dimensional, but it is a subspace of $\mathbb{R}^5$. Therefore, it is not accurate to say that $\text{rowspace}(A) = \mathbb{R}^3$.

14. By the Rank-Nullity Theorem, $\text{rank}(A) = 4 - \text{nullity}(A) = 4 - 0 = 4$, so we conclude that $\text{rowspace}(A)$ is 4-dimensional. Since $\text{rowspace}(A)$ is a subspace of $\mathbb{R}^4$ (since A contains four columns), and it is 4-dimensional, we conclude that $\text{rowspace}(A) = \mathbb{R}^4$. Although $\text{colspace}(A)$ is 4-dimensional, $\text{colspace}(A)$ is a subspace of $\mathbb{R}^6$, and therefore it is not accurate to say that $\text{colspace}(A) = \mathbb{R}^4$.

15. If $\text{rowspace}(A) = \text{nullspace}(A)$, then we know that $\text{rank}(A) = \text{nullity}(A)$. Therefore, $\text{rank}(A) + \text{nullity}(A)$ must be even. But $\text{rank}(A) + \text{nullity}(A)$ is the number of columns of A . Therefore, A contains an even number of columns.

16. We know that $\text{rank}(A) + \text{nullity}(A) = 7$. But since A only has five rows, $\text{rank}(A) \leq 5$. Therefore, $\text{nullity}(A) \geq 2$. However, since $\text{nullspace}(A)$ is a subspace of $\mathbb{R}^7$, $\text{nullity}(A) \leq 7$. Therefore, $2 \leq \text{nullity}(A) \leq 7$. There are many examples of a 5×7 matrix A with $\text{nullity}(A) = 2$; one example is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$
 The only 5×7 matrix with $\text{nullity}(A) = 7$ is $0_{5 \times 7}$, the 5×7 zero matrix.

17. We know that $\text{rank}(A) + \text{nullity}(A) = 8$. But since A only has three rows, $\text{rank}(A) \leq 3$. Therefore, $\text{nullity}(A) \geq 5$. However, since $\text{nullspace}(A)$ is a subspace of $\mathbb{R}^8$, $\text{nullity}(A) \leq 8$. Therefore, $5 \leq \text{nullity}(A) \leq 8$. There are many examples of a 3×8 matrix A with $\text{nullity}(A) = 5$; one example is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
 The only 3×8 matrix with $\text{nullity}(A) = 8$ is $0_{3 \times 8}$, the 3×8 zero matrix.

18. If $B\mathbf{x} = \mathbf{0}$, then $AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$. This observation shows that $\text{nullspace}(B)$ is a subspace of $\text{nullspace}(AB)$. On the other hand, if $AB\mathbf{x} = \mathbf{0}$, then $B\mathbf{x} = (A^{-1}A)B\mathbf{x} = A^{-1}(AB)\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$, so $B\mathbf{x} = \mathbf{0}$. Therefore, $\text{nullspace}(AB)$ is a subspace of $\text{nullspace}(B)$. As a result, since $\text{nullspace}(B)$ and $\text{nullspace}(AB)$ are subspaces of each other, they must be equal: $\text{nullspace}(AB) = \text{nullspace}(B)$. Therefore, $\text{nullity}(AB) = \text{nullity}(B)$.

Solutions to Section 4.10

True-False Review:

- 1. TRUE.** This follows from the equivalence of (a) and (m) in the Invertible Matrix Theorem.
- 2. FALSE.** If the matrix has n linearly independent rows, then by the equivalence of (a) and (m) in the Invertible Matrix Theorem, such a matrix would be invertible. But if that were so, then by part (j) of the Invertible Matrix Theorem, such a matrix would have to have n linearly independent columns.
- 3. FALSE.** If the matrix has n linearly independent columns, then by the equivalence of (a) and (j) in the Invertible Matrix Theorem, such a matrix would be invertible. But if that were so, then by part (m) of the Invertible Matrix Theorem, such a matrix would have to have n linearly independent rows.
- 4. FALSE.** An $n \times n$ matrix A with $\det(A) = 0$ is not invertible by part (g) of the Invertible Matrix Theorem. Therefore, by the equivalence of (a) and (l) in the Invertible Matrix Theorem, the columns of A do not form a basis for $\mathbb{R}^n$.

5. TRUE. If $\text{rowspace}(A) \neq \mathbb{R}^n$, then by the equivalence of (a) and (n) in the Invertible Matrix Theorem, A is not invertible. Therefore, A is not row-equivalent to the identity matrix. Since B is row-equivalent to A , then B is not row-equivalent to the identity matrix, and therefore, B is not invertible. Hence, by part (k) of the Invertible Matrix Theorem, we conclude that $\text{colspace}(B) \neq \mathbb{R}^n$.

6. FALSE. If $\text{nullspace}(A) = \{\mathbf{0}\}$, then A is invertible by the equivalence of (a) and (c) in the Invertible Matrix Theorem. Since E is an elementary matrix, it is also invertible. Therefore, EA is invertible. By part (g) of the Invertible Matrix Theorem, $\det(EA) \neq 0$, contrary to the statement given.

7. FALSE. The matrix $[A|B]$ has $2n$ columns, but only n rows, and therefore $\text{rank}([A|B]) \leq n$. Hence, by the Rank-Nullity Theorem, $\text{nullity}([A|B]) \geq n > 0$.

8. TRUE. The first and third rows are proportional, and hence statement (m) in the Invertible Matrix Theorem is false. Therefore, by the equivalence with (a), the matrix is not invertible.

9. FALSE. For instance, the matrix $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ is of the form given, but satisfies any (and all) of the statements of the Invertible Matrix Theorem.

10. FALSE. For instance, the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 2 \\ 1 & 0 & 0 \end{bmatrix}$ is of the form given, but has a nonzero determinant, and so by part (g) of the Invertible Matrix Theorem, it is invertible.

Solutions to Section 4.11

1. FALSE. The converse of this statement is true, but for the given statement, many counterexamples exist. For instance, the vectors $\mathbf{v} = (1, 1)$ and $\mathbf{w} = (1, 0)$ in $\mathbb{R}^2$ are linearly independent, but they are not orthogonal.

2. FALSE. We have

$$\langle k\mathbf{v}, k\mathbf{w} \rangle = k\langle \mathbf{v}, k\mathbf{w} \rangle = k\langle k\mathbf{w}, \mathbf{v} \rangle = k^2\langle \mathbf{w}, \mathbf{v} \rangle = k^2\langle \mathbf{v}, \mathbf{w} \rangle,$$

where we have used the axioms of an inner product to carry out these steps. Therefore, the result conflicts with the given statement, which must therefore be a false statement.

3. TRUE. We have

$$\langle c_1\mathbf{v}_1 + c_2\mathbf{v}_2, \mathbf{w} \rangle = \langle c_1\mathbf{v}_1, \mathbf{w} \rangle + \langle c_2\mathbf{v}_2, \mathbf{w} \rangle = c_1\langle \mathbf{v}_1, \mathbf{w} \rangle + c_2\langle \mathbf{v}_2, \mathbf{w} \rangle = c_1 \cdot 0 + c_2 \cdot 0 = 0.$$

4. TRUE. We have

$$\langle \mathbf{x} + \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle - \langle \mathbf{y}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle - \langle \mathbf{y}, \mathbf{y} \rangle = \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2.$$

This will be negative if and only if $\|\mathbf{x}\|^2 < \|\mathbf{y}\|^2$, and since $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ are nonnegative real numbers, $\|\mathbf{x}\|^2 < \|\mathbf{y}\|^2$ if and only if $\|\mathbf{x}\| < \|\mathbf{y}\|$.

5. FALSE. For example, if V is the inner product space of integrable functions on $(-\infty, \infty)$, then the formula

$$\langle f, g \rangle = \int_a^b f(t)g(t)dt$$

is a valid any product for any choice of real numbers $a < b$. See also Problem 9 in this section, in which a “non-standard” inner product on $\mathbb{R}^2$ is given.

6. TRUE. The angle between the vectors $-2\mathbf{v}$ and $-2\mathbf{w}$ is

$$\cos \theta = \frac{(-2\mathbf{v}) \cdot (-2\mathbf{w})}{\| -2\mathbf{v} \| \| -2\mathbf{w} \|} = \frac{(-2)^2(\mathbf{v} \cdot \mathbf{w})}{(-2)^2 \|\mathbf{v}\| \|\mathbf{w}\|} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|},$$

and this is the angle between the vectors $\mathbf{v}$ and $\mathbf{w}$.

7. FALSE. This definition of $\langle p, q \rangle$ will not satisfy the requirements of an inner product. For instance, if we take $p = x$, then $\langle p, p \rangle = 0$, but $p \neq 0$.

Problems:

1. $\langle \mathbf{v}, \mathbf{w} \rangle = 8$, $\|\mathbf{v}\| = 3\sqrt{3}$, $\|\mathbf{w}\| = \sqrt{7}$. Hence,

$$\cos \theta = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\| \|\mathbf{w}\|} = \frac{8}{3\sqrt{21}} \implies \theta \approx 0.95 \text{ radians.}$$

2. $\langle f, g \rangle = \int_0^\pi x \sin x dx = \pi$, $\|f\| = \sqrt{\left(\int_0^\pi \sin^2 x dx\right)} = \sqrt{\left(\frac{\pi}{2}\right)}$, $\|g\| = \sqrt{\left(\int_0^\pi x^2 dx\right)} = \sqrt{\left(\frac{\pi^3}{3}\right)}$. Hence,

$$\cos \theta = \frac{\pi}{\left(\frac{\pi}{2}\right)^{1/2} \left(\frac{\pi^3}{3}\right)^{1/2}} = \frac{\sqrt{6}}{\pi} \approx 0.68 \text{ radians.}$$

3. $\langle \mathbf{v}, \mathbf{w} \rangle = (2+i)(-1-i) + (3-2i)(1+3i) + (4+i)(3+i) = 19 + 11i$. $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{35}$. $\|\mathbf{w}\| = \sqrt{\langle \mathbf{w}, \mathbf{w} \rangle} = \sqrt{22}$.

4. Let $A, B, C \in M_2(\mathbb{R})$.

(1): $\langle A, A \rangle = a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 \geq 0$, and $\langle A, A \rangle = 0 \iff a_{11} = a_{12} = a_{21} = a_{22} = 0 \iff A = 0$.

(2): $\langle A, B \rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22} = b_{11}a_{11} + b_{12}a_{12} + b_{21}a_{21} + b_{22}a_{22} = \langle B, A \rangle$.

(3): Let $k \in \mathbb{R}$.

$\langle kA, B \rangle = ka_{11}b_{11} + ka_{12}b_{12} + ka_{21}b_{21} + ka_{22}b_{22} = k(a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}) = k\langle A, B \rangle$.

(4):

$$\begin{aligned} \langle (A+B), C \rangle &= (a_{11}+b_{11})c_{11} + (a_{12}+b_{12})c_{12} + (a_{21}+b_{21})c_{21} + (a_{22}+b_{22})c_{22} \\ &= (a_{11}c_{11} + b_{11}c_{11}) + (a_{12}c_{12} + b_{12}c_{12}) + (a_{21}c_{21} + b_{21}c_{21}) + (a_{22}c_{22} + b_{22}c_{22}) \\ &= (a_{11}c_{11} + a_{12}c_{12} + a_{21}c_{21} + a_{22}c_{22}) + (b_{11}c_{11} + b_{12}c_{12} + b_{21}c_{21} + b_{22}c_{22}) \\ &= \langle A, C \rangle + \langle B, C \rangle. \end{aligned}$$

5. We need only demonstrate one example showing that some property of an inner product is violated by the given formula. Set $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then according to the given formula, we have $\langle A, A \rangle = -2$, violating the requirement that $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ for all vectors $\mathbf{u}$.

6. $\langle A, B \rangle = 2 \cdot 3 + (-1)1 + 3(-1) + 5 \cdot 2 = 12$.

$\|A\| = \sqrt{\langle A, A \rangle} = \sqrt{2 \cdot 2 + (-1)(-1) + 3 \cdot 3 + 5 \cdot 5} = \sqrt{39}$.

$\|B\| = \sqrt{\langle B, B \rangle} = \sqrt{3 \cdot 3 + 1 \cdot 1 + (-1)(-1) + 2 \cdot 2} = \sqrt{15}$.

7. $\langle A, B \rangle = 13$, $\|A\| = \sqrt{33}$, $\|B\| = \sqrt{7}$.

8. Let $p_1, p_2, p_3 \in P_1$ where $p_1(x) = a + bx$, $p_2(x) = c + dx$, and $p_3(x) = e + fx$. Define

$$\langle p_1, p_2 \rangle = ac + bd. \quad (8.1)$$

The properties 1 through 4 of Definition 4.11.3 must be verified.

(1): $\langle p_1, p_1 \rangle = a^2 + b^2 \geq 0$ and $\langle p_1, p_1 \rangle = 0 \iff a = b = 0 \iff p_1(x) = 0$.

(2): $\langle p_1, p_2 \rangle = ac + bd = ca + db = \langle p_2, p_1 \rangle$.

(3): Let $k \in \mathbb{R}$.

$\langle kp_1, p_2 \rangle = kac + kbd = k(ac + bd) = k\langle p_1, p_2 \rangle$.

(4):

$$\begin{aligned} \langle p_1 + p_2, p_3 \rangle &= (a + c)e + (b + d)f = ae + ce + bf + df \\ &= (ae + bf) + (ce + df) \\ &= \langle p_1, p_3 \rangle + \langle p_2, p_3 \rangle. \end{aligned}$$

Hence, the mapping defined by (8.1) is an inner product in P_1 .

9. Property 1:

$$\langle \mathbf{u}, \mathbf{u} \rangle = 2u_1u_1 + u_1u_2 + u_2u_1 + 2u_2u_2 = 2u_1^2 + 2u_1u_2 + 2u_2^2 = (u_1 + u_2)^2 + u_1^2 + u_2^2 \geq 0.$$

$$\langle \mathbf{u}, \mathbf{u} \rangle = 0 \iff (u_1 + u_2)^2 + u_1^2 + u_2^2 = 0 \iff u_1 = 0 = u_2 \iff \mathbf{u} = \mathbf{0}.$$

Property 2:

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + u_1v_2 + u_2v_1 + 2u_2v_2 = 2u_2v_2 + u_2v_1 + u_1v_2 + 2u_1v_1 = \langle \mathbf{v}, \mathbf{u} \rangle.$$

Property 3:

$$\begin{aligned} k\langle \mathbf{u}, \mathbf{v} \rangle &= k(2u_1v_1 + u_1v_2 + u_2v_1 + 2u_2v_2) = 2ku_1v_1 + ku_1v_2 + ku_2v_1 + 2ku_2v_2 \\ &= 2(ku_1)v_1 + (ku_1)v_2 + (ku_2)v_1 + 2(ku_2)v_2 = \langle k\mathbf{u}, \mathbf{v} \rangle \\ &= 2ku_1v_1 + ku_1v_2 + ku_2v_1 + 2ku_2v_2 \\ &= 2u_1(kv_1) + u_1(kv_2) + u_2(kv_1) + 2u_2(kv_2) = \langle \mathbf{u}, k\mathbf{v} \rangle. \end{aligned}$$

Property 4:

$$\begin{aligned} \langle (\mathbf{u} + \mathbf{v}), \mathbf{w} \rangle &= \langle (u_1 + v_1, u_2 + v_2), (w_1, w_2) \rangle \\ &= 2(u_1 + v_1)w_1 + (u_1 + v_1)w_2 + (u_2 + v_2)w_1 + 2(u_2 + v_2)w_2 \\ &= 2u_1w_1 + 2v_1w_1 + u_1w_2 + v_1w_2 + u_2w_1 + v_2w_1 + 2u_2w_2 + 2v_2w_2 \\ &= 2u_1w_1 + u_1w_2 + u_2w_1 + 2u_2w_2 + 2v_1w_1 + v_1w_2 + v_2w_1 + 2v_2w_2 \\ &= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle. \end{aligned}$$

Therefore $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + u_1v_2 + u_2v_1 + 2u_2v_2$ defines an inner product on $\mathbb{R}^2$.

10. (a) Using the defined inner product:

$$\langle \mathbf{v}, \mathbf{w} \rangle = \langle (1, 0), (-1, 2) \rangle = 2 \cdot 1(-1) + 1 \cdot 2 + 0(-1) + 2 \cdot 0 \cdot 2 = 0.$$

(b) Using the standard inner product:

$$\langle \mathbf{v}, \mathbf{w} \rangle = \langle (1, 0), (-1, 2) \rangle = 1(-1) + 0 \cdot 2 = -1 \neq 0.$$

11. (a) Using the defined inner product:

$$\langle \mathbf{v}, \mathbf{w} \rangle = 2 \cdot 2 \cdot 3 + 2 \cdot 6 + (-1)3 + 2(-1)6 = 9 \neq 0.$$

(b) Using the standard inner product:

$$\langle \mathbf{v}, \mathbf{w} \rangle = 2 \cdot 3 + (-1)6 = 0.$$

12. (a) Using the defined inner product:

$$\langle \mathbf{v}, \mathbf{w} \rangle = 2 \cdot 1 \cdot 2 + 1 \cdot 1 + (-2) \cdot 2 + 2 \cdot (-2) \cdot 1 = -3 \neq 0.$$

(b) Using the standard inner product:

$$\langle \mathbf{v}, \mathbf{w} \rangle = 1 \cdot 2 + (-2)(1) = 0.$$

13. (a) Show symmetry: $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$.

$$\langle \mathbf{v}, \mathbf{w} \rangle = \langle (v_1, v_2), (w_1, w_2) \rangle = v_1 w_1 - v_2 w_2 = w_1 v_1 - w_2 v_2 = \langle (w_1, w_2), (v_1, v_2) \rangle = \langle \mathbf{w}, \mathbf{v} \rangle.$$

(b) Show $\langle k\mathbf{v}, \mathbf{w} \rangle = k\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, k\mathbf{w} \rangle$.

Note that $k\mathbf{v} = k(v_1, v_2) = (kv_1, kv_2)$ and $k\mathbf{w} = k(w_1, w_2) = (kw_1, kw_2)$.

$$\langle k\mathbf{v}, \mathbf{w} \rangle = \langle (kv_1, kv_2), (w_1, w_2) \rangle = (kv_1)w_1 - (kv_2)w_2 = k(v_1 w_1 - v_2 w_2) = k\langle (v_1, v_2), (w_1, w_2) \rangle = k\langle \mathbf{v}, \mathbf{w} \rangle.$$

Also,

$$\langle \mathbf{v}, k\mathbf{w} \rangle = \langle (v_1, v_2), (kw_1, kw_2) \rangle = v_1(kw_1) - v_2(kw_2) = k(v_1 w_1 - v_2 w_2) = k\langle (v_1, v_2), (w_1, w_2) \rangle = k\langle \mathbf{v}, \mathbf{w} \rangle.$$

(c) Show $\langle (\mathbf{u} + \mathbf{v}), \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$.

Let $\mathbf{w} = (w_1, w_2)$ and note that $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$.

$$\begin{aligned} \langle (\mathbf{u} + \mathbf{v}), \mathbf{w} \rangle &= \langle (u_1 + v_1, u_2 + v_2), (w_1, w_2) \rangle = (u_1 + v_1)w_1 - (u_2 + v_2)w_2 \\ &= u_1 w_1 + v_1 w_1 - u_2 w_2 - v_2 w_2 = u_1 w_1 - u_2 w_2 + v_1 w_1 - v_2 w_2 \\ &= \langle (u_1, v_2), (w_1, w_2) \rangle + \langle (v_1, v_2), (w_1, w_2) \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle. \end{aligned}$$

Property 1 fails since, for example, $\langle \mathbf{u}, \mathbf{u} \rangle < 0$ whenever $|u_2| > |u_1|$.

14. $\langle \mathbf{v}, \mathbf{v} \rangle = 0 \implies \langle (v_1, v_2), (v_1, v_2) \rangle = 0 \implies v_1^2 - v_2^2 = 0 \implies v_1^2 = v_2^2 \implies |v_1| = |v_2|$. Thus, in this space, null vectors are given by $\{(v_1, v_2) \in \mathbb{R}^2 : |v_1| = |v_2|\}$ or equivalently, $\mathbf{v} = r(1, 1)$ or $\mathbf{v} = s(1, -1)$ where $r, s \in \mathbb{R}$.

15. $\langle \mathbf{v}, \mathbf{v} \rangle < 0 \implies \langle (v_1, v_2), (v_1, v_2) \rangle < 0 \implies v_1^2 - v_2^2 < 0 \implies v_1^2 < v_2^2$. In this space, timelike vectors are given by $\{(v_1, v_2) \in \mathbb{R}^2 : v_1^2 < v_2^2\}$.

16. $\langle \mathbf{v}, \mathbf{v} \rangle > 0 \implies \langle (v_1, v_2), (v_1, v_2) \rangle > 0 \implies v_1^2 - v_2^2 > 0 \implies v_1^2 > v_2^2$. In this space, spacelike vectors are given by $\{(v_1, v_2) \in \mathbb{R}^2 : v_1^2 > v_2^2\}$.

17.

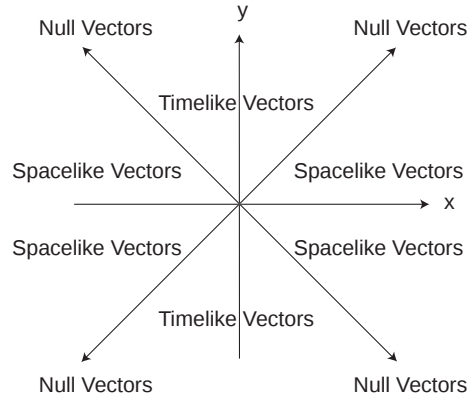


Figure 65: Figure for Problem 17

18. Suppose that some $k_i \leq 0$. If $\mathbf{e}_i$ denotes the standard basis vector in $\mathbb{R}^n$ with 1 in the i th position and zeros elsewhere, then the given formula yields $\langle \mathbf{e}_i, \mathbf{e}_i \rangle = k_i \leq 0$, violating the first axiom of an inner product. Therefore, if the given formula defines a valid inner product, then we must have that $k_i > 0$ for all i .

Now we prove the converse. Suppose that each $k_i > 0$. We verify the axioms of an inner product for the given proposed inner product. We have

$$\langle \mathbf{v}, \mathbf{v} \rangle = k_1 v_1^2 + k_2 v_2^2 + \cdots + k_n v_n^2 \geq 0,$$

and we have equality if and only if $v_1 = v_2 = \cdots = v_n = 0$. For the second axiom, we have

$$\langle \mathbf{w}, \mathbf{v} \rangle = k_1 w_1 v_1 + k_2 w_2 v_2 + \cdots + k_n w_n v_n = k_1 v_1 w_1 + k_2 v_2 w_2 + \cdots + k_n v_n w_n = \langle \mathbf{v}, \mathbf{w} \rangle.$$

For the third axiom, we have

$$\langle k\mathbf{v}, \mathbf{w} \rangle = k_1(kv_1)w_1 + k_2(kv_2)w_2 + \cdots + k_n(kv_n)w_n = k[k_1 v_1 w_1 + k_2 v_2 w_2 + \cdots + k_n v_n w_n] = k\langle \mathbf{v}, \mathbf{w} \rangle.$$

Finally, for the fourth axiom, we have

$$\begin{aligned} \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle &= k_1(u_1 + v_1)w_1 + k_2(u_2 + v_2)w_2 + \cdots + k_n(u_n + v_n)w_n \\ &= [k_1 u_1 w_1 + k_2 u_2 w_2 + \cdots + k_n u_n w_n] + [k_1 v_1 w_1 + k_2 v_2 w_2 + \cdots + k_n v_n w_n] \\ &= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle. \end{aligned}$$

19. We have

$$\langle \mathbf{v}, \mathbf{0} \rangle = \langle \mathbf{v}, \mathbf{0} + \mathbf{0} \rangle = \langle \mathbf{0} + \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{0}, \mathbf{v} \rangle + \langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle + \langle \mathbf{v}, \mathbf{0} \rangle = 2\langle \mathbf{v}, \mathbf{0} \rangle,$$

which implies that $\langle \mathbf{v}, \mathbf{0} \rangle = 0$.

20. (a) For all $\mathbf{v}, \mathbf{w} \in V$.

$$\begin{aligned} \|\mathbf{v} + \mathbf{w}\|^2 &= \langle (\mathbf{v} + \mathbf{w}), (\mathbf{v} + \mathbf{w}) \rangle \\ &= \langle \mathbf{v}, \mathbf{v} + \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle \text{ by Property 4} \\ &= \langle \mathbf{v} + \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{v} + \mathbf{w}, \mathbf{w} \rangle \text{ by Property 2} \\ &= \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle \text{ by Property 4} \\ &= \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle \text{ by Property 2} \\ &= \|\mathbf{v}\|^2 + 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2. \end{aligned}$$

(b) This follows immediately by substituting $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ in the formula given in part (a).

(c) (i) From part (a), it follows that $\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2$ and $\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v} + (-\mathbf{w})\|^2 = \|\mathbf{v}\|^2 + 2\langle \mathbf{v}, -\mathbf{w} \rangle + \|-\mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2$.

Thus, $\|\mathbf{v} + \mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2 = (\|\mathbf{v}\|^2 + 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2) - (\|\mathbf{v}\|^2 - 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2) = 4\langle \mathbf{v}, \mathbf{w} \rangle$.

(ii) $\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 = (\|\mathbf{v}\|^2 + 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2) + (\|\mathbf{v}\|^2 - 2\langle \mathbf{v}, \mathbf{w} \rangle + \|\mathbf{w}\|^2) = 2\|\mathbf{v}\|^2 + 2\|\mathbf{w}\|^2 = 2(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2)$.

21. For all $\mathbf{v}, \mathbf{w} \in V$ and $v_i, w_i \in \mathbb{C}$.

$$\begin{aligned} \|\mathbf{v} + \mathbf{w}\|^2 &= \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle \\ &= \langle \mathbf{v}, \mathbf{v} + \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle \text{ by Property 4} \\ &= \overline{\langle \mathbf{v} + \mathbf{w}, \mathbf{v} \rangle} + \overline{\langle \mathbf{v} + \mathbf{w}, \mathbf{w} \rangle} \text{ by Property 2} \\ &= \overline{\langle \mathbf{v}, \mathbf{v} \rangle} + \overline{\langle \mathbf{w}, \mathbf{v} \rangle} + \overline{\langle \mathbf{v}, \mathbf{w} \rangle} + \overline{\langle \mathbf{w}, \mathbf{w} \rangle} \text{ by Property 4} \\ &= \overline{\langle \mathbf{v}, \mathbf{v} \rangle} + \overline{\langle \mathbf{w}, \mathbf{v} \rangle} + \overline{\langle \mathbf{v}, \mathbf{w} \rangle} + \overline{\langle \mathbf{w}, \mathbf{w} \rangle} \\ &= \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle + \overline{\langle \mathbf{v}, \mathbf{w} \rangle} \\ &= \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 + 2\operatorname{Re}\{\langle \mathbf{v}, \mathbf{w} \rangle\} \\ &= \|\mathbf{v}\|^2 + 2\operatorname{Re}\{\langle \mathbf{v}, \mathbf{w} \rangle\} + \|\mathbf{v}\|^2. \end{aligned}$$

Solutions to Section 4.12

1. TRUE. An orthonormal basis is simply an orthogonal basis consisting of unit vectors.

2. FALSE. The converse of this statement is true, but for the given statement, many counterexamples exist. For instance, the set of vectors $\{(1, 1), (1, 0)\}$ in $\mathbb{R}^2$ is linearly independent, but does not form an orthogonal set.

3. TRUE. We can verify easily that

$$\int_0^\pi \cos t \sin t dt = \frac{\sin^2 t}{2} \Big|_0^\pi = 0,$$

which means that $\{\cos x, \sin x\}$ is an orthogonal set. Moreover, since they are non-proportional functions, they are linearly independent. Therefore, they comprise an orthogonal basis for the 2-dimensional inner product space $\text{span}\{\cos x, \sin x\}$.

4. FALSE. For instance, in $\mathbb{R}^3$ we can take the vectors $\mathbf{x}_1 = (1, 0, 0)$, $\mathbf{x}_2 = (1, 1, 0)$, and $\mathbf{x}_3 = (0, 0, 1)$. Applying the Gram-Schmidt process to the ordered set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ yields the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. However, applying the Gram-Schmidt process to the ordered set $\{\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1\}$ yields the basis $\{\mathbf{x}_3, \mathbf{x}_2, (\frac{1}{2}, -\frac{1}{2}, 0)\}$ instead.

5. TRUE. This is the content of Theorem 4.12.7.

6. TRUE. The vector $\mathbf{P}(\mathbf{w}, \mathbf{v})$ is a scalar multiple of the vector $\mathbf{v}$, and since $\mathbf{v}$ is orthogonal to $\mathbf{u}$, $\mathbf{P}(\mathbf{w}, \mathbf{v})$ is also orthogonal to $\mathbf{u}$, so that its projection onto $\mathbf{u}$ must be $\mathbf{0}$.

7. TRUE. We have

$$\mathbf{P}(\mathbf{w}_1 + \mathbf{w}_2, \mathbf{v}) = \frac{\langle \mathbf{w}_1 + \mathbf{w}_2, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v} = \frac{\langle \mathbf{w}_1, \mathbf{v} \rangle + \langle \mathbf{w}_2, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v} = \frac{\langle \mathbf{w}_1, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v} + \frac{\langle \mathbf{w}_2, \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \mathbf{v} = \mathbf{P}(\mathbf{w}_1, \mathbf{v}) + \mathbf{P}(\mathbf{w}_2, \mathbf{v}).$$

Problems:

1. $\langle (2, -1, 1), (1, 1, -1) \rangle = 2 + (-1) + (-1) = 0;$

$\langle (2, -1, 1), (0, 1, 1) \rangle = 0 + (-1) + 1 = 0;$

$\langle (1, 1, -1), (0, 1, 1) \rangle = 0 + 1 + (-1) = 0.$

Since each vector in the set is orthogonal to every other vector in the set, the vectors form an orthogonal set. To generate an orthonormal set, we divide each vector by its norm:

$\|(2, -1, 1)\| = \sqrt{4 + 1 + 1} = \sqrt{6},$

$\|(1, 1, -1)\| = \sqrt{1 + 1 + 1} = \sqrt{3},$ and

$\|(0, 1, 1)\| = \sqrt{0 + 1 + 1} = \sqrt{2}.$ Thus, the corresponding orthonormal set is:

$$\left\{ \frac{\sqrt{6}}{6}(2, -1, 1), \frac{\sqrt{3}}{3}(1, 1, -1), \frac{\sqrt{2}}{2}(0, 1, 1) \right\}.$$

2. $\langle (1, 3, -1, 1), (-1, 1, 1, -1) \rangle = -1 + 3 + (-1) + (-1) = 0;$

$\langle (1, 3, -1, 1), (1, 0, 2, 1) \rangle = 1 + 0 + (-2) + 1 = 0;$

$\langle (-1, 1, 1, -1), (1, 0, 2, 1) \rangle = -1 + 0 + 2 + (-1) = 0.$

Since all vectors in the set are orthogonal to each other, they form an orthogonal set. To generate an orthonormal set, we divide each vector by its norm:

$\|(1, 3, -1, 1)\| = \sqrt{1 + 9 + 1 + 1} = \sqrt{12} = 2\sqrt{3},$

$\|(-1, 1, 1, -1)\| = \sqrt{1 + 1 + 1 + 1} = \sqrt{4} = 2,$ and

$\|(1, 0, 2, 1)\| = \sqrt{1 + 0 + 4 + 1} = \sqrt{6}.$ Thus, an orthonormal set is:

$$\left\{ \frac{\sqrt{3}}{6}(1, 3, -1, 1), \frac{1}{2}(-1, 1, 1, -1), \frac{\sqrt{6}}{6}(1, 0, 2, 1) \right\}.$$

$$\begin{aligned}
\mathbf{3.} \quad & \langle (1, 2, -1, 0), (1, 0, 1, 2) \rangle = 1 + 0 + (-1) + 0 = 0; \\
& \langle (1, 2, -1, 0), (-1, 1, 1, 0) \rangle = -1 + 2 + (-1) + 0 = 0; \\
& \langle (1, 2, -1, 0), (1, -1, -1, 0) \rangle = 1 + (-2) + 1 + 0 = 0; \\
& \langle (1, 0, 1, 2), (-1, 1, 1, 0) \rangle = -1 + 0 + 1 + 0 = 0; \\
& \langle (1, 0, 1, 2), (1, -1, -1, 0) \rangle = 1 + 0 + (-1) + 0 = 0; \\
& \langle (-1, 1, 1, 0), (1, -1, -1, 0) \rangle = -1 + (-1) + (-1) = -3.
\end{aligned}$$

Hence, this is not an orthogonal set of vectors.

$$\begin{aligned}
\mathbf{4.} \quad & \langle (1, 2, -1, 0, 3), (1, 1, 0, 2, -1) \rangle = 1 + 2 + 0 + 0 + (-3) = 0; \\
& \langle (1, 2, -1, 0, 3), (4, 2, -4, -5, -4) \rangle = 4 + 4 + 4 + 0 + (-12) = 0; \\
& \langle (1, 1, 0, 2, -1), (4, 2, -4, -5, -4) \rangle = 4 + 2 + 0 + (-10) + 4 = 0.
\end{aligned}$$

Since all vectors in the set are orthogonal to each other, they form an orthogonal set. To generate an orthonormal set, we divide each vector by its norm:

$$\begin{aligned}
& \|(1, 2, -1, 0, 3)\| = \sqrt{1 + 4 + 1 + 0 + 9} = \sqrt{15}, \\
& \|(1, 1, 0, 2, -1)\| = \sqrt{1 + 1 + 0 + 4 + 1} = \sqrt{7}, \text{ and} \\
& \|(4, 2, -4, -5, -4)\| = \sqrt{16 + 4 + 16 + 25 + 16} = \sqrt{77}. \text{ Thus, an orthonormal set is:}
\end{aligned}$$

$$\left\{ \frac{\sqrt{15}}{15}(1, 2, -1, 0, 3), \frac{\sqrt{7}}{7}(1, 1, 0, 2, -1), \frac{\sqrt{77}}{77}(4, 2, -4, -5, -4) \right\}.$$

5. We require that $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_1, \mathbf{w} \rangle = \langle \mathbf{v}_2, \mathbf{w} \rangle = 0$. Let $\mathbf{w} = (a, b, c)$ where $a, b, c \in \mathbb{R}$.

$$\begin{aligned}
& \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle (1, 2, 3), (1, 1, -1) \rangle = 0. \\
& \langle \mathbf{v}_1, \mathbf{w} \rangle = \langle (1, 2, 3), (a, b, c) \rangle \implies a + 2b + 3c = 0. \\
& \langle \mathbf{v}_2, \mathbf{w} \rangle = \langle (1, 1, -1), (a, b, c) \rangle \implies a + b - c = 0.
\end{aligned}$$

Letting the free variable $c = t \in \mathbb{R}$, the system has the solution $a = 5t$, $b = -4t$, and $c = t$. Consequently, $\{(1, 2, 3), (1, 1, -1), (5t, -4t, t)\}$ will form an orthogonal set whenever $t \neq 0$. To determine the corresponding orthonormal set, we must divide each vector by its norm:

$$\|\mathbf{v}_1\| = \sqrt{1 + 4 + 9} = \sqrt{14}, \|\mathbf{v}_2\| = \sqrt{1 + 1 + 1} = \sqrt{3}, \|\mathbf{w}\| = \sqrt{25t^2 + 16t^2 + t^2} = \sqrt{42t^2} = |t|\sqrt{42} = t\sqrt{42} \text{ if } t \geq 0. \text{ Setting } t = 1, \text{ an orthonormal set is:}$$

$$\left\{ \frac{\sqrt{14}}{14}(1, 2, 3), \frac{\sqrt{3}}{3}(1, 1, -1), \frac{\sqrt{42}}{42}(5, -4, 1) \right\}.$$

6. $\langle (1 - i, 3 + 2i), (2 + 3i, 1 - i) \rangle = (1 - i)\overline{(2 + 3i)} + (3 + 2i)\overline{(1 - i)} = (1 - i)(2 - 3i) + (3 + 2i)(1 + i) = (2 - 3i - 2i - 3) + (3 + 3i + 2i - 2) = (-1 - 5i) + (1 + 5i) = 0$. The vectors are orthogonal.

$$\|(1 - i, 3 + 2i)\| = \sqrt{(1 - i)(1 + i) + (3 + 2i)(3 - 2i)} = \sqrt{1 + 1 + 9 + 4} = \sqrt{15}.$$

$\|(2 + 3i, 1 - i)\| = \sqrt{(2 + 3i)(2 - 3i) + (1 - i)(1 + i)} = \sqrt{4 + 9 + 1 + 1} = \sqrt{15}$. Thus, the corresponding orthonormal set is:

$$\left\{ \frac{\sqrt{15}}{15}(1 - i, 3 + 2i), \frac{\sqrt{15}}{15}(2 + 3i, 1 - i) \right\}.$$

7. $\langle (1 - i, 1 + i, i), (0, i, 1 - i) \rangle = (1 - i) \cdot 0 + (1 + i)(-i) + i(1 + i) = 0$.

$$\langle (1 - i, 1 + i, i), (-3 + 3i, 2 + 2i, 2i) \rangle = (1 - i)(-3 - 3i) + (1 + i)(2 - 2i) + i(-2i) = 0.$$

$$\langle (0, i, 1 - i), (-3 + 3i, 2 + 2i, 2i) \rangle = 0 + i(2 - 2i) + (1 - i)(-2i) = (2i + 2) + (-2i - 2) = 0.$$

Hence, the vectors are orthogonal. To obtain a corresponding orthonormal set, we divide each vector by its norm.

$$\|(1 - i, 1 + i, i)\| = \sqrt{(1 - i)(1 + i) + (1 + i)(1 - i) + i(-i)} = \sqrt{1 + 1 + 1 + 1 + 1} = \sqrt{5}.$$

$\|(0, i, 1 - i)\| = \sqrt{0 + i(-i) + (1 - i)(1 + i)} = \sqrt{1 + 1 + 1} = \sqrt{3}.$
 $\|(-3 + 3i, 2 + 2i, 2i)\| = \sqrt{(-3 + 3i)(-3 - 3i) + (2 + 2i)(2 - 2i) + 2i(-2i)} = \sqrt{9 + 9 + 4 + 4 + 4} = \sqrt{30}.$
 Consequently, an orthonormal set is:

$$\left\{ \frac{\sqrt{5}}{5}(1 - i, 1 + i, i), \frac{\sqrt{3}}{3}(0, i, 1 - i), \frac{\sqrt{30}}{30}(-3 + 3i, 2 + 2i, 2i) \right\}.$$

8. Let $z = a + bi$ where $a, b \in \mathbb{R}$. We require that $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.

$$\langle (1 - i, 1 + 2i), (2 + i, a + bi) \rangle = 0 \implies (1 - i)(2 - i) + (1 + 2i)(a - bi) = 0 \implies 1 - 3i + a + 2b + (2a - b)i = 0.$$

Equating real parts and imaginary parts from the last equality results in the system:
$$\begin{cases} a + 2b = -1 \\ 2a - b = 3. \end{cases}$$

This system has the solution $a = 1$ and $b = -1$; hence $z = 1 - i$. Our desired orthogonal set is given by $\{(1 - i, 1 + 2i), (2 + i, 1 - i)\}$.

$$\|(1 - i, 1 + 2i)\| = \sqrt{(1 - i)(1 + i) + (1 + 2i)(1 - 2i)} = \sqrt{1 + 1 + 1 + 4} = \sqrt{7}.$$

$$\|(2 + i, 1 - i)\| = \sqrt{(2 + i)(2 - i) + (1 - i)(1 + i)} = \sqrt{4 + 1 + 1 + 1} = \sqrt{7}.$$

The corresponding orthonormal set is given by:

$$\left\{ \frac{\sqrt{7}}{7}(1 - i, 1 + 2i), \frac{\sqrt{7}}{7}(2 + i, 1 - i) \right\}.$$

9. $\langle f_1, f_2 \rangle = \langle 1, \sin \pi x \rangle = \int_{-1}^1 \sin \pi x dx = \left[\frac{-\cos \pi x}{\pi} \right]_{-1}^1 = 0.$

$$\langle f_1, f_3 \rangle = \langle 1, \cos \pi x \rangle = \int_{-1}^1 \cos \pi x dx = \left[\frac{\sin \pi x}{\pi} \right]_{-1}^1 = 0.$$

$$\langle f_2, f_3 \rangle = \langle \sin \pi x, \cos \pi x \rangle = \int_{-1}^1 \sin \pi x \cos \pi x dx = \frac{1}{2} \int_{-1}^1 \sin 2\pi x dx = \left[\frac{-1}{4\pi} \cos 2\pi x \right]_{-1}^1 = 0. \quad \text{Thus, the vectors are orthogonal.}$$

$$\|f_1\| = \sqrt{\int_{-1}^1 1 dx} = \sqrt{[x]_{-1}^1} = \sqrt{2}.$$

$$\|f_2\| = \sqrt{\int_{-1}^1 \sin^2 \pi x dx} = \sqrt{\int_{-1}^1 \frac{1 - \cos 2\pi x}{2} dx} = \sqrt{\left[\frac{x}{2} \right]_{-1}^1 - \left[\frac{1}{4\pi} \sin 2\pi x \right]_{-1}^1} = 1.$$

$$\|f_3\| = \sqrt{\int_{-1}^1 \cos^2 \pi x dx} = \sqrt{\int_{-1}^1 \frac{1 + \cos 2\pi x}{2} dx} = \sqrt{\left[\frac{x}{2} \right]_{-1}^1 + \left[\frac{1}{4\pi} \sin 2\pi x \right]_{-1}^1} = 1.$$

Consequently, $\left\{ \frac{\sqrt{2}}{2}, \sin \pi x, \cos \pi x \right\}$ is an orthonormal set of functions on $[-1, 1]$.

10. $\langle f_1, f_2 \rangle = \langle 1, x \rangle = \int_{-1}^1 1 \cdot x dx = \left[\frac{x^2}{2} \right]_{-1}^1 = 0.$

$$\langle f_1, f_3 \rangle = \langle 1, \frac{3x^2 - 1}{2} \rangle = \int_{-1}^1 \frac{3x^2 - 1}{2} dx = \left[\frac{x^3 - x}{2} \right]_{-1}^1 = 0.$$

$\langle f_2, f_3 \rangle = \langle x, \frac{3x^2 - 1}{2} \rangle = \int_{-1}^1 x \cdot \frac{3x^2 - 1}{2} dx = \frac{1}{2} \left[\frac{3x^4}{4} - \frac{x^2}{2} \right]_{-1}^1 = 0$. Thus, the vectors are orthogonal.

$$\|f_1\| = \sqrt{\int_{-1}^1 dx} = \sqrt{[x]_{-1}^1} = \sqrt{2}.$$

$$\|f_2\| = \sqrt{\int_{-1}^1 x^2 dx} = \sqrt{\left[\frac{x^3}{3} \right]_{-1}^1} = \frac{\sqrt{6}}{3}.$$

$$\|f_3\| = \sqrt{\int_{-1}^1 \left(\frac{3x^2 - 1}{2} \right)^2 dx} = \sqrt{\frac{1}{4} \int_{-1}^1 (9x^4 - 6x^2 + 1) dx} = \sqrt{\frac{1}{4} \left[\frac{9x^5}{5} - 2x^3 + x \right]_{-1}^1} = \frac{\sqrt{10}}{5}.$$

To obtain a set of orthonormal vectors, we divide each vector by its norm:

$$\frac{f_1}{\|f_1\|} = \frac{\sqrt{2}}{2} f_1, \quad \frac{f_2}{\|f_2\|} = \frac{\sqrt{6}}{2} f_2, \quad \frac{f_3}{\|f_3\|} = \frac{\sqrt{10}}{2} f_3.$$

Thus, $\left\{ \frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}x, \frac{\sqrt{10}}{4}(3x^2 - 1) \right\}$ is an orthonormal set of vectors.

$$\mathbf{11.} \quad \langle f_1, f_2 \rangle = \int_{-1}^1 \sin \pi x \sin 2\pi x dx = \frac{1}{2} \int_{-1}^1 (\cos 3\pi x - \cos \pi x) dx = 0.$$

$$\langle f_1, f_3 \rangle = \int_{-1}^1 \sin \pi x \sin 3\pi x dx = \frac{1}{2} \int_{-1}^1 (\cos 4\pi x - \cos 2\pi x) dx = 0.$$

$$\langle f_2, f_3 \rangle = \int_{-1}^1 \sin 2\pi x \sin 3\pi x dx = \frac{1}{2} \int_{-1}^1 (\cos 5\pi x - \cos \pi x) dx = 0.$$

Therefore, $\{f_1, f_2, f_3\}$ is an orthogonal set.

$$\|f_1\| = \sqrt{\int_{-1}^1 \sin^2 \pi x dx} = \sqrt{\int_{-1}^1 \frac{1}{2} (1 - \cos 2\pi x) dx} = \sqrt{\frac{1}{2} \left[x - \frac{\sin 2\pi x}{2\pi} \right]_{-1}^1} = 1.$$

$$\|f_2\| = \sqrt{\int_{-1}^1 \sin^2 2\pi x dx} = \sqrt{\int_{-1}^1 \frac{1}{2} (1 - \cos 4\pi x) dx} = \sqrt{\frac{1}{2} \left[x - \frac{\sin 4\pi x}{4\pi} \right]_{-1}^1} = 1.$$

$$\|f_3\| = \sqrt{\int_{-1}^1 \sin^2 3\pi x dx} = \sqrt{\int_{-1}^1 \frac{1}{2} (1 - \cos 6\pi x) dx} = \sqrt{\frac{1}{2} \left[x - \frac{\sin 6\pi x}{6\pi} \right]_{-1}^1} = 1.$$

Thus, it follows that $\{f_1, f_2, f_3\}$ is an orthonormal set of vectors on $[-1, 1]$.

$$\mathbf{12.} \quad \langle f_1, f_2 \rangle = \langle \cos \pi x, \cos 2\pi x \rangle = \int_{-1}^1 \cos \pi x \cos 2\pi x dx = \frac{1}{2} \int_{-1}^1 (\cos 3\pi x + \cos \pi x) dx = 0.$$

$$\langle f_1, f_3 \rangle = \langle \cos \pi x, \cos 3\pi x \rangle = \int_{-1}^1 \cos \pi x \cos 3\pi x dx = \frac{1}{2} \int_{-1}^1 (\cos 4\pi x + \cos 2\pi x) dx = 0.$$

$$\langle f_2, f_3 \rangle = \langle \cos 2\pi x, \cos 3\pi x \rangle = \int_{-1}^1 \cos 2\pi x \cos 3\pi x dx = \frac{1}{2} \int_{-1}^1 (\cos 5\pi x + \cos \pi x) dx = 0.$$

Therefore, $\{f_1, f_2, f_3\}$ is an orthogonal set.

$$\|f_1\| = \sqrt{\int_{-1}^1 \cos^2 \pi x dx} = \sqrt{\frac{1}{2} \int_{-1}^1 (1 + \cos 2\pi x) dx} = \sqrt{\frac{1}{2} \left[x + \frac{\sin 2\pi x}{2\pi} \right]_{-1}^1} = 1.$$

$$\|f_2\| = \sqrt{\int_{-1}^1 \cos^2 2\pi x dx} = \sqrt{\frac{1}{2} \int_{-1}^1 (1 + \cos 4\pi x) dx} = \sqrt{\frac{1}{2} \left[x + \frac{\sin 4\pi x}{4\pi} \right]_{-1}^1} = 1.$$

$$\|f_3\| = \sqrt{\int_{-1}^1 \cos^2 3\pi x dx} = \sqrt{\frac{1}{2} \int_{-1}^1 (1 + \cos 6\pi x) dx} = \sqrt{\frac{1}{2} \left[x + \frac{\sin 6\pi x}{6\pi} \right]_{-1}^1} = 1.$$

Thus, it follows that $\{f_1, f_2, f_3\}$ is an orthonormal set of vectors on $[-1, 1]$.

13. It is easily verified that $\langle A_1, A_2 \rangle = 0$, $\langle A_1, A_3 \rangle = 0$, $\langle A_2, A_3 \rangle = 0$. Thus we require a, b, c, d such that
 $\langle A_1, A_4 \rangle = 0 \implies a + b - c + 2d = 0$,
 $\langle A_2, A_4 \rangle = 0 \implies -a + b + 2c + d = 0$,
 $\langle A_3, A_4 \rangle = 0 \implies a - 3b + 2d = 0$.

Solving this system for a, b, c , and d , we obtain: $a = \frac{3}{2}c$, $b = -\frac{1}{2}c$, $d = 0$. Thus,

$$A_4 = \begin{bmatrix} \frac{3}{2}c & -\frac{1}{2}c \\ c & 0 \end{bmatrix} = 2c \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} = k \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix},$$

where k is any nonzero real number.

14. Let $\mathbf{v}_1 = (1, -1, -1)$ and $\mathbf{v}_2 = (2, 1, -1)$.

$$\mathbf{u}_1 = \mathbf{v}_1 = (1, -1, -1), \|\mathbf{u}_1\| = \sqrt{1+1+1} = \sqrt{3}.$$

$$\langle \mathbf{v}_2, \mathbf{u}_1 \rangle = \langle (2, 1, -1), (1, -1, -1) \rangle = 2 - 1 + 1 = 2.$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 = (2, 1, -1) - \frac{2}{3}(1, -1, -1) = \frac{1}{3}(4, 5, -1).$$

$$\|\mathbf{u}_2\| = \sqrt{(16+25+1)/9} = \frac{1}{3}\sqrt{42}. \text{ Hence, an orthonormal basis is:}$$

$$\left\{ \frac{\sqrt{3}}{3}(1, -1, -1), \frac{\sqrt{42}}{42}(4, 5, -1) \right\}.$$

15. Let $\mathbf{v}_1 = (2, 1, -2)$ and $\mathbf{v}_2 = (1, 3, -1)$.

$$\mathbf{u}_1 = \mathbf{v}_1 = (2, 1, -2), \|\mathbf{u}_1\| = \sqrt{4+1+4} = \sqrt{9} = 3.$$

$$\langle \mathbf{v}_2, \mathbf{u}_1 \rangle = \langle (1, 3, -1), (2, 1, -2) \rangle = 2 \cdot 1 + 1 \cdot 3 + (-2)(-1) = 7.$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 = (1, 3, -1) - \frac{7}{9}(2, 1, -2) = \frac{5}{9}(-1, 4, 1).$$

$$\|\mathbf{u}_2\| = \frac{5}{9}\sqrt{1+16+1} = \frac{5\sqrt{2}}{3}. \text{ Hence, an orthonormal basis is:}$$

$$\left\{ \frac{1}{3}(2, 1, -2), \frac{\sqrt{2}}{6}(-1, 4, 1) \right\}.$$

16. Let $\mathbf{v}_1 = (-1, 1, 1, 1)$ and $\mathbf{v}_2 = (1, 2, 1, 2)$.

$$\mathbf{u}_1 = \mathbf{v}_1 = (-1, 1, 1, 1), \|\mathbf{u}_1\| = \sqrt{1+1+1+1} = 2.$$

$$\langle \mathbf{v}_2, \mathbf{u}_1 \rangle = \langle (1, 2, 1, 2), (-1, 1, 1, 1) \rangle = 1(-1) + 2 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 = 4.$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 = (1, 2, 1, 2) - (-1, 1, 1, 1) = (2, 1, 0, 1).$$

$$\|\mathbf{u}_2\| = \sqrt{4+1+0+1} = \sqrt{6}. \text{ Hence, an orthonormal basis is:}$$

$$\left\{ \frac{1}{2}(-1, 1, 1, 1), \frac{\sqrt{6}}{6}(2, 1, 0, 1) \right\}.$$

17. Let $\mathbf{v}_1 = (1, 0, -1, 0)$, $\mathbf{v}_2 = (1, 1, -1, 0)$ and $\mathbf{v}_3 = (-1, 1, 0, 1)$.

$$\mathbf{u}_1 = \mathbf{v}_1 = (1, 0, -1, 0), \|\mathbf{u}_1\| = \sqrt{1+0+1+0} = \sqrt{2}.$$

$$\langle \mathbf{v}_2, \mathbf{u}_1 \rangle = \langle (1, 1, -1, 0), (1, 0, -1, 0) \rangle = 1 \cdot 1 + 1 \cdot 0 + (-1)(-1) + 0 \cdot 0 = 2.$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 = (1, 1, -1, 0) - (1, 0, -1, 0) = (0, 1, 0, 0).$$

$$\|\mathbf{u}_2\| = \sqrt{0+1+0+0} = 1.$$

$$\langle \mathbf{v}_3, \mathbf{u}_1 \rangle = \langle (-1, 1, 0, 1), (1, 0, -1, 0) \rangle = (-1)1 + 1 \cdot 0 + 0(-1) + 1 \cdot 0 = -1.$$

$$\langle \mathbf{v}_3, \mathbf{u}_2 \rangle = \langle (-1, 1, 0, 1), (0, 1, 0, 0) \rangle = (-1)0 + 1 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 = 1.$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2} \mathbf{u}_2 = (-1, 1, 0, 1) + \frac{1}{2}(1, 0, -1, 0) - (0, 1, 0, 0) = \frac{1}{2}(-1, 0, -1, 2);$$

$$\|\mathbf{u}_3\| = \frac{1}{2}\sqrt{1+0+1+4} = \frac{\sqrt{6}}{2}. \text{ Hence, an orthonormal basis is:}$$

$$\left\{ \frac{\sqrt{2}}{2}(1, 0, -1, 0), (0, 1, 0, 0), \frac{\sqrt{6}}{6}(-1, 0, -1, 2) \right\}.$$

18. Let $\mathbf{v}_1 = (1, 2, 0, 1)$, $\mathbf{v}_2 = (2, 1, 1, 0)$ and $\mathbf{v}_3 = (1, 0, 2, 1)$.

$$\mathbf{u}_1 = \mathbf{v}_1 = (1, 2, 0, 1), \|\mathbf{u}_1\| = \sqrt{1+4+0+1} = \sqrt{6}.$$

$$\langle \mathbf{v}_2, \mathbf{u}_1 \rangle = \langle (2, 1, 1, 0), (1, 2, 0, 1) \rangle = 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 0 + 0 \cdot 1 = 4.$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 = (2, 1, 1, 0) - \frac{2}{3}(1, 2, 0, 1) = \frac{1}{3}(4, -1, 3, -2).$$

$$\|\mathbf{u}_2\| = \frac{1}{3}\sqrt{16+1+9+4} = \frac{1}{3}\sqrt{30}.$$

$$\langle \mathbf{v}_3, \mathbf{u}_1 \rangle = \langle (1, 0, 2, 1), (1, 2, 0, 1) \rangle = 1 \cdot 1 + 0 \cdot 2 + 2 \cdot 0 + 1 \cdot 1 = 2.$$

$$\langle \mathbf{v}_3, \mathbf{u}_2 \rangle = \langle (1, 0, 2, 1), (4/3, -1/3, 1, -2/3) \rangle = 1(4/3) + 0(-1/3) + 2 \cdot 1 + 1(-2/3) = \frac{8}{3}.$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2} \mathbf{u}_2 = (1, 0, 2, 1) - \frac{1}{3}(1, 2, 0, 1) - \frac{4}{15}(4, -1, 3, -2) = \frac{2}{5}(-1, -1, 3, 3);$$

$$\|\mathbf{u}_3\| = \frac{2}{5}\sqrt{1+1+9+9} = \frac{4\sqrt{5}}{5}. \text{ Hence, an orthonormal basis is:}$$

$$\left\{ \frac{\sqrt{6}}{6}(1, 2, 0, 1), \frac{\sqrt{30}}{30}(4, -1, 3, -2), \frac{\sqrt{5}}{10}(-1, -1, 3, 3) \right\}.$$

19. Let $\mathbf{v}_1 = (1, 1, -1, 0)$, $\mathbf{v}_2 = (-1, 0, 1, 1)$ and $\mathbf{v}_3 = (2, -1, 2, 1)$.

$$\mathbf{u}_1 = \mathbf{v}_1 = (1, 1, -1, 0), \|\mathbf{u}_1\| = \sqrt{1+1+1+0} = \sqrt{3}.$$

$$\langle \mathbf{v}_2, \mathbf{u}_1 \rangle = \langle (-1, 0, 1, 1), (1, 1, -1, 0) \rangle = -1 \cdot 1 + 0 \cdot 1 + 1(-1) + 1 \cdot 0 = -2.$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 = (-1, 0, 1, 1) - \frac{-2}{3}(1, 1, -1, 0) = \frac{1}{3}(-1, 2, 1, 3).$$

$$\|\mathbf{u}_2\| = \frac{1}{3}\sqrt{1+4+1+9} = \frac{\sqrt{15}}{3}.$$

$$\langle \mathbf{v}_3, \mathbf{u}_1 \rangle = \langle (2, -1, 2, 1), (1, 1, -1, 0) \rangle = 2 \cdot 1 + -1 \cdot 1 + 2(-1) + 1 \cdot 0 = -1.$$

$$\langle \mathbf{v}_3, \mathbf{u}_2 \rangle = \langle (2, -1, 2, 1), (-1/3, 2/3, 1/3, 1) \rangle = 2(-1/3) + (-1)(2/3) + 2(1/3) + 1 \cdot 1 = \frac{1}{3}.$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2} \mathbf{u}_2 = (2, -1, 2, 1) + \frac{1}{3}(1, 1, -1, 0) - \frac{1}{15}(-1, 2, 1, 3) = \frac{4}{5}(3, -1, 2, 1);$$

$$\|\mathbf{u}_3\| = \frac{4\sqrt{15}}{5}. \text{ Hence, an orthonormal basis is:}$$

$$\left\{ \frac{\sqrt{3}}{3}(1, 1, -1, 0), \frac{\sqrt{15}}{15}(-1, 2, 1, 3), \frac{\sqrt{15}}{15}(3, -1, 2, 1) \right\}.$$

20. A row-echelon form of A is $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 0 \end{bmatrix}$. Hence, a basis for $\text{rowspace}(A)$ is $\{(1, -2, 1), (0, 7, 1)\}$.

The vectors in this basis are not orthogonal. We therefore use the Gram-Schmidt process to determine an orthogonal basis. Let $\mathbf{v}_1 = (1, -2, 1)$, $\mathbf{v}_2 = (0, 7, 1)$. Then an orthogonal basis for $\text{rowspace}(A)$ is $\{\mathbf{u}_1, \mathbf{u}_2\}$, where

$$\mathbf{u}_1 = (1, -2, 1), \quad \mathbf{u}_2 = (0, 7, 1) + \frac{13}{6}(1, -2, 1) = \frac{1}{6}(13, 16, 19).$$

21. Let $\mathbf{v}_1 = (1 - i, 0, i)$ and $\mathbf{v}_2 = (1, 1 + i, 0)$.

$$\mathbf{u}_1 = \mathbf{v}_1 = (1 - i, 0, i), \quad \|\mathbf{u}_1\| = \sqrt{(1 - i)(1 + i) + 0 + i(-i)} = \sqrt{3}.$$

$$\langle \mathbf{v}_2, \mathbf{u}_1 \rangle = \langle (1, 1 + i, 0), (1 - i, 0, i) \rangle = 1(1 + i) + (1 + i)0 + 0(-i) = 1 + i.$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 = (1, 1 + i, 0) - \frac{1 + i}{3}(1 - i, 0, i) = \frac{1}{3}(1, 3 + 3i, 1 - i).$$

$$\|\mathbf{u}_2\| = \frac{1}{3} \sqrt{1 + (3 + 3i)(3 - 3i) + (1 - i)(1 + i)} = \frac{\sqrt{21}}{3}. \quad \text{Hence, an orthonormal basis is:}$$

$$\left\{ \frac{\sqrt{3}}{3}(1 - i, 0, i), \frac{\sqrt{21}}{21}(1, 3 + 3i, 1 - i) \right\}.$$

22. Let $\mathbf{v}_1 = (1 + i, i, 2 - i)$ and $\mathbf{v}_2 = (1 + 2i, 1 - i, i)$.

$$\mathbf{u}_1 = \mathbf{v}_1 = (1 + i, i, 2 - i), \quad \|\mathbf{u}_1\| = \sqrt{(1 + i)(1 - i) + i(-i) + (2 - i)(2 + i)} = 2\sqrt{2}.$$

$$\langle \mathbf{v}_2, \mathbf{u}_1 \rangle = (1 + 2i)(1 - i) + (1 - i)(-i) + i(2 + i) = 1 + 2i.$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 = (1 + 2i, 1 - i, i) - \frac{1}{8}(1 + 2i)(1 + i, i, 2 - i) = \frac{1}{8}(9 + 13i, 10 - 9i, -4 + 5i).$$

$$\|\mathbf{u}_2\| = \frac{1}{8} \sqrt{(9 + 13i)(9 - 13i) + (10 - 9i)(10 + 9i) + (-4 + 5i)(-4 - 5i)} = \frac{\sqrt{118}}{4}. \quad \text{Hence, an orthonormal basis is:}$$

$$\left\{ \frac{\sqrt{2}}{4}(1 + i, i, 2 - i), \frac{\sqrt{118}}{236}(9 + 13i, 10 - 9i, -4 + 5i) \right\}.$$

23. Let $f_1 = 1$, $f_2 = x$ and $f_3 = x^2$.

$$g_1 = f_1 = 1;$$

$$\|g_1\|^2 = \int_0^1 dx = 1; \quad \langle f_2, g_1 \rangle = \int_0^1 x dx = \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2}.$$

$$g_2 = f_2 - \frac{\langle f_2, g_1 \rangle}{\|g_1\|^2} g_1 = x - \frac{1}{2} = \frac{1}{2}(2x - 1).$$

$$\|g_2\|^2 = \int_0^1 \left(x - \frac{1}{2} \right)^2 dx = \int_0^1 \left(x^2 - x + \frac{1}{4} \right) dx = \left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4} \right]_0^1 = \frac{1}{12}.$$

$$\langle f_3, g_1 \rangle = \int_0^1 x^2 dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}.$$

$$\langle f_3, g_2 \rangle = \int_0^1 x^2 \left(x - \frac{1}{2} \right) dx = \left[\frac{x^4}{4} - \frac{x^3}{6} \right]_0^1 = \frac{1}{12}.$$

$$g_3 = f_3 - \frac{\langle f_3, g_1 \rangle}{\|g_1\|^2} g_1 - \frac{\langle f_3, g_2 \rangle}{\|g_2\|^2} g_2 = x^2 - \frac{1}{3} - \left(x - \frac{1}{2} \right) = \frac{1}{6}(6x^2 - 6x + 1). \quad \text{Thus, an orthogonal basis is given by:}$$

$$\left\{ 1, \frac{1}{2}(2x - 1), \frac{1}{6}(6x^2 - 6x + 1) \right\}.$$

24. Let $f_1 = 1$, $f_2 = x^2$ and $f_3 = x^4$ for all x in $[-1, 1]$.

$$\langle f_1, f_2 \rangle = \int_{-1}^1 x^2 dx = \frac{2}{3}, \quad \langle f_1, f_3 \rangle = \int_{-1}^1 x^4 dx = \frac{2}{5}, \quad \text{and} \quad \langle f_2, f_3 \rangle = \int_{-1}^1 x^6 dx = \frac{2}{7}.$$

$$\|f_1\|^2 = \int_{-1}^1 dx = 2, \quad \|f_2\|^2 = \int_{-1}^1 x^4 dx = \frac{2}{5}.$$

Let $g_1 = f_1 = 1$.

$$g_2 = f_2 - \frac{\langle f_2, g_1 \rangle}{\|g_1\|^2} g_1 = x^2 - \frac{\langle x^2, 1 \rangle}{\|1\|^2} \cdot 1 = x^2 - \frac{1}{3} = \frac{1}{3}(3x^2 - 1).$$

$$\begin{aligned} g_3 &= f_3 - \frac{\langle f_3, g_1 \rangle}{\|g_1\|^2} g_1 - \frac{\langle f_3, g_2 \rangle}{\|g_2\|^2} g_2 = x^4 - \frac{\langle x^4, 1 \rangle}{\|1\|^2} \cdot 1 - \frac{\langle x^4, (x^2 - \frac{1}{3}) \rangle}{\|x^2 - \frac{1}{3}\|^2} \left(x^2 - \frac{1}{3}\right) \\ &= x^4 - \frac{6x^2}{7} + \frac{3}{35} = \frac{1}{35}(35x^4 - 30x^2 + 3). \end{aligned}$$

Thus, an orthogonal basis is given by:

$$\left\{ 1, \frac{1}{3}(3x^2 - 1), \frac{1}{35}(35x^4 - 30x^2 + 3) \right\}.$$

25. Let $f_1 = 1$, $f_2 = \sin x$ and $f_3 = \cos x$ for all x in $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$$\langle f_1, f_2 \rangle = \int_{-\pi/2}^{\pi/2} \sin x dx = [-\cos 2x]_{-\pi/2}^{\pi/2} = 0. \quad \text{Therefore, } f_1 \text{ and } f_2 \text{ are orthogonal.}$$

$$\langle f_2, f_3 \rangle = \int_{-\pi/2}^{\pi/2} \sin x \cos x dx = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \sin 2x dx = \left[-\frac{1}{4} \cos 2x\right]_{-\pi/2}^{\pi/2} = 0. \quad \text{Therefore, } f_2 \text{ and } f_3 \text{ are orthogonal.}$$

$$\langle f_1, f_3 \rangle = \int_{-\pi/2}^{\pi/2} \cos x dx = [\sin x]_{-\pi/2}^{\pi/2} = 2.$$

$$\text{Let } g_1 = f_1 = 1 \text{ so that } \|g_1\|^2 = \int_{-\pi/2}^{\pi/2} dx = \pi.$$

$$g_2 = f_2 = \sin x, \text{ and } \|g_2\|^2 = \int_{-\pi/2}^{\pi/2} \sin^2 x dx = \int_{-\pi/2}^{\pi/2} \frac{1 - \cos 2x}{2} dx = \frac{\pi}{2}.$$

$$g_3 = f_3 - \frac{\langle f_3, g_1 \rangle}{\|g_1\|^2} g_1 - \frac{\langle f_3, g_2 \rangle}{\|g_2\|^2} g_2 = \cos x - \frac{2}{\pi} \cdot 1 - 0 \cdot \sin x = \frac{1}{\pi}(\pi \cos x - 2). \quad \text{Thus, an orthogonal basis for the subspace of } C^0[-\pi/2, \pi/2] \text{ spanned by } \{1, \sin x, \cos x\} \text{ is:}$$

$$\left\{ 1, \sin x, \frac{1}{\pi}(\pi \cos x - 2) \right\}.$$

26. Given $A_1 = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$. Using the Gram-Schmidt procedure:

$$B_1 = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad \langle A_2, B_1 \rangle = 10 + 6 + 24 + 5 = 45, \quad \text{and} \quad \|B_1\|^2 = 5 + 2 + 12 + 5 = 24.$$

$$B_2 = A_2 - \frac{\langle A_2, B_1 \rangle}{\|B_1\|^2} B_1 = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} - \frac{15}{8} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & -\frac{9}{8} \\ \frac{1}{4} & -\frac{7}{8} \end{bmatrix}. \quad \text{Thus, an orthogonal basis for the subspace of } M_2(\mathbb{R}) \text{ spanned by } A_1 \text{ and } A_2 \text{ is:}$$

$$\left\{ \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{8} & -\frac{9}{8} \\ \frac{1}{4} & -\frac{7}{8} \end{bmatrix} \right\}.$$

27. Given $A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Using the Gram-Schmidt procedure:

$$B_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \langle A_2, B_1 \rangle = 5, \text{ and } \|B_1\|^2 = 5.$$

$$B_2 = A_2 - \frac{\langle A_2, B_1 \rangle}{\|B_1\|^2} B_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Also, $\langle A_3, B_1 \rangle = 5$, $\langle A_3, B_2 \rangle = 0$, and $\|B_2\|^2 = 5$, so that

$$B_3 = A_3 - \frac{\langle A_3, B_1 \rangle}{\|B_1\|^2} B_1 - \frac{\langle A_3, B_2 \rangle}{\|B_2\|^2} B_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 0 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \text{ Thus, an orthogonal basis for the subspace of } M_2(\mathbb{R}) \text{ spanned by } A_1, A_2, \text{ and } A_3 \text{ is:}$$

$$\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

which is the subspace of all symmetric matrices in $M_2(\mathbb{R})$.

28. Given $p_1(x) = 1 - 2x + 2x^2$ and $p_2(x) = 2 - x - x^2$. Using the Gram-Schmidt procedure:

$$q_1 = 1 - 2x + 2x^2, \langle p_2, q_1 \rangle = 2 \cdot 1 + (-1)(-2) + (-1)2 = 2, \|q_1\|^2 = 1^2 + (-2)^2 + 2^2 = 9. \text{ So,}$$

$$q_2 = p_2 - \frac{\langle p_2, q_1 \rangle}{\|q_1\|^2} q_1 = 2 - x - x^2 - \frac{2}{9}(1 - 2x + 2x^2) = \frac{1}{9}(16 - 5x - 13x^2). \text{ Thus, an orthogonal basis for the subspace spanned by } p_1 \text{ and } p_2 \text{ is } \{1 - 2x + 2x^2, 16 - 5x - 13x^2\}.$$

29. Given $p_1(x) = 1 + x^2$, $p_2(x) = 2 - x + x^3$, and $p_3(x) = -x + 2x^2$. Using the Gram-Schmidt procedure:

$$q_1 = 1 + x^2, \langle p_2, q_1 \rangle = 2 \cdot 1 + (-1)(0) + 0 \cdot 1 + 1 \cdot 2 = 2, \text{ and } \|q_1\|^2 = 1^2 + 1^2 = 2. \text{ So,}$$

$$q_2 = p_2 - \frac{\langle p_2, q_1 \rangle}{\|q_1\|^2} q_1 = 2 - x + x^3 - (1 + x^2) = 1 - x - x^2 + x^3. \text{ Also,}$$

$$\langle p_3, q_1 \rangle = 0 \cdot 1 + (-1)0 + 2 \cdot 1 + 0^2 = 2$$

$$\langle p_3, q_2 \rangle = 0 \cdot 1 + (-1)^2 + 2(-1) + 0 \cdot 1 = -1, \text{ and}$$

$$\|q_2\|^2 = 1^2 + (-1)^2 + (-1)^2 + 1^2 = 4 \text{ so that}$$

$$q_3 = p_3 - \frac{\langle p_3, q_1 \rangle}{\|q_1\|^2} q_1 - \frac{\langle p_3, q_2 \rangle}{\|q_2\|^2} q_2 = -x + 2x^2 - (1 + x^2) + \frac{1}{4}(1 - x - x^2 + x^3) = \frac{1}{4}(-3 - 5x + 3x^2 + x^3). \text{ Thus, an orthogonal basis for the subspace spanned by } p_1, p_2, \text{ and } p_3 \text{ is } \{1 + x^2, 1 - x - x^2 + x^3, -3 - 5x + 3x^2 + x^3\}.$$

30. $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}\}$ is a linearly independent set of vectors, and $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$. If we let $\mathbf{u}_3 = \mathbf{v} + \lambda \mathbf{u}_1 + \mu \mathbf{u}_2$, then it must be the case that $\langle \mathbf{u}_3, \mathbf{u}_1 \rangle = 0$ and $\langle \mathbf{u}_3, \mathbf{u}_2 \rangle = 0$. $\langle \mathbf{u}_3, \mathbf{u}_1 \rangle = 0 \implies \langle \mathbf{v} + \lambda \mathbf{u}_1 + \mu \mathbf{u}_2, \mathbf{u}_1 \rangle = 0$

$$\implies \langle \mathbf{v}, \mathbf{u}_1 \rangle + \lambda \langle \mathbf{u}_1, \mathbf{u}_1 \rangle + \mu \langle \mathbf{u}_2, \mathbf{u}_1 \rangle = 0 \implies \langle \mathbf{v}, \mathbf{u}_1 \rangle + \lambda \langle \mathbf{u}_1, \mathbf{u}_1 \rangle + \mu \cdot 0 = 0$$

$$\implies \lambda = -\frac{\langle \mathbf{v}, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}.$$

$$\langle \mathbf{u}_3, \mathbf{u}_2 \rangle = 0 \implies \langle \mathbf{v} + \lambda \mathbf{u}_1 + \mu \mathbf{u}_2, \mathbf{u}_2 \rangle = 0$$

$$\implies \langle \mathbf{v}, \mathbf{u}_2 \rangle + \lambda \langle \mathbf{u}_1, \mathbf{u}_2 \rangle + \mu \langle \mathbf{u}_2, \mathbf{u}_2 \rangle = 0 \implies \langle \mathbf{v}, \mathbf{u}_2 \rangle + \lambda \cdot 0 + \mu \langle \mathbf{u}_2, \mathbf{u}_2 \rangle = 0$$

$$\implies \mu = -\frac{\langle \mathbf{v}, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}.$$

Hence, if $\mathbf{u}_3 = \mathbf{v} - \frac{\langle \mathbf{v}, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 - \frac{\langle \mathbf{v}, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2} \mathbf{u}_2$, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for the subspace spanned by $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}\}$.

31. It was shown in Remark 3 following Definition 4.12.1 that each vector $\mathbf{u}_i$ is a unit vector. Moreover, for $i \neq j$,

$$\langle \mathbf{u}_i, \mathbf{u}_j \rangle = \left\langle \frac{1}{\|\mathbf{v}_i\|} \mathbf{v}_i, \frac{1}{\|\mathbf{v}_j\|} \mathbf{v}_j \right\rangle = \frac{1}{\|\mathbf{v}_i\| \|\mathbf{v}_j\|} \langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0,$$

since $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$. Therefore $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$ is an orthonormal set of vectors.

32. Set

$$\mathbf{z} = \mathbf{x} - P(\mathbf{x}, \mathbf{v}_1) - P(\mathbf{x}, \mathbf{v}_2) - \cdots - P(\mathbf{x}, \mathbf{v}_k).$$

To verify that $\mathbf{z}$ is orthogonal to $\mathbf{v}_i$, we compute the inner product of $\mathbf{z}$ with $\mathbf{v}_i$:

$$\begin{aligned}\langle \mathbf{z}, \mathbf{v}_i \rangle &= \langle \mathbf{x} - P(\mathbf{x}, \mathbf{v}_1) - P(\mathbf{x}, \mathbf{v}_2) - \cdots - P(\mathbf{x}, \mathbf{v}_k), \mathbf{v}_i \rangle \\ &= \langle \mathbf{x}, \mathbf{v}_i \rangle - \langle P(\mathbf{x}, \mathbf{v}_1), \mathbf{v}_i \rangle - \langle P(\mathbf{x}, \mathbf{v}_2), \mathbf{v}_i \rangle - \cdots - \langle P(\mathbf{x}, \mathbf{v}_k), \mathbf{v}_i \rangle.\end{aligned}$$

Since $P(\mathbf{x}, \mathbf{v}_j)$ is a multiple of $\mathbf{v}_j$ and the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is orthogonal, all of the subtracted terms on the right-hand side of this expression are zero except

$$\langle P(\mathbf{x}, \mathbf{v}_i), \mathbf{v}_i \rangle = \left\langle \frac{\langle \mathbf{x}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2} \mathbf{v}_i, \mathbf{v}_i \right\rangle = \frac{\langle \mathbf{x}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2} \langle \mathbf{v}_i, \mathbf{v}_i \rangle = \langle \mathbf{x}, \mathbf{v}_i \rangle.$$

Therefore,

$$\langle \mathbf{z}, \mathbf{v}_i \rangle = \langle \mathbf{x}, \mathbf{v}_i \rangle - \langle \mathbf{x}, \mathbf{v}_i \rangle = 0,$$

which implies that $\mathbf{z}$ is orthogonal to $\mathbf{v}_i$.

33. We must show that $W^\perp$ is closed under addition and closed under scalar multiplication.

Closure under addition: Let $\mathbf{v}_1$ and $\mathbf{v}_2$ belong to $W^\perp$. This means that $\langle \mathbf{v}_1, \mathbf{w} \rangle = \langle \mathbf{v}_2, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in W$. Therefore,

$$\langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w} \rangle = \langle \mathbf{v}_1, \mathbf{w} \rangle + \langle \mathbf{v}_2, \mathbf{w} \rangle = 0 + 0 = 0$$

for all $\mathbf{w} \in W$. Therefore, $\mathbf{v}_1 + \mathbf{v}_2 \in W^\perp$.

Closure under scalar multiplication: Let $\mathbf{v}$ belong to $W^\perp$ and let c be a scalar. This means that $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{w} \in W$. Therefore,

$$\langle c\mathbf{v}, \mathbf{w} \rangle = c\langle \mathbf{v}, \mathbf{w} \rangle = c \cdot 0 = 0,$$

which shows that $c\mathbf{v} \in W^\perp$.

34. In this case, $W^\perp$ consists of all $(x, y, z) \in \mathbb{R}^3$ such that $\langle (x, y, z), (r, r, -r) \rangle = 0$ for all $r \in \mathbb{R}$. That is, $rx + ry - rz = 0$. In particular, we must have $x + y - z = 0$. Therefore, $W^\perp$ is the plane $x + y - z = 0$, which can also be expressed as

$$W^\perp = \text{span}\{(-1, 0, 1), (0, 1, 1)\}.$$

35. In this case, $W^\perp$ consists of all $(x, y, z, w) \in \mathbb{R}^4$ such that $\langle (x, y, z, w), (0, 1, -1, 3) \rangle = \langle (x, y, z, w), (1, 0, 0, 3) \rangle = 0$. This requires that $y - z + 3w = 0$ and $x + 3w = 0$. We can associated an augmented matrix with this system of linear equations: $\left[\begin{array}{cccc|c} 0 & 1 & -1 & 3 & 0 \\ 1 & 0 & 0 & 3 & 0 \end{array} \right]$. Note that z and w are free variables: $z = s$ and $w = t$. Then $y = s - 3t$ and $x = -3t$. Thus,

$$W^\perp = \{(-3t, s-3t, s, t) : s, t \in \mathbb{R}\} = \{t(-3, -3, 0, 1) + s(0, 1, 1, 0) : s, t \in \mathbb{R}\} = \text{span}\{(-3, -3, 0, 1), (0, 1, 1, 0)\}.$$

36. In this case, $W^\perp$ consists of all 2×2 matrices $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ that are orthogonal to all symmetric matrices.

The set of symmetric matrices is spanned by the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Thus, we must have

$$\left\langle \begin{bmatrix} x & y \\ z & w \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} x & y \\ z & w \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} x & y \\ z & w \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle = 0.$$

Thus, $x = 0$, $y + z = 0$ and $w = 0$. Therefore

$$W^\perp = \left\{ \begin{bmatrix} 0 & -z \\ z & 0 \end{bmatrix} : z \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\},$$

which is precisely the set of 2×2 skew-symmetric matrices.

37. Suppose $\mathbf{v}$ belongs to both W and $W^\perp$. Then $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ by definition of $W^\perp$, which implies that $\mathbf{v} = \mathbf{0}$ by the first axiom of an inner product. Therefore W and $W^\perp$ can contain no common elements aside from the zero vector.

38. Suppose that W_1 is a subset of W_2 and let $\mathbf{v}$ be a vector in $(W_2)^\perp$. This means that $\langle \mathbf{v}, \mathbf{w}_2 \rangle = 0$ for all $\mathbf{w}_2 \in W_2$. In particular, $\mathbf{v}$ is orthogonal to all vectors in W_1 (since W_1 is merely a subset of W_2). Thus, $\mathbf{v} \in (W_1)^\perp$. Hence, we have shown that every vector belonging to $(W_2)^\perp$ also belongs to $(W_1)^\perp$.

39. (a) Using technology we find that

$$\int_{-\pi}^{\pi} \sin nx \, dx = 0, \quad \int_{-\pi}^{\pi} \cos nx \, dx = 0, \quad \int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0.$$

Further, for $m \neq n$,

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0.$$

Consequently the given set of vectors is orthogonal on $[-\pi, \pi]$.

(b) Multiplying (4.12.7) by $\cos mx$ and integrating over $[-\pi, \pi]$ yields

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \frac{1}{2} a_0 \int_{-\pi}^{\pi} \cos mx \, dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \cos mx \, dx.$$

Assuming that interchange of the integral and infinite summation is permissible, this can be written

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \frac{1}{2} a_0 \int_{-\pi}^{\pi} \cos mx \, dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n \cos nx + b_n \sin nx) \cos mx \, dx.$$

which reduces to

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \frac{1}{2} a_0 \int_{-\pi}^{\pi} \cos mx \, dx + a_m \int_{-\pi}^{\pi} \cos^2 mx \, dx$$

where we have used the results from part (a). When $m = 0$, this gives

$$\int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2} a_0 \int_{-\pi}^{\pi} dx = \pi a_0 \implies a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx,$$

whereas for $m \neq 0$,

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = a_m \int_{-\pi}^{\pi} \cos^2 mx \, dx = \pi a_m \implies a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, dx.$$

(c) Multiplying (4.12.7) by $\sin(mx)$, integrating over $[-\pi, \pi]$, and interchanging the integration and summation yields

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = \frac{1}{2} a_0 \int_{-\pi}^{\pi} \sin mx \, dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n \cos nx + b_n \sin nx) \sin mx \, dx.$$

Using the results from (a), this reduces to

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = b_n \int_{-\pi}^{\pi} \sin^2 mx \, dx = \pi b_n \implies b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx.$$

(d) The Fourier coefficients for f are

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x \, dx = 0, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx \, dx = 0,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx = -\frac{2}{n} \cos n\pi = \frac{2}{n}(-1)^{n+1}.$$

The Fourier series for f is

$$\sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx.$$

(e) The approximations using the first term, the first three terms, the first five terms, and the first ten terms in the Fourier series for f are shown in the accompanying figures. These figures suggest that the Fourier series is converging to the function $f(x)$ at all points in the interval $(-\pi, \pi)$.

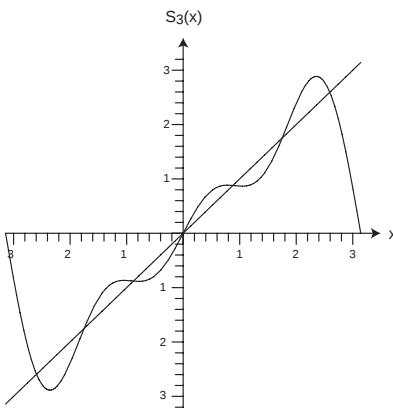


Figure 66: Figure for Problem 39(e) - 3 terms included

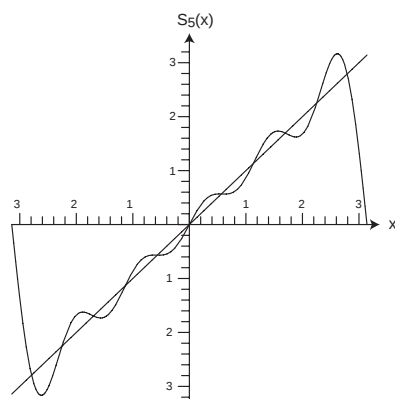


Figure 67: Figure for Problem 39(e) - 5 terms included

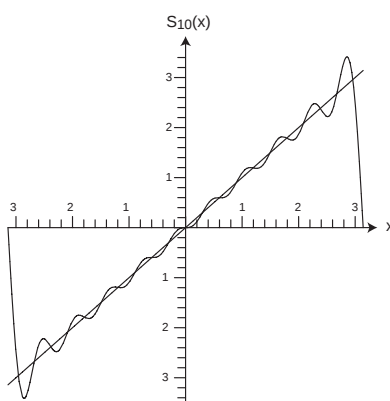


Figure 68: Figure for Problem 39(e) - 10 terms included

Solutions to Section 4.13

Problems:

1. Write $\mathbf{v} = (a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5$. Then we have

$$\begin{aligned}
 (r+s)\mathbf{v} &= (r+s)(a_1, a_2, a_3, a_4, a_5) \\
 &= ((r+s)a_1, (r+s)a_2, (r+s)a_3, (r+s)a_4, (r+s)a_5) \\
 &= (ra_1 + sa_1, ra_2 + sa_2, ra_3 + sa_3, ra_4 + sa_4, ra_5 + sa_5) \\
 &= (ra_1, ra_2, ra_3, ra_4, ra_5) + (sa_1, sa_2, sa_3, sa_4, sa_5) \\
 &= r(a_1, a_2, a_3, a_4, a_5) + s(a_1, a_2, a_3, a_4, a_5) \\
 &= r\mathbf{v} + s\mathbf{v}.
 \end{aligned}$$

2. Write $\mathbf{v} = (a_1, a_2, a_3, a_4, a_5)$ and $\mathbf{w} = (b_1, b_2, b_3, b_4, b_5)$ in $\mathbb{R}^5$. Then we have

$$\begin{aligned} r(\mathbf{v} + \mathbf{w}) &= r((a_1, a_2, a_3, a_4, a_5) + (b_1, b_2, b_3, b_4, b_5)) \\ &= r(a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5) \\ &= (r(a_1 + b_1), r(a_2 + b_2), r(a_3 + b_3), r(a_4 + b_4), r(a_5 + b_5)) \\ &= (ra_1 + rb_1, ra_2 + rb_2, ra_3 + rb_3, ra_4 + rb_4, ra_5 + rb_5) \\ &= (ra_1, ra_2, ra_3, ra_4, ra_5) + (rb_1, rb_2, rb_3, rb_4, rb_5) \\ &= r(a_1, a_2, a_3, a_4, a_5) + r(b_1, b_2, b_3, b_4, b_5) \\ &= r\mathbf{v} + r\mathbf{w}. \end{aligned}$$

3. NO. This set of polynomials is not closed under scalar multiplication. For example, the polynomial $p(x) = 2x$ belongs to the set, but $\frac{1}{3}p(x) = \frac{2}{3}x$ does not belong to the set (since $\frac{2}{3}$ is not an even integer).

4. YES. This set of polynomials forms a subspace of the vector space P_5 . To confirm this, we will check that this set is closed under addition and scalar multiplication:

Closure under Addition: Let $p(x) = a_0 + a_1x + a_4x^4 + a_5x^5$ and $q(x) = b_0 + b_1x + b_4x^4 + b_5x^5$ be polynomials in the set under consideration (their x^2 and x^3 terms are zero). Then

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_4 + b_4)x^4 + (a_5 + b_5)x^5$$

is again in the set (since it still has no x^2 or x^3 terms). So closure under addition holds.

Closure under Scalar Multiplication: Let $p(x) = a_0 + a_1x + a_4x^4 + a_5x^5$ be in the set, and let k be a scalar. Then

$$kp(x) = (ka_0) + (ka_1)x + (ka_4)x^4 + (ka_5)x^5,$$

which is again in the set (since it still has no x^2 or x^3 terms). So closure under scalar multiplication holds.

5. NO. We can see immediately that the zero vector $(0, 0, 0)$ is not a solution to this linear system (the first equation is not satisfied by the zero vector), and therefore, we know at once that this set cannot be a vector space.

6. YES. The set of solutions to this linear system forms a subspace of $\mathbb{R}^3$. To confirm this, we will check that this set is closed under addition and scalar multiplication:

Closure under Addition: Let (a_1, a_2, a_3) and (b_1, b_2, b_3) be solutions to the linear system. This means that

$$4a_1 - 7a_2 + 2a_3 = 0, \quad 5a_1 - 2a_2 + 9a_3 = 0$$

and

$$4b_1 - 7b_2 + 2b_3 = 0, \quad 5b_1 - 2b_2 + 9b_3 = 0.$$

Adding the equations on the left, we get

$$4(a_1 + b_1) - 7(a_2 + b_2) + 2(a_3 + b_3) = 0,$$

so the vector $(a_1 + b_1, a_2 + b_2, a_3 + b_3)$ satisfies the first equation in the linear system. Likewise, adding the equations on the right, we get

$$5(a_1 + b_1) - 2(a_2 + b_2) + 9(a_3 + b_3) = 0,$$

so $(a_1 + b_1, a_2 + b_2, a_3 + b_3)$ also satisfies the second equation in the linear system. Therefore, $(a_1 + b_1, a_2 + b_2, a_3 + b_3)$ is in the solution set for the linear system, and closure under addition therefore holds.

Closure under Scalar Multiplication: Let (a_1, a_2, a_3) be a solution to the linear system, and let k be a scalar. We have

$$4a_1 - 7a_2 + 2a_3 = 0, \quad 5a_1 - 2a_2 + 9a_3 = 0,$$

and so, multiplying both equations by k , we have

$$k(4a_1 - 7a_2 + 2a_3) = 0, \quad k(5a_1 - 2a_2 + 9a_3) = 0,$$

or

$$4(ka_1) - 7(ka_2) + 2(ka_3) = 0, \quad 5(ka_1) - 2(ka_2) + 9(ka_3) = 0.$$

Thus, the vector (ka_1, ka_2, ka_3) is a solution to the linear system, and closure under scalar multiplication therefore holds.

7. NO. This set is not closed under addition. For example, the vectors $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ both belong to the set (their entries are all nonzero), but

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix},$$

which does not belong to the set (some entries are zero, and some are nonzero). So closure under addition fails, and therefore, this set does not form a vector space.

8. YES. The set of 2×2 real matrices that commute with $C = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ forms a subspace of $M_2(\mathbb{R})$. To confirm this, we will check that this set is closed under addition and scalar multiplication:

Closure under Addition: Let A and B be 2×2 real matrices that commute with C . That is, $AC = CA$ and $BC = CB$. Then

$$(A + B)C = AC + BC = CA + CB = C(A + B),$$

so $A + B$ commutes with C , and therefore, closure under addition holds.

Closure under Scalar Multiplication: Let A be a 2×2 real matrix that commutes with C , and let k be a scalar. Then since $AC = CA$, we have

$$(kA)C = k(AC) = k(CA) = C(kA),$$

so kA is still in the set. Thus, the set is also closed under scalar multiplication.

9. YES. The set of functions $f : [0, 1] \rightarrow [0, 1]$ such that $f(0) = f(\frac{1}{4}) = f(\frac{1}{2}) = f(\frac{3}{4}) = f(1) = 0$ is a subspace of the vector space of all functions $[0, 1] \rightarrow [0, 1]$. We confirm this by checking that this set is closed under addition and scalar multiplication:

Closure under Addition: Let g and h be functions such that

$$g(0) = g\left(\frac{1}{4}\right) = g\left(\frac{1}{2}\right) = g\left(\frac{3}{4}\right) = g(1) = 0$$

and

$$h(0) = h\left(\frac{1}{4}\right) = h\left(\frac{1}{2}\right) = h\left(\frac{3}{4}\right) = h(1) = 0.$$

Now

$$\begin{aligned}
 (g+h)(0) &= g(0) + h(0) = 0 + 0 = 0, \\
 (g+h)\left(\frac{1}{4}\right) &= g\left(\frac{1}{4}\right) + h\left(\frac{1}{4}\right) = 0 + 0 = 0, \\
 (g+h)\left(\frac{1}{2}\right) &= g\left(\frac{1}{2}\right) + h\left(\frac{1}{2}\right) = 0 + 0 = 0, \\
 (g+h)\left(\frac{3}{4}\right) &= g\left(\frac{3}{4}\right) + h\left(\frac{3}{4}\right) = 0 + 0 = 0, \\
 (g+h)(1) &= g(1) + h(1) = 0 + 0 = 0.
 \end{aligned}$$

Thus, $g+h$ belongs to the set, and so the set is closed under addition.

Closure under Scalar Multiplication: Let g be a function with

$$g(0) = g\left(\frac{1}{4}\right) = g\left(\frac{1}{2}\right) = g\left(\frac{3}{4}\right) = g(1) = 0,$$

and let k be a scalar. Then

$$\begin{aligned}
 (kg)(0) &= k \cdot g(0) = k \cdot 0 = 0, \\
 (kg)\left(\frac{1}{4}\right) &= k \cdot g\left(\frac{1}{4}\right) = k \cdot 0 = 0, \\
 (kg)\left(\frac{1}{2}\right) &= k \cdot g\left(\frac{1}{2}\right) = k \cdot 0 = 0, \\
 (kg)\left(\frac{3}{4}\right) &= k \cdot g\left(\frac{3}{4}\right) = k \cdot 0 = 0, \\
 (kg)(1) &= k \cdot g(1) = k \cdot 0 = 0.
 \end{aligned}$$

Thus, kg belongs to the set, and so the set is closed under scalar multiplication.

10. NO. This set is not closed under addition. For example, the function f defined by $f(x) = x$ belongs to the set, and the function g defined by

$$g(x) = \begin{cases} x, & \text{if } x \leq 1/2 \\ 0, & \text{if } x > 1/2 \end{cases}$$

belongs to the set. But

$$(f+g)(x) = \begin{cases} 2x, & \text{if } x \leq 1/2 \\ x, & \text{if } x > 1/2. \end{cases}$$

Then $f+g : [0, 1] \rightarrow [0, 1]$, but for $0 < x \leq \frac{1}{2}$, $|(f+g)(x)| = 2x > x$, so $f+g$ is not in the set. Therefore, the set is not closed under addition.

11. NO. This set is not closed under addition. For example, if we let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

then $A^2 = A$ is symmetric, and $B^2 = 0_2$ is symmetric, but

$$(A+B)^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

is not symmetric, so $A + B$ is not in the set. Thus, the set in question is not closed under addition.

12. YES. Let us describe geometrically the set of points equidistant from $(-1, 2)$ and $(1, -2)$. If (x, y) is such a point, then using the distance formula and equating distances to $(-1, 2)$ and $(1, -2)$, we have

$$\sqrt{(x+1)^2 + (y-2)^2} = \sqrt{(x-1)^2 + (y+2)^2}$$

or

$$(x+1)^2 + (y-2)^2 = (x-1)^2 + (y+2)^2$$

or

$$x^2 + 2x + 1 + y^2 - 4y + 4 = x^2 - 2x + 1 + y^2 + 4y + 4.$$

Cancelling like terms and rearranging this equation, we have $4x = 8y$. So the points that are equidistant from $(-1, 2)$ and $(1, -2)$ lie on the line through the origin with equation $y = \frac{1}{2}x$. Any line through the origin of $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$. Therefore, this line forms a vector space.

13. NO. This set is not closed under addition, nor under scalar multiplication. For instance, the point $(5, -3, 4)$ is a distance 5 from $(0, -3, 4)$, so the point $(5, -3, 4)$ lies in the set. But the point $(10, -6, 8) = 2(5, -3, 4)$ is a distance

$$\sqrt{(10-0)^2 + (-6+3)^2 + (8-4)^2} = \sqrt{100 + 9 + 16} = \sqrt{125} \neq 5$$

from $(0, -3, 4)$, so $(10, -6, 8)$ is not in the set. So the set is not closed under scalar multiplication, and hence does not form a subspace.

14. We must check each of the vector space axioms (A1)-(A10).

Axiom (A1): Assume that (a_1, a_2) and (b_1, b_2) belong to V . Then $a_2, b_2 > 0$. Hence,

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2) \in V,$$

since $a_2 b_2 > 0$. Thus, V is closed under addition.

Axiom (A2): Assume that $(a_1, a_2) \in V$, and let k be a scalar. Note that since $a_2 > 0$, the expression $a_2^k > 0$ for every $k \in \mathbb{R}$. Hence, $k(a_1, a_2) = (ka_1, a_2^k) \in V$, thereby showing that V is closed under scalar multiplication.

Axiom (A3): Let $(a_1, a_2), (b_1, b_2) \in V$. We have

$$\begin{aligned} (a_1, a_2) + (b_1, b_2) &= (a_1 + b_1, a_2 b_2) \\ &= (b_1 + a_1, b_2 a_2) \\ &= (b_1, b_2) + (a_1, a_2), \end{aligned}$$

as required.

Axiom (A4): Let $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in V$. We have

$$\begin{aligned} ((a_1, a_2) + (b_1, b_2)) + (c_1, c_2) &= (a_1 + b_1, a_2 b_2) + (c_1, c_2) \\ &= ((a_1 + b_1) + c_1, (a_2 b_2) c_2) \\ &= (a_1 + (b_1 + c_1), a_2 (b_2 c_2)) \\ &= (a_1, a_2) + (b_1 + c_1, b_2 c_2) \\ &= (a_1, a_2) + ((b_1, b_2) + (c_1, c_2)), \end{aligned}$$

as required.

Axiom (A5): We claim that $(0, 1)$ is the zero vector in V . To see this, let $(b_1, b_2) \in V$. Then

$$(0, 1) + (b_1, b_2) = (0 + b_1, 1 \cdot b_2) = (b_1, b_2).$$

Since this holds for every $(b_1, b_2) \in V$, we conclude that $(0, 1)$ is the zero vector.

Axiom (A6): We claim that the additive inverse of a vector (a_1, a_2) in V is the vector $(-a_1, a_2^{-1})$ (Note that $a_2^{-1} > 0$ since $a_2 > 0$.) To check this, we compute as follows:

$$(a_1, a_2) + (-a_1, a_2^{-1}) = (a_1 + (-a_1), a_2 a_2^{-1}) = (0, 1).$$

Axiom (A7): We have

$$1 \cdot (a_1, a_2) = (a_1, a_2^1) = (a_1, a_2)$$

for all $(a_1, a_2) \in V$.

Axiom (A8): Let $(a_1, a_2) \in V$, and let r and s be scalars. Then we have

$$\begin{aligned} (rs)(a_1, a_2) &= ((rs)a_1, a_2^{rs}) \\ &= (r(sa_1), (a_2^s)^r) \\ &= r(sa_1, a_2^s) \\ &= r(s(a_1, a_2)), \end{aligned}$$

as required.

Axiom (A9): Let (a_1, a_2) and (b_1, b_2) be members of V , and let r be a scalar. We have

$$\begin{aligned} r((a_1, a_2) + (b_1, b_2)) &= r(a_1 + b_1, a_2 b_2) \\ &= (r(a_1 + b_1), (a_2 b_2)^r) \\ &= (ra_1 + rb_1, a_2^r b_2^r) \\ &= (ra_1, a_2^r) + (rb_1, b_2^r) \\ &= r(a_1, a_2) + r(b_1, b_2), \end{aligned}$$

as required.

Axiom (A10): Let $(a_1, a_2) \in V$, and let r and s be scalars. We have

$$\begin{aligned} (r + s)(a_1, a_2) &= ((r + s)a_1, a_2^{r+s}) \\ &= (ra_1 + sa_1, a_2^r a_2^s) \\ &= (ra_1, a_2^r) + (sa_1, a_2^s) \\ &= r(a_1, a_2) + s(a_1, a_2), \end{aligned}$$

as required.

15. We must show that W is closed under addition and closed under scalar multiplication:

Closure under Addition: Let $(a, 2^a)$ and $(b, 2^b)$ be elements of W . Now consider the sum of these elements:

$$(a, 2^a) + (b, 2^b) = (a + b, 2^a 2^b) = (a + b, 2^{a+b}) \in W,$$

which shows that W is closed under addition.

Closure under Scalar Multiplication: Let $(a, 2^a)$ be an element of W , and let k be a scalar. Then

$$k(a, 2^a) = (ka, (2^a)^k) = (ka, 2^{ka}) \in W,$$

which shows that W is closed under scalar multiplication.

Thus, W is a subspace of V .

16. Note that $3(1, 2) = (3, 2^3) = (3, 8)$, so the second vector is a multiple of the first one under the vector space operations of V from Problem 14. Therefore, $\{(1, 2), (3, 8)\}$ is linearly dependent.

17. We show that $S = \{(1, 4), (2, 1)\}$ is linearly independent and spans V .

S is linearly independent: Assume that

$$c_1(1, 4) + c_2(2, 1) = (0, 1).$$

This can be written

$$(c_1, 4^{c_1}) + (2c_2, 1^{c_2}) = (0, 1)$$

or

$$(c_1 + 2c_2, 4^{c_1}) = (0, 1).$$

In order for $4^{c_1} = 1$, we must have $c_1 = 0$. And then in order for $c_1 + 2c_2 = 0$, we must have $c_2 = 0$. Therefore, S is linearly independent.

S spans V: Consider an arbitrary vector $(a_1, a_2) \in V$, where $a_2 > 0$. We must find constants c_1 and c_2 such that

$$c_1(1, 4) + c_2(2, 1) = (a_1, a_2).$$

Thus,

$$(c_1, 4^{c_1}) + (2c_2, 1^{c_2}) = (a_1, a_2)$$

or

$$(c_1 + 2c_2, 4^{c_1}) = (a_1, a_2).$$

Hence,

$$c_1 + 2c_2 = a_1 \quad \text{and} \quad 4^{c_1} = a_2.$$

From the second equation, we conclude that

$$c_1 = \log_4(a_2).$$

Thus, from the first equation,

$$c_2 = \frac{1}{2}(a_1 - \log_4(a_2)).$$

Hence, since we were able to find constants c_1 and c_2 in order that

$$c_1(1, 4) + c_2(2, 1) = (a_1, a_2),$$

we conclude that $\{(1, 4), (2, 1)\}$ spans V .

18. This really hinges on whether or not the given vectors are linearly dependent or linearly independent. If we assume that

$$c_1(2 + x^2) + c_2(4 - 2x + 3x^2) + c_3(1 + x) = 0,$$

then

$$(2c_1 + 4c_2 + c_3) + (-2c_2 + c_3)x + (c_1 + 3c_2)x^2 = 0.$$

Thus, we have

$$2c_1 + 4c_2 + c_3 = 0 \quad -2c_2 + c_3 = 0 \quad c_1 + 3c_2 = 0.$$

Since the matrix of coefficients

$$\begin{bmatrix} 2 & 4 & 1 \\ 0 & -2 & 1 \\ 1 & 3 & 0 \end{bmatrix}$$

fails to be invertible, we conclude that there will be non-trivial solutions for c_1, c_2 , and c_3 . Thus, the polynomials are linearly dependent. We can therefore remove a vector from the set without decreasing the span. We remove $1 + x$, leaving us with $\{2 + x^2, 4 - 2x + 3x^2\}$. Since these polynomials are not proportional, they are now linearly independent, and hence, they are a basis for their span. Hence,

$$\dim\{2 + x^2, 4 - 2x + 3x^2, 1 + x\} = 2.$$

19. NO. This set is not closed under scalar multiplication. For example, $(1, 1)$ belongs to W , but $2 \cdot (1, 1) = (2, 2)$ does not belong to W .

20. NO. This set is not closed under scalar multiplication. For example, $(1, 1)$ belongs to W , but $2 \cdot (1, 1) = (2, 2)$ does not belong to W .

21. NO. The zero vector (zero matrix) is not an orthogonal matrix. Any subspace must contain a zero vector.

22. YES. We show that W is closed under addition and closed under scalar multiplication.

Closure under Addition: Assume that f and g belong to W . Thus, $f(a) = 2f(b)$ and $g(a) = 2g(b)$. We must show that $f + g$ belongs to W . We have

$$(f + g)(a) = f(a) + g(a) = 2f(b) + 2g(b) = 2[f(b) + g(b)] = 2(f + g)(b),$$

so $f + g \in W$. So W is closed under addition.

Closure under Scalar Multiplication: Assume that f belongs to W and k is a scalar. Thus, $f(a) = 2f(b)$. Moreover,

$$(kf)(a) = kf(a) = k(2f(b)) = 2(kf(b)) = 2(kf)(b),$$

so $kf \in W$. Thus, W is closed under scalar multiplication.

23. YES. We show that W is closed under addition and closed under scalar multiplication.

Closure under Addition: Assume that f and g belong to W . Thus,

$$\int_a^b f(x)dx = 0 \quad \text{and} \quad \int_a^b g(x)dx = 0.$$

Hence,

$$\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx = 0 + 0 = 0,$$

so $f + g \in W$.

Closure under Scalar Multiplication: Assume that f belongs to W and k is a scalar. Thus,

$$\int_a^b f(x)dx = 0.$$

Therefore,

$$\int_a^b (kf)(x)dx = \int_a^b kf(x)dx = k \int_a^b f(x)dx = k \cdot 0 = 0,$$

so $kf \in W$.

24. YES. We show that W is closed under addition and scalar multiplication.

Closure under Addition: Assume that

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \\ e_1 & f_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \\ e_2 & f_2 \end{bmatrix} \in W.$$

Thus,

$$a_1 + b_1 = c_1 + f_1, \quad a_1 - c_1 = e_1 - f_1 - d_1, \quad a_2 + b_2 = c_2 + f_2, \quad a_2 - c_2 = e_2 - f_2 - d_2.$$

Hence,

$$(a_1 + a_2) + (b_1 + b_2) = (c_1 + c_2) + (f_1 + f_2) \quad \text{and} \quad (a_1 + a_2) - (c_1 + c_2) = (e_1 + e_2) - (f_1 + f_2) - (d_1 + d_2).$$

Thus,

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \\ e_1 & f_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \\ e_2 & f_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \\ e_1 + e_2 & f_1 + f_2 \end{bmatrix} \in W,$$

which means W is closed under addition.

Closure under Scalar Multiplication: Assume that

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \in W$$

and k is a scalar. Then we have $a + b = c + f$ and $a - c = e - f - d$. Thus, $ka + kb = kc + kf$ and $ka - kc = ke - kf - kd$. Therefore,

$$k \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \\ ke & kf \end{bmatrix} \in W,$$

so W is closed under scalar multiplication.

25. (a) NO, (b) YES. Since 3 vectors are required to span $\mathbb{R}^3$, S cannot span V . However, since the vectors are not proportional, they are linearly independent.

26. (a) YES, (b) YES. If we place the three vectors into the columns of a 3×3 matrix $\begin{bmatrix} 6 & -3 & 2 \\ 1 & 1 & 1 \\ 1 & -8 & -1 \end{bmatrix}$, we observe that the matrix is invertible. Hence, its columns are linearly independent. Since we have 3 linearly independent vectors in the 3-dimensional space $\mathbb{R}^3$, we have a basis for $\mathbb{R}^3$.

27. (a) NO, (b) YES. Since we have only 3 vectors in a 4-dimensional vector space, they cannot possibly span $\mathbb{R}^4$. To check linear independence, we place the vectors into the columns of a matrix:

$$\begin{bmatrix} 6 & 1 & 1 \\ -3 & 1 & -8 \\ 2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The first three rows for the same invertible matrix as in the previous problem, so the reduced row-echelon form

is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, so there are no free variables, and hence, the column vectors form a linearly independent set.

28. (a) YES, (b) NO. Since $(0, 0, 0)$ is a member of S , S cannot be linearly independent. However, the set $\{(10, -6, 5), (3, -3, 2), (6, 4, -1)\}$ is linearly independent (these vectors can be made to form a 3×3 matrix that is invertible), and thus, S must span at least a 3-dimensional space. Since $\dim[\mathbb{R}^3] = 3$, we know that S spans $\mathbb{R}^3$.

29. (a) YES, (b) YES. Consider the linear equation

$$c_1(2x - x^3) + c_2(1 + x + x^2) + 3c_3 + c_4x = 0.$$

Then

$$(c_2 + 3c_3) + (2c_1 + c_2 + c_4)x + c_2x^2 - c_1x^3 = 0.$$

From the latter equation, we see that $c_1 = c_2 = 0$ (looking at the x^2 and x^3 coefficients) and thus, $c_3 = c_4 = 0$ (looking at the constant term and x coefficient). Thus, $c_1 = c_2 = c_3 = c_4 = 0$, and hence, S is linearly independent. Since we have four linearly independent vectors in the 4-dimensional vector space P_3 , we conclude that these vectors also span P_3 .

30. (a) NO, (b) NO. The set S only contains four vectors, although $\dim[P_4] = 5$, so it is impossible for S to span P_4 . Alternatively, simply note that none of the polynomials in S contain an x^3 term.

To check linear independence, consider the equation

$$c_1(x^4 + x + 2 + 1) + c_2(x^2 + x + 1) + c_3(x + 1) + c_4(x^4 + 2x + 3) = 0.$$

Rearranging this, we have

$$(c_1 + c_4)x^4 + (c_1 + c_2)x^2 + (c_2 + c_3 + 2c_4)x + (c_1 + c_2 + 3c_4) = 0,$$

and so we look to solve the linear system with augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array} \right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & -1 & -3 & 0 \\ 0 & -1 & -1 & -2 & 0 \end{array} \right] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right],$$

where we have added -1 times the first row to each of the third and fourth rows in the first step, and zeroed at the last row in the second step. We see that c_4 is a free variable, which means that a nontrivial solution to the linear system exists, and therefore, the original vectors are linearly dependent.

31. (a) NO, (b) YES. The vector space $M_{2 \times 3}(\mathbb{R})$ is 6-dimensional, and since only four vectors belong to S , S cannot possibly span $M_{2 \times 3}(\mathbb{R})$. On the other hand, if we form the linear system with augmented matrix

$$\left[\begin{array}{cccc|c} -1 & 3 & -1 & -11 & 0 \\ 0 & 2 & -2 & -6 & 0 \\ 0 & 1 & -3 & -5 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 1 & 2 & 2 & -2 & 0 \\ 1 & 3 & 1 & -5 & 0 \end{array} \right],$$

row reduction shows that each column of a row-echelon form contains a pivot, and therefore, the vectors are linearly independent.

32. (a) NO, (b) NO. Since $M_2(\mathbb{R})$ is only 4-dimensional and S contains 5 vectors, S cannot possibly be linearly independent. Moreover, each matrix in S is symmetric, and therefore, only symmetric matrices can be found in the $\text{span}(S)$. Thus, S fails to span $M_2(\mathbb{R})$.

33. Assume that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, and that $\mathbf{v}_4$ does not lie in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. We will show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent. To do this, assume that

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 + c_4 \mathbf{v}_4 = \mathbf{0}.$$

We must prove that $c_1 = c_2 = c_3 = c_4 = 0$.

If $c_4 \neq 0$, then we rearrange the above equation to show that

$$\mathbf{v}_4 = -\frac{c_1}{c_4} \mathbf{v}_1 - \frac{c_2}{c_4} \mathbf{v}_2 - \frac{c_3}{c_4} \mathbf{v}_3,$$

which implies that $\mathbf{v}_4 \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, contrary to our assumption. Therefore, we know that $c_4 = 0$. Hence the equation above reduces to

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = \mathbf{0},$$

and the linear independence of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ now implies that $c_1 = c_2 = c_3 = 0$. Therefore, $c_1 = c_2 = c_3 = c_4 = 0$, as required.

34. Note that $\mathbf{v}$ and $\mathbf{w}$ are column vectors in $\mathbb{R}^m$. We have $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w}$. Since $\mathbf{w} \in \text{nullspace}(A^T)$, we have $A^T \mathbf{w} = \mathbf{0}$, and since $\mathbf{v} \in \text{colspace}(A)$, we can write $\mathbf{v} = A\mathbf{v}_1$ for some $\mathbf{v}_1 \in \mathbb{R}^m$. Therefore,

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T \mathbf{w} = (A\mathbf{v}_1)^T \mathbf{w} = (\mathbf{v}_1^T A^T) \mathbf{w} = \mathbf{v}_1^T (A^T \mathbf{w}) = \mathbf{v}_1^T (\mathbf{0}) = 0,$$

as desired.

35.

(a): Our proof here actually shows that the set of $n \times n$ skew-symmetric matrices forms a subspace of $M_n(\mathbb{R})$ for all positive integers n . We show that W is closed under addition and scalar multiplication:

Closure under Addition: Suppose that A and B are in W . This means that $A^T = -A$ and $B^T = -B$. Then

$$(A + B)^T = A^T + B^T = (-A) + (-B) = -(A + B),$$

so $A + B$ is skew-symmetric. Therefore, $A + B$ belongs to W , and W is closed under addition.

Closure under Scalar Multiplication: Suppose that A is in W and k is a scalar. We know that $A^T = -A$. Then

$$(kA)^T = k(A^T) = k(-A) = -(kA),$$

so kA is skew-symmetric. Therefore, kA belongs to W , and W is closed under scalar multiplication.

Therefore, W is a subspace.

(b): An arbitrary 3×3 skew-symmetric matrix takes the form

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} = a \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

This results in 0_3 if and only if $a = b = c = 0$, so the three matrices appearing on the right-hand side are linearly independent. Moreover, the equation above also demonstrates that W is spanned by the three matrices appearing on the right-hand side. Therefore, these matrices are a basis for W :

$$\text{Basis} = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}.$$

Hence, $\dim[W] = 3$.

(c): Since $\dim[M_3(\mathbb{R})] = 9$, we must add an additional six (linearly independent) 3×3 matrices to form a basis for $M_3(\mathbb{R})$. Using the notation prior to Example 4.6.3, we can use the matrices $E_{11}, E_{22}, E_{33}, E_{12}, E_{13}, E_{23}$ to extend the basis in part (b) to a basis for $M_3(\mathbb{R})$:

$$\text{Basis for } M_3(\mathbb{R}) = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, E_{11}, E_{22}, E_{33}, E_{12}, E_{13}, E_{23} \right\}$$

36.

(a): We show that W is closed under addition and closed under scalar multiplication. Arbitrary elements of W have the form

$$\begin{bmatrix} a & b & -a-b \\ c & d & -c-d \\ -a-c & -b-d & a+b+c+d \end{bmatrix}.$$

Closure under Addition: Let

$$X_1 = \begin{bmatrix} a_1 & b_1 & -a_1-b_1 \\ c_1 & d_1 & -c_1-d_1 \\ -a_1-c_1 & -b_1-d_1 & a_1+b_1+c_1+d_1 \end{bmatrix} \quad \text{and} \quad X_2 = \begin{bmatrix} a_2 & b_2 & -a_2-b_2 \\ c_2 & d_2 & -c_2-d_2 \\ -a_2-c_2 & -b_2-d_2 & a_2+b_2+c_2+d_2 \end{bmatrix}$$

be elements of W . Then

$$X_1 + X_2 = \begin{bmatrix} a_1+a_2 & b_1+b_2 & -a_1-b_1-a_2-b_2 \\ c_1+c_2 & d_1+d_2 & -c_1-d_1-c_2-d_2 \\ -a_1-c_1-a_2-c_2 & -b_1-d_1-b_2-d_2 & a_1+b_1+c_1+d_1+a_2+b_2+c_2+d_2 \end{bmatrix},$$

which is immediately seen to have row sums and column sums of zero. Thus, $X_1 + X_2$ belongs to W , and W is closed under addition.

Closure under Scalar Multiplication: Let

$$X_1 = \begin{bmatrix} a_1 & b_1 & -a_1-b_1 \\ c_1 & d_1 & -c_1-d_1 \\ -a_1-c_1 & -b_1-d_1 & a_1+b_1+c_1+d_1 \end{bmatrix},$$

and let k be a scalar. Then

$$kX_1 = \begin{bmatrix} ka_1 & kb_1 & k(-a_1 - b_1) \\ kc_1 & kd_1 & k(-c_1 - d_1) \\ k(-a_1 - c_1) & k(-b_1 - d_1) & k(a_1 + b_1 + c_1 + d_1) \end{bmatrix},$$

and we see by inspection that all row and column sums of kX_1 are zero, so kX_1 belongs to W . Therefore, W is closed under scalar multiplication.

Therefore, W is a subspace of $M_3(\mathbb{R})$.

(b): We may write

$$\begin{bmatrix} a & b & -a-b \\ c & d & -c-d \\ -a-c & -b-d & a+b+c+d \end{bmatrix} = a \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

This results in 0_3 if and only if $a = b = c = d = 0$, so the matrices appearing on the right-hand side are linearly independent. Moreover, the equation above also demonstrates that W is spanned by the four matrices appearing on the right-hand side. Therefore, these matrices are a basis for W :

$$\text{Basis} = \left\{ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \right\}.$$

Hence, $\dim[W] = 4$.

(c): Since $\dim[M_3(\mathbb{R})] = 9$, we must add an additional five (linearly independent) 3×3 matrices to form a basis for $M_3(\mathbb{R})$. Using the notation prior to Example 4.6.3, we can use the matrices E_{11} , E_{12} , E_{13} , E_{21} , and E_{31} to extend the basis from part (b) to a basis for $M_3(\mathbb{R})$:

Basis for $M_3(\mathbb{R}) =$

$$\left\{ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, E_{11}, E_{12}, E_{13}, E_{21}, E_{31} \right\}.$$

37.

(a): We must verify the axioms (A1)-(A10) for a vector space:

Axiom (A1): Assume that (v_1, w_1) and (v_2, w_2) belong to $V \oplus W$. Since $v_1 +_V v_2 \in V$ and $w_1 +_W w_2 \in W$, then the sum

$$(v_1, w_1) + (v_2, w_2) = (v_1 +_V v_2, w_1 +_W w_2)$$

lies in $V \oplus W$.

Axiom (A2): Assume that (v, w) belongs to $V \oplus W$, and let k be a scalar. Since $k \cdot_V v \in V$ and $k \cdot_W w \in W$, the scalar multiplication

$$k \cdot (v, w) = (k \cdot_V v, k \cdot_W w)$$

lies in $V \oplus W$.

For the remainder of the axioms, we will omit the $\cdot_V$ and $\cdot_W$ notations. They are to be understood.

Axiom (A3): Assume that $(v_1, w_1), (v_2, w_2) \in V \oplus W$. Then

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (v_2 + v_1, w_2 + w_1) = (v_2, w_2) + (v_1, w_1),$$

as required.

Axiom (A4): Assume that $(v_1, w_1), (v_2, w_2), (v_3, w_3) \in V \oplus W$. Then

$$\begin{aligned} ((v_1, w_1) + (v_2, w_2)) + (v_3, w_3) &= (v_1 + v_2, w_1 + w_2) + (v_3, w_3) \\ &= ((v_1 + v_2) + v_3, (w_1 + w_2) + w_3) \\ &= (v_1 + (v_2 + v_3), w_1 + (w_2 + w_3)) \\ &= (v_1, w_1) + (v_2 + v_3, w_2 + w_3) \\ &= (v_1, w_1) + ((v_2, w_2) + (v_3, w_3)), \end{aligned}$$

as required.

Axiom (A5): We claim that the zero vector in $V \oplus W$ is $(0_V, 0_W)$, where 0_V is the zero vector in the vector space V and 0_W is the zero vector in the vector space W . To check this, let $(v, w) \in V \oplus W$. Then

$$(0_V, 0_W) + (v, w) = (0_V + v, 0_W + w) = (v, w),$$

which confirms that $(0_V, 0_W)$ is the zero vector for $V \oplus W$.

Axiom (A6): We claim that the additive inverse of the vector $(v, w) \in V \oplus W$ is the vector $(-v, -w)$, where $-v$ is the additive inverse of v in the vector space V and $-w$ is the additive inverse of w in the vector space W . We check this:

$$(v, w) + (-v, -w) = (v + (-v), w + (-w)) = (0_V, 0_W),$$

as required.

Axiom (A7): For every vector $(v, w) \in V \oplus W$, we have

$$1 \cdot (v, w) = (1 \cdot v, 1 \cdot w) = (v, w),$$

where in the last step we have used the fact that Axiom (A7) holds in each of the vector spaces V and W .

Axiom (A8): Let (v, w) be a vector in $V \oplus W$, and let r and s be scalars. Using the fact that Axiom (A8) holds in V and W , we have

$$\begin{aligned} (rs)(v, w) &= ((rs)v, (rs)w) \\ &= (r(sv), r(sw)) \\ &= r(sv, sw) \\ &= r(s(v, w)). \end{aligned}$$

Axiom (A9): Let (v_1, w_1) and (v_2, w_2) be vectors in $V \oplus W$, and let r be a scalar. Then

$$\begin{aligned} r((v_1, w_1) + (v_2, w_2)) &= r(v_1 + v_2, w_1 + w_2) \\ &= (r(v_1 + v_2), r(w_1 + w_2)) \\ &= (rv_1 + rv_2, rw_1 + rw_2) \\ &= (rv_1, rw_1) + (rv_2, rw_2) \\ &= r(v_1, w_1) + r(v_2, w_2), \end{aligned}$$

as required.

Axiom (A10): Let $(v, w) \in V \oplus W$, and let r and s be scalars. Then

$$\begin{aligned}(r + s)(v, w) &= ((r + s)v, (r + s)w) \\ &= (rv + sv, rw + sw) \\ &= (rv, rw) + (sv, sw) \\ &= r(v, w) + s(v, w),\end{aligned}$$

as required.

(b): We show that $\{(v, 0) : v \in V\}$ is a subspace of $V \oplus W$, by checking closure under addition and closure under scalar multiplication:

Closure under Addition: Suppose $(v_1, 0)$ and $(v_2, 0)$ belong to $\{(v, 0) : v \in V\}$, where $v_1, v_2 \in V$. Then

$$(v_1, 0) + (v_2, 0) = (v_1 + v_2, 0) \in \{(v, 0) : v \in V\},$$

which shows that the set is closed under addition.

Closure under Scalar Multiplication: Suppose $(v, 0) \in \{(v, 0) : v \in V\}$ and k is a scalar. Then $k(v, 0) = (kv, 0)$ is again in the set. Thus, $\{(v, 0) : v \in V\}$ is closed under scalar multiplication.

Therefore, $\{(v, 0) : v \in V\}$ is a subspace of $V \oplus W$.

(c): Let $\{v_1, v_2, \dots, v_n\}$ be a basis for V , and let $\{w_1, w_2, \dots, w_m\}$ be a basis for W . We claim that

$$S = \{(v_i, 0) : 1 \leq i \leq n\} \cup \{(0, w_j) : 1 \leq j \leq m\}$$

is a basis for $V \oplus W$. To show this, we will verify that S is a linearly independent set that spans $V \oplus W$:

Check that S is linearly independent: Assume that

$$c_1(v_1, 0) + c_2(v_2, 0) + \dots + c_n(v_n, 0) + d_1(0, w_1) + d_2(0, w_2) + \dots + d_m(0, w_m) = (0, 0).$$

We must show that

$$c_1 = c_2 = \dots = c_n = d_1 = d_2 = \dots = d_m = 0.$$

Adding the vectors on the left-hand side, we have

$$(c_1v_1 + c_2v_2 + \dots + c_nv_n, d_1w_1 + d_2w_2 + \dots + d_mw_m) = (0, 0),$$

so that

$$c_1v_1 + c_2v_2 + \dots + c_nv_n = 0 \quad \text{and} \quad d_1w_1 + d_2w_2 + \dots + d_mw_m = 0.$$

Since $\{v_1, v_2, \dots, v_n\}$ is linearly independent,

$$c_1 = c_2 = \dots = c_n = 0,$$

and since $\{w_1, w_2, \dots, w_m\}$ is linearly independent,

$$d_1 = d_2 = \dots = d_m = 0.$$

Thus, S is linearly independent.

Check that S spans $V \oplus W$: Let $(v, w) \in V \oplus W$. We must express (v, w) as a linear combination of the vectors in S . Since $\{v_1, v_2, \dots, v_n\}$ spans V , there exist scalars $c_1, c_2, \dots, c_n$ such that

$$v = c_1v_1 + c_2v_2 + \dots + c_nv_n,$$

and since $\{w_1, w_2, \dots, w_m\}$ spans W , there exist scalars $d_1, d_2, \dots, d_m$ such that

$$w = d_1 w_1 + d_2 w_2 + \dots + d_m w_m.$$

Then

$$(v, w) = c_1(v_1, 0) + c_2(v_2, 0) + \dots + c_n(v_n, 0) + d_1(0, w_1) + d_2(0, w_2) + \dots + d_m(0, w_m).$$

Therefore, (v, w) is a linear combination of vectors in S , so S spans $V \oplus W$.

Therefore, S is a basis for $V \oplus W$. Since S contains $n + m$ vectors, $\dim[V \oplus W] = n + m$.

38. There are many examples here. One such example is $S = \{x^3, x^3 - x^2, x^3 - x, x^3 - 1\}$, a basis for P_3 whose vectors all have degree 3. To see that S is a basis, note that S is linearly independent, since if

$$c_1 x^3 + c_2(x^3 - x^2) + c_3(x^3 - x) + c_4(x^3 - 1) = 0,$$

then

$$(c_1 + c_2 + c_3 + c_4)x^3 - c_2 x^2 - c_3 x - c_4 = 0,$$

and so $c_1 = c_2 = c_3 = c_4 = 0$. Since S is a linearly independent set of 4 vectors and $\dim[P_3] = 4$, S is a basis for P_3 .

39. Let A be an $m \times n$ matrix. By the Rank-Nullity Theorem,

$$\dim[\text{colspace}(A)] + \dim[\text{nullspace}(A)] = n.$$

Since, by assumption, $\text{colspace}(A) = \text{nullspace}(A) = r$, $n = 2r$ must be even.

40. The i th row of the matrix A is $b_i(c_1 \ c_2 \ \dots \ c_n)$. Therefore, each row of A is a multiple of the first row, and so $\text{rank}(A) = 1$. Thus, by the Rank-Nullity Theorem, $\text{nullity}(A) = n - 1$.

41. A row-echelon form of A is given by $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. Thus, a basis for the row space of A is given by $\{(1, 2)\}$, a basis for the column space of A is given by $\left\{ \begin{bmatrix} -3 \\ -6 \end{bmatrix} \right\}$, and a basis for the nullspace of A is given by $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$. All three subspaces are one-dimensional.

42. A row-echelon form of A is given by $\begin{bmatrix} 1 & -6 & -2 & 0 \\ 0 & 1 & 1/3 & 5/21 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus, a basis for the row space of A is $\{(1, -6, -2, 0), (0, 1, \frac{1}{3}, \frac{5}{21})\}$, and a basis for the column space of A is given by $\left\{ \begin{bmatrix} -1 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ 21 \end{bmatrix} \right\}$. For the nullspace, we observe that the equations corresponding to the row-echelon form of A can be written as

$$x - 6y - 2z = 0 \quad \text{and} \quad y + \frac{1}{3}z + \frac{5}{21}w = 0.$$

Set $w = t$ and $z = s$. Then $y = -\frac{1}{3}s - \frac{5}{21}t$ and $x = -\frac{10}{7}t$. Thus,

$$\text{nullspace}(A) = \left\{ \left(-\frac{10}{7}t, -\frac{1}{3}s - \frac{5}{21}t, s, t \right) : s, t \in \mathbb{R} \right\} = \left\{ t \left(-\frac{10}{7}, -\frac{5}{21}, 0, 1 \right) + s \left(0, -\frac{1}{3}, 1, 0 \right) \right\}.$$

Hence, a basis for the nullspace of A is given by $\{(-\frac{10}{7}, -\frac{5}{21}, 0, 1), (0, -\frac{1}{3}, 1, 0)\}$.

43. A row-echelon form of A is given by $\begin{bmatrix} 1 & -2.5 & -5 \\ 0 & 1 & 1.3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, a basis for the row space of A is given by $\{(1, -2.5, -5), (0, 1, 1.3), (0, 0, 1)\}$, and a basis for the column space of A is given by

$$\left\{ \begin{bmatrix} -4 \\ 0 \\ 6 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 10 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 13 \\ 2 \\ 10 \end{bmatrix} \right\}.$$

Moreover, we have that the nullspace of A is 0-dimensional, and so the basis is empty.

44. A row-echelon form of A is given by $\begin{bmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & -1 & -4 & -3 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. Thus, a basis for the row space of A is given by

$$\{(1, 0, 2, 2, 1), (0, 1, -1, -4, -3), (0, 0, 1, 4, 3), (0, 0, 0, 1, 0)\},$$

a basis for the column space of A is given by

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix} \right\}.$$

For the nullspace, if the variables corresponding the columns are (x, y, z, u, v) , then the row-echelon form tells us that $v = t$ is a free variable, $u = 0$, $z = -3t$, $y = 0$, and $x = 5t$. Thus,

$$\text{nullspace}(A) = \{(5t, 0, -3t, 0, t) : t \in \mathbb{R}\} = \{t(5, 0, -3, 0, 1) : t \in \mathbb{R}\},$$

and so a basis for the nullspace of A is $\{(5, 0, -3, 0, 1)\}$.

45. We will obtain bases for the row space, column space, and nullspace and orthonormalize them. A row-echelon form of A is given by $\begin{bmatrix} 1 & 2 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We see that a basis for the row space of A is given by

$$\{(1, 2, 6), (0, 1, 2)\}.$$

We apply Gram-Schmidt to this set, and thus we need to replace $(0, 1, 2)$ by

$$(0, 1, 2) - \frac{14}{41}(1, 2, 6) = \left(-\frac{14}{41}, \frac{13}{41}, -\frac{2}{41}\right).$$

So an orthogonal basis for the row space of A is given by

$$\left\{ (1, 2, 6), \left(-\frac{14}{41}, \frac{13}{41}, -\frac{2}{41}\right) \right\}.$$

To replace this with an orthonormal basis, we must normalize each vector. The first one has norm $\sqrt{41}$ and the second one has norm $\frac{3}{\sqrt{41}}$. Hence, an orthonormal basis for the row space of A is

$$\left\{ \frac{\sqrt{41}}{41}(1, 2, 6), \frac{\sqrt{41}}{3} \left(-\frac{14}{41}, \frac{13}{41}, -\frac{2}{41} \right) \right\}.$$

Returning to the row-echelon form of A obtained above, we see that a basis for the column space of A is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

We apply Gram-Schmidt to this set, and thus we need to replace $\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ by

$$\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{4}{6} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 4 \\ -1 \\ 3 \\ -2 \end{bmatrix}.$$

So an orthogonal basis for the column space of A is given by

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 4 \\ -1 \\ 3 \\ -2 \end{bmatrix} \right\}.$$

The norms of these vectors are, respectively, $\sqrt{6}$ and $\sqrt{30}/3$. Hence, we normalize the above orthogonal basis to obtain the orthonormal basis for the column space:

$$\left\{ \frac{\sqrt{6}}{6} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \frac{\sqrt{30}}{30} \begin{bmatrix} 4 \\ -1 \\ 3 \\ -2 \end{bmatrix} \right\}.$$

Returning once more to the row-echelon form of A obtained above, we see that, in order to find the nullspace of A , we must solve the equations $x + 2y + 6z = 0$ and $y + 2z = 0$. Setting $z = t$ as a free variable, we find that $y = -2t$ and $x = -2t$. Thus, a basis for the nullspace of A is $\{(-2, -2, 1)\}$, which can be normalized to

$$\left\{ \left(-\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right) \right\}.$$

46. A row-echelon form for A is $\begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Note that since $\text{rank}(A) = 3$, $\text{nullity}(A) = 0$, and so

there is no basis for $\text{nullspace}(A)$. Moreover, $\text{row space}(A)$ is a 3-dimensional subspace of $\mathbb{R}^3$, and therefore, $\text{row space}(A) = \mathbb{R}^3$. An orthonormal basis for this is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

Finally, consider the columnspace of A . We must apply the Gram-Schmidt process to the three columns of A . Thus, we replace the second column vector by

$$\mathbf{v}_2 = \begin{bmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{bmatrix} - 4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}.$$

Next, we replace the third column vector by

$$\mathbf{v}_3 = \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix} - \frac{7}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix}.$$

Hence, an orthogonal basis for the columnspace of A is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix} \right\}.$$

Normalizing each vector yields the orthonormal basis

$$\left\{ \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{8}} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{6} \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix} \right\}.$$

47. Let $\mathbf{x}_1 = (5, -1, 2)$ and let $\mathbf{x}_2 = (7, 1, 1)$. Using the Gram-Schmidt process, we have

$$\mathbf{v}_1 = \mathbf{x}_1 = (5, -1, 2)$$

and

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = (7, 1, 1) - \frac{36}{30} (5, -1, 2) = (7, 1, 1) - (6, -\frac{6}{5}, \frac{12}{5}) = (1, \frac{11}{5}, -\frac{7}{5}).$$

Hence, an orthogonal basis is given by

$$\{(5, -1, 2), (1, \frac{11}{5}, -\frac{7}{5})\}.$$

48. We already saw in Problem 26 that S spans $\mathbb{R}^3$, so therefore an obvious orthogonal basis for $\text{span}(S)$ is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Alternatively, for practice with Gram-Schmidt, we would proceed as follows: Let $\mathbf{x}_1 = (6, -3, 2)$, $\mathbf{x}_2 = (1, 1, 1)$, and $\mathbf{x}_3 = (1, -8, -1)$. Using the Gram-Schmidt process, we have

$$\mathbf{v}_1 = \mathbf{x}_1 = (6, -3, 2),$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = (1, 1, 1) - \frac{5}{49}(6, -3, 2) = \left(\frac{19}{49}, \frac{64}{49}, \frac{39}{49}\right),$$

and

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = (1, -8, -1) - \frac{4}{7}(6, -3, 2) + \frac{38}{427}(19, 64, 39) = \left(-\frac{45}{61}, -\frac{36}{61}, \frac{81}{61}\right).$$

Hence, an orthogonal basis is given by

$$\left\{ (6, -3, 2), \left(\frac{19}{49}, \frac{64}{49}, \frac{39}{49}\right), \left(-\frac{45}{61}, -\frac{36}{61}, \frac{81}{61}\right) \right\}.$$

49. We already saw in Problem 29 that S spans P_3 , so therefore we can apply Gram-Schmidt to the basis $\{1, x, x^2, x^3\}$ for P_3 , instead of the given set of polynomials. Let $\mathbf{x}_1 = 1$, $\mathbf{x}_2 = x$, $\mathbf{x}_3 = x^2$, and $\mathbf{x}_4 = x^3$. Using the Gram-Schmidt process, we have

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{x}_1 = 1, \\ \mathbf{v}_2 &= \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 = x - \frac{1}{2}, \\ \mathbf{v}_3 &= \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 = x^2 - (x - \frac{1}{2}) - \frac{1}{3} = x^2 - x + \frac{1}{6}, \end{aligned}$$

and

$$\mathbf{v}_4 = \mathbf{x}_4 - \frac{\langle \mathbf{x}_4, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 - \frac{\langle \mathbf{x}_4, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 - \frac{\langle \mathbf{x}_4, \mathbf{v}_3 \rangle}{\|\mathbf{v}_3\|^2} \mathbf{v}_3 = x^3 - \frac{1}{4} - \frac{9}{10}(x - \frac{1}{2}) - \frac{3}{2}(x^2 - x + \frac{1}{6}) = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}.$$

Hence, an orthogonal basis is given by

$$\left\{ 1, x - \frac{1}{2}, x^2 - x + \frac{1}{6}, x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20} \right\}.$$

50. It is easy to see that the span of the set of vectors in Problem 32 is the set of all 2×2 symmetric matrices. Therefore, we can simply give the orthogonal basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

for the set of all 2×2 symmetric matrices.

51. We have

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle (2, 3), (4, -1) \rangle = 2 \cdot 4 + 3 \cdot (-1) = 5, \|\mathbf{u}\| = \sqrt{13}, \|\mathbf{v}\| = \sqrt{17},$$

and so

$$\theta = \cos^{-1} \left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} \right) = \cos^{-1} \left(\frac{5}{\sqrt{13}\sqrt{17}} \right) \approx 1.23 \text{ radians.}$$

52. We have

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle (-2, -1, 2, 4), (-3, 5, 1, 1) \rangle = (-2) \cdot (-3) + (-1) \cdot 5 + 2 \cdot 1 + 4 \cdot 1 = 7, \|\mathbf{u}\| = 5, \|\mathbf{v}\| = 6,$$

and so

$$\theta = \cos^{-1} \left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} \right) = \cos^{-1} \left(\frac{7}{5 \cdot 6} \right) = \cos^{-1}(7/30) \approx 1.34 \text{ radians.}$$

53. For Problem 51, we have

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle (2, 3), (4, -1) \rangle = 2 \cdot 2 \cdot 4 + 3 \cdot (-1) = 13, \|\mathbf{u}\| = \sqrt{17}, \|\mathbf{v}\| = \sqrt{33},$$

and so

$$\theta = \cos^{-1} \left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} \right) = \cos^{-1} \left(\frac{13}{\sqrt{17} \sqrt{33}} \right) \approx 0.99 \text{ radians.}$$

For Problem 52, we have

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle (-2, -1, 2, 4), (-3, 5, 1, 1) \rangle = 2 \cdot (-2) \cdot (-3) + (-1) \cdot 5 + 2 \cdot 1 + 4 \cdot 1 = 13, \|\mathbf{u}\| = \sqrt{29}, \|\mathbf{v}\| = \sqrt{45},$$

and so

$$\theta = \cos^{-1} \left(\frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \|\mathbf{v}\|} \right) = \cos^{-1} \left(\frac{13}{\sqrt{29} \cdot \sqrt{45}} \right) \approx 1.20 \text{ radians.}$$

54.

(a): We must verify the four axioms for an inner product given in Definition 4.11.3.

Axiom 1: We have

$$p \cdot p = p(t_0)p(t_0) + p(t_1)p(t_1) + \cdots + p(t_n)p(t_n) = p(t_0)^2 + p(t_1)^2 + \cdots + p(t_n)^2 \geq 0.$$

Moreover,

$$p(t_0)^2 + p(t_1)^2 + \cdots + p(t_n)^2 = 0 \iff p(t_0) = p(t_1) = \cdots = p(t_n) = 0.$$

But the only polynomial of degree $\leq n$ which has more than n roots is the zero polynomial. Thus,

$$p \cdot p = 0 \iff p = 0.$$

Axiom 2: We have

$$p \cdot q = p(t_0)q(t_0) + p(t_1)q(t_1) + \cdots + p(t_n)q(t_n) = q(t_0)p(t_0) + q(t_1)p(t_1) + \cdots + q(t_n)p(t_n) = q \cdot p$$

for all $p, q \in P_n$.

Axiom 3: Let k be a scalar, and let $p, q \in P_n$. Then

$$\begin{aligned} (kp) \cdot q &= (kp)(t_0)q(t_0) + (kp)(t_1)q(t_1) + \cdots + (kp)(t_n)q(t_n) \\ &= kp(t_0)q(t_0) + kp(t_1)q(t_1) + \cdots + kp(t_n)q(t_n) \\ &= k[p(t_0)q(t_0) + p(t_1)q(t_1) + \cdots + p(t_n)q(t_n)] \\ &= k[p \cdot q], \end{aligned}$$

as required.

Axiom 4: Let $p_1, p_2, q \in P_n$. Then we have

$$\begin{aligned} (p_1 + p_2) \cdot q &= (p_1 + p_2)(t_0)q(t_0) + (p_1 + p_2)(t_1)q(t_1) + \cdots + (p_1 + p_2)(t_n)q(t_n) \\ &= [p_1(t_0) + p_2(t_0)]q(t_0) + [p_1(t_1) + p_2(t_1)]q(t_1) + \cdots + [p_1(t_n) + p_2(t_n)]q(t_n) \\ &= [p_1(t_0)q(t_0) + p_1(t_1)q(t_1) + \cdots + p_1(t_n)q(t_n)] + [p_2(t_0)q(t_0) + p_2(t_1)q(t_1) + \cdots + p_2(t_n)q(t_n)] \\ &= (p_1 \cdot q) + (p_2 \cdot q), \end{aligned}$$

as required.

(b): The projection of p_2 onto $\text{span}\{p_0, p_1\}$ is given by

$$\frac{\langle p_2, p_0 \rangle}{\|p_0\|^2} p_0 + \frac{\langle p_2, p_1 \rangle}{\|p_1\|^2} p_1 = \frac{20}{4} p_0 + \frac{0}{11} p_1 = 5.$$

(c): We take $q = t^2 - 5$.

55. Let $\mathbf{x} = (2, 3, 4)$ and let $\mathbf{v} = (6, -1, -4)$. We must find the length of

$$\mathbf{x} - \mathbf{P}(\mathbf{x}, \mathbf{v}) = (2, 3, 4) - \frac{\langle (2, 3, 4), (6, -1, -4) \rangle}{\|(6, -1, -4)\|^2} (6, -1, -4) = (2, 3, 4) + \frac{7}{53} (6, -1, -4) = \left(\frac{148}{53}, \frac{152}{53}, \frac{184}{53} \right),$$

which is

$$\|\mathbf{x} - \mathbf{P}(\mathbf{x}, \mathbf{v})\| = \left\| \left(\frac{148}{53}, \frac{152}{53}, \frac{184}{53} \right) \right\| \approx 5.30.$$

56. Note that

$$\langle \mathbf{x} - \mathbf{y}, \mathbf{v}_i \rangle = \langle \mathbf{x}, \mathbf{v}_i \rangle - \langle \mathbf{y}, \mathbf{v}_i \rangle = 0,$$

by assumption. Let $\mathbf{v}$ be an arbitrary vector in V , and write

$$\mathbf{v} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_n \mathbf{v}_n,$$

for some scalars $a_1, a_2, \dots, a_n$. Observe that

$$\begin{aligned} \langle \mathbf{x} - \mathbf{y}, \mathbf{v} \rangle &= \langle \mathbf{x} - \mathbf{y}, a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_n \mathbf{v}_n \rangle = a_1 \langle \mathbf{x} - \mathbf{y}, \mathbf{v}_1 \rangle + a_2 \langle \mathbf{x} - \mathbf{y}, \mathbf{v}_2 \rangle + \cdots + a_n \langle \mathbf{x} - \mathbf{y}, \mathbf{v}_n \rangle \\ &= a_1 \cdot 0 + a_2 \cdot 0 + \cdots + a_n \cdot 0 = 0. \end{aligned}$$

Thus, $\mathbf{x} - \mathbf{y}$ is orthogonal to every vector in V . In particular

$$\langle \mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle = 0,$$

and hence, $\mathbf{x} - \mathbf{y} = \mathbf{0}$. Therefore $\mathbf{x} = \mathbf{y}$.

57. Any of the conditions (a)-(p) appearing in the Invertible Matrix Theorem would be appropriate at this point in the text.

Solutions to Section 5.1

True-False Review:

1. FALSE. The conditions $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ and $T(c \cdot \mathbf{v}) = c \cdot T(\mathbf{v})$ must hold *for all* vectors $\mathbf{u}, \mathbf{v}$ in V and *for all* scalars c , not just “for some”.

2. FALSE. The dimensions of the matrix A should be $m \times n$, not $n \times m$, as stated in the question.

3. FALSE. This will only necessarily hold for a *linear transformation*, not for more general mappings.

4. TRUE. This is precisely the definition of the matrix associated with a linear transformation, as given in the text.

5. TRUE. Since

$$\mathbf{0} = T(\mathbf{0}) = T(\mathbf{v} + (-\mathbf{v})) = T(\mathbf{v}) + T(-\mathbf{v}),$$

we conclude that $T(-\mathbf{v}) = -T(\mathbf{v})$.

6. TRUE. Using the properties of a linear transformation, we have

$$T((c+d)\mathbf{v}) = T(c\mathbf{v} + d\mathbf{v}) = T(c\mathbf{v}) + T(d\mathbf{v}) = cT(\mathbf{v}) + dT(\mathbf{v}),$$

as stated.

Problems:

1. Let $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ and $c \in \mathbb{R}$.

$$\begin{aligned} T((x_1, x_2) + (y_1, y_2)) &= T(x_1 + y_1, x_2 + y_2) \\ &= (x_1 + y_1 + 2x_2 + 2y_2, 2x_1 + 2y_1 - x_2 - y_2) \\ &= (x_1 + 2x_2, 2x_1 - x_2) + (y_1 + 2y_2, 2y_1 - y_2) \\ &= T(x_1, x_2) + T(y_1, y_2). \end{aligned}$$

$T(c(x_1, x_2)) = T(cx_1, cx_2) = (cx_1 + 2cx_2, 2cx_1 - cx_2) = c(x_1 + 2x_2, 2x_1 - x_2) = cT(x_1, x_2)$.
Thus, T is a linear transformation.

2. Let $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$ and $c \in \mathbb{R}$.

$$\begin{aligned} T((x_1, x_2, x_3) + (y_1, y_2, y_3)) &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= (x_1 + y_1 + 3x_2 + 3y_2 + x_3 + y_3, x_1 + y_1 - x_2 - y_2) \\ &= (x_1 + 3x_2 + x_3, x_1 - x_2) + (y_1 + 3y_2 + y_3, y_1 - y_2) \\ &= T(x_1, x_2, x_3) + T(y_1, y_2, y_3). \end{aligned}$$

$T(c(x_1, x_2, x_3)) = T(cx_1, cx_2, cx_3) = (cx_1 + 3cx_2 + cx_3, cx_1 - cx_2) = c(x_1 + 3x_2 + x_3, x_1 - x_2) = cT(x_1, x_2, x_3)$.
Thus, T is a linear transformation.

3. Let $y_1, y_2 \in C^2(I)$ and $c \in \mathbb{R}$. Then,

$$T(y_1 + y_2) = (y_1 + y_2)'' - 16(y_1 + y_2) = (y_1'' - 16y_1) + (y_2'' - 16y_2) = T(y_1) + T(y_2).$$

$$T(cy_1) = (cy_1)'' - 16(cy_1) = c(y_1'' - 16y_1) = cT(y_1).$$

Consequently, T is a linear transformation.

4. Let $y_1, y_2 \in C^2(I)$ and $c \in \mathbb{R}$. Then,

$$\begin{aligned} T(y_1 + y_2) &= (y_1 + y_2)'' + a_1(y_1 + y_2)' + a_2(y_1 + y_2) \\ &= (y_1'' + a_1y_1' + a_2y_1) + (y_2'' + a_1y_2' + a_2y_2) = T(y_1) + T(y_2). \end{aligned}$$

$$T(cy_1) = (cy_1)'' + a_1(cy_1)' + a_2(cy_1) = c(y_1'' + a_1y_1' + a_2y_1) = cT(y_1).$$

Consequently, T is a linear transformation.

5. Let $f, g \in V$ and $c \in \mathbb{R}$. Then $T(f+g) = \int_a^b (f+g)(x)dx = \int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx = T(f) + T(g)$.

$T(cf) = \int_a^b [cf(x)]dx = c \int_a^b f(x)dx = cT(f)$. Therefore, T is a linear transformation.

6. Let $A_1, A_2, B \in M_n(\mathbb{R})$ and $c \in \mathbb{R}$.

$$\begin{aligned} T(A_1 + A_2) &= (A_1 + A_2)B - B(A_1 + A_2) = A_1B + A_2B - BA_1 - BA_2 \\ &= (A_1B - BA_1) + (A_2B - BA_2) = T(A_1) + T(A_2). \end{aligned}$$

$$T(cA_1) = (cA_1B) - B(cA_1) = c(A_1B - BA_1) = cT(A_1).$$

Consequently, T is a linear transformation.

7. Let $A, B \in M_n(\mathbb{R})$ and $c \in \mathbb{R}$. Then,

$$S(A + B) = (A + B) + (A + B)^T = A + A^T + B + B^T = S(A) + S(B).$$

$$S(cA) = (cA) + (cA)^T = c(A + A^T) = cS(A).$$

Consequently, S is a linear transformation.

8. Let $A, B \in M_n(\mathbb{R})$ and $c \in \mathbb{R}$. Then,

$$T(A + B) = \text{tr}(A + B) = \sum_{k=1}^n (a_{kk} + b_{kk}) = \sum_{k=1}^n a_{kk} + \sum_{k=1}^n b_{kk} = \text{tr}(A) + \text{tr}(B) = T(A) + T(B).$$

$$T(cA) = \text{tr}(cA) = \sum_{k=1}^n ca_{kk} = c \sum_{k=1}^n a_{kk} = c\text{tr}(A) = cT(A).$$

Consequently, T is a linear transformation.

9. Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ be in $\mathbb{R}^2$. Then

$$T(\mathbf{x} + \mathbf{y}) = T(x_1 + y_1, x_2 + y_2) = (x_1 + x_2 + y_1 + y_2, 2),$$

whereas

$$T(\mathbf{x}) + T(\mathbf{y}) = (x_1 + x_2, 2) + (y_1 + y_2, 2) = (x_1 + x_2 + y_1 + y_2, 4).$$

We see that $T(\mathbf{x} + \mathbf{y}) \neq T(\mathbf{x}) + T(\mathbf{y})$, hence T is not a linear transformation.

10. Let $A \in M_2(\mathbb{R})$ and $c \in \mathbb{R}$. Then

$$T(cA) = \det(cA) = c^2 \det(A) = c^2 T(A).$$

Since $T(cA) \neq cT(A)$ in general, it follows that T is not a linear transformation.

11. If $T(x_1, x_2) = (3x_1 - 2x_2, x_1 + 5x_2)$, then $A = [T(\mathbf{e}_1), T(\mathbf{e}_2)] = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$.

12. If $T(x_1, x_2) = (x_1 + 3x_2, 2x_1 - 7x_2)$, then $A = [T(\mathbf{e}_1), T(\mathbf{e}_2)] = \begin{bmatrix} 1 & 3 \\ 2 & -7 \\ 1 & 0 \end{bmatrix}$.

13. If $T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, x_3 - x_1)$, then $A = [T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)] = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$.

14. If $T(x_1, x_2, x_3) = x_1 + 5x_2 - 3x_3$, then $A = [T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)] = \begin{bmatrix} 1 & 5 & -3 \end{bmatrix}$.

15. If $T(x_1, x_2, x_3) = (x_3 - x_1, -x_1, 3x_1 + 2x_3, 0)$, then $A = [T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)] = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 0 \\ 3 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

16. $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 3 \\ -4 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ -4x_1 + 7x_2 \end{bmatrix}$, which we write as

$$T(x_1, x_2) = (x_1 + 3x_2, -4x_1 + 7x_2).$$

$$17. T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & -1 & 5 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 + 5x_3 \\ 3x_1 + x_2 - 2x_3 \end{bmatrix}, \text{ which we write as}$$

$$T(x_1, x_2, x_3) = (2x_1 - x_2 + 5x_3, 3x_1 + x_2 - 2x_3).$$

$$18. T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 2 & -3 \\ 4 & -1 & 2 \\ 5 & 7 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 2x_2 - 3x_3 \\ 4x_1 - x_2 + 2x_3 \\ 5x_1 + 7x_2 - 8x_3 \end{bmatrix}, \text{ which we write as}$$

$$T(x_1, x_2, x_3) = (2x_1 + 2x_2 - 3x_3, 4x_1 - x_2 + 2x_3, 5x_1 + 7x_2 - 8x_3).$$

$$19. T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} [x] = \begin{bmatrix} -3x \\ -2x \\ 0 \\ x \end{bmatrix}, \text{ which we write as}$$

$$T(x) = (-3x, -2x, 0, x).$$

$$20. T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -4 & -6 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = [x_1 - 4x_2 - 6x_3 + 2x_5], \text{ which we write as}$$

$$T(x_1, x_2, x_3, x_4, x_5) = x_1 - 4x_2 - 6x_3 + 2x_5.$$

21. Let $\mathbf{u}$ be a fixed vector in V , $\mathbf{v}_1, \mathbf{v}_2 \in V$, and $c \in \mathbb{R}$. Then

$$T(\mathbf{v}_1 + \mathbf{v}_2) = \langle \mathbf{u}, \mathbf{v}_1 + \mathbf{v}_2 \rangle = \langle \mathbf{u}, \mathbf{v}_1 \rangle + \langle \mathbf{u}, \mathbf{v}_2 \rangle = T(\mathbf{v}_1) + T(\mathbf{v}_2).$$

$$T(c\mathbf{v}_1) = \langle \mathbf{u}, c\mathbf{v}_1 \rangle = c\langle \mathbf{u}, \mathbf{v}_1 \rangle = cT(\mathbf{v}_1). \text{ Thus, } T \text{ is a linear transformation.}$$

22. We must show that the linear transformation T respects addition and scalar multiplication:

T respects addition: Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be vectors in V . Then we have

$$\begin{aligned} T(\mathbf{v}_1 + \mathbf{v}_2) &= (\langle \mathbf{u}_1, \mathbf{v}_1 + \mathbf{v}_2 \rangle, \langle \mathbf{u}_2, \mathbf{v}_1 + \mathbf{v}_2 \rangle) \\ &= (\langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{u}_1 \rangle, \langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{u}_2 \rangle) \\ &= (\langle \mathbf{v}_1, \mathbf{u}_1 \rangle + \langle \mathbf{v}_2, \mathbf{u}_1 \rangle, \langle \mathbf{v}_1, \mathbf{u}_2 \rangle + \langle \mathbf{v}_2, \mathbf{u}_2 \rangle) \\ &= (\langle \mathbf{v}_1, \mathbf{u}_1 \rangle, \langle \mathbf{v}_1, \mathbf{u}_2 \rangle) + (\langle \mathbf{v}_2, \mathbf{u}_1 \rangle, \langle \mathbf{v}_2, \mathbf{u}_2 \rangle) \\ &= (\langle \mathbf{u}_1, \mathbf{v}_1 \rangle, \langle \mathbf{u}_2, \mathbf{v}_1 \rangle) + (\langle \mathbf{u}_1, \mathbf{v}_2 \rangle, \langle \mathbf{u}_2, \mathbf{v}_2 \rangle) \\ &= T(\mathbf{v}_1) + T(\mathbf{v}_2). \end{aligned}$$

T respects scalar multiplication: Let $\mathbf{v}$ be a vector in V and let c be a scalar. Then we have

$$\begin{aligned} T(c\mathbf{v}) &= (\langle \mathbf{u}_1, c\mathbf{v} \rangle, \langle \mathbf{u}_2, c\mathbf{v} \rangle) \\ &= (\langle c\mathbf{v}, \mathbf{u}_1 \rangle, \langle c\mathbf{v}, \mathbf{u}_2 \rangle) \\ &= (c\langle \mathbf{v}, \mathbf{u}_1 \rangle, c\langle \mathbf{v}, \mathbf{u}_2 \rangle) \\ &= c(\langle \mathbf{v}, \mathbf{u}_1 \rangle, \langle \mathbf{v}, \mathbf{u}_2 \rangle) \\ &= c(\langle \mathbf{u}_1, \mathbf{v} \rangle, \langle \mathbf{u}_2, \mathbf{v} \rangle) \\ &= cT(\mathbf{v}). \end{aligned}$$

23. (a) If $D = [\mathbf{v}_1, \mathbf{v}_2] = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$ then $\det(D) = -2 \neq 0$, so by Corollary 4.5.15 the vectors $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (1, -1)$ are linearly independent. Since $\dim[\mathbb{R}^2] = 2$, it follows from Theorem 4.6.10 that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^2$.

(b) Let $\mathbf{x} = (x_1, x_2)$ be an arbitrary vector in $\mathbb{R}^2$. Since $\{\mathbf{v}_1, \mathbf{v}_2\}$ forms a basis for $\mathbb{R}^2$, there exist c_1 and c_2 such that

$$(x_1, x_2) = c_1(1, 1) + c_2(1, -1),$$

that is, such that

$$c_1 + c_2 = x_1, \quad c_1 - c_2 = x_2.$$

Solving this system yields $c_1 = \frac{1}{2}(x_1 + x_2)$, $c_2 = \frac{1}{2}(x_1 - x_2)$. Thus,

$$(x_1, x_2) = \frac{1}{2}(x_1 + x_2)\mathbf{v}_1 + \frac{1}{2}(x_1 - x_2)\mathbf{v}_2,$$

so that

$$\begin{aligned} T[(x_1, x_2)] &= T\left[\frac{1}{2}(x_1 + x_2)\mathbf{v}_1 + \frac{1}{2}(x_1 - x_2)\mathbf{v}_2\right] \\ &= \frac{1}{2}(x_1 + x_2)T(\mathbf{v}_1) + \frac{1}{2}(x_1 - x_2)T(\mathbf{v}_2) \\ &= \frac{1}{2}(x_1 + x_2)(2, 3) + \frac{1}{2}(x_1 - x_2)(-1, 1) \\ &= \left(\frac{x_1}{2} + \frac{3x_2}{2}, 2x_1 + x_2\right). \end{aligned}$$

In particular, when $(4, -2)$ is substituted for (x_1, x_2) , it follows that $T(4, -2) = (-1, 6)$.

24. The matrix of T is the 4×2 matrix $[T(\mathbf{e}_1), T(\mathbf{e}_2)]$. Therefore, we must determine $T(1, 0)$ and $T(0, 1)$, which we can determine from the given information by using the linear transformation properties. A quick calculation shows that $(1, 0) = -\frac{2}{3}(-1, 1) + \frac{1}{3}(1, 2)$, so

$$T(1, 0) = T\left(-\frac{2}{3}(-1, 1) + \frac{1}{3}(1, 2)\right) = -\frac{2}{3}T(-1, 1) + \frac{1}{3}T(1, 2) = -\frac{2}{3}(1, 0, -2, 2) + \frac{1}{3}(-3, 1, 1, 1) = \left(-\frac{5}{3}, \frac{1}{3}, \frac{5}{3}, -1\right).$$

Similarly, we have $(0, 1) = \frac{1}{3}(-1, 1) + \frac{1}{3}(1, 2)$, so

$$T(0, 1) = T\left(\frac{1}{3}(-1, 1) + \frac{1}{3}(1, 2)\right) = \frac{1}{3}T(-1, 1) + \frac{1}{3}T(1, 2) = \frac{1}{3}(1, 0, -2, 2) + \frac{1}{3}(-3, 1, 1, 1) = \left(-\frac{2}{3}, \frac{1}{3}, -\frac{1}{3}, 1\right).$$

Therefore, we have the matrix of T :

$$\begin{bmatrix} -5/3 & -2/3 \\ 1/3 & 1/3 \\ 5/3 & -1/3 \\ -1 & 1 \end{bmatrix}.$$

25. The matrix of T is the 2×4 matrix $[T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3), T(\mathbf{e}_4)]$. Therefore, we must determine $T(1, 0, 0, 0)$, $T(0, 1, 0, 0)$, $T(0, 0, 1, 0)$, and $T(0, 0, 0, 1)$, which we can determine from the given information by using the linear transformation properties. We are given that

$$T(1, 0, 0, 0) = (3, -2).$$

Next,

$$\begin{aligned} T(0, 1, 0, 0) &= T(1, 1, 0, 0) - T(1, 0, 0, 0) = (5, 1) - (3, -2) = (2, 3), \\ T(0, 0, 1, 0) &= T(1, 1, 1, 0) - T(1, 1, 0, 0) = (-1, 0) - (5, 1) = (-6, -1), \end{aligned}$$

and

$$T(0, 0, 0, 1) = T(1, 1, 1, 1) - T(1, 1, 1, 0) = (2, 2) - (-1, 0) = (3, 2).$$

Therefore, we have the matrix of T :

$$\begin{bmatrix} 3 & 2 & -6 & 3 \\ -2 & 3 & -1 & 2 \end{bmatrix}.$$

26. The matrix of T is the 3×3 matrix $[T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)]$. Therefore, we must determine $T(1, 0, 0)$, $T(0, 1, 0)$, and $T(0, 0, 1)$, which we can determine from the given information by using the linear transformation properties. A quick calculation shows that $(1, 0, 0) = (1, 2, 0) - 6(0, 1, 1) + 2(0, 2, 3)$, so

$$T(1, 0, 0) = T(1, 2, 0) - 6T(0, 1, 1) + 2T(0, 2, 3) = (2, -1, 1) - 6(3, -1, -1) + 2(6, -5, 4) = (32, -5, 4).$$

Similarly, $(0, 1, 0) = 3(0, 1, 1) - (0, 2, 3)$, so

$$T(0, 1, 0) = 3T(0, 1, 1) - T(0, 2, 3) = 3(3, -1, -1) - (6, -5, 4) = (3, 2, -7).$$

Finally, $(0, 0, 1) = -2(0, 1, 1) + (0, 2, 3)$, so

$$T(0, 0, 1) = -2T(0, 1, 1) + T(0, 2, 3) = -2(3, -1, -1) + (6, -5, 4) = (0, -3, 6).$$

Therefore, we have the matrix of T :

$$\begin{bmatrix} 32 & 3 & 0 \\ -5 & 2 & -3 \\ 4 & -7 & 6 \end{bmatrix}.$$

27. The matrix of T is the 4×3 matrix $[T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)]$. Therefore, we must determine $T(1, 0, 0)$, $T(0, 1, 0)$, and $T(0, 0, 1)$, which we can determine from the given information by using the linear transformation properties. A quick calculation shows that $(1, 0, 0) = \frac{1}{4}(0, -1, 4) - \frac{1}{4}(0, 3, 3) + \frac{1}{4}(4, 4, -1)$, so

$$T(1, 0, 0) = \frac{1}{4}T(0, -1, 4) - \frac{1}{4}T(0, 3, 3) + \frac{1}{4}T(4, 4, -1) = \frac{1}{4}(2, 5, -2, 1) - \frac{1}{4}(-1, 0, 0, 5) + \frac{1}{4}(-3, 1, 1, 3) = (0, \frac{3}{2}, -\frac{1}{4}, -\frac{1}{4}).$$

Similarly, $(0, 1, 0) = -\frac{1}{5}(0, -1, 4) + \frac{4}{15}(0, 3, 3)$, so

$$T(0, 1, 0) = -\frac{1}{5}(2, 5, -2, 1) + \frac{4}{15}(-1, 0, 0, 5) = (-\frac{2}{3}, -1, \frac{2}{5}, \frac{17}{15}).$$

Finally, $(0, 0, 1) = \frac{1}{5}(0, -1, 4) + \frac{1}{15}(0, 3, 3)$, so

$$T(0, 0, 1) = \frac{1}{5}T(0, -1, 4) + \frac{1}{15}T(0, 3, 3) = \frac{1}{5}(2, 5, -2, 1) + \frac{1}{15}(-1, 0, 0, 5) = (\frac{1}{3}, 1, -\frac{2}{5}, \frac{8}{15}).$$

Therefore, we have the matrix of T :

$$\begin{bmatrix} 0 & -2/3 & 1/3 \\ 3/2 & -1 & 1 \\ -1/4 & 2/5 & -2/5 \\ -1/4 & 17/15 & 8/15 \end{bmatrix}.$$

28. $T(ax^2 + bx + c) = aT(x^2) + bT(x) + cT(1) = a(3x + 2) + b(x^2 - 1) + c(x + 1) = bx^2 + (3a + c)x + (2a - b + c)$.

29. Using the linearity of T , we have $T(2\mathbf{v}_1 + 3\mathbf{v}_2) = \mathbf{v}_1 + \mathbf{v}_2$ and $T(\mathbf{v}_1 + \mathbf{v}_2) = 3\mathbf{v}_1 - \mathbf{v}_2$. That is, $2T(\mathbf{v}_1) + 3T(\mathbf{v}_2) = \mathbf{v}_1 + \mathbf{v}_2$ and $T(\mathbf{v}_1) + T(\mathbf{v}_2) = 3\mathbf{v}_1 - \mathbf{v}_2$. Solving this system for the unknowns $T(\mathbf{v}_1)$ and $T(\mathbf{v}_2)$, we obtain $T(\mathbf{v}_2) = 3\mathbf{v}_2 - 5\mathbf{v}_1$ and $T(\mathbf{v}_1) = 8\mathbf{v}_1 - 4\mathbf{v}_2$.

30. Since T is a linear transformation we obtain:

$$T(x^2) - T(1) = x^2 + x - 3, \quad (30.1)$$

$$2T(x) = 4x, \quad (30.2)$$

$$3T(x) + 2T(1) = 2(x + 3) = 2x + 6. \quad (30.3)$$

From Equation (30.2) it follows that $T(x) = 2x$, so upon substitution into Equation (30.3) we have $3(2x) + 2T(1) = 2(x + 3)$ or $T(1) = -2x + 3$. Substituting this last result into Equation (30.1) yields $T(x^2) - (-2x + 3) = x^2 + x - 3$ so $T(x^2) = x^2 - x$. Now if a , b and c are arbitrary real numbers, then $T(ax^2 + bx + c) = aT(x^2) + bT(x) + cT(1) = a(x^2 - x) + b(2x) + c(-2x + 3)$

$$= ax^2 - ax + 2bx - 2cx + 3c = ax^2 + (-a + 2b - 2c)x + 3c.$$

31. Let $\mathbf{v} \in V$. Since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for V , there exists $a, b \in \mathbb{R}$ such that $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$. Hence

$$\begin{aligned} T(\mathbf{v}) &= T(a\mathbf{v}_1 + b\mathbf{v}_2) = aT(\mathbf{v}_1) + bT(\mathbf{v}_2) = a(3\mathbf{v}_1 - \mathbf{v}_2) + b(\mathbf{v}_1 + 2\mathbf{v}_2) \\ &= 3a\mathbf{v}_1 - a\mathbf{v}_2 + b\mathbf{v}_1 + 2b\mathbf{v}_2 = (3a + b)\mathbf{v}_1 + (2b - a)\mathbf{v}_2. \end{aligned}$$

32. Let $\mathbf{v}$ be any vector in V . Since $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ spans V , we can write $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k$ for suitable scalars $c_1, c_2, \dots, c_k$. Then

$$\begin{aligned} T(\mathbf{v}) &= T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \dots + c_kT(\mathbf{v}_k) \\ &= c_1S(\mathbf{v}_1) + c_2S(\mathbf{v}_2) + \dots + c_kS(\mathbf{v}_k) \\ &= S(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k) = S(\mathbf{v}), \end{aligned}$$

as required.

33. Let $\mathbf{v}$ be any vector in V . Since $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a basis for V , we can write $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k$ for suitable scalars $c_1, c_2, \dots, c_k$. Then

$$T(\mathbf{v}) = T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \dots + c_kT(\mathbf{v}_k) = c_1\mathbf{0} + c_2\mathbf{0} + \dots + c_k\mathbf{0} = \mathbf{0},$$

as required.

34. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be arbitrary vectors in V . Then,

$$\begin{aligned} (T_1 + T_2)(\mathbf{v}_1 + \mathbf{v}_2) &= T_1(\mathbf{v}_1 + \mathbf{v}_2) + T_2(\mathbf{v}_1 + \mathbf{v}_2) \\ &= T_1(\mathbf{v}_1) + T_1(\mathbf{v}_2) + T_2(\mathbf{v}_1) + T_2(\mathbf{v}_2) \\ &= T_1(\mathbf{v}_1) + T_2(\mathbf{v}_1) + T_1(\mathbf{v}_2) + T_2(\mathbf{v}_2) \\ &= (T_1 + T_2)(\mathbf{v}_1) + (T_1 + T_2)(\mathbf{v}_2). \end{aligned}$$

Further, if k is any scalar, then

$$(T_1 + T_2)(k\mathbf{v}) = T_1(k\mathbf{v}) + T_2(k\mathbf{v}) = kT_1(\mathbf{v}) + kT_2(\mathbf{v}) = k[T_1(\mathbf{v}) + T_2(\mathbf{v})] = k(T_1 + T_2)(\mathbf{v}).$$

It follows that $T_1 + T_2$ is a linear transformation.

Now consider the transformation cT , where c is an arbitrary scalar.

$$(cT)(\mathbf{v}_1 + \mathbf{v}_2) = cT(\mathbf{v}_1 + \mathbf{v}_2) = c[T(\mathbf{v}_1) + T(\mathbf{v}_2)] = cT(\mathbf{v}_1) + cT(\mathbf{v}_2) = (cT)(\mathbf{v}_1) + (cT)(\mathbf{v}_2).$$

$$(cT)(k\mathbf{v}_1) = cT(k\mathbf{v}_1) = c[kT(\mathbf{v}_1)] = (ck)T(\mathbf{v}_1) = (kc)T(\mathbf{v}_1) = k[cT(\mathbf{v}_1)].$$

Thus, cT is a linear transformation.

$$\mathbf{35.} \quad (T_1 + T_2)(\mathbf{x}) = T_1(\mathbf{x}) + T_2(\mathbf{x}) = A\mathbf{x} + B\mathbf{x} = (A + B)\mathbf{x} = \begin{bmatrix} 5 & 6 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 + 6x_2 \\ 2x_1 - 2x_2 \end{bmatrix}.$$

Hence, $(T_1 + T_2)(x_1, x_2) = (5x_1 + 6x_2, 2x_1 - 2x_2)$.

$$(cT_1)(\mathbf{x}) = cT_1(\mathbf{x}) = c(A\mathbf{x}) = (cA)\mathbf{x} = \begin{bmatrix} 3c & c \\ -c & 2c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3cx_1 + cx_2 \\ -cx_1 + 2cx_2 \end{bmatrix}.$$

Hence, $(cT_1)(x_1, x_2) = (3cx_1 + cx_2, -cx_1 + 2cx_2)$.

$$\mathbf{36.} \quad (T_1 + T_2)(\mathbf{x}) = T_1(\mathbf{x}) + T_2(\mathbf{x}) = A\mathbf{x} + B\mathbf{x} = (A + B)\mathbf{x}.$$

$$(cT_1)(\mathbf{x}) = cT_1(\mathbf{x}) = c(A\mathbf{x}) = (cA)\mathbf{x}.$$

37. Problem 34 establishes that if T_1 and T_2 are in $L(V, W)$ and c is any scalar, then $T_1 + T_2$ and cT_1 are in $L(V, W)$. Consequently, Axioms (A1) and (A2) are satisfied.

A3: Let $\mathbf{v}$ be any vector in $L(V, W)$. Then

$$(T_1 + T_2)(\mathbf{v}) = T_1(\mathbf{v}) + T_2(\mathbf{v}) = T_2(\mathbf{v}) + T_1(\mathbf{v}) = (T_2 + T_1)(\mathbf{v}).$$

Hence $T_1 + T_2 = T_2 + T_1$, therefore the addition operation is commutative.

A4: Let $T_3 \in L(V, W)$. Then

$$\begin{aligned} [(T_1 + T_2) + T_3](\mathbf{v}) &= (T_1 + T_2)(\mathbf{v}) + T_3(\mathbf{v}) = [T_1(\mathbf{v}) + T_2(\mathbf{v})] + T_3(\mathbf{v}) \\ &= T_1(\mathbf{v}) + [T_2(\mathbf{v}) + T_3(\mathbf{v})] = T_1(\mathbf{v}) + (T_2 + T_3)(\mathbf{v}) = [T_1 + (T_2 + T_3)](\mathbf{v}). \end{aligned}$$

Hence $(T_1 + T_2) + T_3 = T_1 + (T_2 + T_3)$, therefore the addition operation is associative.

A5: The zero vector in $L(V, W)$ is the zero transformation, $O : V \rightarrow W$, defined by

$$O(\mathbf{v}) = \mathbf{0}, \text{ for all } \mathbf{v} \text{ in } V,$$

where $\mathbf{0}$ denotes the zero vector in V . To show that O is indeed the zero vector in $L(V, W)$, let T be any transformation in $L(V, W)$. Then

$$(T + O)(\mathbf{v}) = T(\mathbf{v}) + O(\mathbf{v}) = T(\mathbf{v}) + \mathbf{0} = T(\mathbf{v}) \text{ for all } \mathbf{v} \in V,$$

so that $T + O = T$.

A6: The additive inverse of the transformation $T \in L(V, W)$ is the linear transformation $-T$ defined by $-T = (-1)T$, since

$$[T + (-T)](\mathbf{v}) = T(\mathbf{v}) + (-T)(\mathbf{v}) = T(\mathbf{v}) + (-1)T(\mathbf{v}) = T(\mathbf{v}) - T(\mathbf{v}) = \mathbf{0},$$

for all $\mathbf{v} \in V$, so that $T + (-T) = O$.

A7-A10 are all straightforward verifications.

Solutions to Section 5.2

True-False Review:

1. FALSE. For example, $T(x_1, x_2) = (0, 0)$ is a linear transformation that maps every line to the origin.

2. TRUE. All of the matrices R_x , R_y , R_{xy} , LS_x , LS_y , S_x , and S_y discussed in this section are elementary matrices.

3. FALSE. A shear parallel to the x -axis composed with a shear parallel to the y -axis is given by matrix

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix} = \begin{bmatrix} 1+kl & k \\ l & 1 \end{bmatrix}, \text{ which is not a shear.}$$

4. TRUE. This is explained prior to Example 5.2.1.

5. FALSE. For example,

$$R_{xy} \cdot R_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and this matrix is not in the form of a stretch.

6. FALSE. For example,

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & l \end{bmatrix} = \begin{bmatrix} k & 0 \\ 0 & l \end{bmatrix}$$

is not a stretch.

Problems:

1. $T(1,1) = (1, -1)$, $T(2,1) = (1, -2)$, $T(2,2) = (2, -2)$, $T(1,2) = (2, -1)$.

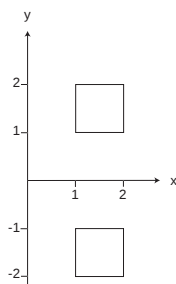


Figure 69: Figure for Problem 1

2. $T(1,1) = (0, 3)$, $T(2,1) = (1, 4)$, $T(2,2) = (0, 6)$, $T(1,2) = (-1, 5)$.

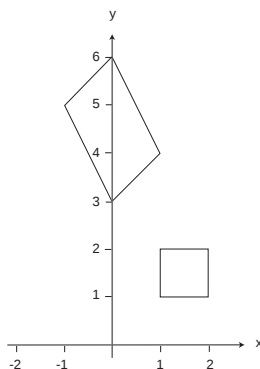


Figure 70: Figure for Problem 2

3. $T(1,1) = (2, 0)$, $T(2,1) = (3, -1)$, $T(2,2) = (4, 0)$, $T(1,2) = (3, 1)$.

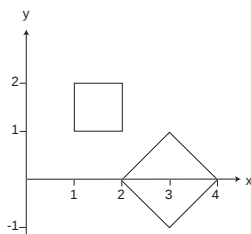


Figure 71: Figure for Problem 3

4. $T(1, 1) = (-4, -2)$, $T(2, 1) = (-6, -4)$, $T(2, 2) = (-8, -4)$, $T(1, 2) = (-6, -2)$.

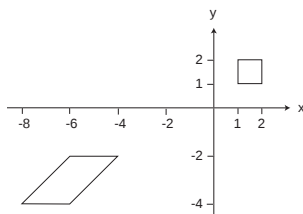


Figure 72: Figure for Problem 4

5. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \Rightarrow T(\mathbf{x}) = A\mathbf{x}$ corresponds to a shear parallel to the x -axis.

6. $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

1. P_{12} 2. $M_1(1/2)$ 3. $M_2(1/2)$. So,

$$T(\mathbf{x}) = A\mathbf{x} = P_{12}M_1(2)M_2(2)\mathbf{x}$$

which corresponds to a stretch in the y -direction, followed by a stretch in the x -direction, followed by a reflection in $y = x$.

7. $A = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \Rightarrow T(\mathbf{x}) = A\mathbf{x}$ corresponds to a shear parallel to the y -axis.

8. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 1. $M_1(-1)$ 2. $M_2(-1)$ So,

$$T(\mathbf{x}) = A\mathbf{x} = M_1(-1)M_2(-1)\mathbf{x}$$

which corresponds to a reflection in the x -axis, followed by a reflection in the y -axis.

9. $\begin{bmatrix} 1 & -3 \\ -2 & 8 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 1. $A_{12}(2)$ 2. $M_2(1/2)$ 3. $A_{21}(3)$. So,

$$T(\mathbf{x}) = A\mathbf{x} = A_{12}(-2)M_2(2)A_{21}(-3)\mathbf{x}$$

which corresponds to a shear parallel to the x -axis, followed by a stretch in the y -direction, followed by a shear parallel to the y -axis.

10. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 1. $A_{12}(-3)$ 2. $A_{21}(1)$ 3. $M_2(-1/2)$. So,

$$T(\mathbf{x}) = A\mathbf{x} = A_{12}(3)A_{21}(-1)M_2(-2)\mathbf{x}$$

which corresponds to a reflection in the x -axis followed by a stretch in the y -direction, followed by a shear parallel to the x -axis, followed by a shear parallel to the y -axis.

11. $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. So $T(\mathbf{x}) = A\mathbf{x}$ corresponds to a reflection in the x -axis followed by a stretch in the y -direction.

12. $\begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 1. $A_{12}(-1)$ 2. $M_1(-1)$ 3. $A_{21}(-1)$.
So,

$$T(\mathbf{x}) = A\mathbf{x} = A_{12}(1)M_1(-1)A_{21}(1)\mathbf{x}$$

which corresponds to a shear parallel to the x -axis, followed by a reflection in the y -axis, followed by a shear parallel to the y -axis.

13.

$$\begin{aligned} R(\theta) &= \begin{bmatrix} \cos \theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \sin \theta & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sec \theta \end{bmatrix} \begin{bmatrix} 1 & -\tan \theta \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \sin \theta & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan \theta \\ 0 & \sec \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan \theta \\ \sin \theta & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \end{aligned}$$

which coincides with the matrix of the transformation of $\mathbb{R}^2$ corresponding to a rotation through an angle θ in the counter-clockwise direction.

14. The matrix for a counter-clockwise rotation through an angle $\theta = \frac{\pi}{2}$ is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Now,

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
 1. P_{12} 2. $M_2(-1)$

So,

$$T(\mathbf{x}) = A\mathbf{x} = P_{12}M_2(-1)\mathbf{x}$$

which corresponds to a reflection in the x -axis followed by a reflection in $y = x$.

Solutions to Section 5.3

True-False Review:

1. **FALSE.** The statement should read

$$\dim[\text{Ker}(T)] + \dim[\text{Rng}(T)] = \dim[V],$$

not $\dim[W]$ on the right-hand side.

2. **FALSE.** As a specific illustration, we could take $T : P_4 \rightarrow \mathbb{R}^7$ defined by

$$T(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4) = (a_0, a_1, a_2, a_3, a_4, 0, 0),$$

and it is easy to see that T is a linear transformation with $\text{Ker}(T) = \{0\}$. Therefore, $\text{Ker}(T)$ is 0-dimensional.

3. FALSE. The solution set to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is $\text{Ker}(T)$, not $\text{Rng}(T)$.

4. FALSE. $\text{Rng}(T)$ is a subspace of W , not V , since it consists of vectors of the form $T(\mathbf{v})$, and these belong to W .

5. TRUE. From the given information, we see that $\text{Ker}(T)$ is at least 2-dimensional, and therefore, since M_{23} is 6-dimensional, the Rank-Nullity Theorem requires that $\text{Rng}(T)$ have dimension at most $6 - 2 = 4$.

6. TRUE. Any vector of the form $T(\mathbf{v})$ where $\mathbf{v}$ belongs to $\mathbb{R}^n$ can be written as $A\mathbf{v}$, and this in turn can be expressed as a linear combination of the columns of A . Therefore, $T(\mathbf{v})$ belongs to $\text{colspace}(A)$.

Problems:

$$1. T(7, 5, -1) = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies (7, 5, -1) \in \text{Ker}(T).$$

$$T(-21, -15, 2) = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} -21 \\ -15 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \implies (-21, -15, 2) \notin \text{Ker}(T).$$

$$T(35, 25, -5) = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 35 \\ 25 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies (35, 25, -5) \in \text{Ker}(T).$$

2. $\text{Ker}(T) = \{\mathbf{x} \in \mathbb{R}^2 : T(\mathbf{x}) = \mathbf{0}\} = \{\mathbf{x} \in \mathbb{R}^2 : A\mathbf{x} = \mathbf{0}\}$. The augmented matrix of the system $A\mathbf{x} = \mathbf{0}$ is: $\left[\begin{array}{cc|c} 3 & 6 & 0 \\ 1 & 2 & 0 \end{array} \right]$, with reduced row-echelon form of $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right]$. It follows that

$$\text{Ker}(T) = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = (-2t, t), t \in \mathbb{R}\} = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = t(-2, 1), t \in \mathbb{R}\}.$$

Geometrically, this is a line in $\mathbb{R}^2$. It is the subspace of $\mathbb{R}^2$ spanned by the vector $(-2, 1)$.

$$\dim[\text{Ker}(T)] = 1.$$

For the given transformation, $\text{Rng}(T) = \text{colspace}(A)$. From the preceding reduced row-echelon form of A , we see that $\text{colspace}(A)$ is generated by the first column vector of A . Consequently,

$$\text{Rng}(T) = \{\mathbf{y} \in \mathbb{R}^2 : \mathbf{y} = r(3, 1), r \in \mathbb{R}\}.$$

Geometrically, this is a line in $\mathbb{R}^2$. It is the subspace of $\mathbb{R}^2$ spanned by the vector $(3, 1)$.

$$\dim[\text{Rng}(T)] = 1.$$

Since $\dim[\text{Ker}(T)] + \dim[\text{Rng}(T)] = 2 = \dim[\mathbb{R}^2]$, Theorem 5.3.8 is satisfied.

3. $\text{Ker}(T) = \{\mathbf{x} \in \mathbb{R}^3 : T(\mathbf{x}) = \mathbf{0}\} = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{0}\}$. The augmented matrix of the system $A\mathbf{x} = \mathbf{0}$ is: $\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & -1 & 1 & 0 \end{array} \right]$, with reduced row-echelon form $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$. Thus $x_1 = x_2 = x_3 = 0$, so

$\text{Ker}(T) = \{\mathbf{0}\}$. Geometrically, this describes a point (the origin) in $\mathbb{R}^3$.

$$\dim[\text{Ker}(T)] = 0.$$

For the given transformation, $\text{Rng}(T) = \text{colspace}(A)$. From the preceding reduced row-echelon form of A , we see that $\text{colspace}(A)$ is generated by the first three column vectors of A . Consequently,

$\text{Rng}(T) = \mathbb{R}^3$, $\dim[\text{Rng}(T)] = \dim[\mathbb{R}^3] = 3$, and Theorem 5.3.8 is satisfied since $\dim[\text{Ker}(T)] + \dim[\text{Rng}(T)] = 0 + 3 = \dim[\mathbb{R}^3]$.

4. $\text{Ker}(T) = \{\mathbf{x} \in \mathbb{R}^3 : T(\mathbf{x}) = \mathbf{0}\} = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{0}\}$. The augmented matrix of the system $A\mathbf{x} = \mathbf{0}$ is:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -3 & -1 & 0 \\ 5 & -8 & -1 & 0 \end{array} \right], \text{ with reduced row-echelon form of } \left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]. \text{ Thus}$$

$$\text{Ker}(T) = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = t(5, 3, 1), t \in \mathbb{R}\}.$$

Geometrically, this describes the line in $\mathbb{R}^3$ through the origin, spanned by $(5, 3, 1)$.
 $\dim[\text{Ker}(T)] = 1$.

For the given transformation, $\text{Rng}(T) = \text{colspace}(A)$. From the preceding reduced row-echelon form of A , we see that a basis for $\text{colspace}(A)$ is given by the first two column vectors of A . Consequently,

$$\text{Rng}(T) = \{\mathbf{y} \in \mathbb{R}^3 : \mathbf{y} = r(1, 2, 5) + s(-2, -3, -8), r, s \in \mathbb{R}\}.$$

Geometrically, this is a plane through the origin in $\mathbb{R}^3$.

$\dim[\text{Rng}(T)] = 2$ and Theorem 5.3.8 is satisfied since $\dim[\text{Ker}(T)] + \dim[\text{Rng}(T)] = 1 + 2 = 3 = \dim[\mathbb{R}^3]$.

5. $\text{Ker}(T) = \{\mathbf{x} \in \mathbb{R}^3 : T(\mathbf{x}) = \mathbf{0}\} = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{0}\}$. The augmented matrix of the system $A\mathbf{x} = \mathbf{0}$ is:
 $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ -3 & 3 & -6 & 0 \end{array} \right]$, with reduced row-echelon form of $\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$. Thus

$$\text{Ker}(T) = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = r(1, 1, 0) + s(-2, 0, 1), r, s \in \mathbb{R}\}.$$

Geometrically, this describes the plane through the origin in $\mathbb{R}^3$, which is spanned by the linearly independent set $\{(1, 1, 0), (-2, 0, 1)\}$.

$\dim[\text{Ker}(T)] = 2$.

For the given transformation, $\text{Rng}(T) = \text{colspace}(A)$. From the preceding reduced row-echelon form of A , we see that a basis for $\text{colspace}(A)$ is given by the first column vector of A . Consequently,

$$\text{Rng}(T) = \{\mathbf{y} \in \mathbb{R}^2 : \mathbf{y} = t(1, -3), t \in \mathbb{R}\}.$$

Geometrically, this is the line through the origin in $\mathbb{R}^2$ spanned by $(1, -3)$.

$\dim[\text{Rng}(T)] = 1$ and Theorem 5.3.8 is satisfied since $\dim[\text{Ker}(T)] + \dim[\text{Rng}(T)] = 2 + 1 = 3 = \dim[\mathbb{R}^3]$.

6. $\text{Ker}(T) = \{\mathbf{x} \in \mathbb{R}^3 : T(\mathbf{x}) = \mathbf{0}\} = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{0}\}$. The augmented matrix of the system $A\mathbf{x} = \mathbf{0}$ is:
 $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 2 & 6 & 5 & 0 \end{array} \right]$, with reduced row-echelon form of $\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$. Thus

$$\text{Ker}(T) = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = r(-3, 1, 0), r \in \mathbb{R}\}.$$

Geometrically, this describes the line through the origin in $\mathbb{R}^3$, which is spanned by $(-3, 1, 0)$.

$\dim[\text{Ker}(T)] = 1$.

For the given transformation, $\text{Rng}(T) = \text{colspace}(A)$. From the preceding reduced row-echelon form of A , we see that a basis for $\text{colspace}(A)$ is given by the first and third column vectors of A . Consequently, $\text{Rng}(T) = \text{span}\{(1, 2), (2, 5)\} = \mathbb{R}^2$, so that $\dim[\text{Rng}(T)] = 2$. Geometrically, $\text{Rng}(T)$ is the xy -plane, and Theorem 5.3.8 is satisfied since $\dim[\text{Ker}(T)] + \dim[\text{Rng}(T)] = 1 + 2 = 3 = \dim[\mathbb{R}^3]$.

7. The matrix of T in Problem 24 of Section 5.1 is $A = \begin{bmatrix} -5/3 & -2/3 \\ 1/3 & 1/3 \\ 5/3 & -1/3 \\ -1 & 1 \end{bmatrix}$. Thus,

$$\text{Ker}(T) = \text{nullspace}(A) = \{\mathbf{0}\}$$

and

$$\text{Rng}(T) = \text{colspace}(A) = \text{span} \left\{ \begin{bmatrix} -5/3 \\ 1/3 \\ 5/3 \\ -1 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ -1/3 \\ 1 \end{bmatrix} \right\}.$$

8. The matrix of T in Problem 25 of Section 5.1 is $A = \begin{bmatrix} 3 & 2 & -6 & 3 \\ -2 & 3 & -1 & 2 \end{bmatrix}$. Thus,

$$\text{Ker}(T) = \text{nullspace}(A) = \text{span}\{(16/13, 15/13, 1, 0), (-5/13, -12/13, 0, 1)\}$$

and

$$\text{Rng}(T) = \text{colspace}(A) = \mathbb{R}^2.$$

9. The matrix of T in Problem 26 of Section 5.1 is $A = \begin{bmatrix} 32 & 3 & 0 \\ -5 & 2 & -3 \\ 4 & -7 & 6 \end{bmatrix}$. Thus,

$$\text{Ker}(T) = \text{nullspace}(A) = \{\mathbf{0}\}$$

and

$$\text{Rng}(T) = \text{colspace}(A) = \mathbb{R}^3.$$

10. The matrix of T in Problem 27 of Section 5.1 is $A = \begin{bmatrix} 0 & -2/3 & 1/3 \\ 3/2 & -1 & 1 \\ 0 & 2/5 & -2/5 \\ -1/4 & 17/15 & 8/15 \end{bmatrix}$. Thus,

$$\text{Ker}(T) = \text{nullspace}(A) = \{\mathbf{0}\}$$

and

$$\text{Rng}(T) = \text{colspace}(A) = \text{span} \left\{ \begin{bmatrix} 0 \\ 3/2 \\ 0 \\ -1/4 \end{bmatrix}, \begin{bmatrix} -2/3 \\ -1 \\ 2/5 \\ 17/15 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 1 \\ -2/5 \\ 8/15 \end{bmatrix} \right\}.$$

11. (a) $\text{Ker}(T) = \{\mathbf{v} \in \mathbb{R}^3 : \langle \mathbf{u}, \mathbf{v} \rangle = 0\}$. For $\mathbf{v}$ to be in the kernel of T , $\mathbf{u}$ and $\mathbf{v}$ must be orthogonal. Since $\mathbf{u}$ is any fixed vector in $\mathbb{R}^3$, then $\mathbf{v}$ must lie in the plane orthogonal to $\mathbf{u}$. Hence $\dim[\text{Ker}(T)] = 2$.

(b) $\text{Rng}(T) = \{y \in \mathbb{R} : y = \langle \mathbf{u}, \mathbf{v} \rangle, \mathbf{v} \in \mathbb{R}^3\}$, and $\dim[\text{Rng}(T)] = 1$.

12. (a) $\text{Ker}(S) = \{A \in M_n(\mathbb{R}) : A - A^T = 0\} = \{A \in M_n(\mathbb{R}) : A = A^T\}$. Hence, any matrix in $\text{Ker}(S)$ is symmetric by definition.

(b) Since any matrix in $\text{Ker}(S)$ is symmetric, it has been shown that $\{A_1, A_2, A_3\}$ is a spanning set for the set of all symmetric matrices in $M_2(\mathbb{R})$, where

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, $\dim[\text{Ker}(S)] = 3$.

13. $\text{Ker}(T) = \{A \in M_n(\mathbb{R}) : AB - BA = 0\} = \{A \in M_2(\mathbb{R}) : AB = BA\}$. This is the set of matrices that commute with B .

14. (a)

$\text{Ker}(T) = \{p \in P_2 : T(p) = 0\} = \{ax^2 + bx + c \in P_2 : ax^2 + (a + 2b + c)x + (3a - 2b - c) = 0, \text{ for all } x\}$. Thus, for $p(x) = ax^2 + bx + c$ to be in $\text{Ker}(T)$, a, b , and c must satisfy the system:

$$\begin{aligned} a &= 0 \\ a + 2b + c &= 0 \\ 3a - 2b - c &= 0 \end{aligned}$$

Solving this system, we obtain that $a = 0$ and $c = -2b$. Consequently, all polynomials of the form $0x^2 + bx + (-2b)$ are in $\text{Ker}(T)$, so

$$\text{Ker}(T) = \{b(x - 2) : b \in \mathbb{R}\}.$$

Since $\text{Ker}(T)$ is spanned by the nonzero vector $x - 2$, it follows that $\dim[\text{Rng}(T)] = 1$.

(b) In this case,

$$\begin{aligned} \text{Rng}(T) &= \{T(ax^2 + bx + c) : a, b, c \in \mathbb{R}\} \\ &= \{ax^2 + (a + 2b + c)x + (3a - 2b - c) : a, b, c \in \mathbb{R}\} \\ &= \{a(x^2 + x + 3) + b(2x - 2) + c(x - 1) : a, b, c \in \mathbb{R}\} \\ &= \{a(x^2 + x + 3) + (2b + c)(x - 1) : a, b, c \in \mathbb{R}\} \\ &= \text{span}\{x^2 + x + 3, x - 1\}. \end{aligned}$$

Since the vectors in this spanning set are linearly independent on any interval, it follows that the spanning set is a basis for $\text{Rng}(T)$. Hence, $\dim[\text{Rng}(T)] = 2$.

15. $\text{Ker}(T) = \{p \in P_2 : T(p) = 0\} = \{ax^2 + bx + c \in P_2 : (a + b) + (b - c)x = 0, \text{ for all } x\}$. Thus, a, b , and c must satisfy: $a + b = 0$ and $b - c = 0 \implies a = -b$ and $b = c$. Letting $c = r \in \mathbb{R}$, we have $ax^2 + bx + c = r(-x^2 + x + 1)$. Thus, $\text{Ker}(T) = \{r(-x^2 + x + 1) : r \in \mathbb{R}\}$ and $\dim[\text{Ker}(T)] = 1$.

$$\text{Rng}(T) = \{T(ax^2 + bx + c) : a, b, c \in \mathbb{R}\} = \{(a + b) + (b - c)x : a, b, c \in \mathbb{R}\} = \{c_1 + c_2x : c_1, c_2 \in \mathbb{R}\}.$$

Consequently, a basis for $\text{Rng}(T)$ is $\{1, x\}$, so that $\text{Rng}(T) = P_1$, and $\dim[\text{Rng}(T)] = 2$.

16. $\text{Ker}(T) = \{p \in P_1 : T(p) = 0\} = \{ax + b \in P_1 : (b - a) + (2b - 3a)x + bx^2 = 0, \text{ for all } x\}$. Thus, a and b must satisfy: $b - a = 0$, $2b - 3a = 0$, and $b = 0 \implies a = b = 0$. Thus, $\text{Ker}(T) = \{0\}$ and $\dim[\text{Ker}(T)] = 0$.

$$\begin{aligned} \text{Rng}(T) &= \{T(ax + b) : a, b \in \mathbb{R}\} = \{(b - a) + (2b - 3a)x + bx^2 : a, b \in \mathbb{R}\} \\ &= \{-a(1 + 3x) + b(1 + 2x + x^2) : a, b \in \mathbb{R}\} \\ &= \text{span}\{1 + 3x, 1 + 2x + x^2\}. \end{aligned}$$

Since the vectors in this spanning set are linearly independent on any interval, it follows that the spanning set is a basis for $\text{Rng}(T)$, and $\dim[\text{Rng}(T)] = 2$.

17.

$$\begin{aligned} T(\mathbf{v}) = \mathbf{0} &\iff T(a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3) = \mathbf{0} \\ &\iff aT(\mathbf{v}_1) + bT(\mathbf{v}_2) + cT(\mathbf{v}_3) = \mathbf{0} \\ &\iff a(2\mathbf{w}_1 - \mathbf{w}_2) + b(\mathbf{w}_1 - \mathbf{w}_2) + c(\mathbf{w}_1 + 2\mathbf{w}_2) = \mathbf{0} \\ &\iff (2a + b + c)\mathbf{w}_1 + (-a - b + 2c)\mathbf{w}_2 = \mathbf{0} \\ &\iff 2a + b + c = 0 \text{ and } -a - b + 2c = 0. \end{aligned}$$

Reducing the augmented matrix of the system yields:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 2 & 1 & 1 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right].$$

Setting $c = r \implies b = r$, $a = -r$. Thus,

$$\text{Ker}(T) = \{\mathbf{v} \in V : \mathbf{v} = r(-\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3), r \in \mathbb{R}\} \text{ and } \dim[\text{Ker}(T)] = 1.$$

$$\text{Rng}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\} = \{(2a + b + c)\mathbf{w}_1 + (-a - b + 2c)\mathbf{w}_2 : a, b, c \in V\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\} = W.$$

Consequently, $\dim[\text{Rng}(T)] = 2$.

18. (a) If $\mathbf{w} \in \text{Rng}(T)$, then $T(\mathbf{v}) = \mathbf{w}$ for some $\mathbf{v} \in V$, and since $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V , there exist $c_1, c_2, \dots, c_n \in \mathbb{R}$ for which $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$.

Accordingly, $\mathbf{w} = T(\mathbf{v}) = a_1T(\mathbf{v}_1) + a_2T(\mathbf{v}_2) + \dots + a_nT(\mathbf{v}_n)$. Thus,

$$\text{Rng}(T) = \text{span}\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}.$$

We must show that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}$ is also linearly independent. Suppose that

$$b_1T(\mathbf{v}_1) + b_2T(\mathbf{v}_2) + \dots + b_nT(\mathbf{v}_n) = \mathbf{0}.$$

Then $T(b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \dots + b_n\mathbf{v}_n) = \mathbf{0}$, so since $\text{Ker}(T) = \{\mathbf{0}\}$, $b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \dots + b_n\mathbf{v}_n = \mathbf{0}$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a linearly independent set, the preceding equation implies that $b_1 = b_2 = \dots = b_n = 0$. Consequently, $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}$ is a linearly independent set in W . Therefore, since we have already shown that it is a spanning set for $\text{Rng}(T)$, $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_n)\}$ is a basis for $\text{Rng}(T)$.

(b) As an example, let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be defined by $T((a, b, c)) = (a, b)$ for all (a, b, c) in $\mathbb{R}^3$. Then for the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$, we have $\{T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)\} = \{(1, 0), (0, 1), (0, 0)\}$, which is clearly not a basis for $\mathbb{R}^2$.

Solutions to Section 5.4

True-False Review:

1. FALSE. Many one-to-one linear transformations $T : P_3 \rightarrow M_{32}$ can be constructed. One possible example would be to define

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = \begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \\ 0 & 0 \end{bmatrix}.$$

It is easy to check that with T so defined, T is a one-to-one linear transformation.

2. TRUE. We can define an isomorphism $T : V \rightarrow M_{32}$ via

$$T\left(\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}.$$

With T so defined, it is easy to check that T is an isomorphism.

3. TRUE. Both $\text{Ker}(T_1)$ and $\text{Ker}(T_2T_1)$ are subspaces of V_1 , and since if $T_1(\mathbf{v}_1) = \mathbf{0}$, then $(T_2T_1)\mathbf{v}_1 = T_2(T_1(\mathbf{v}_1)) = T_2(\mathbf{0}) = \mathbf{0}$, we see that every vector in $\text{Ker}(T_1)$ belongs to $\text{Ker}(T_2T_1)$. Therefore, $\text{Ker}(T_1)$ is a subspace of $\text{Ker}(T_2T_1)$.

4. TRUE. Observe that T is not one-to-one since $\text{Ker}(T) \neq \{\mathbf{0}\}$ (because $\text{Ker}(T)$ is 1-dimensional). Moreover, since M_{22} is 4-dimensional, then by the Rank-Nullity Theorem, $\text{Rng}(T)$ is 3-dimensional. Since P_2 is 3-dimensional, we conclude that $\text{Rng}(T) = P_2$; that is, T is onto.

5. TRUE. Since $M_2(\mathbb{R})$ is 4-dimensional, $\text{Rng}(T)$ can be at most 4-dimensional. However, P_4 is 5-dimensional. Therefore, any such linear transformation T cannot be onto.

6. TRUE. If we assume that $(T_2T_1)\mathbf{v} = (T_2T_1)\mathbf{w}$, then $T_2(T_1(\mathbf{v})) = T_2(T_1(\mathbf{w}))$. Since T_2 is one-to-one, then $T_1(\mathbf{v}) = T_1(\mathbf{w})$. Next, since T_1 is one-to-one, we conclude that $\mathbf{v} = \mathbf{w}$. Therefore, T_2T_1 is one-to-one.

7. FALSE. This linear transformation is onto one-to-one. The reason is essentially because the derivative of any constant is zero. Therefore, $\text{Ker}(T)$ consists of all constant functions, and therefore, $\text{Ker}(T) \neq \{0\}$.

8. TRUE. Since M_{23} is 6-dimensional and $\text{Rng}(T)$ is only 4-dimensional, T is not onto. Moreover, since P_3 is 4-dimensional, the Rank-Nullity Theorem implies that $\text{Ker}(T)$ is 0-dimensional. Therefore, $\text{Ker}(T) = \{0\}$, and this means that T is one-to-one.

9. TRUE. Recall that $\dim[\mathbb{R}^n] = n$ and $\dim[\mathbb{R}^m] = m$. In order for such an isomorphism to exist, $\mathbb{R}^n$ and $\mathbb{R}^m$ must have the same dimension; that is, $m = n$.

10. FALSE. For example, the vector space of all polynomials with real coefficients is an infinite-dimensional real vector space, and since $\mathbb{R}^n$ is finite-dimensional for all positive integers n , this statement is false.

11. FALSE. In order for this to be true, it would also have to be assumed that T_1 is onto. For example, suppose $V_1 = V_2 = V_3 = \mathbb{R}^2$. If we define $T_2(x, y) = (x, y)$ for all (x, y) in $\mathbb{R}^2$, then T_2 is onto. However, if we define $T_1(x, y) = (0, 0)$ for all (x, y) in $\mathbb{R}^2$, then $(T_2T_1)(x, y) = T_2(T_1(x, y)) = T_2(0, 0) = (0, 0)$ for all (x, y) in $\mathbb{R}^2$. Therefore T_2T_1 is not onto, since $\text{Rng}(T_2T_1) = \{(0, 0)\}$, even though T_2 itself is onto.

12. TRUE. This is a direct application of the Rank-Nullity Theorem. Since T is assumed to be onto, $\text{Rng}(T) = \mathbb{R}^3$, which is 3-dimensional. Therefore the dimension of $\text{Ker}(T)$ is $8 - 3 = 5$.

Problems:

1. $T_1T_2(\mathbf{x}) = T_1(T_2(\mathbf{x})) = T_1(B\mathbf{x}) = (AB)\mathbf{x}$, so $T_1T_2 = AB$. Similarly, $T_2T_1 = BA$.

$$AB = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -5 & -5 \\ 1 & 15 \end{bmatrix}, \quad BA = \begin{bmatrix} 1 & 5 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 2 & -4 \end{bmatrix}.$$

$$T_1T_2(\mathbf{x}) = (AB)\mathbf{x} = \begin{bmatrix} -5 & -5 \\ 1 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5x_1 - 5x_2 \\ x_1 + 15x_2 \end{bmatrix} = (-5(x_1 + x_2), x_1 + 15x_2) \text{ and}$$

$$T_2T_1(\mathbf{x}) = (BA)\mathbf{x} = \begin{bmatrix} 14 & 7 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14x_1 + 7x_2 \\ 2x_1 - 4x_2 \end{bmatrix} = (7(2x_1 + x_2), 2(x_1 - 2x_2)).$$

Clearly, $T_1T_2 \neq T_2T_1$.

$$\mathbf{2.} \quad T_2(T_1(\mathbf{x})) = B(A(\mathbf{x})) = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 - x_2 \\ 3x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \end{bmatrix} \mathbf{x}.$$

T_1T_2 does not exist because T_1 must have a domain of $\mathbb{R}^2$, yet the range of T_2 , which must be the domain of T_1 , is $\mathbb{R}$.

3. $\text{Ker}(T_1) = \{\mathbf{x} \in \mathbb{R}^2 : A\mathbf{x} = \mathbf{0}\}$. $A\mathbf{x} = \mathbf{0} \implies \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This matrix equation results in the system: $x_1 - x_2 = 0$ and $2x_1 - 2x_2 = 0$, or equivalently, $x_1 = x_2$. Thus,

$$\text{Ker}(T_1) = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = r(1, 1) \text{ where } r \in \mathbb{R}\}.$$

Geometrically, this is the line through the origin in $\mathbb{R}^2$ spanned by $(1, 1)$.

$\text{Ker}(T_2) = \{\mathbf{x} \in \mathbb{R}^2 : B\mathbf{x} = \mathbf{0}\}$. $B\mathbf{x} = \mathbf{0} \implies \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This matrix equation results in the system: $2x_1 + x_2 = 0$ and $3x_1 - x_2 = 0$, or equivalently, $x_1 = x_2 = 0$. Thus, $\text{Ker}(T_2) = \{\mathbf{0}\}$. Geometrically, this is a point (the origin).

$\text{Ker}(T_1T_2) = \{\mathbf{x} \in \mathbb{R}^2 : (AB)\mathbf{x} = \mathbf{0}\}$. $(AB)\mathbf{x} = \mathbf{0} \implies \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
 $\implies \begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This matrix equation results in the system:

$$-x_1 + 2x_2 = 0 \text{ and } -2x_1 + 4x_2 = 0,$$

or equivalently, $x_1 = 2x_2$.

Thus, $\text{Ker}(T_1T_2) = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = s(2, 1) \text{ where } s \in \mathbb{R}\}$. Geometrically, this is the line through the origin in $\mathbb{R}^2$ spanned by the vector $(2, 1)$.

$\text{Ker}(T_2T_1) = \{\mathbf{x} \in \mathbb{R}^2 : (BA)\mathbf{x} = \mathbf{0}\}$. $(BA)\mathbf{x} = \mathbf{0} \implies \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
 $\implies \begin{bmatrix} 4 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This matrix equation results in the system:

$$4x_1 - 4x_2 = 0 \text{ and } x_1 - x_2 = 0,$$

or equivalently, $x_1 = x_2$.

Thus, $\text{Ker}(T_2T_1) = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = t(1, 1) \text{ where } t \in \mathbb{R}\}$. Geometrically, this is the line through the origin in $\mathbb{R}^2$ spanned by the vector $(1, 1)$.

4.

$$(T_2T_1)(A) = T_2(T_1(A)) = T_2(A - A^T) = (A - A^T) + (A - A^T)^T = (A - A^T) + (A^T - A) = 0_n.$$

5. (a) $(T_1(f))(x) = \frac{d}{dx}[\sin(x - a)] = \cos(x - a).$

$$(T_2(f))(x) = \int_a^x \sin(t - a) dt = [-\cos(t - a)]_a^x = 1 - \cos(x - a).$$

$$(T_1T_2)(f)(x) = \frac{d}{dx}[1 - \cos(x - a)] = \sin(x - a) = f(x).$$

$$(T_2T_1)(f)(x) = \int_a^x \cos(t - a) dt = [\sin(t - a)]_a^x = \sin(x - a) = f(x). \text{ Consequently,}$$

$$(T_1T_2)(f) = (T_2T_1)(f) = f.$$

(b) $(T_1T_2)(f) = T_1(T_2(f)) = T_1\left(\int_a^x f(x) dx\right) = \frac{d}{dx}\left(\int_a^x f(x) dx\right) = f(x).$

$$(T_2T_1)(g) = T_2(T_1(g)) = T_2\left(\frac{dg(x)}{dx}\right) = \int_a^x \frac{dg(x)}{dx} dx = g(x) - g(a).$$

6. Let $\mathbf{v} \in V$. There exists $a, b \in \mathbb{R}$ such that $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$. Then

$$\begin{aligned} T_2T_1(\mathbf{v}) &= T_2[aT_1(\mathbf{v}_1) + bT_1(\mathbf{v}_2)] = T_2[a(\mathbf{v}_1 - \mathbf{v}_2) + b(2\mathbf{v}_1 + \mathbf{v}_2)] \\ &= T_2[(a + 2b)\mathbf{v}_1 + (b - a)\mathbf{v}_2] = (a + 2b)T_2(\mathbf{v}_1) + (b - a)T_2(\mathbf{v}_2) \\ &= (a + 2b)(\mathbf{v}_1 + 2\mathbf{v}_2) + (b - a)(3\mathbf{v}_1 - \mathbf{v}_2) = (5b - 2a)\mathbf{v}_1 + 3(a + b)\mathbf{v}_2. \end{aligned}$$

7. Let $\mathbf{v} \in V$. There exists $a, b \in \mathbb{R}$ such that $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$. Then

$$\begin{aligned} T_2T_1(\mathbf{v}) &= T_2[aT_1(\mathbf{v}_1) + bT_1(\mathbf{v}_2)] = T_2[a(3\mathbf{v}_1 + \mathbf{v}_2)] \\ &= 3a(-5\mathbf{v}_2) + a(-\mathbf{v}_1 + 6\mathbf{v}_2) = -a\mathbf{v}_1 - 9a\mathbf{v}_2. \end{aligned}$$

8. $\text{Ker}(T) = \{\mathbf{x} \in \mathbb{R}^2 : T(\mathbf{x}) = \mathbf{0}\} = \{\mathbf{x} \in \mathbb{R}^2 : A\mathbf{x} = \mathbf{0}\}$. The augmented matrix of the system $A\mathbf{x} = \mathbf{0}$ is: $\left[\begin{array}{cc|c} 4 & 2 & 0 \\ 1 & 3 & 0 \end{array} \right]$, with reduced row-echelon form of $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right]$. It follows that $\text{Ker}(T) = \{\mathbf{0}\}$. Hence T is one-to-one by Theorem 5.4.7.

For the given transformation, $\text{Rng}(T) = \{\mathbf{y} \in \mathbb{R}^2 : A\mathbf{x} = \mathbf{y} \text{ is consistent}\}$. The augmented matrix of the system $A\mathbf{x} = \mathbf{y}$ is $\left[\begin{array}{cc|c} 4 & 2 & y_1 \\ 1 & 3 & y_2 \end{array} \right]$ with reduced row-echelon form of $\left[\begin{array}{cc|c} 1 & 0 & (3y_1 - y_2)/5 \\ 0 & 1 & (2y_2 - y_1)/5 \end{array} \right]$. The system is, therefore, consistent for all (y_1, y_2) , so that $\text{Rng}(T) = \mathbb{R}^2$. Consequently, T is onto. T^{-1} exists since T has been shown to be both one-to-one and onto. Using the Gauss-Jordan method for computing A^{-1} , we have

$$\left[\begin{array}{cc|cc} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{5}{2} & -\frac{1}{4} & 1 \end{array} \right] \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{10} & -\frac{1}{5} \\ 0 & 1 & -\frac{1}{10} & \frac{3}{5} \end{array} \right].$$

Thus, $A^{-1} = \left[\begin{array}{cc} \frac{3}{10} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{5} \end{array} \right]$, so $T^{-1}(\mathbf{y}) = A^{-1}\mathbf{y}$.

9. $\text{Ker}(T) = \{\mathbf{x} \in \mathbb{R}^2 : T(\mathbf{x}) = \mathbf{0}\} = \{\mathbf{x} \in \mathbb{R}^2 : A\mathbf{x} = \mathbf{0}\}$. The augmented matrix of the system $A\mathbf{x} = \mathbf{0}$ is: $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ -2 & -4 & 0 \end{array} \right]$, with reduced row-echelon form of $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right]$. It follows that

$$\text{Ker}(T) = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = t(-2, 1), t \in \mathbb{R}\}.$$

By Theorem 5.4.7, since $\text{Ker}(T) \neq \{\mathbf{0}\}$, T is not one-to-one. This also implies that T^{-1} does not exist. For the given transformation, $\text{Rng}(T) = \{\mathbf{y} \in \mathbb{R}^2 : A\mathbf{x} = \mathbf{y} \text{ is consistent}\}$. The augmented matrix of the system $A\mathbf{x} = \mathbf{y}$ is $\left[\begin{array}{cc|c} 1 & 2 & y_1 \\ -2 & -4 & y_2 \end{array} \right]$ with reduced row-echelon form of $\left[\begin{array}{cc|c} 1 & 2 & y_1 \\ 0 & 0 & 2y_1 + y_2 \end{array} \right]$. The last row of this matrix implies that $2y_1 + y_2 = 0$ is required for consistency. Therefore, it follows that

$$\text{Rng}(T) = \{(y_1, y_2) \in \mathbb{R}^2 : 2y_1 + y_2 = 0\} = \{\mathbf{y} \in \mathbb{R}^2 : \mathbf{y} = s(1, -2), s \in \mathbb{R}\}.$$

T is not onto because $\dim[\text{Rng}(T)] = 1 \neq 2 = \dim[\mathbb{R}^2]$.

10. $\text{Ker}(T) = \{\mathbf{x} \in \mathbb{R}^3 : T(\mathbf{x}) = \mathbf{0}\} = \{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{0}\}$. The augmented matrix of the system $A\mathbf{x} = \mathbf{0}$ is: $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 5 & 1 & 0 \end{array} \right]$, with reduced row-echelon form $\left[\begin{array}{ccc|c} 1 & 0 & -7 & 0 \\ 0 & 1 & 3 & 0 \end{array} \right]$. It follows that

$$\text{Ker}(T) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = 7t, x_2 = -3t, x_3 = t, t \in \mathbb{R}\} = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = t(7, -3, 1), t \in \mathbb{R}\}.$$

By Theorem 5.4.7, since $\text{Ker}(T) \neq \{\mathbf{0}\}$, T is not one-to-one. This also implies that T^{-1} does not exist. For the given transformation, $\text{Rng}(T) = \{\mathbf{y} \in \mathbb{R}^2 : A\mathbf{x} = \mathbf{y} \text{ is consistent}\}$. The augmented matrix of the system $A\mathbf{x} = \mathbf{y}$ is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & y_1 \\ 2 & 5 & 1 & y_2 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -7 & 5y_1 - 2y_2 \\ 0 & 1 & 3 & y_2 - 2y_1 \end{array} \right].$$

We clearly have a consistent system for all $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$, thus $\text{Rng}(T) = \mathbb{R}^2$. Therefore, T is onto by Definition 5.3.3.

11. Reducing A to row-echelon form, we obtain

$$\text{REF}(A) = \left[\begin{array}{ccc} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right].$$

We quickly find that

$$\text{Ker}(T) = \text{nullspace}(A) = \text{span}\{(1, -2, 1)\}.$$

Moreover,

$$\text{Rng}(T) = \text{colspace}(A) = \text{span} \left\{ \begin{bmatrix} 0 \\ 3 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 4 \\ 1 \end{bmatrix} \right\}.$$

Based on these calculations, we see that T is neither one-to-one nor onto.

12. We have

$$(0, 0, 1) = -\frac{1}{3}(2, 1, -3) + \frac{2}{3}(1, 0, 0) + \frac{1}{3}(0, 1, 0),$$

and so

$$\begin{aligned} T(0, 0, 1) &= -\frac{1}{3}T(2, 1, -3) + \frac{2}{3}T(1, 0, 0) + \frac{1}{3}T(0, 1, 0) \\ &= -\frac{1}{3}(7, -1) + \frac{2}{3}(4, 5) + \frac{1}{3}(-1, 1) \\ &= (0, 4). \end{aligned}$$

(a) From the given information and the above calculation, we find that the matrix of T is $A = \begin{bmatrix} 4 & -1 & 0 \\ 5 & 1 & 4 \end{bmatrix}$.

(b) Because A has more columns than rows, $\text{REF}(A)$ must have an unpivoted column, which implies that $\text{nullspace}(A) \neq \{\mathbf{0}\}$. Hence, T is not one-to-one. On the other hand, $\text{colspace}(A) = \mathbb{R}^2$ since the first two columns, for example, are linearly independent vectors in $\mathbb{R}^2$. Thus, T is onto.

13. Show T is a linear transformation:

Let $\mathbf{x}, \mathbf{y} \in V$ and $c, \lambda \in \mathbb{R}$ where $\lambda \neq 0$.

$T(\mathbf{x} + \mathbf{y}) = \lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y} = T(\mathbf{x}) + T(\mathbf{y})$, and

$T(c\mathbf{x}) = \lambda(c\mathbf{x}) = c(\lambda\mathbf{x}) = cT(\mathbf{x})$. Thus, T is a linear transformation.

Show T is one-to-one:

$\mathbf{x} \in \text{Ker}(T) \iff T(\mathbf{x}) = \mathbf{0} \iff \lambda\mathbf{x} = \mathbf{0} \iff \mathbf{x} = \mathbf{0}$, since $\lambda \neq 0$. Thus, $\text{Ker}(T) = \{\mathbf{0}\}$. By Theorem 5.4.7, T is one-to-one.

Show T is onto:

$$\dim[\text{Ker}(T)] + \dim[\text{Rng}(T)] = \dim[V] \implies \dim[\{\mathbf{0}\}] + \dim[\text{Rng}(T)] = \dim[V]$$

$$\implies 0 + \dim[\text{Rng}(T)] = \dim[V] \implies \dim[\text{Rng}(T)] = \dim[V] \implies \text{Rng}(T) = V. \text{ Thus, } T \text{ is onto by Definition 5.3.3.}$$

Find T^{-1} :

$$T^{-1}(\mathbf{x}) = \frac{1}{\lambda}\mathbf{x} \text{ since } T(T^{-1}(\mathbf{x})) = T\left(\frac{1}{\lambda}\mathbf{x}\right) = \lambda\left(\frac{1}{\lambda}\mathbf{x}\right) = \mathbf{x}.$$

14. $T: P_1 \rightarrow P_1$ where $T(ax + b) = (2b - a)x + (b + a)$.

Show T is one-to-one:

$\text{Ker}(T) = \{p \in P_1 : T(p) = 0\} = \{ax + b \in P_1 : (2b - a)x + (b + a) = 0\}$. Thus, a and b must satisfy the system $2b - a = 0$, $b + a = 0$. The only solution to this system is $a = b = 0$. Consequently, $\text{Ker}(T) = \{0\}$, so T is one-to-one.

Show T is onto:

$$\text{Since } \text{Ker}(T) = \{0\}, \dim[\text{Ker}(T)] = 0. \text{ Thus, } \dim[\text{Ker}(T)] + \dim[\text{Rng}(T)] = \dim[P_1]$$

$\implies \dim[\text{Rng}(T)] = \dim[P_1]$, and since $\text{Rng}(T)$ is a subspace of P_1 , it follows that $\text{Rng}(T) = P_1$. Consequently, T is onto by Theorem 5.4.7.

Determine T^{-1} :

Since T is one-to-one and onto, T^{-1} exists. $T(ax + b) = (2b - a)x + (b + a)$

$\implies T^{-1}[(2b - a)x + (b + a)] = ax + b$. If we let $A = 2b - a$ and $B = a + b$, then $b = \frac{A + B}{3}$ and $a = \frac{2B - A}{3}$ so that $T^{-1}[Ax + B] = \frac{1}{3}(2B - A)x + \frac{1}{3}(A + B)$.

15. T is not one-to-one:

$\text{Ker}(T) = \{p \in P_2 : T(p) = 0\} = \{ax^2 + bx + c : c + (a - b)x = 0\}$. Thus, a, b , and c must satisfy the system $c = 0$ and $a - b = 0$. Consequently, $\text{Ker}(T) = \{r(x^2 + x) : r \in \mathbb{R}\} \neq \{0\}$. Thus, by Theorem 5.4.7, T is not one-to-one. T^{-1} does not exist because T is not one-to-one.

T is onto:

Since $\text{Ker}(T) = \{r(x^2 + x) : r \in \mathbb{R}\}$, we see that $\dim[\text{Ker}(T)] = 1$.

Thus, $\dim[\text{Ker}(T)] + \dim[\text{Rng}(T)] = \dim[P_2] \implies 1 + \dim[\text{Rng}(T)] = 3 \implies \dim[\text{Rng}(T)] = 2$. Since $\text{Rng}(T)$ is a subspace of P_1 ,

$$2 = \dim[\text{Rng}(T)] \leq \dim[P_1] = 2,$$

and so equality holds: $\text{Rng}(T) = P_1$. Thus, T is onto by Theorem 5.4.7.

16. $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for V and $T : V \rightarrow V$ is a linear transformation.

Show T is one-to-one:

Any $\mathbf{v} \in V$ can be expressed as $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$ where $a, b \in \mathbb{R}$.

$$T(\mathbf{v}) = \mathbf{0} \iff T(a\mathbf{v}_1 + b\mathbf{v}_2) = \mathbf{0} \iff aT(\mathbf{v}_1) + bT(\mathbf{v}_2) = \mathbf{0} \iff a(\mathbf{v}_1 + 2\mathbf{v}_2) + b(2\mathbf{v}_1 - 3\mathbf{v}_2) = \mathbf{0}$$

$$\iff (a + 2b)\mathbf{v}_1 + (2a - 3b)\mathbf{v}_2 = \mathbf{0} \iff a + 2b = 0 \text{ and } 2a - 3b = 0 \iff a = b = 0 \iff \mathbf{v} = \mathbf{0}. \text{ Hence } \text{Ker}(T) = \{\mathbf{0}\}. \text{ Therefore } T \text{ is one-to-one.}$$

Show T is onto:

$\dim[\text{Ker}(T)] = \dim[\{\mathbf{0}\}] = 0$ and $\dim[\text{Ker}(T)] + \dim[\text{Rng}(T)] = \dim[V]$ implies that $0 + \dim[\text{Rng}(T)] = 2$ or $\dim[\text{Rng}(T)] = 2$. Since $\text{Rng}(T)$ is a subspace of V , it follows that $\text{Rng}(T) = V$. Thus, T is onto by Theorem 5.4.7.

Determine T^{-1} :

Since T is one-to-one and onto, T^{-1} exists.

$$T(a\mathbf{v}_1 + b\mathbf{v}_2) = (a + 2b)\mathbf{v}_1 + (2a - 3b)\mathbf{v}_2$$

$$\implies T^{-1}[(a + 2b)\mathbf{v}_1 + (2a - 3b)\mathbf{v}_2] = (a\mathbf{v}_1 + b\mathbf{v}_2). \text{ If we let } A = a + 2b \text{ and } B = 2a - 3b, \text{ then } a = \frac{1}{7}(3A + 2B) \text{ and } b = \frac{1}{7}(2A - B). \text{ Hence, } T^{-1}(A\mathbf{v}_1 + B\mathbf{v}_2) = \frac{1}{7}(3A + 2B)\mathbf{v}_1 + \frac{1}{7}(2A - B)\mathbf{v}_2.$$

17. Let $\mathbf{v} \in V$. Then there exists $a, b \in \mathbb{R}$ such that $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$.

$$\begin{aligned} (T_1 T_2)\mathbf{v} &= T_1[aT_2(\mathbf{v}_1) + bT_2(\mathbf{v}_2)] = T_1\left[\frac{a}{2}(\mathbf{v}_1 + \mathbf{v}_2) + \frac{b}{2}(\mathbf{v}_1 - \mathbf{v}_2)\right] \\ &= \frac{a+b}{2}T_1(\mathbf{v}_1) + \frac{a-b}{2}T_1(\mathbf{v}_2) = \frac{a+b}{2}(\mathbf{v}_1 + \mathbf{v}_2) + \frac{a-b}{2}(\mathbf{v}_1 - \mathbf{v}_2) \\ &= \left(\frac{a+b}{2} + \frac{a-b}{2}\right)\mathbf{v}_1 + \left(\frac{a+b}{2} - \frac{a-b}{2}\right)\mathbf{v}_2 = a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{v}. \\ (T_2 T_1)\mathbf{v} &= T_2[aT_1(\mathbf{v}_1) + bT_1(\mathbf{v}_2)] = T_2[a(\mathbf{v}_1 + \mathbf{v}_2) + b(\mathbf{v}_1 - \mathbf{v}_2)] \\ &= T_2[(a+b)\mathbf{v}_1 + (a-b)\mathbf{v}_2] = (a+b)T_2(\mathbf{v}_1) + (a-b)T_2(\mathbf{v}_2) \\ &= \frac{a+b}{2}(\mathbf{v}_1 + \mathbf{v}_2) + \frac{a-b}{2}(\mathbf{v}_1 - \mathbf{v}_2) = a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{v}. \end{aligned}$$

Since $(T_1 T_2)\mathbf{v} = \mathbf{v}$ and $(T_2 T_1)\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V$, it follows that T_2 is the inverse of T_1 , thus $T_2 = T_1^{-1}$.

18. An arbitrary vector in P_1 can be written as

$$p(x) = ap_0(x) + bp_1(x),$$

where $p_0(x) = 1$, and $p_1(x) = x$ denote the standard basis vectors in P_1 . Hence, we can define an isomorphism $T : \mathbb{R}^2 \rightarrow P_1$ by

$$T(a, b) = a + bx.$$

19. Let S denote the subspace of $M_2(\mathbb{R})$ consisting of all upper triangular matrices. An arbitrary vector in S can be written as

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, we define an isomorphism $T : \mathbb{R}^3 \rightarrow S$ by

$$T(a, b, c) = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}.$$

20. Let S denote the subspace of $M_2(\mathbb{R})$ consisting of all skew-symmetric matrices. An arbitrary vector in S can be written as

$$\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = a \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Therefore, we can define an isomorphism $T : \mathbb{R} \rightarrow S$ by

$$T(a) = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}.$$

21. Let S denote the subspace of $M_2(\mathbb{R})$ consisting of all symmetric matrices. An arbitrary vector in S can be written as

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, we can define an isomorphism $T : \mathbb{R}^3 \rightarrow S$ by

$$T(a, b, c) = \begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

22. A typical vector in V takes the form $A = \begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix}$. Therefore, we can define $T : V \rightarrow \mathbb{R}^{10}$ via

$$T(A) = (a, b, c, d, e, f, g, h, i, j).$$

It is routine to verify that T is an invertible linear transformation. Therefore, we have $n = 10$.

23. A typical vector in V takes the form $p = a_0 + a_2x^2 + a_4x^4 + a_6x^6 + a_8x^8$. Therefore, we can define $T : V \rightarrow \mathbb{R}^5$ via

$$T(p) = (a_0, a_2, a_4, a_6, a_8).$$

It is routine to verify that T is an invertible linear transformation. Therefore, we have $n = 5$.

24. We have

$$T_1^{-1}(\mathbf{x}) = A^{-1}\mathbf{x} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \mathbf{x}.$$

25. We have

$$T_2^{-1}(\mathbf{x}) = A^{-1}\mathbf{x} = \begin{bmatrix} -1/3 & -1/6 \\ 1/3 & 2/3 \end{bmatrix} \mathbf{x}.$$

26. We have

$$T_3^{-1}(\mathbf{x}) = A^{-1}\mathbf{x} = \begin{bmatrix} 11/14 & -8/7 & 1/14 \\ -3/7 & 5/7 & 1/7 \\ 3/14 & 1/7 & -1/14 \end{bmatrix} \mathbf{x}.$$

27. We have

$$T_4^{-1}(\mathbf{x}) = A^{-1}\mathbf{x} = \begin{bmatrix} 8 & -29 & 3 \\ -5 & 19 & -2 \\ 2 & -8 & 1 \end{bmatrix} \mathbf{x}.$$

28. The matrix of T_2T_1 is $\begin{bmatrix} -4 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -6 & -7 \\ 6 & 8 \end{bmatrix}$. The matrix of T_1T_2 is $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & 4 \end{bmatrix}$.

29. The matrix of T_4T_3 is $\begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 3 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 13 & 18 \\ 4 & 8 & 6 \\ 23 & 43 & 11 \end{bmatrix}$. The matrix of T_3T_4 is $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 3 & 5 & -1 \end{bmatrix} \begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 10 & 25 & 23 \\ 5 & 14 & 15 \\ 12 & 19 & 1 \end{bmatrix}$.

30. We have

$$\begin{aligned} (T_2T_1)(c\mathbf{u}_1) &= T_2(T_1(c\mathbf{u}_1)) \\ &= T_2(cT_1(\mathbf{u}_1)) \\ &= c(T_2(T_1(\mathbf{u}_1))) \\ &= c(T_2T_1)(\mathbf{u}_1), \end{aligned}$$

as needed.

31. We first prove that if T is one-to-one, then T is onto. The assumption that T is one-to-one implies that $\dim[\text{Ker}(T)] = 0$. Hence, by Theorem 5.3.8,

$$\dim[W] = \dim[V] = \dim[\text{Rng}(T)],$$

which implies that $\text{Rng}(T) = W$. That is, T is onto.

Next we show that if T is onto, then T is one-to-one. We have $\text{Rng}(T) = W$, so that $\dim[\text{Rng}(T)] = \dim[W] = \dim[V]$. Hence, by Theorem 5.3.8, we have $\dim[\text{Ker}(T)] = 0$. Hence, $\text{Ker}(T) = \{\mathbf{0}\}$, which implies that T is one-to-one.

32. If $T : V \rightarrow W$ is linear, then show that $T^{-1} : W \rightarrow V$ is also linear.

Let $\mathbf{y}, \mathbf{z} \in W$, $c \in \mathbb{R}$, and $T(\mathbf{u}) = \mathbf{y}$ and $T(\mathbf{v}) = \mathbf{z}$ where $\mathbf{u}, \mathbf{v} \in V$. Then $T^{-1}(\mathbf{y}) = \mathbf{u}$ and $T^{-1}(\mathbf{z}) = \mathbf{v}$. Thus,

$$T^{-1}(\mathbf{y} + \mathbf{z}) = T^{-1}(T(\mathbf{u}) + T(\mathbf{v})) = T^{-1}(T(\mathbf{u} + \mathbf{v})) = \mathbf{u} + \mathbf{v} = T^{-1}(\mathbf{y}) + T^{-1}(\mathbf{z}),$$

and

$$T^{-1}(c\mathbf{y}) = T^{-1}(cT(\mathbf{u})) = T^{-1}(T(c\mathbf{u})) = c\mathbf{u} = cT^{-1}(\mathbf{y}).$$

Hence, T^{-1} is a linear transformation.

33. Let $T : V \rightarrow V$ be a one-to-one linear transformation. Since T is one-to-one, it follows from Theorem 5.4.7 that $\text{Ker}(T) = \{\mathbf{0}\}$, so $\dim[\text{Ker}(T)] = 0$. By Theorem 5.3.8 and substitution, $\dim[\text{Ker}(T)] + \dim[\text{Rng}(T)] = \dim[V] \implies 0 + \dim[\text{Rng}(T)] = \dim[V] \implies \dim[\text{Rng}(T)] = \dim[V]$, and since $\text{Rng}(T)$ is a subspace of V , it follows that $\text{Rng}(T) = V$, thus T is onto. T^{-1} exists because T is both one-to-one and onto.

34. To show that $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_k)\}$ is linearly independent, assume that

$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \dots + c_kT(\mathbf{v}_k) = \mathbf{0}.$$

We must show that $c_1 = c_2 = \dots = c_k = 0$. Using the linearity properties of T , we can write

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k) = \mathbf{0}.$$

Now, since T is one-to-one, we can conclude that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k = \mathbf{0}$, and since $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is linearly independent, we conclude that $c_1 = c_2 = \dots = c_k = 0$ as desired.

35. To prove that T is onto, let $\mathbf{w}$ be an arbitrary vector in W . We must find a vector $\mathbf{v}$ in V such that $T(\mathbf{v}) = \mathbf{w}$. Since $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$ spans W , we can write

$$\mathbf{w} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \dots + c_m\mathbf{w}_m$$

for some scalars $c_1, c_2, \dots, c_m$. Therefore

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \dots + c_mT(\mathbf{v}_m) = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \dots + c_m\mathbf{w}_m = \mathbf{w},$$

which shows that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_m\mathbf{v}_m$ maps under T to $\mathbf{w}$, as desired.

36. Since T is a linear transformation, $\text{Rng}(T)$ is a subspace of W , but $\dim[W] = \dim[\text{Rng}(T)] = n$, so $W = \text{Rng}(T)$, which means, by definition, that T is onto.

37. CORRECTION: ASSUME THAT T_1 IS ONE-TO-ONE. Let $\mathbf{v}$ be an arbitrary vector in V . Since $(T_1T_2)(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v}$ in V , we can use $T_1(\mathbf{v})$ instead of $\mathbf{v}$. Thus, $(T_1T_2)(T_1(\mathbf{v})) = T_1(\mathbf{v})$. Since T_1 is one-to-one, we conclude that $T_2(T_1(\mathbf{v})) = \mathbf{v}$. That is, $(T_2T_1)(\mathbf{v}) = \mathbf{v}$, as required.

38. Suppose that $\mathbf{x}$ belongs to $\text{Rng}(T)$. This means that there exists a vector $\mathbf{v}$ in V such that $T(\mathbf{v}) = \mathbf{x}$. Applying T to both sides of this equation, we have $T(T(\mathbf{v})) = T(\mathbf{x})$, or $T^2(\mathbf{v}) = T(\mathbf{x})$. However, since $T^2 = 0$, we conclude that $T^2(\mathbf{v}) = \mathbf{0}$, and hence, $T(\mathbf{x}) = \mathbf{0}$. By definition, this means that $\mathbf{x}$ belongs to $\text{Ker}(T)$. Thus, since every vector in $\text{Rng}(T)$ belongs to $\text{Ker}(T)$, $\text{Rng}(T)$ is a subset of $\text{Ker}(T)$. In addition, $\text{Rng}(T)$ is closed under addition and scalar multiplication, by Theorem 5.3.5, and therefore, $\text{Rng}(T)$ forms a subspace of $\text{Ker}(T)$.

39.

(a) To show that $T_2T_1 : V_1 \rightarrow V_3$ is one-to-one, we show that $\text{Ker}(T_2T_1) = \{\mathbf{0}\}$. Suppose that $\mathbf{v}_1 \in \text{Ker}(T_2T_1)$. This means that $(T_2T_1)(\mathbf{v}_1) = \mathbf{0}$. Hence, $T_2(T_1(\mathbf{v}_1)) = \mathbf{0}$. However, since T_2 is one-to-one, we conclude that $T_1(\mathbf{v}_1) = \mathbf{0}$. Next, since T_1 is one-to-one, we conclude that $\mathbf{v}_1 = \mathbf{0}$, which shows that the only vector in $\text{Ker}(T_2T_1)$ is $\mathbf{0}$, as expected.

(b) To show that $T_2T_1 : V_1 \rightarrow V_3$ is onto, we begin with an arbitrary vector $\mathbf{v}_3$ in V_3 . Since $T_2 : V_2 \rightarrow V_3$ is onto, there exists $\mathbf{v}_2$ in V_2 such that $T_2(\mathbf{v}_2) = \mathbf{v}_3$. Moreover, since $T_1 : V_1 \rightarrow V_2$ is onto, there exists $\mathbf{v}_1$ in V_1 such that $T_1(\mathbf{v}_1) = \mathbf{v}_2$. Therefore,

$$(T_2T_1)(\mathbf{v}_1) = T_2(T_1(\mathbf{v}_1)) = T_2(\mathbf{v}_2) = \mathbf{v}_3,$$

and therefore, we have found a vector, namely $\mathbf{v}_1$, in V_1 that is mapped under T_2T_1 to $\mathbf{v}_3$. Hence, T_2T_1 is onto.

(c) This follows immediately from parts (a) and (b).

Solutions to Section 5.5

True-False Review:

1. **FALSE.** The matrix representation is an $m \times n$ matrix, not an $n \times m$ matrix.
2. **FALSE.** The matrix $[T]_C^B$ would only make sense if C was a basis for V and B was a basis for W , and this would only be true if V and W were the same vector space. Of course, in general V and W can be different, and so this statement does not hold.
3. **FALSE.** The correct equation is given in (5.5.2).
4. **FALSE.** The correct statement is $([T]_B^C)^{-1} = [T^{-1}]_C^B$.
5. **TRUE.** Many examples are possible. A fairly simple one is the following. Let $T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $T_1(x, y) = (x, y)$, and let $T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $T_2(x, y) = (y, x)$. Clearly, T_1 and T_2 are different linear transformations. Now if $B_1 = \{(1, 0), (0, 1)\} = C_1$, then $[T_1]_{B_1}^{C_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If $B_2 = \{(1, 0), (0, 1)\}$ and $C_2 = \{(0, 1), (1, 0)\}$, then $[T_2]_{B_2}^{C_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus, although $T_1 \neq T_2$, we found suitable bases B_1, C_1, B_2, C_2 such that $[T_1]_{B_1}^{C_1} = [T_2]_{B_2}^{C_2}$.
6. **TRUE.** This is the content of part (b) of Corollary 5.5.10.

Problems:

1.

(a): We must determine $T(1)$, $T(x)$, and $T(x^2)$, and find the components of the resulting vectors in $\mathbb{R}^2$ relative to the basis C . We have

$$T(1) = (1, 2), \quad T(x) = (0, 1), \quad T(x^2) = (-3, -2).$$

Therefore, relative to the standard basis C on $\mathbb{R}^2$, we have

$$[T(1)]_C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad [T(x)]_C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad [T(x^2)]_C = \begin{bmatrix} -3 \\ -2 \end{bmatrix}.$$

Therefore,

$$[T]_B^C = \begin{bmatrix} 1 & 0 & -3 \\ 2 & 1 & -2 \end{bmatrix}.$$

(b): We must determine $T(1)$, $T(1+x)$, and $T(1+x+x^2)$, and find the components of the resulting vectors in $\mathbb{R}^2$ relative to the basis C . We have

$$T(1) = (1, 2), \quad T(1+x) = (1, 3), \quad T(1+x+x^2) = (-2, 1).$$

Setting

$$T(1) = (1, 2) = c_1(1, -1) + c_2(2, 1)$$

and solving, we find $c_1 = -1$ and $c_2 = 1$. Thus, $[T(1)]_C = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Next, setting

$$T(1+x) = (1, 3) = c_1(1, -1) + c_2(2, 1)$$

and solving, we find $c_1 = -5/3$ and $c_2 = 4/3$. Thus, $[T(1+x)]_C = \begin{bmatrix} -5/3 \\ 4/3 \end{bmatrix}$. Finally, setting

$$T(1+x+x^2) = (-2, 1) = c_1(1, -1) + c_2(2, 1)$$

and solving, we find $c_1 = -4/3$ and $c_2 = -1/3$. Thus, $[T(1+x+x^2)]_C = \begin{bmatrix} -4/3 \\ -1/3 \end{bmatrix}$. Putting the results for $[T(1)]_C$, $[T(1+x)]_C$, and $[T(1+x+x^2)]_C$ into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} -1 & -5/3 & -4/3 \\ 1 & 4/3 & -1/3 \end{bmatrix}.$$

2.

(a): We must determine $T(E_{11})$, $T(E_{12})$, $T(E_{21})$, and $T(E_{22})$, and find the components of the resulting vectors in P_3 relative to the basis C . We have

$$T(E_{11}) = 1 - x^3, \quad T(E_{12}) = 3x^2, \quad T(E_{21}) = x^3, \quad T(E_{22}) = -1.$$

Therefore, relative to the standard basis C on P_3 , we have

$$[T(E_{11})]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad [T(E_{12})]_C = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \quad [T(E_{21})]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad [T(E_{22})]_C = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Putting these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}.$$

(b): The values of $T(E_{21})$, $T(E_{11})$, $T(E_{22})$, and $T(E_{12})$ were determined in part (a). We must express those results in terms of the *ordered* basis C given in (b). We have

$$[T(E_{21})]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad [T(E_{11})]_C = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad [T(E_{22})]_C = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(E_{12})]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}.$$

Putting these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

3.

(a): We must determine $T(1, 0, 0)$, $T(0, 1, 0)$, and $T(0, 0, 1)$, and find the components of the resulting vectors relative to the basis C . We have

$$T(1, 0, 0) = \cos x, \quad T(0, 1, 0) = 3 \sin x, \quad T(0, 0, 1) = -2 \cos x + \sin x.$$

Therefore, relative to the basis C , we have

$$[T(1, 0, 0)]_C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad [T(0, 1, 0)]_C = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad [T(0, 0, 1)]_C = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Putting these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & 1 \end{bmatrix}.$$

(b): We must determine $T(2, -1, -1)$, $T(1, 3, 5)$, and $T(0, 4, -1)$, and find the components of the resulting vectors relative to the basis C . We have

$$T(2, -1, -1) = 4 \cos x - 4 \sin x, \quad T(1, 3, 5) = -9 \cos x + 14 \sin x, \quad T(0, 4, -1) = 2 \cos x + 11 \sin x.$$

Setting

$$4 \cos x - 4 \sin x = c_1(\cos x - \sin x) + c_2(\cos x + \sin x)$$

and solving, we find $c_1 = 4$ and $c_2 = 0$. Therefore $[T(2, -1, -1)]_C = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. Next, setting

$$-9 \cos x + 14 \sin x = c_1(\cos x - \sin x) + c_2(\cos x + \sin x)$$

and solving, we find $c_1 = -23/2$ and $c_2 = 5/2$. Therefore, $[T(1, 3, 5)]_C = \begin{bmatrix} -23/2 \\ 5/2 \end{bmatrix}$. Finally, setting

$$2 \cos x + 11 \sin x = c_1(\cos x - \sin x) + c_2(\cos x + \sin x)$$

and solving, we find $c_1 = -9/2$ and $c_2 = 13/2$. Therefore $[T(0, 4, -1)]_C = \begin{bmatrix} -9/2 \\ 13/2 \end{bmatrix}$. Putting these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} 4 & -\frac{23}{2} & -\frac{9}{2} \\ 0 & \frac{5}{2} & \frac{13}{2} \end{bmatrix}.$$

4.

(a): We must determine $T(1)$, $T(x)$, and $T(x^2)$, and find the components of the resulting vectors relative to the standard basis C on P_3 . We have

$$T(1) = 1 + x, \quad T(x) = x + x^2, \quad T(x^2) = x^2 + x^3.$$

Therefore,

$$[T(1)]_C = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x)]_C = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad [T(x^2)]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Putting these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b): We must determine $T(1)$, $T(x-1)$, and $T((x-1)^2)$, and find the components of the resulting vectors relative to the given basis C on P_3 . We have

$$T(1) = 1 + x, \quad T(x-1) = -1 + x^2, \quad T((x-1)^2) = 1 - 2x - x^2 + x^3.$$

Setting

$$1 + x = c_1(1) + c_2(x-1) + c_3(x-1)^2 + c_4(x-1)^3$$

and solving, we find $c_1 = 2$, $c_2 = 1$, $c_3 = 0$, and $c_4 = 0$. Therefore $[T(1)]_C = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Next, setting

$$-1 + x^2 = c_1(1) + c_2(x-1) + c_3(x-1)^2 + c_4(x-1)^3$$

and solving, we find $c_1 = 0$, $c_2 = 2$, $c_3 = 1$, and $c_4 = 0$. Therefore, $[T(x-1)]_C = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$. Finally, setting

$$1 - 2x - x^2 + x^3 = c_1(1) + c_2(x-1) + c_3(x-1)^2 + c_4(x-1)^3$$

and solving, we find $c_1 = -1$, $c_2 = -1$, $c_3 = 2$, and $c_4 = 1$. Therefore, $[T((x-1)^2)]_C = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$. Putting

these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

5.

(a): We must determine $T(1)$, $T(x)$, $T(x^2)$, and $T(x^3)$, and find the components of the resulting vectors relative to the standard basis C on P_2 . We have

$$T(1) = 0, \quad T(x) = 1, \quad T(x^2) = 2x, \quad T(x^3) = 3x^2.$$

Therefore, if C is the standard basis on P_2 , then we have

$$[T(1)]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x)]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x^2)]_C = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad [T(x^3)]_C = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.$$

Putting these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

(b): We must determine $T(x^3)$, $T(x^3 + 1)$, $T(x^3 + x)$, and $T(x^3 + x^2)$, and find the components of the resulting vectors relative to the given basis C on P_2 . We have

$$T(x^3) = 3x^2, \quad T(x^3 + 1) = 3x^2, \quad T(x^3 + x) = 3x^2 + 1, \quad T(x^3 + x^2) = 3x^2 + 2x.$$

Setting

$$3x^2 = c_1(1) + c_2(1 + x) + c_3(1 + x + x^2)$$

and solving, we find $c_1 = 0$, $c_2 = -3$, and $c_3 = 3$. Therefore $[T(x^3)]_C = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix}$. Likewise, $[T(x^3 + 1)]_C =$

$\begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix}$. Next, setting

$$3x^2 + 1 = c_1(1) + c_2(1 + x) + c_3(1 + x + x^2)$$

and solving, we find $c_1 = 1$, $c_2 = -3$, and $c_3 = 3$. Therefore, $[T(x^3 + x)]_C = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$. Finally, setting

$$3x^2 + 2x = c_1(1) + c_2(1 + x) + c_3(1 + x + x^2)$$

and solving, we find $c_1 = -2$, $c_2 = -1$, and $c_3 = 3$. Therefore, $[T(x^3 + 2x)]_C = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$. Putting these

results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} 0 & 0 & 1 & -2 \\ -3 & -3 & -3 & -1 \\ 3 & 3 & 3 & 3 \end{bmatrix}.$$

6.

(a): We must determine $T(E_{11})$, $T(E_{12})$, $T(E_{21})$, and $T(E_{22})$, and find the components of the resulting vectors relative to the standard basis C on $\mathbb{R}^2$. We have

$$T(E_{11}) = (1, 1), \quad T(E_{12}) = (0, 0), \quad T(E_{21}) = (0, 0), \quad T(E_{22}) = (1, 1).$$

Therefore, since C is the standard basis on $\mathbb{R}^2$, we have

$$[T(E_{11})]_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad [T(E_{12})]_C = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad [T(E_{21})]_C = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad [T(E_{22})]_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Putting these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

(b): Let us denote the four matrices in the ordered basis for B as A_1 , A_2 , A_3 , and A_4 , respectively. We must determine $T(A_1)$, $T(A_2)$, $T(A_3)$, and $T(A_4)$, and find the components of the resulting vectors relative to the standard basis C on $\mathbb{R}^2$. We have

$$T(A_1) = (-4, -4), \quad T(A_2) = (3, 3), \quad T(A_3) = (-2, -2), \quad T(A_4) = (0, 0).$$

Therefore, since C is the standard basis on $\mathbb{R}^2$, we have

$$[T(A_1)]_C = \begin{bmatrix} -4 \\ -4 \end{bmatrix}, \quad [T(A_2)]_C = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad [T(A_3)]_C = \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \quad [T(A_4)]_C = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Putting these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} -4 & 3 & -2 & 0 \\ -4 & 3 & -2 & 0 \end{bmatrix}.$$

7.

(a): We must determine $T(E_{11})$, $T(E_{12})$, $T(E_{21})$, and $T(E_{22})$, and find the components of the resulting vectors relative to the standard basis C on $M_2(\mathbb{R})$. We have

$$T(E_{11}) = E_{11}, \quad T(E_{12}) = 2E_{12} - E_{21}, \quad T(E_{21}) = 2E_{21} - E_{12}, \quad T(E_{22}) = E_{22}.$$

Therefore, we have

$$[T(E_{11})]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [T(E_{12})]_C = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \quad [T(E_{21})]_C = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \quad [T(E_{22})]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Putting these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b): Let us denote the four matrices in the ordered basis for B as A_1 , A_2 , A_3 , and A_4 , respectively. We must determine $T(A_1)$, $T(A_2)$, $T(A_3)$, and $T(A_4)$, and find the components of the resulting vectors relative to the standard basis C on $M_2(\mathbb{R})$. We have

$$T(A_1) = \begin{bmatrix} -1 & -2 \\ -2 & -3 \end{bmatrix}, \quad T(A_2) = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}, \quad T(A_3) = \begin{bmatrix} 0 & -8 \\ 7 & -2 \end{bmatrix}, \quad T(A_4) = \begin{bmatrix} 0 & 7 \\ -2 & 0 \end{bmatrix}.$$

Therefore, we have

$$[T(A_1)]_C = \begin{bmatrix} -1 \\ -2 \\ -2 \\ -3 \end{bmatrix}, \quad [T(A_2)]_C = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \quad [T(A_3)]_C = \begin{bmatrix} 0 \\ -8 \\ 7 \\ -2 \end{bmatrix}, \quad [T(A_4)]_C = \begin{bmatrix} 0 \\ 7 \\ -2 \\ 0 \end{bmatrix}.$$

Putting these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -2 & 0 & -8 & 7 \\ -2 & 3 & 7 & -2 \\ -3 & 2 & -2 & 0 \end{bmatrix}.$$

8.

(a): We must determine $T(e^{2x})$ and $T(e^{-3x})$, and find the components of the resulting vectors relative to the given basis C . We have

$$T(e^{2x}) = 2e^{2x} \quad \text{and} \quad T(e^{-3x}) = -3e^{-3x}.$$

Therefore, we have

$$[T]_B^C = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}.$$

(b): We must determine $T(e^{2x} - 3e^{-3x})$ and $T(2e^{-3x})$, and find the components of the resulting vectors relative to the given basis C . We have

$$T(e^{2x} - 3e^{-3x}) = 2e^{2x} + 9e^{-3x} \quad \text{and} \quad T(2e^{-3x}) = -6e^{-3x}.$$

Now, setting

$$2e^{2x} + 9e^{-3x} = c_1(e^{2x} + e^{-3x}) + c_2(-e^{2x})$$

and solving, we find $c_1 = 9$ and $c_2 = 7$. Therefore, $[T(e^{2x} - 3e^{-3x})]_C = \begin{bmatrix} 9 \\ 7 \end{bmatrix}$. Finally, setting

$$-6e^{-3x} = c_1(e^{2x} + e^{-3x}) + c_2(-e^{2x})$$

and solving, we find $c_1 = c_2 = -6$. Therefore, $[T(2e^{-3x})]_C = \begin{bmatrix} -6 \\ -6 \end{bmatrix}$. Putting these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} 9 & -6 \\ 7 & -6 \end{bmatrix}.$$

9.

(a): Let us first compute $[T]_B^C$. We must determine $T(1)$ and $T(x)$, and find the components of the resulting vectors relative to the standard basis $C = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ on $M_2(\mathbb{R})$. We have

$$T(1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad T(x) = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}.$$

Therefore

$$[T(1)]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad [T(x)]_C = \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

Hence,

$$[T]_B^C = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & -2 \\ -1 & 1 \end{bmatrix}.$$

Now,

$$[p(x)]_B = [-2 + 3x]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

Therefore, we have

$$[T(p(x))]_C = [T]_B^C [p(x)]_B = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ -6 \\ 5 \end{bmatrix}.$$

Thus,

$$T(p(x)) = \begin{bmatrix} -5 & 0 \\ -6 & 5 \end{bmatrix}.$$

(b): We have

$$T(p(x)) = T(-2 + 3x) = \begin{bmatrix} -5 & 0 \\ -6 & 5 \end{bmatrix}.$$

10.

(a): Let us first compute $[T]_B^C$. We must determine $T(1, 0, 0)$, $T(0, 1, 0)$, and $T(0, 0, 1)$, and find the components of the resulting vectors relative to the standard basis $C = \{1, x, x^2, x^3\}$ on P_3 . We have

$$T(1, 0, 0) = 2 - x - x^3, \quad T(0, 1, 0) = -x, \quad T(0, 0, 1) = x + 2x^3.$$

Therefore,

$$[T(1, 0, 0)]_C = \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \quad [T(0, 1, 0)]_C = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(0, 0, 1)]_C = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}.$$

Putting these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix}.$$

Now,

$$[\mathbf{v}]_B = [(2, -1, 5)]_B = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix},$$

and therefore,

$$[T(\mathbf{v})]_C = [T]_B^C [\mathbf{v}]_B = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \\ 8 \end{bmatrix}.$$

Therefore,

$$T(\mathbf{v}) = 4 + 4x + 8x^3.$$

(b): We have

$$T(\mathbf{v}) = T(2, -1, 5) = 4 + 4x + 8x^3.$$

11.

(a): Let us first compute $[T]_B^C$. We must determine $T(E_{11})$, $T(E_{12})$, $T(E_{21})$, and $T(E_{22})$, and find the components of the resulting vectors relative to the standard basis $C = \{E_{11}, E_{12}, E_{21}, E_{22}\}$. We have

$$T(E_{11}) = \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix}, \quad T(E_{12}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad T(E_{21}) = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}, \quad T(E_{22}) = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}.$$

Therefore,

$$[T(E_{11})]_C = \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \quad [T(E_{12})]_C = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad [T(E_{21})]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \quad [T(E_{22})]_C = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}.$$

Putting these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & 0 \end{bmatrix}.$$

Now,

$$[A]_B = \begin{bmatrix} -7 \\ 2 \\ 1 \\ -3 \end{bmatrix},$$

and therefore,

$$[T(A)]_C = [T]_B^C [A]_B = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} -7 \\ 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -19 \\ -2 \\ 0 \\ 8 \end{bmatrix}.$$

Hence,

$$T(A) = \begin{bmatrix} -19 & -2 \\ 0 & 8 \end{bmatrix}.$$

(b): We have

$$T(A) = \begin{bmatrix} -19 & -2 \\ 0 & 8 \end{bmatrix}.$$

12.

(a): Let us first compute $[T]_B^C$. We must determine $T(1)$, $T(x)$, and $T(x^2)$, and find the components of the resulting vectors relative to the standard basis $C = \{1, x, x^2, x^3, x^4\}$. We have

$$T(1) = x^2, \quad T(x) = x^3, \quad T(x^2) = x^4.$$

Therefore,

$$[T(1)]_C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x)]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad [T(x^2)]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Putting these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now,

$$[p(x)]_B = [-1 + 5x - 6x^2]_B = \begin{bmatrix} -1 \\ 5 \\ -6 \end{bmatrix},$$

and therefore,

$$[T(p(x))]_C = [T]_B^C [p(x)]_B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 5 \\ -6 \end{bmatrix}.$$

Hence,

$$T(p(x)) = -x^2 + 5x^3 - 6x^4.$$

(b): We have

$$T(p(x)) = -x^2 + 5x^3 - 6x^4.$$

13.

(a): Let us first compute $[T]_B^C$. We must determine $T(E_{ij})$ for $1 \leq i, j \leq 3$, and find components of the resulting vectors relative to the standard basis $C = \{1\}$. We have

$$T(E_{11}) = T(E_{22}) = T(E_{33}) = 1 \quad \text{and} \quad T(E_{12}) = T(E_{13}) = T(E_{23}) = T(E_{21}) = T(E_{31}) = T(E_{32}) = 0.$$

Therefore

$$[T(E_{11})]_C = [T(E_{22})]_C = [T(E_{33})]_C = [1] \quad \text{and all other component vectors are } [0].$$

Putting these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now,

$$[A]_B = \begin{bmatrix} 2 \\ -6 \\ 0 \\ 1 \\ 4 \\ -4 \\ 0 \\ 0 \\ -3 \end{bmatrix}.$$

and therefore,

$$[T(A)]_C = [T]_B^C [A]_B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -6 \\ 0 \\ 1 \\ 4 \\ -4 \\ 0 \\ 0 \\ -3 \end{bmatrix} = [3].$$

Hence, $T(A) = 3$.

(b): We have $T(A) = 3$.

14.

(a): Let us first compute $[T]_B^C$. We must determine $T(1)$, $T(x)$, $T(x^2)$, $T(x^3)$, and $T(x^4)$, and find the components of the resulting vectors relative to the standard basis $C = \{1, x, x^2, x^3\}$. We have

$$T(1) = 0, \quad T(x) = 1, \quad T(x^2) = 2x, \quad T(x^3) = 3x^2, \quad T(x^4) = 4x^3.$$

Therefore,

$$[T(1)]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x)]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x^2)]_C = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad [T(x^3)]_C = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \quad [T(x^4)]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}.$$

Putting these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

Now,

$$[p(x)]_B = [3 - 4x + 6x^2 + 6x^3 - 2x^4]_B = \begin{bmatrix} 3 \\ -4 \\ 6 \\ 6 \\ -2 \end{bmatrix},$$

and therefore,

$$[T(p(x))]_C = [T]_B^C [p(x)]_B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \\ 6 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 12 \\ 18 \\ -8 \end{bmatrix}.$$

Therefore,

$$T(p(x)) = T(3 - 4x + 6x^2 + 6x^3 - 2x^4) = -4 + 12x + 18x^2 - 8x^3.$$

(b): We have

$$T(p(x)) = p'(x) = -4 + 12x + 18x^2 - 8x^3.$$

15.

(a): Let us first compute $[T]_B^C$. We must determine $T(1)$, $T(x)$, $T(x^2)$, and $T(x^3)$, and find the components of the resulting vectors relative to the standard basis $C = \{1\}$. We have

$$T(1) = 1, \quad T(x) = 2, \quad T(x^2) = 4, \quad T(x^3) = 8.$$

Therefore,

$$[T(1)]_C = [1], \quad [T(x)]_C = [2], \quad [T(x^2)]_C = [4], \quad [T(x^3)]_C = [8].$$

Putting these results into the columns of $[T]_B^C$, we obtain

$$[T]_B^C = \begin{bmatrix} 1 & 2 & 4 & 8 \end{bmatrix}.$$

Now,

$$[p(x)]_B = [2x - 3x^2]_B = \begin{bmatrix} 0 \\ 2 \\ -3 \\ 0 \end{bmatrix},$$

and therefore,

$$[T(p(x))]_C = [T]_B^C [p(x)]_B = \begin{bmatrix} 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ -3 \\ 0 \end{bmatrix} = [-8].$$

Therefore, $T(p(x)) = -8$.

(b): We have $p(2) = 2 \cdot 2 - 3 \cdot 2^2 = -8$.

16. The linear transformation $T_2 T_1 : P_4 \rightarrow \mathbb{R}$ is given by

$$(T_2 T_1)(p(x)) = p'(2).$$

Let A denote the standard basis on P_4 , let B denote the standard basis on P_3 , and let C denote the standard basis on $\mathbb{R}$.

(a): To determine $[T_2 T_1]_A^C$, we compute

$$(T_2 T_1)(1) = 0, \quad (T_2 T_1)(x) = 1, \quad (T_2 T_1)(x^2) = 4, \quad (T_2 T_1)(x^3) = 12, \quad (T_2 T_1)(x^4) = 32.$$

Therefore

$$[(T_2T_1)(1)]_C = [0], \quad [(T_2T_1)(x)]_C = [1], \quad [(T_2T_1)(x^2)]_C = [4], \quad [(T_2T_1)(x^3)]_C = [12], \quad [(T_2T_1)(x^4)]_C = [32].$$

Putting these results into the columns of $[T_2T_1]_A^C$, we obtain

$$[T_2T_1]_A^C = \begin{bmatrix} 0 & 1 & 4 & 12 & 32 \end{bmatrix}.$$

(b): We have

$$[T_2]_B^C [T_1]_A^B = \begin{bmatrix} 1 & 2 & 4 & 8 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 4 & 12 & 32 \end{bmatrix} = [T_2T_1]_A^C.$$

(c): Let $p(x) = 2 + 5x - x^2 + 3x^4$. Then the component vector of $p(x)$ relative to the standard basis A is

$$[p(x)]_A = \begin{bmatrix} 2 \\ 5 \\ -1 \\ 0 \\ 3 \end{bmatrix}. \text{ Thus,}$$

$$[(T_2T_1)(p(x))]_C = [T_2T_1]_A^C [p(x)]_A = \begin{bmatrix} 0 & 1 & 4 & 12 & 32 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ -1 \\ 0 \\ 3 \end{bmatrix} = [97].$$

Therefore,

$$(T_2T_1)(2 + 5x - x^2 + 3x^4) = 97.$$

Of course,

$$p'(2) = 5 - 2 \cdot 2 + 12 \cdot 2^3 = 97$$

by direct calculation as well.

17. The linear transformation $T_2T_1 : P_1 \rightarrow \mathbb{R}^2$ is given by

$$(T_2T_1)(a + bx) = (0, 0).$$

Let A denote the standard basis on P_1 , let B denote the standard basis on $M_2(\mathbb{R})$, and let C denote the standard basis on $\mathbb{R}^2$.

(a): To determine $[T_2T_1]_A^C$, we compute

$$(T_2T_1)(1) = (0, 0) \quad \text{and} \quad (T_2T_1)(x) = (0, 0).$$

Therefore, we obtain $[T_2T_1]_A^C = 0_2$.

(b): We have

$$[T_2]_B^C [T_1]_A^B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & -2 \\ -1 & 1 \end{bmatrix} = 0_2 = [T_2T_1]_A^C.$$

(c): The component vector of $p(x) = -3 + 8x$ relative to the standard basis A is $[p(x)]_A = \begin{bmatrix} -3 \\ 8 \end{bmatrix}$. Thus,

$$[(T_2T_1)(p(x))]_C = [T_2T_1]_A^C[p(x)]_A = 0_2[p(x)]_A = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore,

$$(T_2T_1)(-3 + 8x) = (0, 0).$$

Of course

$$(T_2T_1)(-3 + 8x) = T_1 \left(\begin{bmatrix} -11 & 0 \\ -16 & 11 \end{bmatrix} \right) = (0, 0)$$

by direct calculation as well.

18. The linear transformation $T_2T_1 : P_2 \rightarrow P_2$ is given by

$$(T_2T_1)(p(x)) = [(x+1)p(x)]'.$$

Let A denote the standard basis on P_2 , let B denote the standard basis on P_3 , and let C denote the standard basis on P_2 .

(a): To determine $[T_2T_1]_A^C$, we compute as follows:

$$(T_2T_1)(1) = 1, \quad (T_2T_1)(x) = 1 + 2x, \quad (T_2T_1)(x^2) = 2x + 3x^2.$$

Therefore,

$$[(T_2T_1)(1)]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [(T_2T_1)(x)]_C = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad [(T_2T_1)(x^2)]_C = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}.$$

Putting these results into the columns of $[T_2T_1]_A^C$, we obtain

$$[T_2T_1]_A^C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

(b): We have

$$[T_2]_B^C[T_1]_A^B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} = [T_2T_1]_A^C.$$

(c): The component vector of $p(x) = 7 - x + 2x^2$ relative to the standard basis A is $[p(x)]_A = \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix}$.

Thus,

$$[(T_2T_1)(p(x))]_C = [T_2T_1]_A^C[p(x)]_A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 6 \end{bmatrix}.$$

Therefore,

$$(T_2T_1)(7 - x + 2x^2) = 6 + 2x + 6x^2.$$

Of course

$$(T_2T_1)(7 - x + 2x^2) = T_2((x + 1)(7 - x + 2x^2)) = T_2(7 + 6x + x^2 + 2x^3) = 6 + 2x + 6x^2$$

by direct calculation as well.

(d): YES. Since the matrix $[T_2T_1]_A^C$ computed in part (a) is invertible, T_2T_1 is invertible.

19. NO. The matrices $[T]_B^C$ obtained in Problem 2 are not invertible (they contain rows of zeros), and therefore, the corresponding linear transformation T is not invertible.

20. YES. We can explain this answer by using a matrix representation of T . Let $B = \{1, x, x^2\}$ and let $C = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Then $T(1) = (1, 1, 1)$, $T(x) = (0, 1, 2)$, and $T(x^2) = (0, 1, 4)$, and so

$$[T]_B^C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}.$$

Since this matrix is invertible, the corresponding linear transformation T is invertible.

21. Note that

$$\begin{aligned} \mathbf{w} \text{ belongs to } \text{Rng}(T) &\iff \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \text{ in } V \\ &\iff [\mathbf{w}]_C = [T(\mathbf{v})]_C \text{ for some } \mathbf{v} \text{ in } V \\ &\iff [\mathbf{w}]_C = [T]_B^C[\mathbf{v}]_B \text{ for some } \mathbf{v} \text{ in } V. \end{aligned}$$

The right-hand side of this last expression can be expressed as a linear combination of the columns of $[T]_B^C$, and therefore, $\mathbf{w}$ belongs to $\text{Rng}(T)$ if and only if $[\mathbf{w}]_C$ can be expressed as a linear combination of the columns of $[T]_B^C$. That is, if and only if $[\mathbf{w}]_C$ belongs to $\text{colspace}([T]_B^C)$.

Solutions to Section 5.6

True-False Review:

- 1. FALSE.** If $\mathbf{v} = \mathbf{0}$, then $A\mathbf{v} = \lambda\mathbf{v} = \mathbf{0}$, but by definition, an eigenvector must be a *nonzero* vector.
- 2. TRUE.** When we compute $\det(A - \lambda I)$ for an upper or lower triangular matrix A , the determinant is the product of the entries lying along the main diagonal of $A - \lambda I$:

$$\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda).$$

The roots of this characteristic equation are precisely the values $a_{11}, a_{22}, \dots, a_{nn}$ along the main diagonal of the matrix A .

- 3. TRUE.** The eigenvalues of a matrix are precisely the set of roots of its characteristic equation. Therefore, two matrices A and B that have the same characteristic equation will have the same eigenvalues.

- 4. FALSE.** Many examples of this can be found. As a simple one, consider $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. We have $\det(A - \lambda I) = (-\lambda)^2 = \lambda^2 = \det(B - \lambda I)$. Note that *every* nonzero vector in $\mathbb{R}^2$ is an eigenvector of A corresponding to $\lambda = 0$. However, only vectors of the form $\begin{bmatrix} a \\ 0 \end{bmatrix}$ with $a \neq 0$ are eigenvectors of B . Therefore, A and B do not have precisely the same set of eigenvectors. In this case, every eigenvector of B is also an eigenvector of A , but not conversely.

5. TRUE. Geometrically, all nonzero points $\mathbf{v} = (x, y)$ in $\mathbb{R}^2$ are oriented in a different direction from the origin after a 90° rotation than they are initially. Therefore, the vectors $\mathbf{v}$ and $A\mathbf{v}$ are not parallel.

6. TRUE. The characteristic equation of an $n \times n$ matrix A , $\det(A - \lambda I)$ is a polynomial of degree n in the indeterminate λ . Since such a polynomial always possesses n roots (with possible repeated or complex roots) by the Fundamental Theorem of Algebra, the statement is true.

7. FALSE. This is not true, in general, when the linear combination formed involves eigenvectors corresponding to *different* eigenvalues. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, with eigenvalues $\lambda = 1$ and $\lambda = 2$.

It is easy to see that corresponding eigenvectors to these eigenvalues are, respectively, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. However, note that

$$A(\mathbf{v}_1 + b\mathbf{v}_2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

which is not of the form $\lambda(\mathbf{v}_1 + \mathbf{v}_2)$, and therefore $\mathbf{v}_1 + \mathbf{v}_2$ is not an eigenvector of A . As a more trivial illustration, note that if $\mathbf{v}$ is an eigenvector of A , then $0\mathbf{v}$ is a linear combination of $\{\mathbf{v}\}$ that is no longer an eigenvector of A .

8. TRUE. This is basically a fact about roots of polynomials. Complex roots of real polynomials always occur in complex conjugate pairs. Therefore, if $\lambda = a + ib$ ($b \neq 0$) is an eigenvalue of A , then so is $\bar{\lambda} = a - ib$.

9. TRUE. If λ is an eigenvalue of A , then we have $A\mathbf{v} = \lambda\mathbf{v}$ for some eigenvector $\mathbf{v}$ of A corresponding to λ . Then

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v},$$

which shows that $\mathbf{v}$ is also an eigenvector of A^2 , this time corresponding to the eigenvalue λ^2 .

Problems:

$$1. A\mathbf{v} = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda\mathbf{v}.$$

$$2. A\mathbf{v} = \begin{bmatrix} 1 & -2 & -6 \\ -2 & 2 & -5 \\ 2 & 1 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \lambda\mathbf{v}.$$

$$3. \text{ Since } \mathbf{v} = c_1 \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + 4c_2 \\ -3c_2 \\ -3c_1 \end{bmatrix}, \text{ it follows that}$$

$$A\mathbf{v} = \begin{bmatrix} 1 & 4 & 1 \\ 3 & 2 & 1 \\ 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} c_1 + 4c_2 \\ -3c_2 \\ -3c_1 \end{bmatrix} = \begin{bmatrix} -2c_1 - 8c_2 \\ 6c_2 \\ 6c_1 \end{bmatrix} = -2 \begin{bmatrix} c_1 + 4c_2 \\ -3c_2 \\ -3c_1 \end{bmatrix} = \lambda\mathbf{v}.$$

$$4. A\mathbf{v}_1 = \lambda_1\mathbf{v}_1 \implies \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \implies \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ -2\lambda_1 \end{bmatrix} \implies \lambda_1 = 2.$$

$$A\mathbf{v}_2 = \lambda_2\mathbf{v}_2 \implies \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} \lambda_2 \\ \lambda_2 \end{bmatrix} \implies \lambda_2 = 5.$$

Thus $\lambda_1 = 2$ corresponds to $\mathbf{v}_1$ and $\lambda_2 = 5$ corresponds to $\mathbf{v}_2$.

5. The only vectors that are mapped into a scalar multiple of themselves under a reflection in the x -axis are those vectors that either point along the x -axis, or that point along the y -axis. Hence, the eigenvectors

are of the form $(a, 0)$ or $(0, b)$ where a and b are arbitrary nonzero real numbers. A vector that points along the x -axis will have neither its magnitude nor its direction altered by a reflection in the x -axis. Hence, the eigenvectors of the form $(a, 0)$ correspond to the eigenvalue $\lambda = 1$. A vector of the form $(0, b)$ will be mapped into the vector $(0, -b) = -1(0, b)$ under a reflection in the x -axis. Consequently, the eigenvectors of the form $(0, b)$ correspond to the eigenvalue $\lambda = -1$.

6. Any vectors lying along the line $y = x$ are unmoved by the action of T , and hence, any vector (t, t) with $t \neq 0$ is an eigenvector with corresponding eigenvalue $\lambda = 1$. On the other hand, any vector lying along the line $y = -x$ will be reflected across the line $y = x$, thereby experiencing a 180° change of direction. Therefore, any vector $(t, -t)$ with $t \neq 0$ is an eigenvector with corresponding eigenvalue $\lambda = -1$. All vectors that do not lie on the line $y = x$ or the line $y = -x$ are not eigenvectors of this linear transformation.

7. If $\theta \neq 0, \pi$, there are no vectors that are mapped into scalar multiples of themselves under the rotation, and consequently, there are no real eigenvalues and eigenvectors in this case. If $\theta = 0$, then every vector is mapped onto itself under the rotation, therefore $\lambda = 1$, and every nonzero vector in $\mathbb{R}^2$ is an eigenvector. If $\theta = \pi$, then every vector is mapped onto its negative under the rotation, therefore $\lambda = -1$, and once again, every nonzero vector in $\mathbb{R}^2$ is an eigenvector.

8. Any vectors lying on the y -axis are unmoved by the action of T , and hence, any vector $(0, y, 0)$ with y not zero is an eigenvector of T with corresponding eigenvalue $\lambda = 1$. On the other hand, any vector lying in the xz -plane, say $(x, 0, z)$ is transformed under T to $(0, 0, 0)$. Thus, any vector $(x, 0, z)$ with x and z not both zero is an eigenvector with corresponding eigenvalue $\lambda = 0$.

$$\begin{aligned} 9. \det(A - \lambda I) = 0 &\iff \begin{vmatrix} 3 - \lambda & -1 \\ -5 & -1 - \lambda \end{vmatrix} = 0 \iff \lambda^2 - 2\lambda - 8 = 0 \\ &\iff (\lambda + 2)(\lambda - 4) = 0 \iff \lambda = -2 \text{ or } \lambda = 4. \end{aligned}$$

If $\lambda = -2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 5 & -1 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
 $\implies 5v_1 - v_2 = 0 \implies v_2 = 5v_1$. If we let $v_1 = t \in \mathbb{R}$, then the solution set of this system is $\{(t, 5t) : t \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = -2$ are $\mathbf{v} = t(1, 5)$ where $t \in \mathbb{R}$.

If $\lambda = 4$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -1 & -1 \\ -5 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
 $\implies -v_1 - v_2 = 0 \implies v_2 = -v_1$. If we let $v_2 = r \in \mathbb{R}$, then the solution set of this system is $\{(r, -r) : r \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 4$ are $\mathbf{v} = r(1, -1)$ where $r \in \mathbb{R}$.

$$\begin{aligned} 10. \det(A - \lambda I) = 0 &\iff \begin{vmatrix} 1 - \lambda & 6 \\ 2 & -3 - \lambda \end{vmatrix} = 0 \iff \lambda^2 + 2\lambda - 15 = 0 \\ &\iff (\lambda - 3)(\lambda + 5) = 0 \iff \lambda = 3 \text{ or } \lambda = -5. \end{aligned}$$

If $\lambda = 3$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -2 & 6 \\ 2 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
 $\implies v_1 - 3v_2 = 0$. If we let $v_2 = r \in \mathbb{R}$, then the solution set of this system is $\{(3r, r) : r \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 3$ are $\mathbf{v} = r(3, 1)$ where $r \in \mathbb{R}$.

If $\lambda = -5$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 6 & 6 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
 $\implies v_1 + v_2 = 0$. If we let $v_2 = s \in \mathbb{R}$, then the solution set of this system is $\{(-s, s) : s \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = -5$ are $\mathbf{v} = s(-1, 1)$ where $s \in \mathbb{R}$.

$$\begin{aligned} 11. \det(A - \lambda I) = 0 &\iff \begin{vmatrix} 7 - \lambda & 4 \\ -1 & 3 - \lambda \end{vmatrix} = 0 \iff \lambda^2 - 10\lambda + 25 = 0 \\ &\iff (\lambda - 5)^2 = 0 \iff \lambda = 5 \text{ of multiplicity two.} \end{aligned}$$

If $\lambda = 5$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\implies v_1 + 2v_2 = 0 \implies v_1 = -2v_2$. If we let $v_2 = t \in \mathbb{R}$, then the solution set of this system is $\{(-2t, t) : t \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 5$ are $\mathbf{v} = t(-2, 1)$ where $t \in \mathbb{R}$.

12. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & 0 \\ 0 & 2 - \lambda \end{vmatrix} = 0 \iff (2 - \lambda)^2 = 0 \iff \lambda = 2$ of multiplicity two.

If $\lambda = 2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Thus, if we let $v_1 = s$ and $v_2 = t$ where $s, t \in \mathbb{R}$, then the solution set of this system is $\{(s, t) : s, t \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 2$ are $\mathbf{v} = s(1, 0) + t(0, 1)$ where $s, t \in \mathbb{R}$.

13. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 3 - \lambda & -2 \\ 4 & -1 - \lambda \end{vmatrix} = 0 \iff \lambda^2 - 2\lambda + 5 = 0 \iff \lambda = 1 \pm 2i$.

If $\lambda = 1 - 2i$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 2 + 2i & -2 \\ 4 & -2 + 2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\implies (1 + i)v_1 - v_2 = 0$. If we let $v_1 = s \in \mathbb{C}$, then the solution set of this system is $\{(s, (1 + i)s) : s \in \mathbb{C}\}$ so the eigenvectors corresponding to $\lambda = 1 - 2i$ are $\mathbf{v} = s(1, 1 + i)$ where $s \in \mathbb{C}$. By Theorem 5.6.8, since the entries of A are real, $\lambda = 1 + 2i$ has corresponding eigenvectors of the form $\mathbf{v} = t(1, 1 - i)$ where $t \in \mathbb{C}$.

14. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & 3 \\ -3 & 2 - \lambda \end{vmatrix} = 0 \iff (2 - \lambda)^2 = -9 \iff \lambda = 2 \pm 3i$.

If $\lambda = 2 - 3i$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 3i & 3 \\ -3 & 3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\implies v_2 = -iv_1$. If we let $v_1 = t \in \mathbb{C}$, then the solution set of this system is $\{(t, -it) : t \in \mathbb{C}\}$ so the eigenvectors corresponding to $\lambda = 2 - 3i$ are $\mathbf{v} = t(1, -i)$ where $t \in \mathbb{C}$. By Theorem 5.6.8, since the entries of A are real, $\lambda = 2 + 3i$ has corresponding eigenvectors of the form $\mathbf{v} = r(1, i)$ where $r \in \mathbb{C}$.

15. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 10 - \lambda & -12 & 8 \\ 0 & 2 - \lambda & 0 \\ -8 & 12 & -6 - \lambda \end{vmatrix} = 0 \iff (\lambda - 2)^3 = 0 \iff \lambda = 2$ of multiplicity three.

If $\lambda = 2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 8 & -12 & 8 \\ 0 & 0 & 0 \\ -8 & 12 & -8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies 2v_1 - 3v_2 + 2v_3 = 0$. Thus, if we let $v_2 = 2s$ and $v_3 = t$ where $s, t \in \mathbb{R}$, then the solution set of this system is $\{(3s - t, 2s, t) : s, t \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 2$ are $\mathbf{v} = s(3, 2, 0) + t(-1, 0, 1)$ where $s, t \in \mathbb{R}$.

16. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 3 - \lambda & 0 & 0 \\ 0 & 2 - \lambda & -1 \\ 1 & -1 & 2 - \lambda \end{vmatrix} = 0 \iff (\lambda - 1)(\lambda - 3)^2 = 0 \iff \lambda = 1 \text{ or } \lambda = 3$ of multiplicity two.

If $\lambda = 1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies v_1 = 0$ and $v_2 - v_3 = 0$. Thus, if we let $v_3 = s$ where $s \in \mathbb{R}$, then the solution set of this system is $\{(0, s, s) : s \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 1$ are $\mathbf{v} = s(0, 1, 1)$ where $s \in \mathbb{R}$.

If $\lambda = 3$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies v_1 = 0$ and $v_2 + v_3 = 0$. Thus, if we let $v_3 = t$ where $t \in \mathbb{R}$, then the solution set of this system is

$\{(0, -t, t) : t \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 3$ are $\mathbf{v} = t(0, -1, 1)$ where $t \in \mathbb{R}$.

$$17. \det(A - \lambda I) = 0 \iff \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & 2 \\ 2 & -2 & -1-\lambda \end{vmatrix} = 0 \iff (\lambda - 1)^3 = 0 \iff \lambda = 1 \text{ of multiplicity three.}$$

$$\text{If } \lambda = 1 \text{ then } (A - \lambda I)\mathbf{v} = \mathbf{0} \text{ assumes the form } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 2 & -2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies v_1 = 0$ and $v_2 + v_3 = 0$. Thus, if we let $v_3 = s$ where $s \in \mathbb{R}$, then the solution set of this system is $\{(0, -s, s) : s \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 1$ are $\mathbf{v} = s(0, -1, 1)$ where $s \in \mathbb{R}$.

$$18. \det(A - \lambda I) = 0 \iff \begin{vmatrix} 6-\lambda & 3 & -4 \\ -5 & -2-\lambda & 2 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0 \iff (\lambda - 1)(\lambda + 1)(\lambda - 3) = 0 \iff \lambda = -1, \lambda = 1,$$

or $\lambda = 3$.

$$\text{If } \lambda = -1 \text{ then } (A - \lambda I)\mathbf{v} = \mathbf{0} \text{ assumes the form } \begin{bmatrix} 7 & 3 & -4 \\ -5 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies v_1 = v_3 - v_2$ and $4v_2 - 3v_3 = 0$. Thus, if we let $v_3 = 4r$ where $r \in \mathbb{R}$, then the solution set of this system is $\{(r, 3r, 4r) : r \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = -1$ are $\mathbf{v} = r(1, 3, 4)$ where $r \in \mathbb{R}$.

$$\text{If } \lambda = 1 \text{ then } (A - \lambda I)\mathbf{v} = \mathbf{0} \text{ assumes the form } \begin{bmatrix} 5 & 3 & -4 \\ -5 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies 5v_1 + 3v_2 = 0$ and $v_3 = 0$. Thus, if we let $v_2 = -5s$ where $s \in \mathbb{R}$, then the solution set of this system is $\{(3s, -5s, 0) : s \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 1$ are $\mathbf{v} = s(3, -5, 0)$ where $s \in \mathbb{R}$.

$$\text{If } \lambda = 3 \text{ then } (A - \lambda I)\mathbf{v} = \mathbf{0} \text{ assumes the form } \begin{bmatrix} 3 & 3 & -4 \\ -5 & -5 & 2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies v_1 + v_2 = 0$ and $v_3 = 0$. Thus, if we let $v_2 = t$ where $t \in \mathbb{R}$, then the solution set of this system is $\{(-t, t, 0) : t \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 3$ are $\mathbf{v} = t(-1, 1, 0)$ where $t \in \mathbb{R}$.

$$19. \det(A - \lambda I) = 0 \iff \begin{vmatrix} 7-\lambda & -8 & 6 \\ 8 & -9-\lambda & 6 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0 \iff (\lambda + 1)^3 = 0 \iff \lambda = -1 \text{ of multiplicity}$$

three.

$$\text{If } \lambda = -1 \text{ then } (A - \lambda I)\mathbf{v} = \mathbf{0} \text{ assumes the form } \begin{bmatrix} 8 & -8 & 6 \\ 8 & -8 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies 4v_1 - 4v_2 + 3v_3 = 0$. Thus, if we let $v_2 = r$ and $v_3 = 4s$ where $r, s \in \mathbb{R}$, then the solution set of this system is $\{(r - 3s, r, 4s) : r, s \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = -1$ are $\mathbf{v} = r(1, 1, 0) + s(-3, 0, 4)$ where $r, s \in \mathbb{R}$.

$$20. \det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 1 & -1 \\ 0 & 2-\lambda & 0 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0 \iff (\lambda - 1)(\lambda - 2)^2 = 0 \iff \lambda = 1 \text{ or } \lambda = 2 \text{ of}$$

multiplicity two.

$$\text{If } \lambda = 1 \text{ then } (A - \lambda I)\mathbf{v} = \mathbf{0} \text{ assumes the form } \begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies v_2 = 0$ and $v_1 + v_3 = 0$. Thus, if we let $v_3 = r$ where $r \in \mathbb{R}$, then the solution set of this system is $\{(-r, 0, r) : r \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 1$ are $\mathbf{v} = r(-1, 0, 1)$ where $r \in \mathbb{R}$.

If $\lambda = 2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} -2 & 1 & -1 \\ 0 & 0 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies 2v_1 - v_2 + v_3 = 0$. Thus, if we let $v_1 = s$ and $v_3 = t$ where $s, t \in \mathbb{R}$, then the solution set of this system is $\{(s, 2s + t, t) : s, t \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 2$ are $\mathbf{v} = s(1, 2, 0) + t(0, 1, 1)$ where $s, t \in \mathbb{R}$.

$$\mathbf{21.} \det(A - \lambda I) = 0 \iff \begin{vmatrix} 1 - \lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -1 & -\lambda \end{vmatrix} = 0 \iff (1 - \lambda)(1 + \lambda^2) = 0 \iff \lambda = 1 \text{ or } \lambda = \pm i.$$

If $\lambda = 1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies -v_2 + v_3 = 0$ and $-v_2 - v_3 = 0$. The solution set of this system is $\{(r, 0, 0) : r \in \mathbb{C}\}$ so the eigenvectors corresponding to $\lambda = 1$ are $\mathbf{v} = r(1, 0, 0)$ where $r \in \mathbb{C}$.

If $\lambda = -i$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 1 + i & 0 & 0 \\ 0 & i & 1 \\ 0 & -1 & i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies v_1 = 0$ and $-v_2 + iv_3 = 0$. The solution set of this system is $\{(0, si, s) : s \in \mathbb{C}\}$ so the eigenvectors corresponding to $\lambda = -i$ are $\mathbf{v} = s(0, i, 1)$ where $s \in \mathbb{C}$. By Theorem 5.6.8, since the entries of A are real, $\lambda = i$ has corresponding eigenvectors of the form $\mathbf{v} = t(0, -i, 1)$ where $t \in \mathbb{C}$.

$$\mathbf{22.} \det(A - \lambda I) = 0 \iff \begin{vmatrix} -2 - \lambda & 1 & 0 \\ 1 & -1 - \lambda & -1 \\ 1 & 3 & -3 - \lambda \end{vmatrix} = 0 \iff (\lambda + 2)(\lambda^2 + 4\lambda + 5) = 0 \iff \lambda = -2 \text{ or } \lambda = -2 \pm i.$$

If $\lambda = -2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies v_2 = 0$ and $v_1 - v_3 = 0$. Thus, if we let $v_3 = r$ where $r \in \mathbb{C}$, then the solution set of this system is $\{(r, 0, r) : r \in \mathbb{C}\}$ so the eigenvectors corresponding to $\lambda = -2$ are $\mathbf{v} = r(1, 0, 1)$ where $r \in \mathbb{C}$.

If $\lambda = -2 + i$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} -i & 1 & 0 \\ 1 & 1 - i & -1 \\ 1 & 3 & -1 - i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies v_1 + \frac{-2 + i}{5}v_3 = 0$ and $v_2 + \frac{-1 - 2i}{5}v_3 = 0$. Thus, if we let $v_3 = 5s$ where $s \in \mathbb{C}$, then the solution set of this system is $\{((2 - i)s, (1 + 2i)s, 5s) : s \in \mathbb{C}\}$ so the eigenvectors corresponding to $\lambda = -2 + i$ are $\mathbf{v} = s(2 - i, 1 + 2i, 5)$ where $s \in \mathbb{C}$. By Theorem 5.6.8, since the entries of A are real, $\lambda = -2 - i$ has corresponding eigenvectors of the form $\mathbf{v} = t(2 + i, 1 - 2i, 5)$ where $t \in \mathbb{C}$.

$$\mathbf{23.} \det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & -1 & 3 \\ 3 & 1 - \lambda & 0 \\ 2 & -1 & 3 - \lambda \end{vmatrix} = 0 \iff \lambda(\lambda - 2)(\lambda - 4) = 0 \iff \lambda = 0, \lambda = 2, \text{ or } \lambda = 4.$$

If $\lambda = 0$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies v_1 + 2v_2 - 3v_3 = 0$ and $-5v_2 + 9v_3 = 0$. Thus, if we let $v_3 = 5r$ where $r \in \mathbb{R}$, then the solution set of this system is $\{(-3r, 9r, 5r) : r \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 0$ are $\mathbf{v} = r(-3, 9, 5)$ where $r \in \mathbb{R}$.

If $\lambda = 2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 0 & -1 & 3 \\ 3 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies v_1 - v_3 = 0$ and $v_2 - 3v_3 = 0$. Thus, if we let $v_3 = s$ where $s \in \mathbb{R}$, then the solution set of this system is $\{(s, 3s, s) : s \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 2$ are $\mathbf{v} = s(1, 3, 1)$ where $s \in \mathbb{R}$.

If $\lambda = 4$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} -2 & -1 & 3 \\ 3 & -3 & 0 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies v_1 - v_3 = 0$ and $v_2 - v_3 = 0$. Thus, if we let $v_3 = t$ where $t \in \mathbb{R}$, then the solution set of this system is $\{(t, t, t) : t \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 4$ are $\mathbf{v} = t(1, 1, 1)$ where $t \in \mathbb{R}$.

24. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 5 - \lambda & 0 & 0 \\ 0 & 5 - \lambda & 0 \\ 0 & 0 & 5 - \lambda \end{vmatrix} = 0 \iff (\lambda - 5)^3 = 0 \iff \lambda = 5$ of multiplicity three.

If $\lambda = 5$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, if we let $v_1 = r$, $v_2 = s$, and $v_3 = t$ where $r, s, t \in \mathbb{R}$, then the solution set of this system is $\{(r, s, t) : r, s, t \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 5$ are $\mathbf{v} = r(1, 0, 0) + s(0, 1, 0) + t(0, 0, 1)$ where $r, s, t \in \mathbb{R}$. That is, every nonzero vector in $\mathbb{R}^3$ is an eigenvector of A corresponding to $\lambda = 5$.

25. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 2 & 2 \\ 2 & -\lambda & 2 \\ 2 & 2 & -\lambda \end{vmatrix} = 0 \iff (\lambda - 4)(\lambda + 2)^2 = 0 \iff \lambda = 4$ or $\lambda = -2$ of multiplicity two.

If $\lambda = 4$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies v_1 - v_3 = 0$ and $v_2 - v_3 = 0$. Thus, if we let $v_3 = r$ where $r \in \mathbb{R}$, then the solution set of this system is $\{(r, r, r) : r \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 4$ are $\mathbf{v} = r(1, 1, 1)$ where $r \in \mathbb{R}$.

If $\lambda = -2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies v_1 + v_2 + v_3 = 0$. Thus, if we let $v_2 = s$ and $v_3 = t$ where $s, t \in \mathbb{R}$, then the solution set of this system is $\{(-s - t, s, t) : s, t \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = -2$ are $\mathbf{v} = s(-1, 1, 0) + t(-1, 0, 1)$ where $s, t \in \mathbb{R}$.

26. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1 - \lambda & 2 & 3 & 4 \\ 4 & 3 - \lambda & 2 & 1 \\ 4 & 5 & 6 - \lambda & 7 \\ 7 & 6 & 5 & 4 - \lambda \end{vmatrix} = 0 \iff \lambda^4 - 14\lambda^3 - 32\lambda^2 = 0$

$\iff \lambda^2(\lambda - 16)(\lambda + 2) = 0 \iff \lambda = 16$, $\lambda = -2$, or $\lambda = 0$ of multiplicity two.

If $\lambda = 16$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} -15 & 2 & 3 & 4 \\ 4 & -13 & 2 & 1 \\ 4 & 5 & -10 & 7 \\ 7 & 6 & 5 & -12 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies v_1 - 1841v_3 + 2078v_4 = 0$, $v_2 + 82v_3 - 93v_4 = 0$, and $31v_3 - 35v_4 = 0$. Thus, if we let $v_4 = 31r$ where $r \in \mathbb{R}$, then the solution set of this system is $\{(17r, 13r, 35r, 31r) : r \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 16$ are $\mathbf{v} = r(17, 13, 35, 31)$ where $r \in \mathbb{R}$.

If $\lambda = -2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 3 & 2 & 3 & 4 \\ 4 & 5 & 2 & 1 \\ 4 & 5 & 8 & 7 \\ 7 & 6 & 5 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow v_1 + v_4 = 0$, $v_2 - v_4 = 0$, and $v_3 + v_4 = 0$. Thus, if we let $v_4 = s$ where $s \in \mathbb{R}$, then the solution set of this system is $\{(-s, s, -s, s) : s \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = -2$ are $\mathbf{v} = s(-1, 1, -1, 1)$ where $s \in \mathbb{R}$.

If $\lambda = 0$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow v_1 - v_3 - 2v_4 = 0$ and $v_2 + 2v_3 + 3v_4 = 0$. Thus, if we let $v_3 = a$ and $v_4 = b$ where $a, b \in \mathbb{R}$, then the solution set of this system is $\{(a + 2b, -2a - 3b, a, b) : a, b \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 0$ are $\mathbf{v} = a(1, -2, 1, 0) + b(2, -3, 0, 1)$ where $a, b \in \mathbb{R}$.

27. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ -1 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & -1 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} = 0 \iff (\lambda^2 + 1)^2 = 0 \iff \lambda = \pm i$, where each root is of multiplicity two.

If $\lambda = -i$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} i & 1 & 0 & 0 \\ -1 & i & 0 & 0 \\ 0 & 0 & i & -1 \\ 0 & 0 & 1 & i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow v_1 - iv_2 = 0$ and $v_3 + iv_4 = 0$. Thus, if we let $v_2 = r$ and $v_4 = s$ where $r, s \in \mathbb{C}$, then the solution set of this system is $\{(ir, r, -is, s) : r, s \in \mathbb{C}\}$ so the eigenvectors corresponding to $\lambda = -i$ are $\mathbf{v} = r(i, 1, 0, 0) + s(0, 0, -i, 1)$ where $r, s \in \mathbb{C}$. By Theorem 5.6.8, since the entries of A are real, $\lambda = i$ has corresponding eigenvectors $\mathbf{v} = a(-i, 1, 0, 0) + b(0, 0, i, 1)$ where $a, b \in \mathbb{C}$.

28. This matrix is lower triangular, and therefore, the eigenvalues appear along the main diagonal of the matrix: $\lambda = 1 + i, 1 - 3i, 1$. Note that the eigenvalues do not occur in complex conjugate pairs, but this does not contradict Theorem 5.6.8 because the matrix does not consist entirely of *real* elements.

29. (a) $p(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} 1 - \lambda & -1 \\ 2 & 4 - \lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6$.

(b)

$$\begin{aligned} A^2 - 5A + 6I_2 &= \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} - 5 \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -5 \\ 10 & 14 \end{bmatrix} + \begin{bmatrix} -5 & 5 \\ -10 & -20 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_2. \end{aligned}$$

(c) Using part (b) of this problem:

$$\begin{aligned} A^2 - 5A + 6I_2 = 0_2 &\iff A^{-1}(A^2 - 5A + 6I_2) = A^{-1} \cdot 0_2 \\ &\iff A - 5I_2 + 6A^{-1} = 0_2 \\ &\iff 6A^{-1} = 5I_2 - A \\ &\iff A^{-1} = \frac{1}{6} \left\{ 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \right\} \\ &\iff A^{-1} = \frac{1}{6} \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}, \text{ or } A^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} \end{bmatrix}. \end{aligned}$$

30. (a) $\det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0 \iff \lambda^2 + \lambda - 6 = 0 \iff (\lambda - 2)(\lambda + 3) = 0 \iff \lambda = 2 \text{ or } \lambda = -3.$

(b) $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = B.$

$\det(B - \lambda I_2) = 0 \iff \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0 \iff (\lambda - 1)^2 = 0 \iff \lambda = 1 \text{ of multiplicity two. Matrices } A \text{ and } B \text{ do not have the same eigenvalues.}$

31.

$$\begin{aligned} A(3\mathbf{v}_1 - \mathbf{v}_2) &= 3A\mathbf{v}_1 - A\mathbf{v}_2 = 3(2\mathbf{v}_1) - (-3\mathbf{v}_2) = 6\mathbf{v}_1 + 3\mathbf{v}_2 \\ &= 6 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ -3 \end{bmatrix}. \end{aligned}$$

32. (a) Let $a, b, c \in \mathbb{R}$. If $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3$, then $5, 0, 3 = a(1, -1, 1) + b(2, 1, 3) + c(-1, -1, 2)$, or $(5, 0, 3) = (a + 2b - c, -a + b - c, a + 3b + 2c)$. The last equality results in the system:
$$\begin{cases} a + 2b - c = 5 \\ -a + b - c = 0 \\ a + 3b + 2c = 3. \end{cases}$$

This system has the solution $a = 2$, $b = 1$, and $c = -1$. Consequently, $\mathbf{v} = 2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3$.

(b) Using part (a):

$$\begin{aligned} A\mathbf{v} &= A(2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3) = 2A\mathbf{v}_1 + A\mathbf{v}_2 - A\mathbf{v}_3 = 2(2\mathbf{v}_1) + (-2\mathbf{v}_2) - (3\mathbf{v}_3) \\ &= 4\mathbf{v}_1 - 2\mathbf{v}_2 - 3\mathbf{v}_3 = 4(1, -1, 1) - 2(2, 1, 3) - 3(-1, -1, 2) = (3, -3, -8). \end{aligned}$$

33.

$$\begin{aligned} A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) &= A(c_1\mathbf{v}_1) + A(c_2\mathbf{v}_2) + A(c_3\mathbf{v}_3) \\ &= c_1(A\mathbf{v}_1) + c_2(A\mathbf{v}_2) + c_3(A\mathbf{v}_3) \\ &= c_1(\lambda\mathbf{v}_1) + c_2(\lambda\mathbf{v}_2) + c_3(\lambda\mathbf{v}_3) \\ &= \lambda(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3). \end{aligned}$$

Thus, $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ is an eigenvector of A corresponding to the eigenvalue λ .

34. Recall that the determinant of an upper (lower) triangular matrix is just the product of its main diagonal elements. Let A be an $n \times n$ upper (lower) triangular matrix. It follows that $A - \lambda I_n$ is an upper (lower) triangular matrix with main diagonal element $a_{ii} - \lambda$, $i = 1, 2, \dots, n$. Consequently,

$$\det(A - \lambda I_n) = 0 \iff \prod_{i=1}^n (a_{ii} - \lambda) = 0.$$

This implies that $\lambda = a_{11}, a_{22}, \dots, a_{nn}$.

35. Any scalar λ such that $\det(A - \lambda I) = 0$ is an eigenvalue of A . Therefore, if 0 is an eigenvalue of A , then $\det(A - 0 \cdot I) = 0$, or $\det(A) = 0$, which implies that A is not invertible. On the other hand, if 0 is not an eigenvalue of A , then $\det(A - 0 \cdot I) \neq 0$, or $\det(A) \neq 0$, which implies that A is invertible.

36. A is invertible, so A^{-1} exists. Also, λ is an eigenvalue of A so that $A\mathbf{v} = \lambda\mathbf{v}$. Thus,

$$A^{-1}(A\mathbf{v}) = A^{-1}(\lambda\mathbf{v}) \implies (A^{-1}A)\mathbf{v} = \lambda A^{-1}\mathbf{v} \implies I_n\mathbf{v} = \lambda A^{-1}\mathbf{v} \implies \mathbf{v} = \lambda A^{-1}\mathbf{v} \implies \frac{1}{\lambda}\mathbf{v} = A^{-1}\mathbf{v}.$$

Therefore $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} provided that λ is an eigenvalue of A .

37. By assumption, we have $A\mathbf{v} = \lambda\mathbf{v}$ and $B\mathbf{v} = \mu\mathbf{v}$.

(a) Therefore,

$$(AB)\mathbf{v} = A(B\mathbf{v}) = A(\mu\mathbf{v}) = \mu(A\mathbf{v}) = \mu(\lambda\mathbf{v}) = (\lambda\mu)\mathbf{v},$$

which shows that $\mathbf{v}$ is an eigenvector of AB with corresponding eigenvalue $\lambda\mu$.

(b) Also,

$$(A + B)\mathbf{v} = A\mathbf{v} + B\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v} = (\lambda + \mu)\mathbf{v},$$

which shows that $\mathbf{v}$ is an eigenvector of $A + B$ with corresponding eigenvalue $\lambda + \mu$.

38. Recall that a matrix and its transpose have the same determinant. Thus,

$$\det(A - \lambda I_n) = \det([A - \lambda I_n]^T) = \det(A^T - \lambda I_n).$$

Since A and A^T have the same characteristic polynomial, it follows that both matrices also have the same eigenvalues.

39. (a) $\mathbf{v} = \mathbf{r} + i\mathbf{s}$ is an eigenvector with eigenvalue $\lambda = a + bi$, $b \neq 0 \implies A\mathbf{v} = \lambda\mathbf{v}$

$$\implies A(\mathbf{r} + i\mathbf{s}) = (a + bi)(\mathbf{r} + i\mathbf{s}) = (a\mathbf{r} - b\mathbf{s}) + i(as + b\mathbf{r}) \implies A\mathbf{r} = a\mathbf{r} - b\mathbf{s} \text{ and } A\mathbf{s} = as + b\mathbf{r}.$$

Now if $\mathbf{r} = \mathbf{0}$, then $A\mathbf{0} = a\mathbf{0} - b\mathbf{s} \implies \mathbf{0} = \mathbf{0} - b\mathbf{s} \implies \mathbf{0} = b\mathbf{s} \implies \mathbf{s} = \mathbf{0}$ since $b \neq 0$. This would mean that $\mathbf{v} = \mathbf{0}$ so $\mathbf{v}$ could not be an eigenvector. Thus, it must be that $\mathbf{r} \neq \mathbf{0}$. Similarly, if $\mathbf{s} = \mathbf{0}$, then $\mathbf{r} = \mathbf{0}$, and again, this would contradict the fact that $\mathbf{v}$ is an eigenvector. Hence, it must be the case that $\mathbf{r} \neq \mathbf{0}$ and $\mathbf{s} \neq \mathbf{0}$.

(b) As in part (a), $A\mathbf{r} = a\mathbf{r} - b\mathbf{s}$ and $A\mathbf{s} = as + b\mathbf{r}$.

Let $c_1, c_2 \in \mathbb{R}$. Then if

$$c_1\mathbf{r} + c_2\mathbf{s} = \mathbf{0}, \quad (39.1)$$

$$\text{we have } A(c_1\mathbf{r} + c_2\mathbf{s}) = \mathbf{0} \implies c_1A\mathbf{r} + c_2A\mathbf{s} = \mathbf{0} \implies c_1(a\mathbf{r} - b\mathbf{s}) + c_2(as + b\mathbf{r}) = \mathbf{0}.$$

Hence, $(c_1a + c_2b)\mathbf{r} + (c_2a - c_1b)\mathbf{s} = \mathbf{0} \implies a(c_1\mathbf{r} + c_2\mathbf{s}) + b(c_2\mathbf{r} - c_1\mathbf{s}) = \mathbf{0} \implies b(c_2\mathbf{r} - c_1\mathbf{s}) = \mathbf{0}$ where we have used (39.1). Since $b \neq 0$, we must have $c_2\mathbf{r} - c_1\mathbf{s} = \mathbf{0}$. Combining this with (39.1) yields $c_1 = c_2 = 0$. Therefore, it follows that $\mathbf{r}$ and $\mathbf{s}$ are linearly independent vectors.

40. $\lambda_1 = 2$, $\mathbf{v} = r(-1, 1)$. $\lambda_2 = 5$, $\mathbf{v} = s(1, 2)$.

41. $\lambda_1 = -2$ (multiplicity two), $\mathbf{v} = r(1, 1, 1)$. $\lambda_2 = -5$, $\mathbf{v} = s(20, 11, 14)$.

42. $\lambda_1 = 3$ (multiplicity two), $\mathbf{v} = r(1, 0, -1) + s(0, 1, -1)$. $\lambda_2 = 6$, $\mathbf{v} = t(1, 1, 1)$.

43. $\lambda_1 = 3 - \sqrt{6}$, $\mathbf{v} = r(\sqrt{6}, -1 + \sqrt{6}, -5 + \sqrt{6})$. $\lambda_2 = 3 + \sqrt{6}$, $\mathbf{v} = s(\sqrt{6}, 1 + \sqrt{6}, 5 + \sqrt{6})$, $\lambda_3 = -2$, $\mathbf{v} = t(-1, 3, 0)$.

44. $\lambda_1 = 0$, $\mathbf{v} = r(2, 2, 1)$. $\lambda_2 = 3i$, $\mathbf{v} = s(-4 - 3i, 5, -2 + 6i)$, $\lambda_3 = -3i$, $\mathbf{v} = t(-4 + 3i, 5, -2 - 6i)$.

45. $\lambda_1 = -1$ (multiplicity four), $\mathbf{v} = a(-1, 0, 0, 1, 0) + b(-1, 0, 1, 0, 0) + c(-1, 0, 0, 0, 1) + d(-1, 1, 0, 0, 0)$.

Solutions to Section 5.7

True-False Review:

1. TRUE. This is the definition of a nondefective matrix.

2. TRUE. The eigenspace E_λ is equal to the null space of the $n \times n$ matrix $A - \lambda I$, and this null space is a subspace of $\mathbb{R}^n$.

3. TRUE. The dimension of an eigenspace never exceeds the algebraic multiplicity of the corresponding eigenvalue.

4. TRUE. Eigenvectors corresponding to distinct eigenspaces are linearly independent. Therefore if we choose one (nonzero) vector from each distinct eigenspace, the chosen vectors will form a linearly independent set.

5. TRUE. Since each eigenvalue of the matrix A occurs with algebraic multiplicity 1, we can simply choose one eigenvector from each eigenspace to obtain a basis of eigenvectors for A . Thus, A is nondefective.

6. FALSE. Many examples will show that this statement is false, including the $n \times n$ identity matrix I_n for $n \geq 2$. The matrix I_n is not defective, and yet, has $\lambda = 1$ occurring with algebraic multiplicity n .

7. TRUE. Eigenvectors corresponding to distinct eigenvalues are always linearly independent, as proved in the text in this section.

Problems:

1. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0 \iff \lambda^2 - 4\lambda - 5 = 0 \iff (\lambda - 5)(\lambda + 1) = 0 \iff \lambda = 5 \text{ or } \lambda = -1.$

If $\lambda_1 = 5$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 - v_2 = 0$. The solution set of this system is $\{(r, r) : r \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = 5$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} = r(1, 1), r \in \mathbb{R}\}$. A basis for E_1 is $\{(1, 1)\}$, and $\dim[E_1] = 1$.

If $\lambda_2 = -1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 + 2v_2 = 0$. The solution set of this system is $\{(-2s, s) : s \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_2 = -1$ is $E_2 = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} = s(-2, 1), s \in \mathbb{R}\}$. A basis for E_2 is $\{(-2, 1)\}$, and $\dim[E_2] = 1$.

A complete set of eigenvectors for A is given by $\{(1, 1), (-2, 1)\}$, so A is nondefective.

2. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = 0 \iff (3 - \lambda)^2 = 0 \iff \lambda = 3 \text{ of multiplicity two.}$

If $\lambda_1 = 3$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The solution set of this system is $\{(r, s) : r, s \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = 3$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} = r(1, 0) + s(0, 1), r, s \in \mathbb{R}\}$. A basis for E_1 is $\{(1, 0), (0, 1)\}$, and $\dim[E_1] = 2$. A is nondefective.

3. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1-\lambda & 2 \\ -2 & 5-\lambda \end{vmatrix} = 0 \iff (\lambda - 3)^2 = 0 \iff \lambda = 3 \text{ of multiplicity two.}$

If $\lambda_1 = 3$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 - v_2 = 0$. The solution set of this system is $\{(r, r) : r \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = 3$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} = r(1, 1), r \in \mathbb{R}\}$. A basis for E_1 is $\{(1, 1)\}$, and $\dim[E_1] = 1$.

A is defective since it does not have a complete set of eigenvectors.

4. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 5-\lambda & 5 \\ -2 & -1-\lambda \end{vmatrix} = 0 \iff \lambda^2 - 4\lambda + 5 = 0 \iff \lambda = 2 \pm i.$

If $\lambda_1 = 2 - i$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 3+i & 5 \\ -2 & -3+i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies -2v_1 + (-3+i)v_2 = 0$. The solution set of this system is $\{((-3+i)r, 2r) : r \in \mathbb{C}\}$, so the eigenspace corresponding to $\lambda_1 = 2 - i$ is $E_1 = \{\mathbf{v} \in \mathbb{C}^2 : \mathbf{v} = r(-3+i, 2), r \in \mathbb{C}\}$. A basis for E_1 is $\{(-3+i, 2)\}$, and $\dim[E_1] = 1$.

If $\lambda_2 = 2 + i$ then from Theorem 5.6.8, the eigenvectors corresponding to $\lambda_2 = 2 + i$ are $\mathbf{v} = s(-3 - i, 2)$ where $s \in \mathbb{C}$, so the eigenspace corresponding to $\lambda_2 = 2 + i$ is $E_2 = \{\mathbf{v} \in \mathbb{C}^2 : \mathbf{v} = s(-3 - i, 2), s \in \mathbb{C}\}$. A basis for E_2 is $\{(-3 - i, 2)\}$, and $\dim[E_2] = 1$. A is nondefective since it has a complete set of eigenvectors,

namely $\{(-3+i, 2), (-3-i, 2)\}$.

5. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 3-\lambda & -4 & -1 \\ 0 & -1-\lambda & -1 \\ 0 & -4 & 2-\lambda \end{vmatrix} = 0 \iff (\lambda+2)(\lambda-3)^2 = 0 \iff \lambda = -2 \text{ or } \lambda = 3$ of multiplicity two.

If $\lambda_1 = -2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 5 & -4 & -1 \\ 0 & 1 & -1 \\ 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

The solution set of this system is $\{(r, r, r) : r \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = -2$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = r(1, 1, 1), r \in \mathbb{R}\}$. A basis for E_1 is $\{(1, 1, 1)\}$, and $\dim[E_1] = 1$.

If $\lambda_2 = 3$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 0 & -4 & -1 \\ 0 & -4 & -1 \\ 0 & -4 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies 4v_2 + v_3 = 0$. The

solution set of this system is $\{(s, t, -4t) : s, t \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_2 = 3$ is $E_2 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = s(1, 0, 0) + t(0, 1, -4), s, t \in \mathbb{R}\}$. A basis for E_2 is $\{(1, 0, 0), (0, 1, -4)\}$, and $\dim[E_2] = 2$. A complete set of eigenvectors for A is given by $\{(1, 1, 1), (1, 0, 0), (0, 1, -4)\}$, so A is nondefective.

6. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & 2-\lambda & -3 \\ 0 & -2 & 1-\lambda \end{vmatrix} = 0 \iff (\lambda+1)(\lambda-4)^2 = 0 \iff \lambda = -1 \text{ or } \lambda = 4$ of multiplicity two.

If $\lambda_1 = -1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & -3 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies v_1 = 0$ and $v_2 - v_3 = 0$. The solution set of this system is $\{(0, r, r) : r \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = -1$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = r(0, 1, 1), r \in \mathbb{R}\}$. A basis for E_1 is $\{(0, 1, 1)\}$, and $\dim[E_1] = 1$.

If $\lambda_2 = 4$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & -3 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies 2v_2 + 3v_3 = 0$.

The solution set of this system is $\{(s, 3t, -2t) : s, t \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_2 = 4$ is $E_2 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = s(1, 0, 0) + t(0, 3, -2), s, t \in \mathbb{R}\}$. A basis for E_2 is $\{(1, 0, 0), (0, 3, -2)\}$, and $\dim[E_2] = 2$. A complete set of eigenvectors for A is given by $\{(1, 0, 0), (0, 3, -2), (0, 1, 1)\}$, so A is nondefective.

7. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 3-\lambda & 1 & 0 \\ -1 & 5-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0 \iff (\lambda-4)^3 = 0 \iff \lambda = 4$ of multiplicity three.

If $\lambda_1 = 4$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies v_1 - v_2 = 0$ and $v_3 \in \mathbb{R}$. The solution set of this system is $\{(r, r, s) : r, s \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = 4$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = r(1, 1, 0) + s(0, 0, 1), r, s \in \mathbb{R}\}$. A basis for E_1 is $\{(1, 1, 0), (0, 0, 1)\}$, and $\dim[E_1] = 2$.

A is defective since it does not have a complete set of eigenvectors.

8. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 3-\lambda & 0 & 0 \\ 2 & -\lambda & -4 \\ 1 & 4 & -\lambda \end{vmatrix} = 0 \iff (\lambda-3)(\lambda^2+16) = 0 \iff \lambda = 3 \text{ or } \lambda = \pm 4i$.

If $\lambda_1 = 3$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & -3 & -4 \\ 1 & 4 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies 11v_1 - 25v_3 = 0$ and $11v_2 - 2v_3$. The solution set of this system is $\{(25r, 2r, 11r) : r \in \mathbb{C}\}$, so the eigenspace corresponding to $\lambda_1 = 3$ is $E_1 = \{\mathbf{v} \in \mathbb{C}^3 : \mathbf{v} = r(25, 2, 11), r \in \mathbb{C}\}$. A basis for E_1 is $\{(25, 2, 11)\}$, and $\dim[E_1] = 1$.

If $\lambda_2 = -4i$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 3+4i & 0 & 0 \\ 2 & 4i & -4 \\ 1 & 4 & 4i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies v_1 = 0$ and $iv_2 - v_3 = 0$. The solution set of this system is $\{(0, s, is) : s \in \mathbb{C}\}$, so the eigenspace corresponding to $\lambda_2 = -4i$ is $E_2 = \{\mathbf{v} \in \mathbb{C}^3 : \mathbf{v} = s(0, 1, i), s \in \mathbb{C}\}$. A basis for E_2 is $\{(0, 1, i)\}$, and $\dim[E_2] = 1$.

If $\lambda_3 = 4i$ then from Theorem 5.6.8, the eigenvectors corresponding to $\lambda_3 = 4i$ are $\mathbf{v} = t(0, 1, -i)$ where $t \in \mathbb{C}$, so the eigenspace corresponding to $\lambda_3 = 4i$ is $E_3 = \{\mathbf{v} \in \mathbb{C}^3 : \mathbf{v} = t(0, 1, -i), t \in \mathbb{C}\}$. A basis for E_3 is $\{(0, 1, -i)\}$, and $\dim[E_3] = 1$. A complete set of eigenvectors for A is given by $\{(25, 2, 11), (0, 1, i), (0, 1, -i)\}$, so A is nondefective.

9. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 4-\lambda & 1 & 6 \\ -4 & -\lambda & -7 \\ 0 & 0 & -3-\lambda \end{vmatrix} = 0 \iff (\lambda+3)(\lambda-2)^2 = 0 \iff \lambda = -3 \text{ or } \lambda = 2 \text{ of}$

multiplicity two.

If $\lambda_1 = -3$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 7 & 1 & 6 \\ -4 & 3 & -7 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies v_1 + v_3 = 0$ and $v_2 - v_3 = 0$. The solution set of this system is $\{(-r, r, r) : r \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = -3$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = r(-1, 1, 1), r \in \mathbb{R}\}$. A basis for E_1 is $\{(-1, 1, 1)\}$, and $\dim[E_1] = 1$.

If $\lambda_2 = 2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 2 & 1 & 6 \\ -4 & -2 & -7 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies 2v_1 + v_2 = 0$ and $v_3 = 0$. The solution set of this system is $\{(-s, 2s, 0) : s \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_2 = 2$ is $E_2 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = s(-1, 2, 0), s \in \mathbb{R}\}$. A basis for E_2 is $\{(-1, 2, 0)\}$, and $\dim[E_2] = 1$.

A is defective because it does not have a complete set of eigenvectors.

10. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 \iff (\lambda-2)^3 = 0 \iff \lambda = 2 \text{ of multiplicity three.}$

If $\lambda_1 = 2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$
 The solution set of this

system is $\{(r, s, t) : r, s, t \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = 2$ is

$E_1 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = r(1, 0, 0) + s(0, 1, 0) + t(0, 0, 1), r, s, t \in \mathbb{R}\}$. A basis for E_1 is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, and $\dim[E_1] = 3$.

A is nondefective since it has a complete set of eigenvectors.

11. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 7-\lambda & -8 & 6 \\ 8 & -9-\lambda & 6 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0 \iff (\lambda+1)^3 = 0 \iff \lambda = -1 \text{ of multiplicity}$

three.

If $\lambda_1 = -1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 8 & -8 & 6 \\ 8 & -8 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies 4v_1 - 4v_2 + 3v_3 = 0.$$

The solution set of this system is $\{(r - 3s, r, 4s) : r, s \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = -1$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = r(1, 1, 0) + s(-3, 0, 4), r, s \in \mathbb{R}\}$. A basis for E_1 is $\{(1, 1, 0), (-3, 0, 4)\}$, and $\dim[E_1] = 2$. A is defective since it does not have a complete set of eigenvectors.

12. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & 2 & -1 \\ 2 & 1 - \lambda & -1 \\ 2 & 3 & -1 - \lambda \end{vmatrix} = 0 \iff (\lambda - 2)\lambda^2 = 0 \iff \lambda = 2 \text{ or } \lambda = 0 \text{ of multiplicity two.}$

If $\lambda_1 = 2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 0 & 2 & -1 \\ 2 & -1 & -1 \\ 2 & 3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies 4v_1 - 3v_3 = 0$ and $2v_2 - v_3 = 0$. The solution set of this system is $\{(3r, 2r, 4r) : r \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = 2$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = r(3, 2, 4), r \in \mathbb{R}\}$. A basis for E_1 is $\{(3, 2, 4)\}$, and $\dim[E_1] = 1$.

If $\lambda_2 = 0$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 2 & 2 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies 2v_1 - v_3 = 0$ and $v_2 = 0$. The solution set of this system is $\{(s, 0, 2s) : s \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_2 = 0$ is $E_2 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = s(1, 0, 2), s \in \mathbb{R}\}$. A basis for E_2 is $\{(1, 0, 2)\}$, and $\dim[E_2] = 1$.

A is defective because it does not have a complete set of eigenvectors.

13. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1 - \lambda & -1 & 2 \\ 1 & -1 - \lambda & 2 \\ 1 & -1 & 2 - \lambda \end{vmatrix} = 0 \iff (\lambda - 2)\lambda^2 = 0 \iff \lambda = 2 \text{ or } \lambda = 0 \text{ of multiplicity two.}$

If $\lambda_1 = 2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} -1 & -1 & 2 \\ 1 & -3 & 2 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies v_1 - v_3 = 0$ and $v_2 - v_3 = 0$. The solution set of this system is $\{(r, r, r) : r \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = 2$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = r(1, 1, 1), r \in \mathbb{R}\}$. A basis for E_1 is $\{(1, 1, 1)\}$, and $\dim[E_1] = 1$.

If $\lambda_2 = 0$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 - v_2 + 2v_3 = 0.$$

The solution set of this system is $\{(s - 2t, s, t) : s, t \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_2 = 0$ is $E_2 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = s(1, 1, 0) + t(-2, 0, 1), s, t \in \mathbb{R}\}$. A basis for E_2 is $\{(1, 1, 0), (-2, 0, 1)\}$, and $\dim[E_2] = 2$. A is nondefective because it has a complete set of eigenvectors.

14. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & 3 & 0 \\ -1 & -\lambda & 1 \\ -2 & -1 & 4 - \lambda \end{vmatrix} = 0 \iff (\lambda - 2)^3 = 0 \iff \lambda = 2 \text{ of multiplicity three.}$

If $\lambda_1 = 2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 0 & 3 & 0 \\ -1 & -2 & 1 \\ -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies v_1 - v_3 = 0$ and $v_2 = 0$. The solution set of this system is $\{(r, 0, r) : r \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = 2$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = r(1, 0, 1), r \in \mathbb{R}\}$. A basis for E_1 is $\{(1, 0, 1)\}$, and

$\dim[E_1] = 1$.

A is defective since it does not have a complete set of eigenvectors.

$$15. \det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & -1 & -1 \\ -1 & -\lambda & -1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0 \iff (\lambda + 2)(\lambda - 1)^2 = 0 \iff \lambda = -2 \text{ or } \lambda = 1 \text{ of multiplicity two.}$$

If $\lambda_1 = -2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
 $\implies v_1 - v_3 = 0$ and $v_2 - v_3 = 0$. The solution set of this system is $\{(r, r, r) : r \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = -2$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = r(1, 1, 1), r \in \mathbb{R}\}$. A basis for E_1 is $\{(1, 1, 1)\}$, and $\dim[E_1] = 1$.

$$\text{If } \lambda_2 = 1 \text{ then } (A - \lambda I)\mathbf{v} = \mathbf{0} \text{ assumes the form } \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 + v_2 + v_3 = 0.$$

The solution set of this system is $\{(-s - t, s, t) : s, t \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_2 = 1$ is $E_2 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = s(-1, 1, 0) + t(-1, 0, 1), s, t \in \mathbb{R}\}$. A basis for E_2 is $\{(-1, 1, 0), (-1, 0, 1)\}$, and $\dim[E_2] = 2$.

A is nondefective because it has a complete set of eigenvectors.

16. $(\lambda - 4)(\lambda + 1) = 0 \iff \lambda = 4$ or $\lambda = -1$. Since A has two distinct eigenvalues, it has two linearly independent eigenvectors and is, therefore, nondefective.

17. $(\lambda - 1)^2 = 0 \iff \lambda = 1$ of multiplicity two.

If $\lambda_1 = 1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 5 & 5 \\ -5 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 + v_2 = 0$. The solution set of this system is $\{(-r, r) : r \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda_1 = 1$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} = r(-1, 1), r \in \mathbb{R}\}$. A basis for E_1 is $\{(1, 1)\}$, and $\dim[E_1] = 1$.
 A is defective since it does not have a complete set of eigenvectors.

18. $\lambda^2 - 4\lambda + 13 = 0 \iff \lambda = 2 \pm 3i$. Since A has two distinct eigenvalues, it has two linearly independent eigenvectors and is, therefore, nondefective.

19. $(\lambda - 2)^2(\lambda + 1) = 0 \iff \lambda = -1$ or $\lambda = 2$ of multiplicity two. To determine whether A is nondefective, all we require is the dimension of the eigenspace corresponding to $\lambda = 2$.

$$\text{If } \lambda = 2 \text{ then } (A - \lambda I)\mathbf{v} = \mathbf{0} \text{ assumes the form } \begin{bmatrix} -1 & -3 & 1 \\ -1 & -3 & 1 \\ -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies -v_1 - 3v_2 + v_3 = 0.$$

The solution set of this system is $\{(-3s + t, s, t) : s, t \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda = 2$ is $E = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = s(-3, 1, 0) + t(1, 0, 1), s, t \in \mathbb{R}\}$. Since $\dim[E] = 2$, A is nondefective.

20. $(\lambda - 3)^3 = 0 \iff \lambda = 3$ of multiplicity three.

$$\text{If } \lambda = 3 \text{ then } (A - \lambda I)\mathbf{v} = \mathbf{0} \text{ assumes the form } \begin{bmatrix} -4 & 2 & 2 \\ -4 & 2 & 2 \\ -4 & 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies -2v_1 + v_2 + v_3 = 0 \text{ and}$$

$v_2 = 0$. The solution set of this system is $\{(r, 2r - s, s) : r, s \in \mathbb{R}\}$, so the eigenspace corresponding to $\lambda = 3$ is $E = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = r(1, 2, 0) + s(0, -1, 1), r, s \in \mathbb{R}\}$.

A is defective since it does not have a complete set of eigenvectors.

$$21. \det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & 1 \\ 3 & 4 - \lambda \end{vmatrix} = 0 \iff (\lambda - 1)(\lambda - 5) = 0 \iff \lambda = 1 \text{ or } \lambda = 5.$$

If $\lambda_1 = 1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 + v_2 = 0$. The eigenspace corresponding to $\lambda_1 = 1$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} = r(-1, 1), r \in \mathbb{R}\}$. A basis for E_1 is $\{(-1, 1)\}$.

If $\lambda_2 = 5$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 3v_1 - v_2 = 0$. The eigenspace corresponding to $\lambda_2 = 5$ is $E_2 = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} = s(1, 3), s \in \mathbb{R}\}$. A basis for E_2 is $\{(1, 3)\}$.

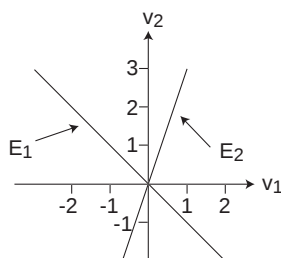


Figure 73: Figure for Problem 21

22. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & 3 \\ 0 & 2 - \lambda \end{vmatrix} = 0 \iff (2 - \lambda)^2 = 0 \iff \lambda = 2$ of multiplicity two.

If $\lambda_1 = 2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 \in \mathbb{R} \text{ and } v_2 = 0$. The eigenspace corresponding to $\lambda_1 = 2$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} = r(1, 0), r \in \mathbb{R}\}$. A basis for E_1 is $\{(1, 0)\}$.

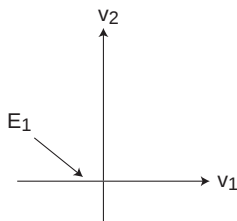


Figure 74: Figure for Problem 22

23. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 5 - \lambda & 0 \\ 0 & 5 - \lambda \end{vmatrix} = 0 \iff (5 - \lambda)^2 = 0 \iff \lambda = 5$ of multiplicity two.

If $\lambda_1 = 5$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1, v_2 \in \mathbb{R}$. The eigenspace corresponding to $\lambda_1 = 5$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} = r(1, 0) + s(0, 1), r, s \in \mathbb{R}\}$. A basis for E_1 is $\{(1, 0), (0, 1)\}$.

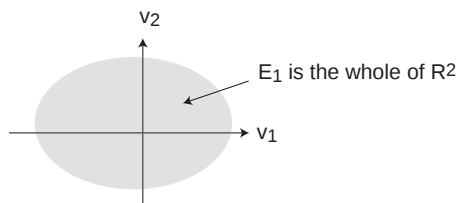


Figure 75: Figure for Problem 23

24. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 3 - \lambda & 1 & -1 \\ 1 & 3 - \lambda & -1 \\ -1 & -1 & 3 - \lambda \end{vmatrix} = 0 \iff (\lambda - 5)(\lambda - 2)^2 = 0 \iff \lambda = 5 \text{ or } \lambda = 2 \text{ of multiplicity two.}$

If $\lambda_1 = 5$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -2 & 1 & -1 \\ 1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies v_1 + v_3 = 0$ and $v_2 + v_3 = 0$. The eigenspace corresponding to $\lambda_1 = 5$ is

$E_1 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = r(1, 1, -1), r \in \mathbb{R}\}$. A basis for E_1 is $\{(1, 1, -1)\}$.

If $\lambda_2 = 2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 + v_2 - v_3 = 0$.

The eigenspace corresponding to $\lambda_2 = 2$ is $E_2 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = s(-1, 1, 0) + t(1, 0, 1), s, t \in \mathbb{R}\}$. A basis for E_2 is $\{(-1, 1, 0), (1, 0, 1)\}$.

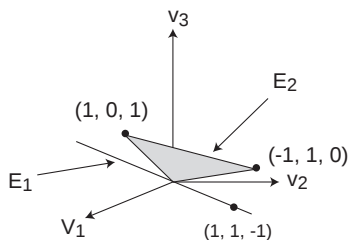


Figure 76: Figure for Problem 24

25. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} -3 - \lambda & 1 & 0 \\ -1 & -1 - \lambda & 2 \\ 0 & 0 & -2 - \lambda \end{vmatrix} = 0 \iff (\lambda + 2)^3 = 0 \iff \lambda = -2 \text{ of multiplicity three.}$

If $\lambda_1 = -2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 - v_2 = 0$ and

$v_3 = 0$. The eigenspace corresponding to $\lambda_1 = -2$ is $E_1 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = r(1, 1, 0), r \in \mathbb{R}\}$. A basis for E_1 is $\{(1, 1, 0)\}$.

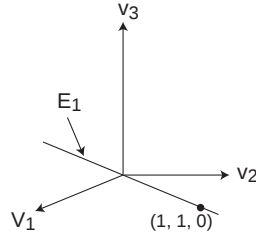


Figure 77: Figure for Problem 25

26. (a) If $\lambda_1 = 1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 1 & -2 & 3 \\ 1 & -2 & 3 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies v_1 - 2v_2 + 3v_3 = 0$. The eigenspace corresponding to $\lambda_1 = 1$ is

$E_1 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = r(2, 1, 0) + s(-3, 0, 1), r, s \in \mathbb{R}\}$. A basis for E_1 is $\{(2, 1, 0), (-3, 0, 1)\}$.

Now apply the Gram-Schmidt process where $\mathbf{v}_1 = (-3, 0, 1)$, and $\mathbf{v}_2 = (2, 1, 0)$. Let $\mathbf{u}_1 = \mathbf{v}_1$ so that $\langle \mathbf{v}_2, \mathbf{u}_1 \rangle = \langle (2, 1, 0), (-3, 0, 1) \rangle = 2(-3) + 1 \cdot 0 + 0 \cdot 1 = -6$ and $\|\mathbf{u}_1\|^2 = (-3)^2 + 0^2 + 1^2 = 10$.

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 = (2, 1, 0) + \frac{6}{10}(-3, 0, 1) = \frac{1}{5}(1, 5, 3).$$

Thus, $\{(-3, 0, 1), (1, 5, 3)\}$ is an orthogonal basis for E_1 .

(b) If $\lambda_2 = 3$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} -1 & -2 & 3 \\ 1 & -4 & 3 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 - v_2 = 0 \text{ and}$$

$v_2 - v_3 = 0$. The eigenspace corresponding to $\lambda_2 = 3$ is $E_2 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = r(1, 1, 1), r \in \mathbb{R}\}$. A basis for E_2 is $\{(1, 1, 1)\}$.

To determine the orthogonality of the vectors, consider the following inner products:

$$\langle (-3, 0, 1), (1, 1, 1) \rangle = -3 + 0 + 1 = -2 \neq 0 \text{ and } \langle (1, 5, 3), (1, 1, 1) \rangle = 1 + 5 + 3 = 9 \neq 0.$$

Thus, the vectors in E_1 are not orthogonal to the vectors in E_2 .

27. (a) If $\lambda_1 = 2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\implies v_1 + v_2 - v_3 = 0$. The eigenspace corresponding to $\lambda_1 = 2$ is

$E_1 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = r(-1, 1, 0) + s(1, 0, 1), r, s \in \mathbb{R}\}$. A basis for E_1 is $\{(-1, 1, 0), (1, 0, 1)\}$.

Now apply the Gram-Schmidt process where $\mathbf{v}_1 = (1, 0, 1)$, and $\mathbf{v}_2 = (-1, 1, 0)$. Let $\mathbf{u}_1 = \mathbf{v}_1$ so that $\langle \mathbf{v}_2, \mathbf{u}_1 \rangle = \langle (-1, 1, 0), (1, 0, 1) \rangle = -1 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 = -1$ and $\|\mathbf{u}_1\|^2 = 1^2 + 0^2 + 1^2 = 2$.

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 = (-1, 1, 0) + \frac{1}{2}(1, 0, 1) = \frac{1}{2}(-1, 2, 1).$$

Thus, $\{(1, 0, 1), (-1, 2, 1)\}$ is an orthogonal basis for E_1 .

(b) If $\lambda_2 = -1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$
 The eigenspace

corresponding to $\lambda_2 = -1$ is $E_2 = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} = r(-1, -1, 1), r \in \mathbb{R}\}$. A basis for E_2 is $\{(-1, -1, 1)\}$.

To determine the orthogonality of the vectors, consider the following inner products:

$$\langle (1, 0, 1), (-1, -1, 1) \rangle = -1 + 0 + 1 = 0 \text{ and } \langle (-1, 2, 1), (-1, -1, 1) \rangle = 1 - 2 + 1 = 0.$$

Thus, the vectors in E_1 are orthogonal to the vectors in E_2 .

28. We are given that the eigenvalues of A are $\lambda_1 = 0$ (multiplicity two), and $\lambda_2 = a + b + c$. There are two cases to consider: $\lambda_1 = \lambda_2$ or $\lambda_1 \neq \lambda_2$.

If $\lambda_1 = \lambda_2$ then $\lambda_1 = 0$ is of multiplicity three, and $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form
$$\begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ or equivalently,}$$

$$av_1 + bv_2 + cv_3 = 0. \quad (28.1)$$

The only way to have three linearly independent eigenvectors for A is if $a = b = c = 0$.

If $\lambda_1 \neq \lambda_2$ then $\lambda_1 = 0$ is of multiplicity two, and $\lambda_2 = a + b + c \neq 0$ are distinct eigenvalues. By Theorem 5.7.11, E_1 must have dimension two for A to possess a complete set of eigenvectors. The system for determining the eigenvectors corresponding to $\lambda_1 = 0$ is once more given by (28.1). Since we can choose two variables freely in (28.1), it follows that there are indeed two corresponding linearly independent eigenvectors. Consequently, A is nondefective in this case.

29. (a) Setting $\lambda = 0$ in (5.7.4), we have $p(\lambda) = \det(A - \lambda I) = \det(A)$, and in (5.7.5), we have $p(\lambda) = p(0) = b_n$. Thus, $b_n = \det(A)$.

The value of $\det(A - \lambda I)$ is the sum of products of its elements, one taken from each row and each column. Expanding $\det(A - \lambda I)$ yields equation (5.7.5). The expression involving λ^n in $p(\lambda)$ comes from the product,
$$\prod_{i=1}^n (a_{ii} - \lambda),$$
 of the diagonal elements. All the remaining products of the determinant have degree not higher than $n - 2$, since, if one of the factors of the product is a_{ij} , where $i \neq j$, then this product cannot contain the factors $\lambda - a_{ii}$ and $\lambda - a_{jj}$. Hence,

$$p(\lambda) = \prod_{i=1}^n (a_{ii} - \lambda) + (\text{terms of degree not higher than } n - 2)$$

so,

$$p(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} (a_{11} + a_{22} + \cdots + a_{nn}) \lambda^{n-1} + \cdots + a_n.$$

Equating like coefficients from (5.7.5), it follows that $b_1 = (-1)^{n-1} (a_{11} + a_{22} + \cdots + a_{nn})$.

(b) Letting $\lambda = 0$, we have from (5.7.6) that $p(0) = \prod_{i=1}^n (\lambda_i - 0)$ or $p(0) = \prod_{i=1}^n \lambda_i$, but from (5.7.5), $p(0) = b_n$,

therefore $b_n = \prod_{i=1}^n \lambda_i$. Letting $\lambda = 1$, we have from (5.7.6) that

$$p(\lambda) = \prod_{i=1}^n (\lambda_i - \lambda) = (-1)^n \prod_{i=1}^n (\lambda - \lambda_i) = (-1)^n [\lambda^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n) \lambda^{n-1} + \cdots + b_n].$$

Equating like coefficients with (5.7.5), it follows that $b_1 = (-1)^{n-1} (\lambda_1 + \lambda_2 + \cdots + \lambda_n)$.

(c) From (a), $b_n = \det(A)$, and from (b), $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$, so $\det(A)$ is the product of the eigenvalues of A .

From (a), $b_1 = (-1)^{n-1} (a_{11} + a_{22} + \cdots + a_{nn})$ and from (b), $b_1 = (-1)^{n-1} (\lambda_1 + \lambda_2 + \cdots + \lambda_n)$, thus $a_{11} + a_{22} + \cdots + a_{nn} = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. That is, $\text{tr}(A)$ is the sum of the eigenvalues of A .

30.

(a) We have $\det(A) = 19$ and $\operatorname{tr}(A) = 3$, so the product of the eigenvalues of A is 19, and the sum of the eigenvalues of A is 3.

(b) We have $\det(A) = -69$ and $\operatorname{tr}(A) = 1$, so the product of the eigenvalues of A is -69, and the sum of the eigenvalues of A is 1.

(c) We have $\det(A) = -607$ and $\operatorname{tr}(A) = 24$, so the product of the eigenvalues of A is -607, and the sum of the eigenvalues of A is 24.

31. Note that $E_i \neq \emptyset$ since $\mathbf{0}$ belongs to E_i .

Closure under Addition: Let $\mathbf{v}_1, \mathbf{v}_2 \in E_i$. Then $A(\mathbf{v}_1 + \mathbf{v}_2) = A\mathbf{v}_1 + A\mathbf{v}_2 = \lambda_i \mathbf{v}_1 + \lambda_i \mathbf{v}_2 = \lambda_i(\mathbf{v}_1 + \mathbf{v}_2) \implies \mathbf{v}_1 + \mathbf{v}_2 \in E_i$.

Closure under Scalar Multiplication: Let $c \in \mathbb{C}$ and $\mathbf{v}_1 \in E_i$. Then $A(c\mathbf{v}_1) = c(A\mathbf{v}_1) = c(\lambda_i \mathbf{v}_1) = \lambda_i(c\mathbf{v}_1) \implies c\mathbf{v}_1 \in E_i$.

Thus, by Theorem 4.3.2, E_i is a subspace of C^n .

32. The condition

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 = \mathbf{0} \quad (32.1)$$

$$\implies A(c_1 \mathbf{v}_1) + A(c_2 \mathbf{v}_2) = \mathbf{0} \implies c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2 = \mathbf{0}$$

$$\implies c_1(\lambda_1 \mathbf{v}_1) + c_2(\lambda_2 \mathbf{v}_2) = \mathbf{0}. \quad (32.2)$$

Substituting $c_2 \mathbf{v}_2$ from (32.1) into (32.2) yields $(\lambda_1 - \lambda_2)c_1 \mathbf{v}_1 = \mathbf{0}$.

Since $\lambda_2 \neq \lambda_1$, we must have $c_1 \mathbf{v}_1 = \mathbf{0}$, but $\mathbf{v}_1 \neq \mathbf{0}$, so $c_1 = 0$. Substituting into (32.1) yields $c_2 = 0$ also. Consequently, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

33. Consider

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = \mathbf{0}. \quad (33.1)$$

If $c_1 \neq 0$, then the preceding equation can be written as

$$\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{0},$$

where $\mathbf{w}_1 = c_1 \mathbf{v}_1$ and $\mathbf{w}_2 = c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3$. But this would imply that $\{\mathbf{w}_1, \mathbf{w}_2\}$ is linearly dependent, which would contradict Theorem 5.7.5 since $\mathbf{w}_1$ and $\mathbf{w}_2$ are eigenvectors corresponding to different eigenvalues. Consequently, we must have $c_1 = 0$. But then (33.1) implies that $c_2 = c_3 = 0$ since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set by assumption. Hence $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

34. $\lambda_1 = 1$ (multiplicity 3), basis: $\{(0, 1, 1)\}$.

35. $\lambda_1 = 0$ (multiplicity 2), basis: $\{(-1, 1, 0), (-1, 0, 1)\}$. $\lambda_2 = 3$, basis: $\{(1, 1, 1)\}$.

36. $\lambda_1 = 2$, basis: $\{(1, -2\sqrt{2}, 1)\}$. $\lambda_2 = 0$, basis: $\{(1, 0, -1)\}$. $\lambda_3 = 7$, basis: $\{(\sqrt{2}, 1, \sqrt{2})\}$.

37. $\lambda_1 = -2$, basis: $\{(2, 1, -4)\}$. $\lambda_2 = 3$ (multiplicity 2), basis: $\{(0, 2, 1), (3, 11, 0)\}$.

38. $\lambda_1 = 0$ (multiplicity 2), basis: $\{(0, 1, 0, -1), (1, 0, -1, 0)\}$. $\lambda_2 = 6$, basis: $\{(1, 1, 1, 1)\}$. $\lambda_3 = -2$, basis: $\{1, -1, 1, -1\}$.

39. A has eigenvalues:

$$\lambda_1 = \frac{3}{2}a + \frac{1}{2}\sqrt{a^2 + 8b^2},$$

$$\lambda_2 = \frac{3}{2}a - \frac{1}{2}\sqrt{a^2 + 8b^2},$$

$$\lambda_3 = 0.$$

Provided $a \neq \pm b$, these eigenvalues are distinct, and therefore the matrix is nondefective.

If $a = b \neq 0$, then the eigenvalue $\lambda = 0$ has multiplicity two. A basis for the corresponding eigenspace is $\{(-1, 0, 1), (-1, 1, 0)\}$. Since this is two-dimensional, the matrix is nondefective in this case.

If $a = -b \neq 0$, then the eigenvalue $\lambda = 0$ once more has multiplicity two. A basis for the corresponding eigenspace is $\{(0, 1, 1), (1, 1, 0)\}$, therefore the matrix is nondefective in this case also.

If $a = b = 0$, then $A = 0_2$, so that $\lambda = 0$ (multiplicity three), and the corresponding eigenspace is all of $\mathbb{R}^3$. Hence A is nondefective.

40. If $a = b = 0$, then $A = 0_3$, which is nondefective. We now assume that at least one of either a or b is nonzero.

A has eigenvalues: $\lambda_1 = b$, $\lambda_2 = a - b$, and $\lambda_3 = 3a + b$.

Provided $a \neq 0, 2b, -b$, the eigenvalues are distinct, and therefore A is nondefective.

If $a = 0$, then the eigenvalue $\lambda = b$ has multiplicity two. In this case, a basis for the corresponding eigenspace is $\{(1, 0, 1), (0, 1, 0)\}$, so that A is nondefective.

If $a = 2b$, then the eigenvalue $\lambda = b$ has multiplicity two. In this case, a basis for the corresponding eigenspace is $\{(-2, 1, 0), (-1, 0, 1)\}$, so that A is nondefective.

If $a = -b$, then the eigenvalue $\lambda = -2b$ has multiplicity two. In this case, a basis for the corresponding eigenspace is $\{(0, 1, 1), (1, 0, -1)\}$, so that A is nondefective.

Solutions to Section 5.8

True-False Review:

1. TRUE. The terms “diagonalizable” and “nondefective” are synonymous. The diagonalizability of a matrix A hinges on the ability to form an invertible matrix S with a full set of linearly independent eigenvectors of the matrix as its columns. This, in turn, requires the original matrix to be nondefective.

2. TRUE. If we assume that A is diagonalizable, then there exists an invertible matrix S and a diagonal matrix D such that $S^{-1}AS = D$. Since A is invertible, we can take the inverse of each side of this equation to obtain

$$D^{-1} = (S^{-1}AS)^{-1} = S^{-1}A^{-1}S,$$

and since D^{-1} is still a diagonal matrix, this equation shows that A^{-1} is diagonalizable.

3. FALSE. For instance, the matrices $A = I_2$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ both have eigenvalue $\lambda = 1$ (with multiplicity 2). However, A and B are not similar. [**Reason:** If A and B were similar, then $S^{-1}AS = B$ for some invertible matrix S , but since $A = I_2$, this would imply that $B = I_2$, contrary to our choice of B above.]

4. FALSE. An $n \times n$ matrix is diagonalizable if and only if it has n *linearly independent* eigenvectors. Besides, every matrix actually has infinitely many eigenvectors, obtained by taking scalar multiples of a single eigenvector $\mathbf{v}$.

5. TRUE. Assume A is an $n \times n$ matrix such that $p(\lambda) = \det(A - \lambda I)$ has no repeated roots. This implies that A has n distinct eigenvalues. Corresponding to each eigenvalue, we can select an eigenvector. Since eigenvectors corresponding to distinct eigenvalues are linearly independent, this yields n linearly independent eigenvectors for A . Therefore, A is nondefective, and hence, diagonalizable.

6. TRUE. Assuming that A is diagonalizable, then there exists an invertible matrix S and a diagonal matrix D such that $S^{-1}AS = D$. Therefore,

$$D^2 = (S^{-1}AS)^2 = (S^{-1}AS)(S^{-1}AS) = S^{-1}ASS^{-1}AS = S^{-1}A^2S.$$

Since D^2 is still a diagonalizable matrix, this equation shows that A^2 is diagonalizable.

7. TRUE. Since $I_n^{-1}AI_n = A$, A is similar to itself.

8. TRUE. The sum of the dimensions of the eigenspaces of such a matrix is even, and therefore not equal to n . This means we cannot obtain n linearly independent eigenvectors for A , and therefore, A is defective (and not diagonalizable).

Problems:

1. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} -1-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = 0 \iff \lambda^2 - \lambda - 6 = 0 \iff (\lambda - 3)(\lambda + 2) = 0 \iff \lambda = 3 \text{ or } \lambda = -2$. A is diagonalizable because it has two distinct eigenvalues.

If $\lambda_1 = 3$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 2v_1 + v_2 = 0$. If we let $v_1 = r \in \mathbb{R}$, then the solution set of this system is $\{(-r, 2r) : r \in \mathbb{R}\}$, so the eigenvectors corresponding to $\lambda_1 = 3$ are $\mathbf{v}_1 = r(-1, 2)$ where $r \in \mathbb{R}$.

If $\lambda_2 = -2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 - 2v_2 = 0$. If we let $v_2 = s \in \mathbb{R}$, then the solution set of this system is $\{(2s, s) : s \in \mathbb{R}\}$, so the eigenvectors corresponding to $\lambda_2 = -2$ are $\mathbf{v}_2 = s(2, 1)$ where $s \in \mathbb{R}$.

Thus, the matrix $S = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ satisfies $S^{-1}AS = \text{diag}(3, -2)$.

2. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} -7-\lambda & 4 \\ -4 & 1-\lambda \end{vmatrix} = 0 \iff \lambda^2 + 6\lambda + 9 = 0 \iff (\lambda + 3)^2 = 0 \iff \lambda = -3$ of multiplicity two.

If $\lambda = -3$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -4 & 4 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 - v_2 = 0$. If we let $v_1 = r \in \mathbb{R}$, then the solution set of this system is $\{(r, r) : r \in \mathbb{R}\}$, so the eigenvectors corresponding to $\lambda = -3$ are $\mathbf{v} = r(1, 1)$ where $r \in \mathbb{R}$.

A has only one linearly independent eigenvector, so by Theorem 5.8.4, A is not diagonalizable.

3. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1-\lambda & -8 \\ 2 & -7-\lambda \end{vmatrix} = 0 \iff \lambda^2 + 6\lambda + 9 = 0 \iff (\lambda + 3)^2 = 0 \iff \lambda = -3$ of multiplicity two.

If $\lambda = -3$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 4 & -8 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 - 2v_2 = 0$. If we let $v_2 = r \in \mathbb{R}$, then the solution set of this system is $\{(2r, r) : r \in \mathbb{R}\}$, so the eigenvectors corresponding to $\lambda = -3$ are $\mathbf{v} = r(2, 1)$ where $r \in \mathbb{R}$.

A has only one linearly independent eigenvector, so by Theorem 5.8.4, A is not diagonalizable.

4. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 4 \\ -4 & -\lambda \end{vmatrix} = 0 \iff \lambda^2 + 16 = 0 \iff \lambda = \pm 4i$. A is diagonalizable because it has two distinct eigenvalues.

If $\lambda = -4i$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 4i & 4 \\ -4 & 4i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 - iv_2 = 0$. If we let $v_2 = r \in \mathbb{C}$, then the solution set of this system is $\{(ir, r) : r \in \mathbb{C}\}$, so the eigenvectors corresponding to $\lambda = -4i$ are $\mathbf{v} = r(i, 1)$ where $r \in \mathbb{C}$. Since the entries of A are real, it follows from Theorem 5.6.8 that $v_2 = (-i, 1)$ is an eigenvector corresponding to $\lambda = 4i$.

Thus, the matrix $S = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ satisfies $S^{-1}AS = \text{diag}(-4i, 4i)$.

$$5. \det(A - \lambda I) = 0 \iff \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & 7 \\ 1 & 1 & -3-\lambda \end{vmatrix} = 0 \iff (1-\lambda)(\lambda+4)(\lambda-4) = 0 \iff \lambda = 1, \lambda = -4 \text{ or } \lambda = 4.$$

If $\lambda = 1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 7 \\ 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
 $\implies 2v_1 - 15v_3 = 0$ and $2v_2 + 7v_3 = 0$. If we let $v_3 = 2r$ where $r \in \mathbb{R}$, then the solution set of this system is $\{(15r, -7r, 2r) : r \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 1$ are $\mathbf{v}_1 = r(15, -7, 2)$ where $r \in \mathbb{R}$.

If $\lambda = -4$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 7 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
 $\implies v_1 = 0$ and $v_2 + v_3 = 0$. If we let $v_2 = s \in \mathbb{R}$, then the solution set of this system is $\{(0, s, -s) : s \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = -4$ are $\mathbf{v}_2 = s(0, 1, -1)$ where $s \in \mathbb{R}$.

If $\lambda = 4$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & 7 \\ 1 & 1 & -7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
 $\implies v_1 = 0$ and $v_2 - 7v_3 = 0$. If we let $v_3 = t \in \mathbb{R}$, then the solution set of this system is $\{(0, 7t, t) : t \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = 4$ are $\mathbf{v}_3 = t(0, 7, 1)$ where $t \in \mathbb{R}$.

Thus, the matrix $S = \begin{bmatrix} 15 & 0 & 0 \\ -7 & 7 & 1 \\ 2 & 1 & -1 \end{bmatrix}$ satisfies $S^{-1}AS = \text{diag}(1, 4, -4)$.

$$6. \det(A - \lambda I) = 0 \iff \begin{vmatrix} 1-\lambda & -2 & 0 \\ 2 & -3-\lambda & 0 \\ 2 & -2 & -2-\lambda \end{vmatrix} = 0 \iff (\lambda+1)^3 = 0 \iff \lambda = -1 \text{ of multiplicity three.}$$

If $\lambda = -1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 2 & -2 & 0 \\ 2 & -2 & 0 \\ 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 - v_2 = 0$ and $v_3 \in \mathbb{R}$. If we let $v_2 = r \in \mathbb{R}$ and $v_3 = s \in \mathbb{R}$, then the solution set of this system is $\{(r, r, s) : r, s \in \mathbb{R}\}$, so the eigenvectors corresponding to $\lambda = -1$ are $\mathbf{v}_1 = r(1, 1, 0)$ and $\mathbf{v}_2 = s(0, 0, 1)$ where $r, s \in \mathbb{R}$.
 A has only two linearly independent eigenvectors, so by Theorem 5.8.4, A is not diagonalizable.

$$7. \det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & -2 & -2 \\ -2 & -\lambda & -2 \\ -2 & -2 & -\lambda \end{vmatrix} = 0 \iff (\lambda-2)^2(\lambda+4) = 0 \iff \lambda = -4 \text{ or } \lambda = 2 \text{ of multiplicity two.}$$

If $\lambda = -4$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
 $\implies v_1 - v_3 = 0$ and $v_2 - v_3 = 0$. If we let $v_3 = r \in \mathbb{R}$, then the solution set of this system is $\{(r, r, r) : r \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = -4$ are $\mathbf{v}_1 = r(1, 1, 1)$ where $r \in \mathbb{R}$.

If $\lambda = 2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 + v_2 + v_3 = 0$. If we let $v_2 = s \in \mathbb{R}$ and $v_3 = t \in \mathbb{R}$, then the solution set of this system is $\{(-s-t, s, t) : s, t \in \mathbb{R}\}$, so two linearly independent eigenvectors corresponding to $\lambda = 2$ are $\mathbf{v}_2 = s(-1, 1, 0)$ and $\mathbf{v}_3 = t(-1, 0, 1)$.

Thus, the matrix $S = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ satisfies $S^{-1}AS = \text{diag}(-4, 2, 2)$.

8. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} -2-\lambda & 1 & 4 \\ -2 & 1-\lambda & 4 \\ -2 & 1 & 4-\lambda \end{vmatrix} = 0 \iff \lambda^2(\lambda-3) = 0 \iff \lambda = 3, \text{ or } \lambda = 0 \text{ of multiplicity two.}$

If $\lambda = 3$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -5 & 1 & 4 \\ -2 & -2 & 4 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies v_1 - v_3 = 0$ and $v_2 - v_3 = 0$. If we let $v_3 = r \in \mathbb{R}$, then the solution set of this system is $\{(r, r, r) : r \in \mathbb{R}\}$, so the eigenvectors corresponding to $\lambda = 3$ are $\mathbf{v}_1 = r(1, 1, 1)$ where $r \in \mathbb{R}$.

If $\lambda = 0$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -2 & 1 & 4 \\ -2 & 1 & 4 \\ -2 & 1 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies -2v_1 + v_2 + 4v_3 = 0$. If

we let $v_1 = s \in \mathbb{R}$ and $v_3 = t \in \mathbb{R}$, then the solution set of this system is $\{(s, 2s - 4t, t) : s, t \in \mathbb{R}\}$, so two linearly independent eigenvectors corresponding to $\lambda = 0$ are $\mathbf{v}_2 = s(1, 2, 0)$ and $\mathbf{v}_3 = t(0, -4, 1)$.

Thus, the matrix $S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -4 \\ 1 & 0 & 1 \end{bmatrix}$ satisfies $S^{-1}AS = \text{diag}(3, 0, 0)$.

9. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 2 & -1 & 1-\lambda \end{vmatrix} = 0 \iff (\lambda-2)(\lambda-1)^2 = 0 \iff \lambda = 2 \text{ or } \lambda = 1 \text{ of multiplicity two.}$

If $\lambda = 1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 = v_2 = 0$ and

$v_3 \in \mathbb{R}$. If we let $v_3 = r \in \mathbb{R}$ then the solution set of this system is $\{(0, 0, r) : r \in \mathbb{R}\}$, so there is only one corresponding linearly independent eigenvector. Hence, by Theorem 5.8.4, A is not diagonalizable.

10. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 4-\lambda & 0 & 0 \\ 3 & -1-\lambda & -1 \\ 0 & 2 & 1-\lambda \end{vmatrix} = 0 \iff (\lambda^2 + 1)(\lambda - 4) = 0 \iff \lambda = 4, \text{ or } \lambda = \pm i.$

If $\lambda = 4$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 0 & 0 & 0 \\ 3 & -5 & -1 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies 6v_1 - 17v_3 = 0$ and $2v_2 - 3v_3 = 0$. If we let $v_3 = 6r \in \mathbb{C}$, then the solution set of this system is $\{(17r, 9r, 6r) : r \in \mathbb{C}\}$, so the eigenvectors corresponding to $\lambda = 4$ are $\mathbf{v}_1 = r(17, 9, 6)$ where $r \in \mathbb{C}$.

If $\lambda = i$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 4-i & 0 & 0 \\ 3 & -1-i & -1 \\ 0 & 2 & 1-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 = 0$ and

$2v_2 + (1-i)v_3 = 0$. If we let $v_3 = -2s \in \mathbb{C}$, then the solution set of this system is $\{(0, (1-i)s, 2s) : s \in \mathbb{C}\}$, so the eigenvectors corresponding to $\lambda = i$ are $v_2 = s(0, 1-i, 2)$ where $s \in \mathbb{C}$. Since the entries of A are real, $v_3 = t(0, 1+i, 2)$ where $t \in \mathbb{C}$ are the eigenvectors corresponding to $\lambda = -i$ by Theorem 5.6.8.

Thus, the matrix $S = \begin{bmatrix} 17 & 0 & 0 \\ 9 & 1-i & 1+i \\ 6 & 2 & 2 \end{bmatrix}$ satisfies $S^{-1}AS = \text{diag}(4, i, -i)$.

11. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 2 & -1 \\ -2 & -\lambda & -2 \\ 1 & 2 & -\lambda \end{vmatrix} = 0 \iff \lambda(\lambda^2 + 9) = 0 \iff \lambda = 0, \text{ or } \lambda = \pm 3i.$

If $\lambda = 0$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies v_1 + v_3 = 0$ and $2v_2 - v_3 = 0$. If we let $v_3 = 2r \in \mathbb{C}$, then the solution set of this system is $\{(-2r, r, 2r) : r \in \mathbb{C}\}$, so the eigenvectors corresponding to $\lambda = 0$ are $\mathbf{v}_1 = r(-2, 1, 2)$ where $r \in \mathbb{C}$.

If $\lambda = -3i$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 3i & 2 & -1 \\ -2 & 3i & -2 \\ 1 & 2 & 3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies 5v_1 + (-4 + 3i)v_3 = 0$ and $5v_2 + (2 + 6i)v_3 = 0$. If we let $v_3 = 5s \in \mathbb{C}$, then the solution set of this system is $\{(4 - 3i)s, (-2 - 6i)s, 5s) : s \in \mathbb{C}\}$, so the eigenvectors corresponding to $\lambda = -3i$ are $\mathbf{v}_2 = s(4 - 3i, -2 - 6i, 5)$ where $s \in \mathbb{C}$. Since the entries of A are real, $\mathbf{v}_3 = t(4 + 3i, -2 + 6i, 5)$ where $t \in \mathbb{C}$ are the eigenvectors corresponding to $\lambda = 3i$ by Theorem 5.6.8.

Thus, the matrix $S = \begin{bmatrix} -2 & 4 + 3i & 4 - 3i \\ 1 & -2 + 6i & -2 - 6i \\ 2 & 5 & 5 \end{bmatrix}$ satisfies $S^{-1}AS = \text{diag}(0, 3i, -3i)$.

12. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1 - \lambda & -2 & 0 \\ -2 & 1 - \lambda & 0 \\ 0 & 0 & 3 - \lambda \end{vmatrix} = 0 \iff (\lambda - 3)^2(\lambda + 1) = 0 \iff \lambda = -1 \text{ or } \lambda = 3$ of multiplicity two.

If $\lambda = -1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies 2v_1 - v_2 = 0$ and $v_3 = 0$. If we let $v_1 = r \in \mathbb{R}$, then the solution set of this system is $\{(r, r, 0) : r \in \mathbb{R}\}$ so the eigenvectors corresponding to $\lambda = -1$ are $\mathbf{v}_1 = r(1, 1, 0)$ where $r \in \mathbb{R}$.

If $\lambda = 3$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 + v_2 = 0$ and

$v_3 \in \mathbb{R}$. If we let $v_2 = s \in \mathbb{R}$ and $v_3 = t \in \mathbb{R}$, then the solution set of this system is $\{(-s, s, t) : s, t \in \mathbb{R}\}$, so the eigenvectors corresponding to $\lambda = 3$ are $\mathbf{v}_2 = s(-1, 1, 0)$ and $\mathbf{v}_3 = t(0, 0, 1)$.

Thus, the matrix $S = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ satisfies $S^{-1}AS = \text{diag}(-1, 3, 3)$.

13. $\lambda_1 = 2$ (multiplicity 2), basis for eigenspace: $\{(-3, 1, 0), (3, 0, 1)\}$.

$\lambda_2 = 1$, basis for eigenspace: $\{(1, 2, 2)\}$.

Set $S = \begin{bmatrix} -3 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$. Then $S^{-1}AS = \text{diag}(2, 2, 1)$.

14. $\lambda_1 = 0$ (multiplicity 2), basis for eigenspace: $\{(0, 1, 0, -1), (1, 0, -1, 0)\}$.

$\lambda_2 = 2$, basis for eigenspace: $\{(1, 1, 1, 1)\}$. $\lambda_3 = 10$, basis for eigenspace: $\{(-1, 1, -1, 1)\}$.

Set $S = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \\ -1 & 0 & 1 & 1 \end{bmatrix}$. Then $S^{-1}AS = \text{diag}(0, 0, 2, 10)$.

15. The given system can be written as $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

A has eigenvalues $\lambda_1 = -1$, $\lambda_2 = 5$ with corresponding linearly independent eigenvectors $\mathbf{v}_1 = (-2, 1)$ and

$\mathbf{v}_2 = (1, 1)$. If we set $S = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$, then $S^{-1}AS = \text{diag}(-1, 5)$, therefore, under the transformation $\mathbf{x} = S\mathbf{y}$, the given system of differential equations simplifies to

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Hence, $y_1' = -y_1$ and $y_2' = 5y_2$. Integrating these equations, we obtain

$$y_1(t) = c_1 e^{-t}, \quad y_2(t) = c_2 e^{5t}.$$

Returning to the original variables, we have

$$\mathbf{x} = S\mathbf{y} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{5t} \end{bmatrix} = \begin{bmatrix} -2c_1 e^{-t} + c_2 e^{5t} \\ c_1 e^{-t} + c_2 e^{5t} \end{bmatrix}.$$

Consequently, $x_1(t) = -2c_1 e^{-t} + c_2 e^{5t}$ and $x_2(t) = c_1 e^{-t} + c_2 e^{5t}$.

16. The given system can be written as $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 6 & -2 \\ -2 & 6 \end{bmatrix}$.

A has eigenvalues $\lambda_1 = 4$, $\lambda_2 = 8$ with corresponding linearly independent eigenvectors $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (-1, 1)$. If we set $S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, then $S^{-1}AS = \text{diag}(4, 8)$, therefore, under the transformation $\mathbf{x} = S\mathbf{y}$, the given system of differential equations simplifies to

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Hence, $y_1' = 4y_1$ and $y_2' = 8y_2$. Integrating these equations, we obtain

$$y_1(t) = c_1 e^{4t}, \quad y_2(t) = c_2 e^{8t}.$$

Returning to the original variables, we have

$$\mathbf{x} = S\mathbf{y} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{4t} \\ c_2 e^{8t} \end{bmatrix} = \begin{bmatrix} c_1 e^{4t} - c_2 e^{8t} \\ c_1 e^{4t} + c_2 e^{8t} \end{bmatrix}.$$

Consequently, $x_1(t) = c_1 e^{4t} - c_2 e^{8t}$ and $x_2(t) = c_1 e^{4t} + c_2 e^{8t}$.

17. The given system can be written as $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 9 & 6 \\ -10 & -7 \end{bmatrix}$.

A has eigenvalues $\lambda_1 = -1$, $\lambda_2 = 3$ with corresponding linearly independent eigenvectors $\mathbf{v}_1 = (3, -5)$ and $\mathbf{v}_2 = (-1, 1)$. If we set $S = \begin{bmatrix} 3 & -1 \\ -5 & 1 \end{bmatrix}$, then $S^{-1}AS = \text{diag}(-1, 3)$, therefore, under the transformation $\mathbf{x} = S\mathbf{y}$, the given system of differential equations simplifies to

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Hence, $y_1' = -y_1$ and $y_2' = 3y_2$. Integrating these equations, we obtain

$$y_1(t) = c_1 e^{-t}, \quad y_2(t) = c_2 e^{3t}.$$

Returning to the original variables, we have

$$\mathbf{x} = S\mathbf{y} = \begin{bmatrix} 3 & -1 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{3t} \end{bmatrix} = \begin{bmatrix} 3c_1 e^{-t} - c_2 e^{3t} \\ -5c_1 e^{-t} + c_2 e^{3t} \end{bmatrix}.$$

Consequently, $x_1(t) = 3c_1 e^{-t} - c_2 e^{3t}$ and $x_2(t) = -5c_1 e^{-t} + c_2 e^{3t}$.

18. The given system can be written as $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} -12 & -7 \\ 16 & 10 \end{bmatrix}$.

A has eigenvalues $\lambda_1 = 2$, $\lambda_2 = -4$ with corresponding linearly independent eigenvectors $\mathbf{v}_1 = (1, -2)$ and $\mathbf{v}_2 = (7, -8)$. If we set $S = \begin{bmatrix} 1 & 7 \\ -2 & -8 \end{bmatrix}$, then $S^{-1}AS = \text{diag}(2, -4)$, therefore, under the transformation $\mathbf{x} = S\mathbf{y}$, the given system of differential equations simplifies to

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Hence, $y_1' = 2y_1$ and $y_2' = -4y_2$. Integrating these equations, we obtain

$$y_1(t) = c_1 e^{2t}, \quad y_2(t) = c_2 e^{-4t}.$$

Returning to the original variables, we have

$$\mathbf{x} = S\mathbf{y} = \begin{bmatrix} 1 & 7 \\ -2 & -8 \end{bmatrix} \begin{bmatrix} c_1 e^{2t} \\ c_2 e^{-4t} \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} + 7c_2 e^{-4t} \\ -2c_1 e^{2t} - 8c_2 e^{-4t} \end{bmatrix}.$$

Consequently, $x_1(t) = c_1 e^{2t} + 7c_2 e^{-4t}$ and $x_2(t) = -2c_1 e^{2t} - 8c_2 e^{-4t}$.

19. The given system can be written as $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

A has eigenvalues $\lambda_1 = i$, $\lambda_2 = -i$ with corresponding linearly independent eigenvectors $\mathbf{v}_1 = (1, i)$ and $\mathbf{v}_2 = (1, -i)$. If we set $S = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$, then $S^{-1}AS = \text{diag}(i, -i)$, therefore, under the transformation $\mathbf{x} = S\mathbf{y}$, the given system of differential equations simplifies to

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Hence, $y_1' = iy_1$ and $y_2' = -iy_2$. Integrating these equations, we obtain

$$y_1(t) = c_1 e^{it}, \quad y_2(t) = c_2 e^{-it}.$$

Returning to the original variables, we have

$$\mathbf{x} = S\mathbf{y} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} c_1 e^{it} \\ c_2 e^{-it} \end{bmatrix} = \begin{bmatrix} c_1 e^{it} + c_2 e^{-it} \\ i(c_1 e^{it} - c_2 e^{-it}) \end{bmatrix}.$$

Consequently, $x_1(t) = c_1 e^{it} + c_2 e^{-it}$ and $x_2(t) = i(c_1 e^{it} - c_2 e^{-it})$. Using Euler's formula, these expressions can be written as

$$\begin{aligned} x_1(t) &= (c_1 + c_2) \cos t + i(c_1 - c_2) \sin t, \\ x_2(t) &= i(c_1 - c_2) \cos t - (c_1 + c_2) \sin t, \end{aligned}$$

or equivalently,

$$x_1(t) = a \cos t + b \sin t, \quad x_2(t) = b \cos t - a \sin t,$$

where $a = c_1 + c_2$, and $b = i(c_1 - c_2)$.

20. The given system can be written as $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 3 & -4 & -1 \\ 0 & -1 & -1 \\ 0 & -4 & 2 \end{bmatrix}$.

A has eigenvalue $\lambda_1 = -2$ with corresponding eigenvector $\mathbf{v}_1 = (1, 1, 1)$, and eigenvalue $\lambda_2 = 3$ with corresponding linearly independent eigenvectors $\mathbf{v}_2 = (1, 0, 0)$ and $\mathbf{v}_3 = (0, 1, -4)$. If we set $S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -4 \end{bmatrix}$, then $S^{-1}AS = \text{diag}(-2, 3, 3)$, therefore, under the transformation $\mathbf{x} = S\mathbf{y}$, the given system of differential equations simplifies to

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Hence, $y_1' = -2y_1$, $y_2' = 3y_2$, and $y_3' = 3y_3$. Integrating these equations, we obtain

$$y_1(t) = c_1 e^{-2t}, \quad y_2(t) = c_2 e^{3t}, \quad y_3(t) = c_3 e^{3t}.$$

Returning to the original variables, we have

$$\mathbf{x} = S\mathbf{y} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -4 \end{bmatrix} \begin{bmatrix} c_1 e^{-2t} \\ c_2 e^{3t} \\ c_3 e^{3t} \end{bmatrix} = \begin{bmatrix} c_1 e^{-2t} + c_2 e^{3t} \\ c_1 e^{-2t} + c_3 e^{3t} \\ c_1 e^{-2t} - 4c_3 e^{3t} \end{bmatrix}.$$

Consequently, $x_1(t) = c_1 e^{-2t} + c_2 e^{3t}$, $x_2(t) = c_1 e^{-2t} + c_3 e^{3t}$, and $x_3(t) = c_1 e^{-2t} - 4c_3 e^{3t}$.

21. The given system can be written as $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$.

A has eigenvalue $\lambda_1 = -1$ with corresponding eigenvector $\mathbf{v}_1 = (-1, 1, -1)$, and eigenvalue $\lambda_2 = 2$ with corresponding linearly independent eigenvectors $\mathbf{v}_2 = (0, 1, 1)$ and $\mathbf{v}_3 = (1, 0, -1)$. If we set $S = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & -1 \end{bmatrix}$, then $S^{-1}AS = \text{diag}(-1, 2, 2)$, therefore, under the transformation $\mathbf{x} = S\mathbf{y}$, the given system of differential equations simplifies to

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Hence, $y_1' = -y_1$, $y_2' = 2y_2$, and $y_3' = 2y_3$. Integrating these equations, we obtain

$$y_1(t) = c_1 e^{-t}, \quad y_2(t) = c_2 e^{2t}, \quad y_3(t) = c_3 e^{2t}.$$

Returning to the original variables, we have

$$\mathbf{x} = S\mathbf{y} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{2t} \\ c_3 e^{2t} \end{bmatrix} = \begin{bmatrix} -c_1 e^{-t} + c_3 e^{2t} \\ c_1 e^{-t} + c_2 e^{2t} \\ -c_1 e^{-t} + c_2 e^{2t} - c_3 e^{2t} \end{bmatrix}.$$

Consequently, $x_1(t) = -c_1e^{-t} + c_3e^{2t}$, $x_2(t) = c_1e^{-t} + c_2e^{2t}$, and $-c_1e^{-t} + (c_2 - c_3)e^{2t}$.

22. $A^2 = (SDS^{-1})(SDS^{-1}) = SD(S^{-1}S)DS^{-1} = SDI_nDS^{-1} = SD^2S^{-1}$.

We now use mathematical induction to establish the general result. Suppose that for $k = m > 2$ that $A^m = SD^mS^{-1}$. Then

$$A^{m+1} = AA^m = (SDS^{-1})(SD^mS^{-1}) = SD(S^{-1}S)D^mS^{-1} = SD^{m+1}S^{-1}.$$

It follows by mathematical induction that $A^k = SD^kS^{-1}$ for $k = 1, 2, \dots$

23. Let $A = \text{diag}(a_1, a_2, \dots, a_n)$ and let $B = \text{diag}(b_1, b_2, \dots, b_n)$. Then from the index form of the matrix product,

$$(AB)_{ij} = \sum_{k=1}^n a_{ik}b_{kj} = a_{ii}b_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ a_i b_i, & \text{if } i = j. \end{cases}$$

Consequently, $AB = \text{diag}(a_1b_1, a_2b_2, \dots, a_nb_n)$. Applying this result to the matrix $D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_k)$, it follows directly that $D^k = \text{diag}(\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k)$.

24. The matrix A has eigenvalues $\lambda_1 = 5$, $\lambda_2 = -1$, with corresponding eigenvectors $\mathbf{v}_1 = (1, -3)$ and $\mathbf{v}_2 = (2, -3)$. Thus, if we set $S = \begin{bmatrix} 1 & 2 \\ -3 & -3 \end{bmatrix}$, then $S^{-1}AS = D$, where $D = \text{diag}(5, -1)$.

Equivalently, $A = SDS^{-1}$. It follows from the results of the previous two examples that

$$A^3 = SD^3S^{-1} = \begin{bmatrix} 1 & 2 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} 125 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -\frac{2}{3} \\ 1 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} -127 & -84 \\ 378 & 251 \end{bmatrix},$$

whereas

$$A^5 = SD^5S^{-1} = \begin{bmatrix} 1 & 2 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} 3125 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -\frac{2}{3} \\ 1 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} -3127 & -2084 \\ 9378 & 6251 \end{bmatrix}.$$

25.

(a) This is self-evident from matrix multiplication. Another perspective on this is that when we multiply a matrix B on the left by a diagonal matrix D , the i th row of B gets multiplied by the i th diagonal element of D . Thus, if we multiply the i th row of $\sqrt{D}$, $\sqrt{\lambda_i}$, by the i th diagonal element of $\sqrt{D}$, $\sqrt{\lambda_i}$, the result in the i th row of the product is $\sqrt{\lambda_i}\sqrt{\lambda_i} = \lambda_i$. Therefore, $\sqrt{D}\sqrt{D} = D$, which means that $\sqrt{D}$ is a square root of D .

(b) We have

$$(S\sqrt{D}S^{-1})^2 = (S\sqrt{D}S^{-1})(S\sqrt{D}S^{-1}) = S(\sqrt{D}I\sqrt{D})S^{-1} = SDS^{-1} = A,$$

as required.

(c) We begin by diagonalizing A . We have

$$\det(A - \lambda I) = \det \begin{bmatrix} 6 - \lambda & -2 \\ -3 & 7 - \lambda \end{bmatrix} = (6 - \lambda)(7 - \lambda) - 6 = \lambda^2 - 13\lambda + 36 = (\lambda - 4)(\lambda - 9),$$

so the eigenvalues of A are $\lambda = 4$ and $\lambda = 9$. An eigenvector of A corresponding to $\lambda = 4$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and an eigenvector of A corresponding to $\lambda = 9$ is $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Thus, we can form

$$S = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}.$$

We take $\sqrt{D} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. A fast computation shows that $S^{-1} = \begin{bmatrix} 3/5 & 2/5 \\ 1/5 & -1/5 \end{bmatrix}$. By part (b), one square root of A is given by

$$\sqrt{A} = S\sqrt{D}S^{-1} = \begin{bmatrix} 12/5 & -2/5 \\ -3/5 & 13/5 \end{bmatrix}.$$

Directly squaring this result confirms this matrix as a square root of A .

26. (a) Show: $A \sim A$.

The identity matrix, I , is invertible, and $I = I^{-1}$. Since $IA = AI \implies A = I^{-1}AI$, it follows that $A \sim A$.

(b) Show: $A \sim B \iff B \sim A$.

$A \sim B \implies$ there exists an invertible matrix S such that $B = S^{-1}AS \implies A = SBS^{-1} = (S^{-1})^{-1}BS^{-1}$. But S^{-1} is invertible since S is invertible. Consequently, $B \sim A$.

(c) Show $A \sim B$ and $B \sim C \iff A \sim C$.

$A \sim B \implies$ there exists an invertible matrix S such that $B = S^{-1}AS$; moreover, $B \sim C \implies$ there exists an invertible matrix P such that $C = P^{-1}BP$. Thus,
 $C = P^{-1}BP = P^{-1}(S^{-1}AS)P = (P^{-1}S^{-1})A(SP) = (SP)^{-1}A(SP)$ where SP is invertible. Therefore $A \sim C$.

27. Let $A \sim B$ mean that A is similar to B . Show: $A \sim B \iff A^T \sim B^T$.

$A \sim B \implies$ there exists an invertible matrix S such that $B = S^{-1}AS$
 $\implies B^T = (S^{-1}AS)^T = S^T A^T (S^{-1})^T = S^T A^T (S^T)^{-1}$. S^T is invertible because S is invertible, [since $\det(S) = \det(S^T)$]. Thus, $A^T \sim B^T$.

28. We are given that $A\mathbf{v} = \lambda\mathbf{v}$ and $B = S^{-1}AS$.

$$B(S^{-1}\mathbf{v}) = (S^{-1}AS)(S^{-1}\mathbf{v}) = S^{-1}A(SS^{-1})\mathbf{v} = S^{-1}AI\mathbf{v} = S^{-1}A\mathbf{v} = S^{-1}(\lambda\mathbf{v}) = \lambda(S^{-1}\mathbf{v}).$$

Hence, $S^{-1}\mathbf{v}$ is an eigenvector of B corresponding to the eigenvalue λ .

29. (a) $S^{-1}AS = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \implies \det(S^{-1}AS) = \lambda_1\lambda_2 \cdots \lambda_n$

$\implies \det(A)\det(S^{-1})\det(S) = \lambda_1\lambda_2 \cdots \lambda_n \implies \det(A) = \lambda_1\lambda_2 \cdots \lambda_n$. Since all eigenvalues are nonzero, it follows that $\det(A) \neq 0$. Consequently, A is invertible.

(b) $S^{-1}AS = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \implies [S^{-1}AS]^{-1} = [\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)]^{-1}$

$$\implies S^{-1}A^{-1}(S^{-1})^{-1} = \text{diag}\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_n}\right)$$

$$\implies S^{-1}A^{-1}S = \text{diag}\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_n}\right).$$

30. (a) $S^{-1}AS = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \implies (S^{-1}AS)^T = [\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)]^T$

$\implies S^T A^T (S^{-1})^T = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \implies S^T A^T (S^T)^{-1} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$. Since we have that $Q = (ST)^{-1}$, this implies that $Q^{-1}A^T Q = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$.

(b) Let $M_C = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n]$ where M_C denotes the matrix of cofactors of S . We see from part (a) that A^T is nondefective, which means it possesses a complete set of eigenvectors. Also from part (a), $Q^{-1}A^T Q = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ where $Q = (S^T)^{-1}$, so

$$A^T Q = Q \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \quad (30.1)$$

If we let M_C denote the matrix of cofactors of S , then

$$S^{-1} = \frac{\text{adj}(S)}{\det(S)} = \frac{M_C^T}{\det(S)} \implies (S^{-1})^T = \left(\frac{M_C^T}{\det(S)}\right)^T \implies (S^T)^{-1} = \left(\frac{M_C}{\det(S)}\right) \implies Q = \frac{M_C}{\det(S)}.$$

Substituting this result into Equation (30.1), we obtain

$$\begin{aligned} A^T \frac{M_C}{\det(S)} &= \frac{M_C}{\det(S)} \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \\ \implies A^T M_C &= M_C \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \\ \implies A^T [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n] &= [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n] \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \\ \implies [A^T \mathbf{v}_1, A^T \mathbf{v}_2, A^T \mathbf{v}_3, \dots, A^T \mathbf{v}_n] &= [\lambda_1 \mathbf{v}_1, \lambda_2 \mathbf{v}_2, \lambda_3 \mathbf{v}_3, \dots, \lambda_n \mathbf{v}_n] \\ \implies A^T \mathbf{v}_i &= \lambda \mathbf{v}_i \text{ for } i \in \{1, 2, 3, \dots, n\}. \end{aligned}$$

Hence, the column vectors of M_C are linearly independent eigenvectors of A^T .

$$\mathbf{31.} \det(A - \lambda I) = 0 \iff \begin{vmatrix} -2 - \lambda & 4 \\ 1 & 1 - \lambda \end{vmatrix} = 0 \iff (\lambda + 3)(\lambda - 2) = 0 \iff \lambda_1 = -3 \text{ or } \lambda_2 = 2.$$

If $\lambda_1 = -3$ the corresponding eigenvectors are of the form $\mathbf{v}_1 = r(-4, 1)$ where $r \in \mathbb{R}$.

If $\lambda_2 = 2$ the corresponding eigenvectors are of the form $\mathbf{v}_2 = s(1, 1)$ where $s \in \mathbb{R}$.

Thus, a complete set of eigenvectors is $\{(-4, 1), (1, 1)\}$ so that $S = \begin{bmatrix} -4 & 1 \\ 1 & 1 \end{bmatrix}$. If M_C denotes the matrix of cofactors of S , then $M_C = \begin{bmatrix} 1 & -1 \\ -1 & -4 \end{bmatrix}$. Consequently, from Problem 30, $(1, -1)$ is an eigenvector corresponding to $\lambda = -3$ and $(-1, -4)$ is an eigenvector corresponding to $\lambda = 2$ for the matrix A^T .

$$\begin{aligned} \mathbf{32.} \quad S^{-1}AS &= J_\lambda \iff AS = SJ_\lambda \iff A[\mathbf{v}_1, \mathbf{v}_2] = [\mathbf{v}_1, \mathbf{v}_2] \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \\ \implies [A\mathbf{v}_1, A\mathbf{v}_2] &= [\lambda\mathbf{v}_1, \mathbf{v}_1 + \lambda\mathbf{v}_2] \implies A\mathbf{v}_1 = \lambda\mathbf{v}_1 \text{ and } A\mathbf{v}_2 = \mathbf{v}_1 + \lambda\mathbf{v}_2 \\ \implies (A - \lambda I)\mathbf{v}_1 &= \mathbf{0} \text{ and } (A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1. \end{aligned}$$

$$\mathbf{33.} \det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & 1 \\ -1 & 4 - \lambda \end{vmatrix} = 0 \iff (\lambda - 3)^2 = 0 \iff \lambda_1 = 3 \text{ of multiplicity two.}$$

If $\lambda_1 = 3$ the corresponding eigenvectors are of the form $\mathbf{v}_1 = r(1, 1)$ where $r \in \mathbb{R}$. Consequently, A does not have a complete set of eigenvectors, so it is a defective matrix.

By the preceding problem, $J_3 = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is similar to A . Hence, there exists $S = [\mathbf{v}_1, \mathbf{v}_2]$ such that $S^{-1}AS = J_3$. From the first part of the problem, we can let $\mathbf{v}_1 = (1, 1)$. Now consider $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$ where $\mathbf{v}_2 = (a, b)$ for $a, b \in \mathbb{R}$. Upon substituting, we obtain

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies -a + b = 1.$$

Thus, S takes the form $S = \begin{bmatrix} 1 & b-1 \\ 1 & b \end{bmatrix}$ where $b \in \mathbb{R}$, and if $b = 0$, then $S = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$.

34.

$$\begin{aligned} S^{-1}AS &= \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \iff A[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \\ &\iff [A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3] = [\lambda\mathbf{v}_1, \mathbf{v}_1 + \lambda\mathbf{v}_2, \mathbf{v}_2 + \lambda\mathbf{v}_3] \\ &\iff A\mathbf{v}_1 = \lambda\mathbf{v}_1, A\mathbf{v}_2 = \mathbf{v}_1 + \lambda\mathbf{v}_2, \text{ and } A\mathbf{v}_3 = \mathbf{v}_2 + \lambda\mathbf{v}_3 \\ &\iff (A - \lambda I)\mathbf{v}_1 = \mathbf{0}, (A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1, \text{ and } (A - \lambda I)\mathbf{v}_3 = \mathbf{v}_2. \end{aligned}$$

$$\mathbf{35. (a)} \text{ From (5.8.15), } c_1\mathbf{f}_1 + c_2\mathbf{f}_2 + \dots + c_n\mathbf{f}_n = \mathbf{0} \iff \sum_{i=1}^n \left[c_i \sum_{j=1}^n s_{ji}\mathbf{e}_j \right] = \mathbf{0}$$

$\iff \sum_{j=1}^n \left[\sum_{i=1}^n s_{ji} c_i \right] \mathbf{e}_j = 0 \iff \sum_{i=1}^n s_{ji} c_i = 0, j = 1, 2, \dots, n$, since $\{\mathbf{e}_i\}$ is a linearly independent set. The latter equation is just the component form of the linear system $S\mathbf{c} = \mathbf{0}$. Since $\{\mathbf{f}_i\}$ is a linearly independent set, the only solution to this system is the trivial solution $\mathbf{c} = \mathbf{0}$, so $\det(S) \neq 0$. Consequently, S is invertible.

(b) From (5.8.14) and (5.8.15), we have

$$T(\mathbf{f}_k) = \sum_{i=1}^n b_{ik} \sum_{j=1}^n s_{ji} \mathbf{e}_j = \sum_{j=1}^n \left[\sum_{i=1}^n s_{ji} b_{ik} \right] \mathbf{e}_j, k = 1, 2, \dots, n.$$

Replacing i with j and j with i yields

$$T(\mathbf{f}_k) = \sum_{i=1}^n \left[\sum_{j=1}^n s_{ij} b_{jk} \right] \mathbf{e}_i, k = 1, 2, \dots, n. \quad (*)$$

(c) From (5.8.15) and (5.8.13), we have

$$T(\mathbf{f}_k) = \sum_{j=1}^n s_{jk} T(\mathbf{e}_j) = \sum_{j=1}^n s_{jk} \sum_{i=1}^n a_{ij} \mathbf{e}_i,$$

that is,

$$T(\mathbf{f}_k) = \sum_{i=1}^n \left[\sum_{j=1}^n a_{ij} s_{jk} \right] \mathbf{e}_i, k = 1, 2, \dots, n. \quad (**)$$

(d) Subtracting (*) from (**) yields

$$\sum_{i=1}^n \left[\sum_{j=1}^n (s_{ij} b_{jk} - a_{ij} s_{jk}) \right] \mathbf{e}_i = \mathbf{0}.$$

Thus, since $\{\mathbf{e}_i\}$ is a linearly independent set,

$$\sum_{j=1}^n s_{ij} b_{jk} = \sum_{j=1}^n a_{ij} s_{jk}, i = 1, 2, \dots, n.$$

But this is just the index form of the matrix equation $SB = AS$. Multiplying both sides of the preceding equation on the left by S^{-1} yields $B = S^{-1}AS$.

Solutions to Section 5.9

True-False Review:

1. TRUE. In the definition of the matrix exponential function e^{At} , we see that powers of the matrix A must be computed:

$$e^{At} = I_n + (At) + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$$

In order to do this, A must be a square matrix.

2. TRUE. We see this by plugging in $t = 1$ into the definition of the matrix exponential function. All terms containing $A^3, A^4, A^5, \dots$ must be zero, leaving us with the result given in this statement.

3. FALSE. The inverse of the matrix exponential function e^{At} is the matrix exponential function e^{-At} , and this will exist for all square matrices A , not just invertible ones.

4. TRUE. The matrix exponential function e^{At} converges to a matrix the same size as A , for all $t \in \mathbb{R}$. This is asserted, but not proven, directly beneath Definition 5.9.1.

5. FALSE. The correct statement is

$$(SDS^{-1})^k = SD^kS^{-1}.$$

The matrices S and S^{-1} on the right-hand side of this equation do not get raised to the power k .

6. FALSE. According to Property 1 of the Matrix Exponential Function, we have

$$(e^{At})^2 = (e^{At})(e^{At}) = e^{2At}.$$

Problems:

1. Note that

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2!}(At)^2 + \dots + \frac{1}{k!}(At)^k + \dots \\ &= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} + \begin{bmatrix} d_1 t & 0 & \dots & 0 \\ 0 & d_2 t & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n t \end{bmatrix} + \dots + \frac{1}{k!} \begin{bmatrix} (d_1 t)^k & 0 & \dots & 0 \\ 0 & (d_2 t)^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (d_n t)^k \end{bmatrix} \\ &= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} (d_1 t)^k & 0 & \dots & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!} (d_2 t)^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sum_{k=0}^{\infty} \frac{1}{k!} (d_n t)^k \end{bmatrix} \\ &= \begin{bmatrix} e^{d_1 t} & 0 & \dots & 0 \\ 0 & e^{d_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{d_n t} \end{bmatrix} \\ &= \text{diag}(e^{d_1 t}, e^{d_2 t}, \dots, e^{d_n t}). \end{aligned}$$

2. Using Problem 1, we have

$$e^{At} = \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{5t} \end{bmatrix} \quad \text{and} \quad e^{-At} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-5t} \end{bmatrix}.$$

3. We have

$$e^{\lambda I_n t} = \begin{bmatrix} e^{\lambda t} & 0 & \dots & 0 \\ 0 & e^{\lambda t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda t} \end{bmatrix} = e^{\lambda t} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = e^{\lambda t} I_n.$$

4.

(a) We have

$$BC = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & ab \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = CB.$$

(b) We have $C^2 = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_2$. Therefore,

$$e^{Ct} = I + Ct = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} t = \begin{bmatrix} 1 & bt \\ 0 & 1 \end{bmatrix}.$$

(c) Using the results in the first two parts of this problem, it follows that

$$e^{At} = e^{(B+C)t} = e^{Bt} e^{Ct} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{at} \end{bmatrix} \begin{bmatrix} 1 & bt \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{at} & bte^{bt} \\ 0 & e^{at} \end{bmatrix}.$$

5. Define $B = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ and $C = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$, so that $A = B + C$. Now $BC = \begin{bmatrix} 0 & ab \\ -ab & 0 \end{bmatrix} = CB$, so Property (1) of the matrix exponential function implies that $e^{At} = e^{(B+C)t} = e^{Bt} e^{Ct}$. Now

$$e^{Bt} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{at} \end{bmatrix} = e^{at} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

by Problem 1. To determine e^{Ct} , observe first that

$$C^2 = \begin{bmatrix} -b^2 & 0 \\ 0 & -b^2 \end{bmatrix}, \quad C^3 = \begin{bmatrix} 0 & b^3 \\ -b^3 & 0 \end{bmatrix}, \quad C^4 = \begin{bmatrix} b^4 & 0 \\ 0 & b^4 \end{bmatrix}, \quad C^5 = \begin{bmatrix} 0 & -b^5 \\ b^5 & 0 \end{bmatrix}, \dots$$

In general, we have

$$C^{2n} = b^{2n} \begin{bmatrix} (-1)^n & 0 \\ 0 & (-1)^n \end{bmatrix} \quad \text{and} \quad C^{2n+1} = b^{2n+1} \begin{bmatrix} 0 & (-1)^n \\ (-1)^{n+1} & 0 \end{bmatrix}$$

for each integer $n \geq 0$. Therefore, from Definition 5.9.1, we have

$$e^{Ct} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} b^{2n} t^{2n} & \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} b^{2n+1} t^{2n+1} \\ -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} b^{2n+1} t^{2n+1} & \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} b^{2n} t^{2n} \end{bmatrix} = \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}.$$

Thus,

$$e^{At} = e^{Bt} e^{Ct} = e^{at} \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}.$$

6. We have

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1 - \lambda & 2 \\ 0 & 3 - \lambda \end{vmatrix} = 0 \iff \lambda^2 - 4\lambda + 3 = 0 \iff \lambda = 1 \text{ or } \lambda = 3.$$

Eigenvalue $\lambda = 1$: We have $A - I = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$, so we can choose the eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Eigenvalue $\lambda = 3$: We have $A - 3I = \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}$, so we can choose the eigenvector $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We form the matrices $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. From Theorem 5.9.3, we have

$$e^{At} = Se^{Dt}S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & e^{3t} - e^t \\ 0 & e^{3t} \end{bmatrix}.$$

7. We have

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{vmatrix} = 0 \iff \lambda^2 - 6\lambda + 8 = 0 \iff \lambda = 2 \text{ or } \lambda = 4.$$

Eigenvalue $\lambda = 4$: We have $A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, so we can choose the eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Eigenvalue $\lambda = 2$: We have $A - 2I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, so we can choose the eigenvector $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

We form the matrices $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$. From Theorem 5.9.3, we have

$$e^{At} = Se^{Dt}S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(e^{4t} + e^{2t}) & \frac{1}{2}(e^{4t} - e^{2t}) \\ \frac{1}{2}(e^{4t} - e^{2t}) & \frac{1}{2}(e^{4t} + e^{2t}) \end{bmatrix}.$$

8. We have

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 2 \\ -2 & -\lambda \end{vmatrix} = 0 \iff \lambda^2 + 4 = 0 \iff \lambda = \pm 2i.$$

Eigenvalue $\lambda = 2i$: We have $A - 2iI = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}$, so we can choose the eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$.

Eigenvalue $\lambda = -2i$: By taking the conjugate of the eigenvector obtained above, we can choose $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

We form the matrices $S = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$ and $D = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}$. From Theorem 5.9.3, we have

$$e^{At} = Se^{Dt}S^{-1} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{2it} & 0 \\ 0 & e^{-2it} \end{bmatrix} \begin{bmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{bmatrix} = \begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix}.$$

The simplification in the last step uses the well-known identities $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin x = -\frac{i}{2}(e^{ix} - e^{-ix})$.

9. We have

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} -1 - \lambda & 3 \\ -3 & -1 - \lambda \end{vmatrix} = 0 \iff \lambda^2 + 2\lambda + 10 = 0 \iff \lambda = -1 \pm 3i.$$

Eigenvalue $\lambda = -1 + 3i$: We have $A - (-1 + 3i)I = \begin{bmatrix} -3i & 3 \\ -3 & -3i \end{bmatrix}$, so we can choose the eigenvector $\mathbf{v}_1 =$

$$\begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Eigenvalue $\lambda = -1 - 3i$: By taking the conjugate of the eigenvector obtained above, we can choose $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

We form the matrices $S = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$ and $D = \begin{bmatrix} -1+3i & 0 \\ 0 & -1-3i \end{bmatrix}$. From Theorem 5.9.3, we have

$$e^{At} = Se^{Dt}S^{-1} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{(-1+3i)t} & 0 \\ 0 & e^{(-1-3i)t} \end{bmatrix} \begin{bmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{bmatrix} = e^{-t} \begin{bmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{bmatrix}.$$

The simplification in the last step uses the well-known identities $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin x = -\frac{i}{2}(e^{ix} - e^{-ix})$.

Alternative Solution: Apply Problem 5 directly.

10. Observe that we are repeating the result of Problem 5 here. This problem gives a traditional solution. We have

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} a - \lambda & b \\ -b & a - \lambda \end{vmatrix} = 0 \iff \lambda^2 - 2a\lambda + a^2 + b^2 = 0 \iff \lambda = a \pm bi.$$

Eigenvalue $\lambda = a + bi$: We have $A - (a + bi)I = \begin{bmatrix} -bi & b \\ -b & -bi \end{bmatrix}$, so we can choose the eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$.

Eigenvalue $\lambda = a - bi$: By taking the conjugate of the eigenvector obtained above, we can choose $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

We form the matrices $S = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$ and $D = \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix}$. From Theorem 5.9.3, we have

$$e^{At} = Se^{Dt}S^{-1} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{(a+bi)t} & 0 \\ 0 & e^{(a-bi)t} \end{bmatrix} \begin{bmatrix} 1/2 & -i/2 \\ 1/2 & i/2 \end{bmatrix} = e^{at} \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}.$$

The simplification in the last step uses the well-known identities $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin x = -\frac{i}{2}(e^{ix} - e^{-ix})$.

11. We have

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 3 - \lambda & -2 & -2 \\ 1 & -\lambda & -2 \\ 0 & 0 & 3 - \lambda \end{vmatrix} = 0 \iff (3 - \lambda)(\lambda^2 - 3\lambda + 2) = 0 \iff \lambda = 1 \text{ or } \lambda = 2 \text{ or } \lambda = 3.$$

Eigenvalue $\lambda = 1$: We have $A - I = \begin{bmatrix} 2 & -2 & -2 \\ 1 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$, and we may choose the vector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ in $\text{nullspace}(A - I)$ as an eigenvector corresponding to $\lambda = 1$.

Eigenvalue $\lambda = 2$: We have $A - 2I = \begin{bmatrix} 1 & -2 & -2 \\ 1 & -2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$, and we may choose the vector $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ in $\text{nullspace}(A - 2I)$ as an eigenvector corresponding to $\lambda = 2$.

Eigenvalue $\lambda = 3$: We have $A - 3I = \begin{bmatrix} 0 & -2 & -2 \\ 1 & -3 & -2 \\ 0 & 0 & 0 \end{bmatrix}$, and we may choose the vector $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ in $\text{nullspace}(A - 3I)$ as an eigenvector corresponding to $\lambda = 3$.

We form the matrices $S = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. From Theorem 5.9.3, we have

$$\begin{aligned} e^{At} &= Se^{Dt}S^{-1} \\ &= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} e^t(2e^t - 1) & -2e^t(e^t - 1) & -e^t(e^{2t} - 1) \\ e^t(e^t - 1) & -e^t(e^t - 2) & -e^t(e^{2t} - 1) \\ 0 & 0 & e^{3t} \end{bmatrix}. \end{aligned}$$

12. We have

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 6 - \lambda & -2 & -1 \\ 8 & -2 - \lambda & -2 \\ 4 & -2 & 1 - \lambda \end{vmatrix} = 0 \iff (\lambda - 2)^2(\lambda - 1) = 0 \iff \lambda = 2 \text{ or } \lambda = 1.$$

Eigenvalue $\lambda = 2$: We have $A - 2I = \begin{bmatrix} 4 & -2 & -1 \\ 8 & -4 & -2 \\ 4 & -2 & -1 \end{bmatrix}$, and we may choose two vectors, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and

$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ in $\text{nullspace}(A - 2I)$ as an eigenvector corresponding to $\lambda = 2$.

Eigenvalue $\lambda = 1$: We have $A - I = \begin{bmatrix} 5 & -2 & -1 \\ 8 & -3 & -2 \\ 4 & -2 & 0 \end{bmatrix}$. The reduced row-echelon form of this matrix is

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$, so we make choose $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ in $\text{nullspace}(A - I)$ as an eigenvector corresponding to $\lambda = 1$.

We form the matrices $S = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 0 & 4 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. From Theorem 5.9.3, we have

$$\begin{aligned} e^{At} &= Se^{Dt}S^{-1} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 4 & -3/2 & -1 \\ 1 & -1/2 & 0 \\ -4 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^t(5e^t - 4) & 2e^t(1 - e^t) & e^t(1 - e^t) \\ 8e^t(e^t - 1) & e^t(4 - 3e^t) & 2e^t(1 - e^t) \\ 4e^t(e^t - 1) & 2e^t(1 - e^t) & e^t \end{bmatrix}. \end{aligned}$$

13. Direct computation shows that $A^2 = 0_2$. Therefore,

$$e^{At} = I + At = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -3 & 9 \\ -1 & 3 \end{bmatrix} t = \begin{bmatrix} -3t+1 & 9t \\ -t & 3t+1 \end{bmatrix}.$$

14. Direct computation shows that $A^2 = 0_2$. Therefore,

$$e^{At} = I + At = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} t = \begin{bmatrix} 1+t & t \\ -t & 1-t \end{bmatrix}.$$

15. Direct computation shows that $A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $A^3 = 0_3$. Therefore,

$$e^{At} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} t + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \frac{t^2}{2} = \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ t^2/2 & t & 1 \end{bmatrix}.$$

16. Direct computation shows that $A^2 = \begin{bmatrix} -4 & 8 & -4 \\ -1 & 2 & -1 \\ 2 & -4 & 2 \end{bmatrix}$ and $A^3 = 0_3$. Therefore,

$$\begin{aligned} e^{At} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -6 & -5 \\ 0 & -2 & -1 \\ 1 & 2 & 3 \end{bmatrix} t + \begin{bmatrix} -4 & 8 & -4 \\ -1 & 2 & -1 \\ 2 & -4 & 2 \end{bmatrix} \frac{t^2}{2} \\ &= \begin{bmatrix} 1-t-2t^2 & -6t+4t^2 & -5t-2t^2 \\ -t^2/2 & 1-2t+t^2 & -t-t^2/2 \\ t+t^2 & 2t-2t^2 & 1+3t+t^2 \end{bmatrix}. \end{aligned}$$

17. Direct computation shows that $A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and $A^4 = 0_4$. Therefore,

$$\begin{aligned} e^{At} &= I_4 + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & t^2/2 & 0 \\ 0 & 0 & 0 & t^2/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & t^3/6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & t & t^2/2 & t^3/6 \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

18. It is easy to see (and can be formally verified by induction on k) that the matrix A^k contains all zero entries except for the $(k+i, i)$ entry for $1 \leq i \leq n-k$. In particular, $A^n = A^{n+1} = A^{n+2} = \dots = 0_n$.

Therefore, by Definition 5.9.1, we have

$$\begin{aligned}
 e^{At} &= I_n + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots + \frac{(At)^{n-1}}{(n-1)!} \\
 &= \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ t & 1 & 0 & 0 & \cdots & 0 \\ \frac{t^2}{2!} & t & 1 & 0 & \cdots & 0 \\ \frac{t^3}{3!} & \frac{t^2}{2!} & t & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{t^{n-1}}{(n-1)!} & \frac{t^{n-2}}{(n-2)!} & \frac{t^{n-3}}{(n-3)!} & \frac{t^{n-4}}{(n-4)!} & \cdots & 1 \end{bmatrix}.
 \end{aligned}$$

19. Assume that A_0 is $m \times m$ and B_0 is $n \times n$. Note that these two matrices must be square in order for $e^{A_0 t}$ and $e^{B_0 t}$ to make sense. The key point is that $A^k = \begin{pmatrix} A_0^k & 0 \\ 0 & B_0^k \end{pmatrix}$. Therefore,

$$\begin{aligned}
 e^{At} &= I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots \\
 &= \begin{pmatrix} I_m & 0 \\ 0 & I_n \end{pmatrix} + \begin{pmatrix} A_0 & 0 \\ 0 & B_0 \end{pmatrix} t + \begin{pmatrix} A_0^2 & 0 \\ 0 & B_0^2 \end{pmatrix} \frac{t^2}{2!} + \begin{pmatrix} A_0^3 & 0 \\ 0 & B_0^3 \end{pmatrix} \frac{t^3}{3!} + \cdots \\
 &= \begin{pmatrix} I_m + A_0 t + \frac{(A_0 t)^2}{2!} + \frac{(A_0 t)^3}{3!} + \cdots & 0 \\ 0 & I_n + B_0 t + \frac{(B_0 t)^2}{2!} + \frac{(B_0 t)^3}{3!} + \cdots \end{pmatrix} \\
 &= \begin{pmatrix} e^{A_0 t} & 0 \\ 0 & e^{B_0 t} \end{pmatrix}.
 \end{aligned}$$

Solutions to Section 5.10

True-False Review:

- 1. TRUE.** This is immediate from Definition 5.10.1.
- 2. TRUE.** The roots of $p(\lambda) = \lambda^3 + \lambda$ are $\lambda = 0$ and $\lambda = \pm i$. Therefore, the matrix has complex eigenvalues. But Theorem 5.10.4 indicates that a real matrix that is symmetric must have only real eigenvalues.
- 3. FALSE.** The zero matrix is a real, symmetric matrix for which both $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors.
- 4. TRUE.** This is the statement of the Principal Axes Theorem (Theorem 5.10.6).
- 5. TRUE.** This is a direct application of Lemma 5.10.9.
- 6. TRUE.** Assuming that A and B are orthogonal matrices, then A and B are invertible, and hence AB is invertible. Moreover,

$$(AB)^{-1} = B^{-1}A^{-1} = B^T A^T = (AB)^T,$$
 so that AB meets the requirements of orthogonality spelled out in Definition 5.10.1.
- 7. TRUE.** This is essentially the content of Theorem 5.10.3, since a set of orthogonal unit vectors are precisely a set of orthonormal vectors.
- 8. TRUE.** Let S be the matrix consisting of a complete set of orthonormal eigenvectors of A . Then $S^T A S = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) = D$, where the λ_i are the corresponding eigenvalues. Then S is an orthogonal matrix: $S^T = S^{-1}$. Hence, $A = SDS^T$ and $A^T = (SDS^T)^T = S D^T S^T = SDS^T = A$, so A is symmetric.

Problems:

1. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0 \iff \lambda^2 - \lambda - 6 = 0 \iff (\lambda + 2)(\lambda - 3) = 0 \iff \lambda = -2 \text{ or } \lambda = 3.$

If $\lambda = -2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 2v_1 + v_2 = 0.$ $\mathbf{v}_1 = (1, -2)$, is an eigenvector corresponding to $\lambda = -2$ and $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$ is a unit eigenvector corresponding to $\lambda = -2$.

If $\lambda = 3$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 + 2v_2 = 0.$ $\mathbf{v}_2 = (2, 1)$, is an eigenvector corresponding to $\lambda = 3$ and $\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ is a unit eigenvector corresponding to $\lambda = 3$.

Thus, $S = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$ and $S^T A S = \text{diag}(-2, 3).$

2. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 4-\lambda & 6 \\ 6 & 9-\lambda \end{vmatrix} = 0 \iff \lambda^2 - 13\lambda = 0 \iff \lambda(\lambda - 13) = 0 \iff \lambda = 0 \text{ or } \lambda = 13.$

If $\lambda = 0$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 2v_1 + 3v_2 = 0.$ $\mathbf{v}_1 = (-3, 2)$, is an eigenvector corresponding to $\lambda = 0$ and $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(-\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\right)$ is a unit eigenvector corresponding to $\lambda = 0$.

If $\lambda = 13$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -9 & 6 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies -3v_1 + 2v_2 = 0.$

$\mathbf{v}_2 = (2, 3)$, is an eigenvector corresponding to $\lambda = 13$ and $\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)$ is a unit eigenvector corresponding to $\lambda = 13$.

Thus, $S = \begin{bmatrix} -\frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{bmatrix}$ and $S^T A S = \text{diag}(0, 13).$

3. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0 \iff (\lambda - 1)^2 - 4 = 0 \iff (\lambda + 1)(\lambda - 3) = 0 \iff \lambda = -1 \text{ or } \lambda = 3.$

If $\lambda = -1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 + v_2 = 0.$ $\mathbf{v}_1 = (-1, 1)$, is an eigenvector corresponding to $\lambda = -1$ and $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is a unit eigenvector corresponding to $\lambda = -1$.

If $\lambda = 3$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 - v_2 = 0.$ $\mathbf{v}_2 = (1, 1)$, is an eigenvector corresponding to $\lambda = 3$ and $\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is a unit eigenvector corresponding to $\lambda = 3$.

Thus, $S = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $S^T AS = \text{diag}(-1, 3)$.

4. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 0 & 3 \\ 0 & -2 - \lambda & 0 \\ 3 & 0 & -\lambda \end{vmatrix} = 0 \iff (\lambda + 2)(\lambda - 3)(\lambda + 3) = 0 \iff \lambda = -3, \lambda = -2, \text{ or } \lambda = 3.$

If $\lambda = -3$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 3 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 + v_3 = 0 \text{ and } v_2 = 0.$

$\mathbf{v}_1 = (-1, 0, 1)$, is an eigenvector corresponding to $\lambda = -3$ and $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ is a unit eigenvector corresponding to $\lambda = -3$.

If $\lambda = -2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 2 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 = v_3 = 0 \text{ and } v_2 \in \mathbb{R}.$

$\mathbf{v}_2 = (0, 1, 0)$, is an eigenvector corresponding to $\lambda = -2$ and $\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = (0, 1, 0)$ is a unit eigenvector corresponding to $\lambda = -2$.

If $\lambda = 3$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -3 & 0 & 3 \\ 0 & -5 & 0 \\ 3 & 0 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 - v_3 = 0 \text{ and } v_2 = 0.$

$\mathbf{v}_3 = (1, 0, 1)$, is an eigenvector corresponding to $\lambda = 3$ and $\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ is a unit eigenvector corresponding to $\lambda = 3$.

Thus, $S = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $S^T AS = \text{diag}(-3, -2, 3)$.

5. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1 - \lambda & 2 & 1 \\ 2 & 4 - \lambda & 2 \\ 1 & 2 & 1 - \lambda \end{vmatrix} = 0 \iff \lambda^3 - 6\lambda^2 = 0 \iff \lambda^2(\lambda - 6) = 0 \iff \lambda = 6 \text{ or } \lambda = 0 \text{ of multiplicity two.}$

If $\lambda = 0$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 + 2v_2 + v_3 = 0.$

$\mathbf{v}_1 = (-1, 0, 1)$ and $\mathbf{v}_2 = (-2, 1, 0)$ are linearly independent eigenvectors corresponding to $\lambda = 0$. $\mathbf{v}_1$ and $\mathbf{v}_2$ are not orthogonal since $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 2 \neq 0$, so we will use the Gram-Schmidt procedure.

Let $\mathbf{u}_1 = \mathbf{v}_1 = (-1, 0, 1)$, so

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 = (-2, 1, 0) - \frac{2}{2}(-1, 0, 1) = (-1, 1, -1).$$

Now $\mathbf{w}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ and $\mathbf{w}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ are orthonormal eigenvectors corresponding to $\lambda = 0$.

If $\lambda = 6$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -5 & 2 & 1 \\ 2 & -2 & 2 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 - v_3 = 0$ and $v_2 - 2v_3 = 0$. $\mathbf{v}_3 = (1, 2, 1)$, is an eigenvector corresponding to $\lambda = 6$ and $\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ is a unit eigenvector corresponding to $\lambda = 6$.

Thus, $S = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$ and $S^T A S = \text{diag}(0, 0, 6)$.

6. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix} = 0 \iff (\lambda - 2)^2(\lambda - 4) = 0 \iff \lambda = 4 \text{ or } \lambda = 2$ of multiplicity two.

If $\lambda = 2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_2 + v_3 = 0$ and $v_1 \in \mathbb{R}$. $\mathbf{v}_1 = (1, 0, 0)$ and $\mathbf{v}_2 = (0, -1, 1)$ are linearly independent eigenvectors corresponding to $\lambda = 2$, and $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = (1, 0, 0)$ and $\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ are unit eigenvectors corresponding to $\lambda = 2$. $\mathbf{w}_1$ and $\mathbf{w}_2$ are also orthogonal because $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = 0$.

If $\lambda = 4$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 = 0$ and

$v_2 - v_3 = 0$. $\mathbf{v}_3 = (0, 1, 1)$, is an eigenvector corresponding to $\lambda = 4$ and $\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is a unit eigenvector corresponding to $\lambda = 4$.

Thus, $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $S^T A S = \text{diag}(2, 2, 4)$.

7. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \iff (\lambda - 1)^2(\lambda + 1) = 0 \iff \lambda = -1 \text{ or } \lambda = 1$ of multiplicity two.

If $\lambda = 1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 - v_2 = 0$ and $v_3 \in \mathbb{R}$. $\mathbf{v}_1 = (1, 1, 0)$ and $\mathbf{v}_2 = (0, 0, 1)$ are linearly independent eigenvectors corresponding to $\lambda = 1$, and $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ and $\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = (0, 0, 1)$ are unit eigenvectors corresponding to $\lambda = 1$. $\mathbf{w}_1$ and $\mathbf{w}_2$ are also orthogonal because $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = 0$.

If $\lambda = -1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 + v_2 = 0 \text{ and } v_3 = 0$.

$\mathbf{v}_3 = (-1, 1, 0)$, is an eigenvector corresponding to $\lambda = -1$ and $\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is a unit eigenvector corresponding to $\lambda = -1$.

Thus, $S = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$ and $S^T A S = \text{diag}(-1, 1, 1)$.

8. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 1-\lambda & 1 \\ -1 & 1 & 1-\lambda \end{vmatrix} = 0 \iff (\lambda-2)^2(\lambda+1) = 0 \iff \lambda = 2$ of multiplicity two
 $\lambda = -1$.

If $\lambda = 2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 - v_2 + v_3 = 0$.

$\mathbf{v}_1 = (1, 1, 0)$ and $\mathbf{v}_2 = (-1, 0, 1)$ are linearly independent eigenvectors corresponding to $\lambda = 2$. $\mathbf{v}_1$ and $\mathbf{v}_2$ are not orthogonal since $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = -1 \neq 0$, so we will use the Gram-Schmidt procedure.

Let $\mathbf{u}_1 = \mathbf{v}_1 = (1, 1, 0)$, so

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 = (-1, 0, 1) + \frac{1}{2}(1, 1, 0) = \left(-\frac{1}{2}, \frac{1}{2}, 1\right).$$

Now $\mathbf{w}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ and $\mathbf{w}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$ are orthonormal eigenvectors corresponding to $\lambda = 2$.

If $\lambda = -1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 - v_3 = 0$ and

$v_2 + v_3 = 0$. $\mathbf{v}_3 = (1, -1, 1)$, is an eigenvector corresponding to $\lambda = -1$ and $\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is a unit eigenvector corresponding to $\lambda = -1$.

Thus, $S = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ and $S^T A S = \text{diag}(2, 2, -1)$.

9. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 1 \\ -1 & 1 & -\lambda \end{vmatrix} = 0 \iff (1-\lambda)(\lambda+1)(\lambda-2) = 0 \iff \lambda = -1, \lambda = 1, \text{ or } \lambda = 2$.

If $\lambda = -1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies 2v_1 - v_3 = 0$ and

$2v_2 + v_3 = 0$. $\mathbf{v}_1 = (1, -1, 2)$, is an eigenvector corresponding to $\lambda = -1$ and $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$ is a unit eigenvector corresponding to $\lambda = -1$.

If $\lambda = 1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_1 - v_2 = 0$ and $v_3 = 0$. $\mathbf{v}_2 = (1, 1, 0)$, is an eigenvector corresponding to $\lambda = 1$ and $\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is a unit eigenvector corresponding to $\lambda = 1$.

If $\lambda = 2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_1 + v_3 = 0$ and $v_2 - v_3 = 0$. $\mathbf{v}_3 = (-1, 1, 1)$, is an eigenvector corresponding to $\lambda = 2$ and $\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is a unit eigenvector corresponding to $\lambda = 2$.

Thus, $S = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$ and $S^T A S = \text{diag}(-1, 1, 2)$.

10. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 3-\lambda & 3 & 4 \\ 3 & 3-\lambda & 0 \\ 4 & 0 & 3-\lambda \end{vmatrix} = 0 \iff (\lambda - 3)(\lambda + 2)(\lambda - 8) = 0 \iff \lambda = -2, \lambda = 3,$
or $\lambda = 8$.

If $\lambda = -2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 5 & 3 & 4 \\ 3 & 5 & 0 \\ 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow 4v_1 + 5v_3 = 0$ and $4v_2 - 3v_3 = 0$. $\mathbf{v}_1 = (-5, 3, 4)$, is an eigenvector corresponding to $\lambda = -2$ and $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(-\frac{1}{\sqrt{2}}, \frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}\right)$ is a unit eigenvector corresponding to $\lambda = -2$.

If $\lambda = 3$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 0 & 3 & 4 \\ 3 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = 0$ and $3v_2 + 4v_3 = 0$. $\mathbf{v}_2 = (0, -4, 3)$, is an eigenvector corresponding to $\lambda = 3$ and $\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \left(0, -\frac{4}{5}, \frac{3}{5}\right)$ is a unit eigenvector corresponding to $\lambda = 3$.

If $\lambda = 8$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -5 & 3 & 4 \\ 3 & -5 & 0 \\ 4 & 0 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow 4v_1 - 5v_3 = 0$ and $4v_2 - 3v_3 = 0$. $\mathbf{v}_3 = (5, 3, 4)$, is an eigenvector corresponding to $\lambda = 8$ and $\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(\frac{1}{\sqrt{2}}, \frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}\right)$ is a unit eigenvector corresponding to $\lambda = 8$.

Thus, $S = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{3}{5\sqrt{2}} & -\frac{4}{5} & \frac{4}{5\sqrt{2}} \\ \frac{4}{5\sqrt{2}} & \frac{3}{5} & \frac{4}{5\sqrt{2}} \end{bmatrix}$ and $S^T A S = \text{diag}(-2, 3, 8)$.

11. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} -3-\lambda & 2 & 2 \\ 2 & -3-\lambda & 2 \\ 2 & 2 & -3-\lambda \end{vmatrix} = 0 \iff (\lambda + 5)^2(\lambda - 1) = 0 \iff \lambda = -5$ of multiplicity two, or $\lambda = 1$.

If $\lambda = 1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 - v_3 = 0$ and

$v_2 - v_3 = 0$. $\mathbf{v}_1 = (1, 1, 1)$ is an eigenvector corresponding to $\lambda = 1$ and $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is a unit eigenvector corresponding to $\lambda = 1$.

If $\lambda = -5$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 + v_2 + v_3 = 0$.

$\mathbf{v}_2 = (-1, 1, 0)$ and $\mathbf{v}_3 = (-1, 0, 1)$ are linearly independent eigenvectors corresponding to $\lambda = -5$. $\mathbf{v}_2$ and $\mathbf{v}_3$ are not orthogonal since $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = -1 \neq 0$, so we will use the Gram-Schmidt procedure.

Let $\mathbf{u}_2 = \mathbf{v}_2 = (-1, 1, 0)$, so

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2} \mathbf{u}_2 = (-1, 0, 1) - \frac{1}{2}(-1, 1, 0) = \left(-\frac{1}{2}, -\frac{1}{2}, 1\right).$$

Now $\mathbf{w}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ and $\mathbf{w}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \left(-\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$ are orthonormal eigenvectors corresponding to $\lambda = -5$.

$$\text{Thus, } S = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix} \text{ and } S^T A S = \text{diag}(1, -5, -5).$$

12. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0 \iff (\lambda + 1)^2(\lambda - 2) = 0 \iff \lambda = -1$ of multiplicity two, or $\lambda = 2$.

If $\lambda = -1$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 + v_2 + v_3 = 0$.

$\mathbf{v}_1 = (-1, 0, 1)$ and $\mathbf{v}_2 = (-1, 1, 0)$ are linearly independent eigenvectors corresponding to $\lambda = -1$. $\mathbf{v}_1$ and $\mathbf{v}_2$ are not orthogonal since $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 1 \neq 0$, so we will use the Gram-Schmidt procedure.

Let $\mathbf{u}_1 = \mathbf{v}_1 = (-1, 0, 1)$, so

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2} \mathbf{u}_1 = (-1, 1, 0) - \frac{1}{2}(-1, 0, 1) = \left(-\frac{1}{2}, 1, -\frac{1}{2}\right).$$

Now $\mathbf{w}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ and $\mathbf{w}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$ are orthonormal eigenvectors corresponding to $\lambda = -1$.

If $\lambda = 2$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_1 - v_3 = 0$ and

$v_2 - v_3 = 0$. $\mathbf{v}_3 = (1, 1, 1)$, is an eigenvector corresponding to $\lambda = 2$ and $\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is a unit eigenvector corresponding to $\lambda = 2$.

Thus, $S = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$ and $S^T A S = \text{diag}(-1, -1, 2)$.

13. A has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = -2$ with corresponding eigenvectors $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (-1, 1)$. Therefore, a set of principal axes is $\left\{\frac{1}{\sqrt{2}}(1, 1), \frac{1}{\sqrt{2}}(-1, 1)\right\}$. Relative to these principal axes, the quadratic form reduces to $4y_1^2 - 2y_2^2$.

14. A has eigenvalues $\lambda_1 = 7$ and $\lambda_2 = 3$ with corresponding eigenvectors $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (1, -1)$. Therefore, a set of principal axes is $\left\{\frac{1}{\sqrt{2}}(1, 1), \frac{1}{\sqrt{2}}(1, -1)\right\}$. Relative to these principal axes, the quadratic form reduces to $7y_1^2 + 3y_2^2$.

15. A has eigenvalue $\lambda = 2$ of multiplicity two with corresponding linearly independent eigenvectors $\mathbf{v}_1 = (1, 0, -1)$ and $\mathbf{v}_2 = (0, 1, 1)$. Using the Gram-Schmidt procedure, an orthogonal basis in this eigenspace is $\{\mathbf{u}_1, \mathbf{u}_2\}$ where

$$\mathbf{u}_1 = (1, 0, -1), \mathbf{u}_2 = (0, 1, 1) + \frac{1}{2}(1, 0, -1) = \frac{1}{2}(1, 2, 1).$$

An orthonormal basis for the eigenspace is $\left\{\frac{1}{\sqrt{2}}(1, 0, -1), \frac{1}{\sqrt{6}}(1, 2, 1)\right\}$. The remaining eigenvalue of A is $\lambda = -1$, with eigenvector $\mathbf{v}_3 = (-1, 1, -1)$. Consequently, a set of principal axes for the given quadratic form is $\left\{\frac{1}{\sqrt{2}}(1, 0, -1), \frac{1}{\sqrt{6}}(1, 2, 1), \frac{1}{\sqrt{3}}(-1, 1, -1)\right\}$. Relative to these principal axes, the quadratic form reduces to $2y_1^2 + 2y_2^2 - y_3^2$.

16. A has eigenvalue $\lambda = 0$ of multiplicity two with corresponding linearly independent eigenvectors $\mathbf{v}_1 = (0, 1, 0, -1)$ and $\mathbf{v}_2 = (1, 0, -1, 0)$. Notice that these are orthogonal, hence we do not need to apply the Gram-Schmidt procedure. The remaining eigenvalues of A are $\lambda = 4$ and $\lambda = 8$, with corresponding eigenvectors $\mathbf{v}_3 = (-1, 1, -1, 1)$ and $\mathbf{v}_4 = (1, 1, 1, 1)$, respectively. Consequently, a set of principal axes for the given quadratic form is

$$\left\{\frac{1}{\sqrt{2}}(0, 1, 0, -1), \frac{1}{\sqrt{2}}(1, 0, -1, 0), \frac{1}{2}(-1, 1, -1, 1), \frac{1}{2}(1, 1, 1, 1)\right\}.$$

Relative to these principal axes, the quadratic form reduces to $4y_3^2 + 8y_4^2$.

17. $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ where $a, b, c \in \mathbb{R}$.

$$\text{Consider } \det(A - \lambda I) = 0 \iff \begin{vmatrix} a - \lambda & b \\ b & c - \lambda \end{vmatrix} = 0 \iff \lambda^2 - (a + c)\lambda + (ac - b^2) = 0$$

$$\iff \lambda = \frac{(a + c) \pm \sqrt{(a - c)^2 + 4b^2}}{2}. \text{ Now } A \text{ has repeated eigenvalues}$$

$$\iff (a - c)^2 + 4b^2 = 0 \iff a = c \text{ and } b = 0 \text{ (since } a, b, c \in \mathbb{R}) \iff A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

$$\iff A = aI_2 \iff A \text{ is a scalar matrix.}$$

18.(a) A is a real $n \times n$ symmetric matrix $\implies A$ possesses a complete set of eigenvectors
 $\implies A$ is similar to a diagonal matrix
 $\implies$ there exists an invertible matrix S such that $S^{-1}AS = \text{diag}(\lambda, \lambda, \lambda, \dots, \lambda)$ where λ occurs n times.
 But $\text{diag}(\lambda, \lambda, \lambda, \dots, \lambda) = \lambda I_n$, so $S^{-1}AS = \lambda I_n \implies AS = S(\lambda I_n) \implies AS = \lambda S \implies A = \lambda SS^{-1}$
 $\implies A = \lambda I_n \implies A$ is a scalar matrix

(b) Theorem: Let A be a nondefective $n \times n$ matrix. If λ is an eigenvalue of multiplicity n , then A is a scalar matrix.

Proof: The proof is the same as that in part (a) since A has a complete set of eigenvectors.

19. Since real eigenvectors of A that correspond to distinct eigenvalues are orthogonal, it must be the case that if $\mathbf{y} = (y_1, y_2)$ corresponds to λ_2 where $A\mathbf{y} = \lambda_2\mathbf{y}$, then

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0 \implies \langle (1, 2), (y_1, y_2) \rangle = 0 \implies y_1 + 2y_2 = 0 \implies y_1 = -2y_2 \implies \mathbf{y} = (-2y_2, y_2) = y_2(-2, 1).$$

Consequently, $(-2, 1)$ is an eigenvector corresponding to λ_2 .

20. (a) Let A be a real symmetric 2×2 matrix with two distinct eigenvalues, λ_1 and λ_2 , where $\mathbf{v}_1 = (a, b)$ is an eigenvector corresponding to λ_1 . Since real eigenvectors of A that correspond to distinct eigenvalues are orthogonal, it follows that if $\mathbf{v}_2 = (c, d)$ corresponds to λ_2 where $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$, then $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0 \implies \langle (a, b), (c, d) \rangle = 0$, that is, $ac + bd = 0$. By inspection, we see that $\mathbf{v}_2 = (-b, a)$. An orthonormal set of eigenvectors for A is

$$\left\{ \frac{1}{\sqrt{a^2 + b^2}} (a, b), \frac{1}{\sqrt{a^2 + b^2}} (-b, a) \right\}.$$

Thus, if $S = \begin{bmatrix} \frac{1}{\sqrt{a^2 + b^2}} a & -\frac{1}{\sqrt{a^2 + b^2}} b \\ \frac{1}{\sqrt{a^2 + b^2}} b & \frac{1}{\sqrt{a^2 + b^2}} a \end{bmatrix}$, then $S^T AS = \text{diag}(\lambda_1, \lambda_2)$.

(b) $S^T AS = \text{diag}(\lambda_1, \lambda_2) \implies AS = S \text{diag}(\lambda_1, \lambda_2)$, since $S^T = S^{-1}$. Thus,

$$\begin{aligned} A &= S \text{diag}(\lambda_1, \lambda_2) S^T = \frac{1}{(a^2 + b^2)} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \\ &= \frac{1}{(a^2 + b^2)} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} \lambda_1 a & \lambda_1 b \\ -\lambda_2 b & \lambda_2 a \end{bmatrix} \\ &= \frac{1}{(a^2 + b^2)} \begin{bmatrix} \lambda_1 a^2 + \lambda_2 b^2 & ab(\lambda_1 - \lambda_2) \\ ab(\lambda_1 - \lambda_2) & \lambda_1 b^2 + \lambda_2 a^2 \end{bmatrix}. \end{aligned}$$

21. A is a real symmetric 3×3 matrix with eigenvalues λ_1 and λ_2 of multiplicity two.

(a) Let $\mathbf{v}_1 = (1, -1, 1)$ be an eigenvector of A that corresponds to λ_1 . Since real eigenvectors of A that correspond to distinct eigenvalues are orthogonal, it must be the case that if $\mathbf{v} = (a, b, c)$ corresponds to λ_2 where $A\mathbf{v} = \lambda_2\mathbf{v}$, then $\langle \mathbf{v}_1, \mathbf{v} \rangle = 0 \implies \langle (1, -1, 1), (a, b, c) \rangle = 0 \implies a - b + c = 0 \implies \mathbf{v} = r(1, 1, 0) + s(-1, 0, 1)$ where r and s are free variables. Consequently, $\mathbf{v}_2 = (1, 1, 0)$ and $\mathbf{v}_3 = (-1, 0, 1)$ are linearly independent eigenvectors corresponding to λ_2 . Thus, $\{(1, 1, 0), (-1, 0, 1)\}$ is a basis for E_2 . $\mathbf{v}_2$ and $\mathbf{v}_3$ are not orthogonal since $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = -1 \neq 0$, so we will apply the Gram-Schmidt procedure to $\mathbf{v}_2$ and $\mathbf{v}_3$. Let $\mathbf{u}_2 = \mathbf{v}_2 = (1, 1, 0)$ and

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2} \mathbf{u}_2 = (-1, 0, 1) + \frac{1}{2}(1, 1, 0) = \left(-\frac{1}{2}, \frac{1}{2}, 1\right).$$

Now, $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ is a unit eigenvector corresponding to λ_1 , and

$\mathbf{w}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)$, $\mathbf{w}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right)$ are orthonormal eigenvectors corresponding to λ_2 .

Consequently, $S = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$ and $S^T A S = \text{diag}(\lambda_1, \lambda_2, \lambda_2)$.

(b) Since S is an orthogonal matrix, $S^T A S = \text{diag}(\lambda_1, \lambda_2, \lambda_2) \implies A S = S \text{diag}(\lambda_1, \lambda_2, \lambda_2) \implies A = S \text{diag}(\lambda_1, \lambda_2, \lambda_2) S^T \implies$

$$\begin{aligned} A &= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{\lambda_1}{\sqrt{3}} & -\frac{\lambda_1}{\sqrt{3}} & \frac{\lambda_1}{\sqrt{3}} \\ \frac{\lambda_2}{\sqrt{2}} & \frac{\lambda_2}{\sqrt{2}} & 0 \\ -\frac{\lambda_2}{\sqrt{6}} & \frac{\lambda_2}{\sqrt{6}} & \frac{2\lambda_2}{\sqrt{6}} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} \lambda_1 + 2\lambda_2 & -\lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ -\lambda_1 + \lambda_2 & \lambda_1 + 2\lambda_2 & -\lambda_1 + \lambda_2 \\ \lambda_1 - \lambda_2 & -\lambda_1 + \lambda_2 & \lambda_1 + 2\lambda_2 \end{bmatrix}. \end{aligned}$$

22. (a) Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{C}^n$ and recall that $\mathbf{v}_1^T \overline{\mathbf{v}_2} = [\langle \mathbf{v}_1, \mathbf{v}_2 \rangle]$.

$[\langle A\mathbf{v}_1, \mathbf{v}_2 \rangle] = (A\mathbf{v}_1)^T \overline{\mathbf{v}_2} = (\mathbf{v}_1^T A^T) \overline{\mathbf{v}_2} = \mathbf{v}_1^T (-A) \overline{\mathbf{v}_2} = -\mathbf{v}_1^T (A) \overline{\mathbf{v}_2} = -\mathbf{v}_1^T (A \overline{\mathbf{v}_2}) = -\mathbf{v}_1^T \overline{A\mathbf{v}_2} = [-\langle \mathbf{v}_1, A\mathbf{v}_2 \rangle]$. Thus,

$$\langle A\mathbf{v}_1, \mathbf{v}_2 \rangle = -\langle \mathbf{v}_1, A\mathbf{v}_2 \rangle. \quad (22.1)$$

(b) Let $\mathbf{v}_1$ be an eigenvector corresponding to the eigenvalue λ_1 , so

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1. \quad (22.2)$$

Taking the inner product of (22.2) with $\mathbf{v}_1$ yields $\langle A\mathbf{v}_1, \mathbf{v}_1 \rangle = \lambda_1 \langle \mathbf{v}_1, \mathbf{v}_1 \rangle$, that is

$$\langle A\mathbf{v}_1, \mathbf{v}_1 \rangle = \lambda_1 \|\mathbf{v}_1\|^2. \quad (22.3)$$

Taking the complex conjugate of (22.3) gives $\overline{\langle A\mathbf{v}_1, \mathbf{v}_1 \rangle} = \overline{\lambda_1} \|\mathbf{v}_1\|^2$, that is

$$\langle \mathbf{v}_1, A\mathbf{v}_1 \rangle = \overline{\lambda_1} \|\mathbf{v}_1\|^2. \quad (22.4)$$

Adding (22.3) and (22.4), and using (22.1) with $\mathbf{v}_2 = \mathbf{v}_1$ yields $(\lambda_1 + \overline{\lambda_1}) \|\mathbf{v}_1\|^2 = 0$. But $\|\mathbf{v}_1\|^2 \neq 0$, so $\lambda_1 + \overline{\lambda_1} = 0$, or equivalently, $\lambda_1 = -\overline{\lambda_1}$, which means that all nonzero eigenvalues of A are pure imaginary.

23. Let A be an $n \times n$ real skew-symmetric matrix where n is odd. Since A is real, the characteristic equation, $\det(A - \lambda I) = 0$, has real coefficients, so its roots come in conjugate pairs. By problem 20, all nonzero solutions of $\det(A - \lambda I) = 0$ are pure imaginary, hence when n is odd, zero will be one of the eigenvalues of A .

$$\mathbf{24.} \det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 4 & -4 \\ -4 & -\lambda & -2 \\ 4 & 2 & -\lambda \end{vmatrix} = 0 \iff \lambda^3 + 36\lambda = 0 \iff \lambda = 0 \text{ or } \lambda = \pm 6i.$$

If $\lambda = 0$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 0 & 4 & -4 \\ -4 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies 2v_1 + v_3 = 0$ and $v_2 - v_3 = 0$. If we let $v_3 = 2r \in \mathbb{C}$, then the solution set of this system is $\{(-r, 2r, 2r) : r \in \mathbb{C}\}$ so the eigenvectors corresponding to $\lambda = 0$ are $\mathbf{v}_1 = r(-1, 2, 2)$ where $r \in \mathbb{C}$.

If $\lambda = -6i$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 6i & 4 & -4 \\ -4 & 6i & -2 \\ 4 & 2 & 6i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies 5v_1 + (-2 + 6i)v_3 = 0$

and $5v_2 + (4 + 3i)v_3 = 0$. If we let $v_3 = 5s \in \mathbb{C}$, then the solution set of this system is $\{(2 - 6i)s, (-4 - 3i)s, 5s : s \in \mathbb{C}\}$, so the eigenvectors corresponding to $\lambda = -6i$ are $\mathbf{v}_2 = s(2 - 6i, -4 - 3i, 5)$ where $s \in \mathbb{C}$. By Theorem 5.6.8, since A is a matrix with real entries, the eigenvectors corresponding to $\lambda = 6i$ are of the form $\mathbf{v}_3 = t(2 + 6i, -4 + 3i, 5)$ where $t \in \mathbb{C}$.

$$25. \det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & -1 & -6 \\ 1 & -\lambda & 5 \\ 6 & -5 & -\lambda \end{vmatrix} = 0 \iff -\lambda^3 - 62\lambda = 0 \iff \lambda = 0 \text{ or } \lambda = \pm\sqrt{62}i.$$

If $\lambda = 0$ then $(A - \lambda I)\mathbf{v} = \mathbf{0}$ assumes the form $\begin{bmatrix} 0 & -1 & -6 \\ 1 & 0 & 5 \\ 6 & -5 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies v_3 = t, v_2 = -6t, v_1 = -5t$, where $t \in \mathbb{C}$. Thus, the solution set of the system is $\{(-5t, -6t, t) : t \in \mathbb{C}\}$, so the eigenvectors corresponding to $\lambda = 0$ are $\mathbf{v} = t(-5, -6, 1)$, where $t \in \mathbb{C}$.

For the other eigenvalues, it is best to use technology to generate the corresponding eigenvectors.

26. A is a real $n \times n$ orthogonal matrix $\iff A^{-1} = A^T \iff A^T A = I_n$. Suppose that $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n]$. Since the i^{th} row of A^T is equal to $\mathbf{v}_i^T$, the matrix multiplication assures us that the ij^{th} entry of $A^T A$ is equal to $\langle \mathbf{v}_i^T, \mathbf{v}_j \rangle$. Thus from the equality $A^T A = I_n = [\delta_{ij}]$, an $n \times n$ matrix $A = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n]$ is orthogonal if and only if $\langle \mathbf{v}_i^T, \mathbf{v}_j \rangle = \delta_{ij}$, that is, if and only if the columns (rows) of A , $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\}$, form an orthonormal set of vectors.

Solutions to Section 5.11

True-False Review:

1. **TRUE.** See Remark 1 following Definition 5.11.1.
2. **TRUE.** Each Jordan block corresponds to a cycle of generalized eigenvectors, and each such cycle contains exactly one eigenvector. By construction, the eigenvectors (and generalized eigenvectors) are chosen to form a linearly independent set of vectors.
3. **FALSE.** For example, in a diagonalizable $n \times n$ matrix, the n linearly independent eigenvectors can be arbitrarily placed in the columns of the matrix S . Thus, an ample supply of invertible matrices S can be constructed.
4. **FALSE.** For instance, if $J_1 = J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then J_1 and J_2 are in Jordan canonical form, but $J_1 + J_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ is not in Jordan canonical form.
5. **TRUE.** This is simply a restatement of the definition of a generalized eigenvector.
6. **FALSE.** The number of Jordan blocks corresponding to λ in the Jordan canonical form of A is the number of linearly independent eigenvectors of A corresponding to λ , which is $\dim[E_\lambda]$, the dimension of the *eigenspace* corresponding to λ , not $\dim[K_\lambda]$.

7. TRUE. This is the content of Theorem 5.11.8.

8. FALSE. For instance, if $J_1 = J_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then J_1 and J_2 are in Jordan canonical form, but $J_1 J_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is not in Jordan canonical form.

9. TRUE. If we place the vectors in a cycle of generalized eigenvectors of A (see Equation (5.11.3)) in the columns of the matrix S formulated in this section in the order they appear in the cycle, then the corresponding columns of the matrix $S^{-1}AS$ will form a Jordan block.

10. TRUE. The assumption here is that all Jordan blocks have size 1×1 , which precisely says that the Jordan canonical form of A is a diagonal matrix. This means that A is diagonalizable.

11. TRUE. Suppose that $S^{-1}AS = B$ and that J is a Jordan canonical form of A . So there exists an invertible matrix T such that $T^{-1}AT = J$. Then $B = S^{-1}AS = S^{-1}(TJT^{-1})S = (T^{-1}S)^{-1}J(T^{-1}S)$, and hence,

$$(T^{-1}S)B(T^{-1}S)^{-1} = J,$$

which shows that J is also a Jordan canonical form of B .

12. FALSE. For instance, if $J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $r = 2$, then J is in Jordan canonical form, but $rJ = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ is not in Jordan canonical form.

Problems:

1. There are 3 possible Jordan canonical forms:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

2. There are 4 possible Jordan canonical forms:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

3. There are 7 possible Jordan canonical forms:

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

4. There are 10 possible Jordan canonical forms:

$$\begin{aligned}
 & \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}, \\
 & \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}, \\
 & \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}.
 \end{aligned}$$

5. Since $\lambda = 2$ occurs with multiplicity 4, it can give rise to the following possible Jordan block sizes:

- (a) 4
- (b) 3,1
- (c) 2,2
- (d) 2,1,1
- (e) 1,1,1,1

Likewise, $\lambda = 6$ occurs with multiplicity 4, so it can give rise to the same five possible Jordan block sizes.

Finally, $\lambda = 8$ occurs with multiplicity 3, so it can give rise to three possible Jordan block sizes:

- (a) 3
- (b) 2,1
- (c) 1,1,1

Since the block sizes for each eigenvalue can be independently determined, we have $5 \cdot 5 \cdot 3 = 75$ possible Jordan canonical forms.

6. Since $\lambda = 2$ occurs with multiplicity 4, it can give rise to the following possible Jordan block sizes:

- (a) 4
- (b) 3,1
- (c) 2,2
- (d) 2,1,1
- (e) 1,1,1,1

Next, $\lambda = 5$ occurs with multiplicity 6, so it can give rise to the following possible Jordan block sizes:

- (a) 6
- (b) 5,1
- (c) 4,2
- (d) 4,1,1
- (e) 3,3

- (f) 3,2,1
- (g) 3,1,1,1
- (h) 2,2,2
- (i) 2,2,1,1
- (j) 2,1,1,1,1
- (k) 1,1,1,1,1,1

There are 5 possible Jordan block sizes corresponding to $\lambda = 2$ and 11 possible Jordan block sizes corresponding to $\lambda = 5$. Multiplying these results, we have $5 \cdot 11 = 55$ possible Jordan canonical forms.

7. Since $(A - 5I)^2 = 0$, no cycles of generalized eigenvectors corresponding to $\lambda = 5$ can have length greater than 2, and hence, only Jordan block sizes 2 or less are possible. Thus, the possible block sizes under this restriction (corresponding to $\lambda = 5$) are:

- 2,2,2
- 2,2,1,1
- 2,1,1,1,1
- 1,1,1,1,1,1

There are four such. There are still five possible block size lists corresponding to $\lambda = 2$. Multiplying these results, we have $5 \cdot 4 = 20$ possible Jordan canonical forms under this restriction.

8.

(a):

$$\begin{aligned} & \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}, \\ & \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \end{aligned}$$

(b): The assumption that $(A - \lambda_1 I)^2 \neq 0$ implies that there can be no Jordan blocks corresponding to λ_1 of size 3×3 (or greater). Thus, the only possible Jordan canonical forms for this matrix now are

$$\begin{aligned} & \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}, \\ & \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \end{aligned}$$

9. The assumption that $(A - \lambda I)^3 = 0$ implies no Jordan blocks of size greater than 3×3 are possible. The fact that $(A - \lambda I)^2 \neq 0$ implies that there is at least one Jordan block of size 3×3 . Thus, the possible block

size combinations for a 6×6 matrix with eigenvalue λ of multiplicity 6 and no blocks of size greater than 3×3 with at least one 3×3 block are:

3,3
3,2,1
3,1,1,1

Thus, there are 3 possible Jordan canonical forms. (We omit the list itself; it can be produced simply from the list of block sizes above.)

10. The eigenvalues of the matrix with this characteristic polynomial are $\lambda = 4, 4, -6$. The possible Jordan canonical forms in this case are therefore:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -6 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -6 \end{bmatrix}.$$

11. The eigenvalues of the matrix with this characteristic polynomial are $\lambda = 4, 4, 4, -1, -1$. The possible Jordan canonical forms in this case are therefore:

$$\begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

12. The eigenvalues of the matrix with this characteristic polynomial are $\lambda = -2, -2, -2, 0, 0, 3, 3$. The possible Jordan canonical forms in this case are therefore:

$$\begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix},$$

$$\begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix},$$

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix},$$

13. The eigenvalues of the matrix with this characteristic polynomial are $\lambda = -2, -2, 6, 6, 6, 6, 6$. The possible Jordan canonical forms in this case are therefore:

[illegible]

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

14. Of the Jordan canonical forms in Problem 13, we are asked here to find the ones that contain exactly five Jordan blocks, since there is a correspondence between the Jordan blocks and the linearly independent eigenvectors. There are three:

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

15. Many examples are possible here. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The only eigenvalue of A is 0. The vector $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not an eigenvector since $A\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \lambda\mathbf{v}$. However $A^2 = 0_2$, so every vector is a generalized eigenvector of A corresponding to $\lambda = 0$.

16. Many examples are possible here. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The only eigenvalue of A is 0. The vector $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is not an eigenvector since $A\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq \lambda\mathbf{v}$. However, $A^2 = 0_3$, so every vector is a generalized eigenvector of A corresponding to $\lambda = 0$.

17. The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{bmatrix} 1 - \lambda & 1 \\ -1 & 3 - \lambda \end{bmatrix} = (1 - \lambda)(3 - \lambda) + 1 = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2,$$

with roots $\lambda = 2, 2$. We have

$$A - 2I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Because there is only one unpivoted column in this latter matrix, we only have one eigenvector for A . Hence, A is not diagonalizable, and therefore

$$\text{JCF}(A) = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

To determine the matrix S , we must find a cycle of generalized eigenvectors of length 2. Therefore, it suffices to find a vector $\mathbf{v}$ in $\mathbb{R}^2$ such that $(A - 2I)\mathbf{v} \neq \mathbf{0}$. Many choices are possible here. We take $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then

$(A - 2I)\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus, we have

$$S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

18. The characteristic polynomial is

$$\det(A - \lambda I) = \det \begin{bmatrix} 1 - \lambda & 1 & 1 \\ 0 & 1 - \lambda & 1 \\ 0 & 1 & 1 - \lambda \end{bmatrix} = (1 - \lambda) [(1 - \lambda)^2 - 1] = (1 - \lambda)(\lambda^2 - 2\lambda) = \lambda(1 - \lambda)(\lambda - 2),$$

with roots $\lambda = 0, 1, 2$. Since A is a 3×3 matrix with three distinct eigenvalues, it is diagonalizable. Therefore, its Jordan canonical form is simply a diagonal matrix with the eigenvalues as its diagonal entries:

$$\text{JCF}(A) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

To determine the invertible matrix S , we must find eigenvectors associated with each eigenvalue.

Eigenvalue $\lambda = 0$: Consider

$$\text{nullspace}(A) = \text{nullspace} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

and this latter matrix can be row reduced to $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The equations corresponding to the rows of this matrix are $x + y + z = 0$ and $y + z = 0$. Setting $z = t$, then $y = -t$ and $x = 0$. With $t = 1$ this gives us the eigenvector $(0, -1, 1)$.

Eigenvalue $\lambda = 1$: Consider

$$\text{nullspace}(A - I) = \text{nullspace} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

By inspection, we see that $z = 0$ and $y = 0$ are required, but $x = t$ is free. Thus, an eigenvector associated with $\lambda = 1$ may be chosen as $(1, 0, 0)$.

Eigenvalue $\lambda = 2$: Consider

$$\text{nullspace}(A - 2I) = \text{nullspace} \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix},$$

which can be row-reduced to $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. Setting $z = t$, we have $y = t$ and $x = 2t$. Thus, with $t = 1$ we obtain the eigenvector $(2, 1, 1)$ associated with $\lambda = 2$.

Placing the eigenvectors obtained as the columns of S (with columns corresponding to the eigenvalues of $\text{JCF}(A)$ above), we have

$$S = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

19. We can get the characteristic polynomial by using cofactor expansion along the second column as follows:

$$\text{nullspace}(A - \lambda I) = \det \begin{bmatrix} 5 - \lambda & 0 & -1 \\ 1 & 4 - \lambda & -1 \\ 1 & 0 & 3 - \lambda \end{bmatrix} = (4 - \lambda)[(5 - \lambda)(3 - \lambda) + 1] = (4 - \lambda)(\lambda^2 - 8\lambda + 16) = (4 - \lambda)(\lambda - 4)^2,$$

with roots $\lambda = 4, 4, 4$.

We have $A - 4I = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}$, and so vectors (x, y, z) in the nullspace of this matrix must satisfy $x - z = 0$. Setting $z = t$ and $y = s$, we have $x = t$. Hence, we obtain two linearly independent eigenvectors of A corresponding to $\lambda = 4$: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Therefore, $\text{JCF}(A)$ contains exactly two Jordan blocks. This uniquely determines $\text{JCF}(A)$, up to a rearrangement of the Jordan blocks:

$$\text{JCF}(A) = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

To determine the matrix S , we must seek a generalized eigenvector. It is easy to verify that $(A - 4I)^2 = 0_3$, so every nonzero vector $\mathbf{v}$ is a generalized eigenvector. We must choose one such that $(A - 4I)\mathbf{v} \neq \mathbf{0}$ in

order to form a cycle of length 2. There are many choices here, but let us choose $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then

$(A - 4I)\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Notice that this is an eigenvector of A corresponding to $\lambda = 4$. To complete the matrix S , we will need a second linearly independent eigenvector. Again, there are a multitude of choices. Let us choose the eigenvector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ found above. Thus,

$$S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

20. We will do cofactor expansion along the first column of the matrix to obtain the characteristic polynomial:

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{bmatrix} 4 - \lambda & -4 & 5 \\ -1 & 4 - \lambda & 2 \\ -1 & 2 & 4 - \lambda \end{bmatrix} \\
&= (4 - \lambda)(\lambda^2 - 8\lambda + 12) + (-4)(4 - \lambda) - 10 - (-8 - 5(4 - \lambda)) \\
&= (4 - \lambda)(\lambda^2 - 8\lambda + 12) + (2 - \lambda) \\
&= (4 - \lambda)(\lambda - 2)(\lambda - 6) + (2 - \lambda) \\
&= (\lambda - 2)[(4 - \lambda)(\lambda - 6) - 1] \\
&= (\lambda - 2)(-\lambda^2 + 10\lambda - 25) \\
&= -(\lambda - 2)(\lambda - 5)^2,
\end{aligned}$$

with eigenvalues $\lambda = 2, 5, 5$.

Since $\lambda = 5$ is a repeated eigenvalue, we consider this eigenvalue first. We must consider the nullspace of the matrix $A - 5I = \begin{bmatrix} -1 & -4 & 5 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix}$, which can be row-reduced to $\begin{bmatrix} 1 & 1 & -2 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$, a matrix that has only one unpivoted column, and hence $\lambda = 5$ only yields one linearly independent eigenvector. Thus,

$$\text{JCF}(A) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}.$$

To determine an invertible matrix S , we first proceed to find a cycle of eigenvectors of length 2 corresponding to $\lambda = 5$. Therefore, we must find a vector $\mathbf{v}$ in $\mathbb{R}^3$ such that $(A - 5I)\mathbf{v} \neq \mathbf{0}$, but $(A - 5I)^2\mathbf{v} = \mathbf{0}$ (in order that $\mathbf{v}$ is a generalized eigenvector). Note that $(A - 5I)^2 = \begin{bmatrix} 0 & 18 & -18 \\ 0 & 9 & -9 \end{bmatrix}$, so if we set $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, then

$$(A - 5I)\mathbf{v} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \text{ and } (A - 5I)^2\mathbf{v} = \mathbf{0}. \text{ Thus, we obtain the cycle}$$

$$\left\{ \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Next, corresponding to $\lambda = 2$, we must find an eigenvector. We need to find a nonzero vector (x, y, z) in nullspace $\begin{bmatrix} 2 & -4 & 5 \\ -1 & 2 & 2 \\ -1 & 2 & 2 \end{bmatrix}$, and this latter matrix can be row-reduced to $\begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. The middle row requires that $z = 0$, and if we set $y = t$, then $x = 2t$. Thus, by using $t = 1$, we obtain the eigenvector $(2, 1, 0)$.

Thus, we can form the matrix

$$S = \begin{bmatrix} 2 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

21. We are given that $\lambda = -5$ occurs with multiplicity 2 as a root of the characteristic polynomial of A . To

search for corresponding eigenvectors, we consider

$$\text{nullspace}(A + 5I) = \text{nullspace} \begin{bmatrix} -1 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix},$$

and this matrix row-reduces to $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Since there is only one unpivoted column in this row-echelon form of A , the eigenspace corresponding to $\lambda = -5$ is only one-dimensional. Thus, based on the eigenvalues $\lambda = -5, -5, -6$, we already know that

$$\text{JCF}(A) = \begin{bmatrix} -5 & 1 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -6 \end{bmatrix}.$$

Next, we seek a cycle of generalized eigenvectors of length 2 corresponding to $\lambda = -5$. The cycle takes the form $\{(A + 5I)\mathbf{v}, \mathbf{v}\}$, where $\mathbf{v}$ is a vector such that $(A + 5I)^2\mathbf{v} = \mathbf{0}$. We readily compute that

$$(A + 5I)^2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}. \text{ An obvious vector that is killed by } (A + 5I)^2 \text{ (although other choices are}$$

also possible) is $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Then $(A + 5I)\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Hence, we have a cycle of generalized eigenvectors corresponding to $\lambda = -5$:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Now consider the eigenspace corresponding to $\lambda = -6$. We need only find one eigenvector (x, y, z) in this eigenspace. To do so, we must compute

$$\text{nullspace}(A + 6I) = \text{nullspace} \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

and this matrix row-reduces to $\begin{bmatrix} 1 & -3 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We see that $y = 0$ and $x - 3y - z = 0$, which is equivalent to $x - z = 0$. Setting $z = t$, we have $x = t$. With $t = 1$, we obtain the eigenvector $(1, 0, 1)$. Hence, we can form the matrix

$$S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

22. Because the matrix is upper triangular, the eigenvalues of A appear along the main diagonal: $\lambda = 2, 2, 3$. Let us first consider the eigenspace corresponding to $\lambda = 2$: We consider

$$\text{nullspace}(A - 2I) = \text{nullspace} \begin{bmatrix} 0 & -2 & 14 \\ 0 & 1 & -7 \\ 0 & 0 & 0 \end{bmatrix},$$

which row reduces to $\begin{bmatrix} 0 & 1 & -7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. There are two unpivoted columns, so this eigenspace is two-dimensional. Therefore, the matrix A is diagonalizable:

$$\text{JCF}(A) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Next, we must determine an invertible matrix S such that $S^{-1}AS$ is the Jordan canonical form just obtained. Using the row-echelon form of $A - 2I$ obtained above, vectors (x, y, z) in $\text{nullspace}(A - 2I)$ must satisfy $y - 7z = 0$. Setting $z = t$, $y = 7t$, and $x = s$, we obtain the eigenvectors $(1, 0, 0)$ and $(0, 7, 1)$.

Next, consider the eigenspace corresponding to $\lambda = 3$. We consider

$$\text{nullspace}(A - 3I) = \text{nullspace} \begin{bmatrix} -1 & -2 & 14 \\ 0 & 0 & -7 \\ 0 & 0 & -1 \end{bmatrix},$$

which row reduces to $\begin{bmatrix} 1 & 2 & -14 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Vectors (x, y, z) in the nullspace of this matrix must satisfy $z = 0$ and $x + 2y - 14z = 0$ or $x + 2y = 0$. Setting $y = t$ and $x = -2t$. Hence, setting $t = 1$ gives the eigenvector $(-2, 1, 0)$.

Thus, using the eigenvectors obtained above, we obtain the matrix

$$S = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 7 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

23. We use the characteristic polynomial to determine the eigenvalues of A :

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 7 - \lambda & -2 & 2 \\ 0 & 4 - \lambda & -1 \\ -1 & 1 & 4 - \lambda \end{bmatrix} \\ &= (4 - \lambda) [(7 - \lambda)(4 - \lambda) + 2] + (7 - \lambda - 2) \\ &= (4 - \lambda)(\lambda^2 - 11\lambda + 30) + (5 - \lambda) \\ &= (4 - \lambda)(\lambda - 5)(\lambda - 6) + (5 - \lambda) \\ &= (5 - \lambda) [1 - (4 - \lambda)(6 - \lambda)] \\ &= (5 - \lambda)(\lambda - 5)^2 \\ &= -(\lambda - 5)^3. \end{aligned}$$

Hence, the eigenvalues are $\lambda = 5, 5, 5$. Let us consider the eigenspace corresponding to $\lambda = 5$. We consider

$$\text{nullspace}(A - 5I) = \text{nullspace} \begin{bmatrix} 2 & -2 & 2 \\ 0 & -1 & -1 \\ -1 & 1 & -1 \end{bmatrix},$$

and this latter matrix row-reduces to $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, which contains one unpivoted column. Therefore, the eigenspace corresponding to $\lambda = 5$ is one-dimensional. Therefore, the Jordan canonical form of A consists

of one Jordan block:

$$\text{JCF}(A) = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}.$$

A corresponding invertible matrix S in this case must have columns that consist of one cycle of generalized eigenvectors, which will take the form $\{(A-5I)^2\mathbf{v}, (A-5I)\mathbf{v}, \mathbf{v}\}$, where $\mathbf{v}$ is a generalized eigenvector. Now, we can verify quickly that

$$A - 5I = \begin{bmatrix} 2 & -2 & 2 \\ 0 & -1 & -1 \\ -1 & 1 & -1 \end{bmatrix}, \quad (A - 5I)^2 = \begin{bmatrix} 2 & 0 & 4 \\ 1 & 0 & 2 \\ -1 & 0 & -2 \end{bmatrix}, \quad (A - 5I)^3 = 0_3.$$

The fact that $(A - 5I)^3 = 0_3$ means that every nonzero vector $\mathbf{v}$ is a generalized eigenvector. Hence, we simply choose $\mathbf{v}$ such that $(A - 5I)^2\mathbf{v} \neq \mathbf{0}$. There are many choices. Let us take $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then

$$(A - 5I)\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \text{ and } (A - 5I)^2\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}. \text{ Thus, we have the cycle of generalized eigenvectors}$$

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Hence, we have

$$S = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \\ -1 & -1 & 0 \end{bmatrix}.$$

24. Because the matrix is upper triangular, the eigenvalues of A appear along the main diagonal: $\lambda = -1, -1, -1$. Let us consider the eigenspace corresponding to $\lambda = -1$. We consider

$$\text{nullspace}(A + I) = \text{nullspace} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix},$$

and it is straightforward to see that the nullspace here consists precisely of vectors that are multiples of $(1, 0, 0)$. Because only one linearly independent eigenvector was obtained, the Jordan canonical form of this matrix consists of only one Jordan block:

$$\text{JCF}(A) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

A corresponding invertible matrix S in this case must have columns that consist of one cycle of generalized eigenvectors, which will take the form $\{(A+I)^2\mathbf{v}, (A+I)\mathbf{v}, \mathbf{v}\}$. Here, we have

$$A + I = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad (A + I)^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad (A + I)^3 = 0_3.$$

Therefore, every nonzero vector is a generalized eigenvector. We wish to choose a vector $\mathbf{v}$ such that $(A + I)^2 \mathbf{v} \neq \mathbf{0}$. There are many choices, but we will choose $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Then $(A + I)\mathbf{v} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$ and $(A + I)^2 \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$. Hence, we form the matrix S as follows:

$$S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

25. We use the characteristic polynomial to determine the eigenvalues of A :

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 2 - \lambda & -1 & 0 & 1 \\ 0 & 3 - \lambda & -1 & 0 \\ 0 & 1 & 1 - \lambda & 0 \\ 0 & -1 & 0 & 3 - \lambda \end{bmatrix} \\ &= (2 - \lambda)(3 - \lambda)[(3 - \lambda)(1 - \lambda) + 1] \\ &= (2 - \lambda)(3 - \lambda)(\lambda^2 - 4\lambda + 4) = (2 - \lambda)(3 - \lambda)(\lambda - 2)^2, \end{aligned}$$

and so the eigenvalues are $\lambda = 2, 2, 2, 3$.

First, consider the eigenspace corresponding to $\lambda = 2$. We consider

$$\text{nullspace}(A - 2I) = \text{nullspace} \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

and this latter matrix can be row-reduced to $\begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. There are two unpivoted columns, and

therefore two linearly independent eigenvectors corresponding to $\lambda = 2$. Thus, we will obtain two Jordan blocks corresponding to $\lambda = 2$, and they necessarily will have size 2×2 and 1×1 . Thus, we are already in a position to write down the Jordan canonical form of A :

$$\text{JCF}(A) = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

We continue in order to obtain an invertible matrix S such that $S^{-1}AS$ is in Jordan canonical form. To this end, we see a generalized eigenvector $\mathbf{v}$ such that $(A - 2I)\mathbf{v} \neq \mathbf{0}$ and $(A - 2I)^2 \mathbf{v} = \mathbf{0}$. Note that

$$A - 2I = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad (A - 2I)^2 = \begin{bmatrix} 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix}.$$

By inspection, we see that by taking $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ (there are many other valid choices, of course), then

$(A - 2I)\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $(A - 2I)^2\mathbf{v} = \mathbf{0}$. We also need a second eigenvector corresponding to $\lambda = 2$ that is

linearly independent from $(A - 2I)\mathbf{v}$ just obtained. From the row-echelon form of $A - 2I$, we see that all eigenvectors corresponding to $\lambda = 2$ take the form $\begin{bmatrix} s \\ t \\ t \\ t \end{bmatrix}$, so for example, we can take $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Next, we consider the eigenspace corresponding to $\lambda = 3$. We consider

$$\text{nullspace}(A - 3I) = \text{nullspace} \begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Now, if (x, y, z, w) is an eigenvector corresponding to $\lambda = 3$, the last three rows of the matrix imply that $y = z = 0$. Thus, the first row becomes $-x + w = 0$. Setting $w = t$, then $x = t$, so we obtain eigenvectors in the form $(t, 0, 0, t)$. Setting $t = 1$ gives the eigenvector $(1, 0, 0, 1)$.

Thus, we can now form the matrix S such that $S^{-1}AS$ is the Jordan canonical form we obtained above:

$$S = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

26. From the characteristic polynomial, we have eigenvalues $\lambda = 2, 2, 4, 4$. Let us consider the associated eigenspaces.

Corresponding to $\lambda = 2$, we seek eigenvectors (x, y, z, w) by computing

$$\text{nullspace}(A - 2I) = \text{nullspace} \begin{bmatrix} 0 & -4 & 2 & 2 \\ -2 & -2 & 1 & 3 \\ -2 & -2 & 1 & 3 \\ -2 & -6 & 3 & 5 \end{bmatrix},$$

and this matrix can be row-reduced to $\begin{bmatrix} 2 & 2 & -1 & -3 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Setting $w = 2t$ and $z = 2s$, we obtain $y = s + t$

and $x = 2t$, so we obtain the eigenvectors $(2, 1, 0, 2)$ and $(0, 1, 2, 0)$ corresponding to $\lambda = 2$.

Next, corresponding to $\lambda = 4$, we seek eigenvectors (x, y, z, w) by computing

$$\text{nullspace}(A - 4I) = \text{nullspace} \begin{bmatrix} -2 & -4 & 2 & 2 \\ -2 & -4 & 1 & 3 \\ -2 & -2 & -1 & 3 \\ -2 & -6 & 3 & 3 \end{bmatrix}.$$

This matrix can be reduced to $\begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since there is only one unpivoted column, this eigenspace is only one-dimensional, despite $\lambda = 4$ occurring with multiplicity 2 as a root of the characteristic equation. Therefore, we must seek a generalized eigenvector $\mathbf{v}$ such that $(A - 4I)\mathbf{v}$ is an eigenvector. This in turn requires that $(A - 4I)^2\mathbf{v} = \mathbf{0}$. We find that

$$A - 4I = \begin{bmatrix} -2 & -4 & 2 & 2 \\ -2 & -4 & 1 & 3 \\ -2 & -2 & -1 & 3 \\ -2 & -6 & 3 & 3 \end{bmatrix} \quad \text{and} \quad (A - 4I)^2 = \begin{bmatrix} 4 & 8 & -4 & -4 \\ 4 & 4 & 0 & -4 \\ 4 & 0 & 4 & -4 \\ 4 & 8 & -4 & -4 \end{bmatrix}.$$

Note that the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ satisfies $(A - 4I)^2\mathbf{v} = \mathbf{0}$ and $(A - 4I)\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Thus, we have the cycle of generalized eigenvectors corresponding to $\lambda = 4$ given by

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Hence, we can form the matrix

$$S = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \text{JCF}(A) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

27. Since A is upper triangular, the eigenvalues appear along the main diagonal: $\lambda = 2, 2, 2, 2, 2$. Looking at

$$\text{nullspace}(A - 2I) = \text{nullspace} \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that the row-echelon form of this matrix will contain two pivots, and therefore, three unpivoted columns. That means that the eigenspace corresponding to $\lambda = 2$ is three-dimensional. Therefore, $\text{JCF}(A)$ consists of three Jordan blocks. The only list of block sizes for a 5×5 matrix with three blocks are (a) 3,1,1 and (b) 2,2,1. In this case, note that

$$(A - 2I)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \neq 0_5,$$

so that it is possible to find a vector $\mathbf{v}$ that generates a cycle of generalized eigenvectors of length 3: $\{(A - 2I)^2\mathbf{v}, (A - 2I)\mathbf{v}, \mathbf{v}\}$. Thus, $\text{JCF}(A)$ contains a Jordan block of size 3×3 . We conclude that the correct list of block sizes for this matrix is 3,1,1:

$$\text{JCF}(A) = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

28. Since A is upper triangular, the eigenvalues appear along the main diagonal: $\lambda = 0, 0, 0, 0, 0$. Looking at $\text{nullspace}(A - 0I) = \text{nullspace}(A)$, we see that eigenvectors (x, y, z, u, v) corresponding to $\lambda = 0$ must satisfy $z = u = v = 0$ (since the third row gives $6u = 0$, the first row gives $u + 4v = 0$, and the second row gives $z + u + v = 0$). Thus, we have only two free variables, and thus $\text{JCF}(A)$ will consist of two Jordan blocks. The only list of block sizes for a 5×5 matrix with two blocks are (a) 4,1 and (b) 3,2. In this case, it is easy to verify that $A^3 = 0$, so that the longest possible cycle of generalized eigenvectors $\{A^2\mathbf{v}, A\mathbf{v}, \mathbf{v}\}$ has length 3. Therefore, case (b) holds: $\text{JCF}(A)$ consists of one Jordan block of size 3×3 and one Jordan block of size 2×2 :

$$\text{JCF}(A) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

29. Since A is upper triangular, the eigenvalues appear along the main diagonal: $\lambda = 1, 1, 1, 1, 1, 1, 1, 1$. Looking at

$$\text{nullspace}(A - I) = \text{nullspace} \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that if (a, b, c, d, e, f, g, h) is an eigenvector of A , then $b = d = f = h = 0$, and a, c, e , and g are free variables. Thus, we have four linearly independent eigenvectors of A , and hence we expect four Jordan blocks. Now, an easy calculation shows that $(A - I)^2 = 0$, and thus, no Jordan blocks of size greater than 2×2 are permissible. Thus, it must be the case that $\text{JCF}(A)$ consists of four Jordan blocks, each of which is a 2×2 matrix:

$$\text{JCF}(A) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

30. NOT SIMILAR. We will compute $\text{JCF}(A)$ and $\text{JCF}(B)$. If they are the same (up to a rearrangement of the Jordan blocks), then A and B are similar; otherwise they are not. Both matrices have eigenvalues $\lambda = 6, 6, 6$. Consider

$$\text{nullspace}(A - 6I) = \text{nullspace} \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We see that eigenvectors (x, y, z) corresponding to $\lambda = 6$ must satisfy $x + y = 0$ and $x = 0$. Therefore, $x = y = 0$, while z is a free variable. Since we obtain only one free variable, $\text{JCF}(A)$ consists of just one Jordan block corresponding to $\lambda = 6$:

$$\text{JCF}(A) = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 6 \end{bmatrix}.$$

Next, consider

$$\text{nullspace}(B - 6I) = \text{nullspace} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In this case, eigenvectors (x, y, z) corresponding to $\lambda = 6$ must satisfy $-y + z = 0$. Therefore, both x and z are free variables, and hence, $\text{JCF}(B)$ consists of two Jordan blocks corresponding to $\lambda = 6$:

$$\text{JCF}(B) = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

Since $\text{JCF}(A) \neq \text{JCF}(B)$, we conclude that A and B are not similar.

31. SIMILAR. We will compute $\text{JCF}(A)$ and $\text{JCF}(B)$. If they are the same (up to a rearrangement of the Jordan blocks), then A and B are similar; otherwise they are not. Both matrices have eigenvalues $\lambda = 5, 5, 5$. (For A , this is easiest to compute by expanding $\det(A - \lambda I)$ along the middle row, and for B , this is easiest to compute by expanding $\det(B - \lambda I)$ along the second column.) In Problem 23, we computed

$$\text{JCF}(A) = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}.$$

Next, consider

$$\text{nullspace}(B - 5I) = \text{nullspace} \begin{bmatrix} -2 & -1 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

and this latter matrix can be row-reduced to $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which has only one unpivoted column. Thus, the eigenspace of B corresponding to $\lambda = 5$ is only one-dimensional, and so

$$\text{JCF}(B) = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}.$$

Since A and B each had the same Jordan canonical form, they are similar matrices.

32. The eigenvalues of A are $\lambda = -1, -1$, and the eigenspace corresponding to $\lambda = -1$ is only one-dimensional. Thus, we seek a generalized eigenvector $\mathbf{v}$ of A corresponding to $\lambda = -1$ such that $\{(A+I)\mathbf{v}, \mathbf{v}\}$ is a cycle of generalized eigenvectors. Note that $A+I = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}$ and $(A+I)^2 = 0_2$. Thus, every nonzero vector $\mathbf{v}$ is a generalized eigenvector of A corresponding to $\lambda = -1$. Let us choose $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $(A+I)\mathbf{v} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Form the matrices

$$S = \begin{bmatrix} -2 & 1 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Via the substitution $\mathbf{x} = S\mathbf{y}$, the system $\mathbf{x}' = A\mathbf{x}$ is transformed into $\mathbf{y}' = J\mathbf{y}$. The corresponding equations are

$$y_1' = -y_1 + y_2 \quad \text{and} \quad y_2' = -y_2.$$

The solution to the second equation is

$$y_2(t) = c_1 e^{-t}.$$

Substituting this solution into $y_1' = -y_1 + y_2$ gives

$$y_1' + y_1 = c_1 e^{-t}.$$

This is a first order linear equation with integrating factor $I(t) = e^t$. When we multiply the differential equation for $y_1(t)$ by $I(t)$, it becomes $(y_1 \cdot e^t)' = c_1$. Integrating both sides yields $y_1 \cdot e^t = c_1 t + c_2$. Thus,

$$y_1(t) = c_1 t e^{-t} + c_2 e^{-t}.$$

Thus, we have

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_1 t e^{-t} + c_2 e^{-t} \\ c_1 e^{-t} \end{bmatrix}.$$

Finally, we solve for $\mathbf{x}(t)$:

$$\begin{aligned} \mathbf{x}(t) = S\mathbf{y}(t) &= \begin{bmatrix} -2 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 t e^{-t} + c_2 e^{-t} \\ c_1 e^{-t} \end{bmatrix} \\ &= \begin{bmatrix} -2(c_1 t e^{-t} + c_2 e^{-t}) + c_1 e^{-t} \\ 2(c_1 t e^{-t} + c_2 e^{-t}) \end{bmatrix} \\ &= c_1 e^{-t} \begin{bmatrix} -2t + 1 \\ 2t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -2 \\ 2 \end{bmatrix}. \end{aligned}$$

This is an acceptable answer, or we can write the individual equations comprising the general solution:

$$x_1(t) = -2(c_1 t e^{-t} + c_2 e^{-t}) + c_1 e^{-t} \quad \text{and} \quad x_2(t) = 2(c_1 t e^{-t} + c_2 e^{-t}).$$

33. The eigenvalues of A are $\lambda = -1, -1, 1$. The eigenspace corresponding to $\lambda = -1$ is

$$\text{nullspace}(A+I) = \text{nullspace} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

which is only one-dimensional, spanned by the vector $(1, -1, 1)$. Therefore, we seek a generalized eigenvector $\mathbf{v}$ of A corresponding to $\lambda = -1$ such that $\{(A + I)\mathbf{v}, \mathbf{v}\}$ is a cycle of generalized eigenvectors. Note that

$$A + I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad (A + I)^2 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

In order that $\mathbf{v}$ be a generalized eigenvector of A corresponding to $\lambda = -1$, we should choose $\mathbf{v}$ such that $(A + I)^2\mathbf{v} = \mathbf{0}$ and $(A + I)\mathbf{v} \neq \mathbf{0}$. There are many valid choices; let us choose $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Then $(A + I)\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Hence, we obtain the cycle of generalized eigenvectors corresponding to $\lambda = -1$:

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

Next, consider the eigenspace corresponding to $\lambda = 1$. For this, we compute

$$\text{nullspace}(A - I) = \text{nullspace} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

This can be row-reduced to $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. We find the eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a basis for this eigenspace. Hence, we are ready to form the matrices S and J :

$$S = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Via the substitution $\mathbf{x} = S\mathbf{y}$, the system $\mathbf{x}' = A\mathbf{x}$ is transformed into $\mathbf{y}' = J\mathbf{y}$. The corresponding equations are

$$y_1' = -y_1 + y_2, \quad y_2' = -y_2, \quad y_3' = y_3.$$

The third equation has solution $y_3(t) = c_3 e^t$, the second equation has solution $y_2(t) = c_2 e^{-t}$, and so the first equation becomes

$$y_1' + y_1 = c_2 e^{-t}.$$

This is a first-order linear equation with integrating factor $I(t) = e^t$. When we multiply the differential equation for $y_1(t)$ by $I(t)$, it becomes $(y_1 \cdot e^t)' = c_2$. Integrating both sides yields $y_1 \cdot e^t = c_2 t + c_1$. Thus,

$$y_1(t) = c_2 t e^{-t} + c_1 e^{-t}.$$

Thus, we have

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} c_2 t e^{-t} + c_1 e^{-t} \\ c_2 e^{-t} \\ c_3 e^t \end{bmatrix}.$$

Finally, we solve for $\mathbf{x}(t)$:

$$\begin{aligned}\mathbf{x}(t) = S\mathbf{y}(t) &= \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_2te^{-t} + c_1e^{-t} \\ c_2e^{-t} \\ c_3e^t \end{bmatrix} \\ &= \begin{bmatrix} c_2te^{-t} + c_1e^{-t} + c_2e^{-t} + c_3e^t \\ -c_2te^{-t} - c_1e^{-t} + c_3e^t \\ c_2te^{-t} + c_1e^{-t} - c_2e^{-t} + c_3e^t \end{bmatrix} \\ &= c_1e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2e^{-t} \begin{bmatrix} t+1 \\ -t \\ t-1 \end{bmatrix} + c_3e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.\end{aligned}$$

34. The eigenvalues of A are $\lambda = -2, -2, -2$. The eigenspace corresponding to $\lambda = -2$ is

$$\text{nullspace}(A + 2I) = \text{nullspace} \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix},$$

and there are two linearly independent vectors in this nullspace, corresponding to the unpivoted columns of the row-echelon form of this matrix. Therefore, the Jordan canonical form of A is

$$J = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

To form an invertible matrix S such that $S^{-1}AS = J$, we must find a cycle of generalized eigenvectors corresponding to $\lambda = -2$ of length 2: $\{(A + 2I)\mathbf{v}, \mathbf{v}\}$. Now

$$A + 2I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad (A + 2I)^2 = 0_3.$$

Since $(A + 2I)^2 = 0_3$, every nonzero vector in $\mathbb{R}^3$ is a generalized eigenvector corresponding to $\lambda = -2$. We need only find a nonzero vector $\mathbf{v}$ such that $(A + 2I)\mathbf{v} \neq \mathbf{0}$. There are many valid choices; let us choose

$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then $(A + 2I)\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, an eigenvector of A corresponding to $\lambda = -2$. We also need a second linearly independent eigenvector corresponding to $\lambda = -2$. There are many choices; let us choose $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Therefore, we can form the matrix

$$S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

Via the substitution $\mathbf{x} = S\mathbf{y}$, the system $\mathbf{x}' = A\mathbf{x}$ is transformed into $\mathbf{y}' = J\mathbf{y}$. The corresponding equations are

$$y'_1 = -2y_1 + y_2, \quad y'_2 = -2y_2, \quad y'_3 = -2y_3.$$

The third equation has solution $y_3(t) = c_3e^{-2t}$, the second equation has solution $y_2(t) = c_2e^{-2t}$, and so the first equation becomes

$$y_1' + 2y_1 = c_2e^{-2t}.$$

This is a first-order linear equation with integrating factor $I(t) = e^{2t}$. When we multiply the differential equation for $y_1(t)$ by $I(t)$, it becomes $(y_1 \cdot e^{2t})' = c_2$. Integrating both sides yields $y_1 \cdot e^{2t} = c_2t + c_1$. Thus,

$$y_1(t) = c_2te^{-2t} + c_1e^{-2t}.$$

Thus, we have

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} c_2te^{-2t} + c_1e^{-2t} \\ c_2e^{-2t} \\ c_3e^{-2t} \end{bmatrix}.$$

Finally, we solve for $\mathbf{x}(t)$:

$$\begin{aligned} \mathbf{x}(t) = S\mathbf{y}(t) &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_2te^{-2t} + c_1e^{-2t} \\ c_2e^{-2t} \\ c_3e^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} c_2e^{-2t} + c_3e^{-2t} \\ c_2te^{-2t} + c_1e^{-2t} + c_3e^{-2t} \\ -(c_2te^{-2t} + c_1e^{-2t}) \end{bmatrix} \\ &= c_1e^{-2t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_2e^{-2t} \begin{bmatrix} 1 \\ t \\ -t \end{bmatrix} + c_3e^{-2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \end{aligned}$$

35. The eigenvalues of A are $\lambda = 4, 4, 4$. The eigenspace corresponding to $\lambda = 4$ is

$$\text{nullspace}(A - 4I) = \text{nullspace} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and there is only one eigenvector. Therefore, the Jordan canonical form of A is

$$J = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

Next, we need to find an invertible matrix S such that $S^{-1}AS = J$. To do this, we must find a cycle of generalized eigenvectors $\{(A - 4I)^2\mathbf{v}, (A - 4I)\mathbf{v}, \mathbf{v}\}$ of length 3 corresponding to $\lambda = 4$. We have

$$A - 4I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad (A - 4I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad (A - 4I)^3 = 0_3.$$

From $(A - 4I)^3 = 0_3$, we know that every nonzero vector is a generalized eigenvector corresponding to $\lambda = 4$. We choose $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (any multiple of the chosen vector $\mathbf{v}$ would be acceptable as well). Thus,

$(A - 4I)\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $(A - 4I)^2\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Thus, we have the cycle of generalized eigenvectors

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Thus, we can form the matrix

$$S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Via the substitution $\mathbf{x} = S\mathbf{y}$, the system $\mathbf{x}' = A\mathbf{x}$ is transformed into $\mathbf{y}' = J\mathbf{y}$. The corresponding equations are

$$y_1' = 4y_1 + y_2, \quad y_2' = 4y_2 + y_3, \quad y_3' = 4y_3.$$

The third equation has solution $y_3(t) = c_3 e^{4t}$, and the second equation becomes

$$y_2' - 4y_2 = c_3 e^{4t}.$$

This is a first-order linear equation with integrating factor $I(t) = e^{-4t}$. When we multiply the differential equation for $y_2(t)$ by $I(t)$, it becomes $(y_2 \cdot e^{-4t})' = c_3$. Integrating both sides yields $y_2 \cdot e^{-4t} = c_3 t + c_2$. Thus,

$$y_2(t) = c_3 t e^{4t} + c_2 e^{4t} = e^{4t}(c_3 t + c_2).$$

Therefore, the differential equation for $y_1(t)$ becomes

$$y_1' - 4y_1 = e^{4t}(c_3 t + c_2).$$

This equation is first-order linear with integrating factor $I(t) = e^{-4t}$. When we multiply the differential equation for $y_1(t)$ by $I(t)$, it becomes

$$(y_1 \cdot e^{-4t})' = c_3 t + c_2.$$

Integrating both sides, we obtain

$$y_1 \cdot e^{-4t} = c_3 \frac{t^2}{2} + c_2 t + c_1.$$

Hence,

$$y_1(t) = e^{4t} \left(c_3 \frac{t^2}{2} + c_2 t + c_1 \right).$$

Thus, we have

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} e^{4t} \left(c_3 \frac{t^2}{2} + c_2 t + c_1 \right) \\ e^{4t}(c_3 t + c_2) \\ c_3 e^{4t} \end{bmatrix}.$$

Finally, we solve for $\mathbf{x}(t)$:

$$\begin{aligned}\mathbf{x}(t) = S\mathbf{y}(t) &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{4t} \left(c_3 \frac{t^2}{2} + c_2 t + c_1 \right) \\ e^{4t} (c_3 t + c_2) \\ c_3 e^{4t} \end{bmatrix} \\ &= \begin{bmatrix} c_3 e^{4t} \\ e^{4t} (c_3 t + c_2) \\ e^{4t} \left(c_3 \frac{t^2}{2} + c_2 t + c_1 \right) \end{bmatrix} \\ &= c_1 e^{4t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} 1 \\ t \\ t^2/2 \end{bmatrix}.\end{aligned}$$

36. The eigenvalues of A are $\lambda = -3, -3$. The eigenspace corresponding to $\lambda = -3$ is only one-dimensional, and therefore the Jordan canonical form of A contains one 2×2 Jordan block:

$$J = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}.$$

Next, we look for a cycle of generalized eigenvectors of the form $\{(A + 3I)\mathbf{v}, \mathbf{v}\}$, where $\mathbf{v}$ is a generalized eigenvector of A . Since $(A + 3I)^2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}^2 = 0_2$, every nonzero vector in $\mathbb{R}^2$ is a generalized eigenvector.

Let us choose $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $(A + 3I)\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus, we form the matrix

$$S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Via the substitution $\mathbf{x} = S\mathbf{y}$, the system $\mathbf{x}' = A\mathbf{x}$ is transformed into $\mathbf{y}' = J\mathbf{y}$. The corresponding equations are

$$y_1' = -3y_1 + y_2 \quad \text{and} \quad y_2' = -3y_2.$$

The second equation has solution $y_2(t) = c_2 e^{-3t}$. Substituting this expression for $y_2(t)$ into the differential equation for $y_1(t)$ yields

$$y_1' + 3y_1 = c_2 e^{-3t}.$$

An integrating factor for this first-order linear differential equation is $I(t) = e^{3t}$. Multiplying the differential equation for $y_1(t)$ by $I(t)$ gives us $(y_1 \cdot e^{3t})' = c_2$. Integrating both sides, we obtain $y_1 \cdot e^{3t} = c_2 t + c_1$. Thus,

$$y_1(t) = c_2 t e^{-3t} + c_1 e^{-3t}.$$

Thus,

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_2 t e^{-3t} + c_1 e^{-3t} \\ c_2 e^{-3t} \end{bmatrix}.$$

Finally, we solve for $\mathbf{x}(t)$:

$$\begin{aligned}\mathbf{x}(t) = S\mathbf{y}(t) &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_2 t e^{-3t} + c_1 e^{-3t} \\ c_2 e^{-3t} \end{bmatrix} \\ &= \begin{bmatrix} c_2 t e^{-3t} + c_1 e^{-3t} + c_2 e^{-3t} \\ c_2 t e^{-3t} + c_1 e^{-3t} \end{bmatrix} \\ &= c_1 e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} t + 1 \\ t \end{bmatrix}.\end{aligned}$$

Now, we must apply the initial condition:

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Therefore, $c_1 = -1$ and $c_2 = 1$. Hence, the unique solution to the given initial-value problem is

$$\mathbf{x}(t) = -e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} t+1 \\ t \end{bmatrix}.$$

37. Let $J = \text{JCF}(A) = \text{JCF}(B)$. Thus, there exist invertible matrices S and T such that

$$S^{-1}AS = J \quad \text{and} \quad T^{-1}BT = J.$$

Thus,

$$S^{-1}AS = T^{-1}BT,$$

and so

$$B = TS^{-1}AST^{-1} = (ST^{-1})^{-1}A(ST^{-1}),$$

which implies by definition that A and B are similar matrices.

38. Since the characteristic polynomial has degree 3, we know that A is a 3×3 matrix. Moreover, the roots of the characteristic equation has roots $\lambda = 0, 0, 0$. Hence, the Jordan canonical form J of A must be one of the three below:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

In all three cases, note that $J^3 = 0_3$. Moreover, there exists an invertible matrix S such that $S^{-1}AS = J$. Thus, $A = SJS^{-1}$, and so

$$A^3 = (SJS^{-1})^3 = SJ^3S^{-1} = S0_3S^{-1} = 0_3,$$

which implies that A is nilpotent.

39.

(a): Let J be an $n \times n$ Jordan block with eigenvalue λ . Then the eigenvalues of J^T are λ (with multiplicity n). The matrix $J^T - \lambda I$ consists of 1's on the subdiagonal (the diagonal parallel and directly beneath the main diagonal) and zeros elsewhere. Hence, the null space of $J^T - \lambda I$ is one-dimensional (with a free variable corresponding to the right-most column of $J^T - \lambda I$). Therefore, the Jordan canonical form of J^T consists of a single Jordan block, since there is only one linearly independent eigenvector corresponding to the eigenvalue λ . However, a single Jordan block with eigenvalue λ is precisely the matrix J . Therefore,

$$\text{JCF}(J^T) = J.$$

(b): Let $\text{JCF}(A) = J$. Then there exists an invertible matrix S such that $S^{-1}AS = J$. Transposing both sides, we obtain $(S^{-1}AS)^T = J^T$, or $S^T A^T (S^{-1})^T = J^T$, or $S^T A^T (S^T)^{-1} = J^T$. Hence, the matrix A^T is similar to J^T . However, by applying part (a) to each block in J^T , we find that $\text{JCF}(J^T) = J$. Hence, J^T is similar to J . By Problem 26 in Section 5.8, we conclude that A^T is similar to J . Hence, A^T and J have the same Jordan canonical form. However, since $\text{JCF}(J) = J$, we deduce that $\text{JCF}(A^T) = J = \text{JCF}(A)$, as required.

Solutions to Section 5.12

Problems:

1. NO. Note that $T(1, 1) = (2, 0, 0, 1)$ and $T(2, 2) = (4, 0, 0, 4) \neq 2T(1, 1)$. Thus, T is not a linear transformation.

2. YES. The function T can be represented by the matrix function

$$T(\mathbf{x}) = A\mathbf{x},$$

where

$$A = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

and $\mathbf{x}$ is a vector in $\mathbb{R}^3$. Every matrix transformation of the form $T(\mathbf{x}) = A\mathbf{x}$ is linear. Since the domain of T has larger dimension than the codomain of T , T cannot be one-to-one. However, since

$$T(0, -1/3, 0) = (1, 0) \quad \text{and} \quad T(-1, -2/3, 0) = (0, 1),$$

we see that T is onto. Thus, $\text{Rng}(T) = \mathbb{R}^2$ (2-dimensional), and so a basis for $\text{Rng}(T)$ is $\{(1, 0), (0, 1)\}$. The kernel of T consists of vectors of the form $(0, 0, z)$, and hence, a basis for $\text{Ker}(T)$ is $\{(0, 0, 1)\}$ and $\text{Ker}(T)$ is 1-dimensional.

3. YES. The function T can be represented by the matrix function

$$T(\mathbf{x}) = A\mathbf{x},$$

where

$$A = \begin{bmatrix} 0 & 0 & -3 \\ 2 & -1 & 5 \end{bmatrix}$$

and $\mathbf{x}$ is a vector in $\mathbb{R}^3$. Every matrix transformation of the form $T(\mathbf{x}) = A\mathbf{x}$ is linear. Since the domain of T has larger dimension than the codomain of T , T cannot be one-to-one. However, since

$$T(1, 1/3, -1/3) = (1, 0) \quad \text{and} \quad T(1, 1, 0) = (0, 1),$$

we see that T is onto. Thus, $\text{Rng}(T) = \mathbb{R}^2$, and so a basis for $\text{Rng}(T)$ is $\{(1, 0), (0, 1)\}$, and $\text{Rng}(T)$ is 2-dimensional. The kernel of T consists of vectors of the form $(t, 2t, 0)$, where $t \in \mathbb{R}$, and hence, a basis for $\text{Ker}(T)$ is $\{(1, 2, 0)\}$. We have that $\text{Ker}(T)$ is 1-dimensional.

4. YES. The function T is a linear transformation, because if $g, h \in C[0, 1]$, then

$$T(g + h) = ((g + h)(0), (g + h)(1)) = (g(0) + h(0), g(1) + h(1)) = (g(0), g(1)) + (h(0), h(1)) = T(g) + T(h),$$

and if c is a scalar,

$$T(cg) = ((cg)(0), (cg)(1)) = (cg(0), cg(1)) = c(g(0), g(1)) = cT(g).$$

Note that any function $g \in C[0, 1]$ for which $g(0) = g(1) = 0$ (such as $g(x) = x^2 - x$) belongs to $\text{Ker}(T)$, and hence, T is not one-to-one. However, given $(a, b) \in \mathbb{R}^2$, note that g defined by

$$g(x) = a + (b - a)x$$

satisfies

$$T(g) = (g(0), g(1)) = (a, b),$$

so T is onto. Thus, $\text{Rng}(T) = \mathbb{R}^2$, with basis $\{(1,0), (0,1)\}$ (2-dimensional). Now, $\text{Ker}(T)$ is infinite-dimensional. We cannot list a basis for this subspace of $C[0,1]$.

5. YES. The function T can be represented by the matrix function

$$T(\mathbf{x}) = A\mathbf{x},$$

where

$$A = \begin{bmatrix} 1/5 & 1/5 \end{bmatrix}$$

and $\mathbf{x}$ is a vector in $\mathbb{R}^2$. Every such matrix transformation is linear. Since the domain of T has larger dimension than the codomain of T , T cannot be one-to-one. However, $T(5,0) = 1$, so we see that T is onto. Thus, $\text{Rng}(T) = \mathbb{R}$, a 1-dimensional space with basis $\{1\}$. The kernel of T consists of vectors of the form $t(1, -1)$, and hence, a basis for $\text{Ker}(T)$ is $\{(1, -1)\}$. We have that $\text{Ker}(T)$ is 1-dimensional.

6. NO. For instance, note that

$$T \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = (1,0) \quad \text{and} \quad T \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = (0,1),$$

and so

$$T \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + T \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = (1,0) + (0,1) = (1,1).$$

However,

$$T \left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) = T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = (2,2).$$

Thus, T does not respect addition, and hence, T is not a linear transformation. Similar work could be given to show that T also fails to respect scalar multiplication.

7. YES. We can verify that T respects addition and scalar multiplication as follows:

T respects addition: Let $a_1 + b_1x + c_1x^2$ and $a_2 + b_2x + c_2x^2$ belong to P_2 . Then

$$\begin{aligned} T((a_1 + b_1x + c_1x^2) + (a_2 + b_2x + c_2x^2)) &= T((a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2) \\ &= \begin{bmatrix} -(a_1 + a_2) - (b_1 + b_2) & 0 \\ 3(c_1 + c_2) - (a_1 + a_2) & -2(b_1 + b_2) \end{bmatrix} \\ &= \begin{bmatrix} -a_1 - b_1 & 0 \\ 3c_1 - a_1 & -2b_1 \end{bmatrix} + \begin{bmatrix} -a_2 - b_2 & 0 \\ 3c_2 - a_2 & -2b_2 \end{bmatrix} \\ &= T(a_1 + b_1x + c_1x^2) + T(a_2 + b_2x + c_2x^2). \end{aligned}$$

T respects scalar multiplication: Let $a + bx + cx^2$ belong to P_2 and let k be a scalar. Then we have

$$\begin{aligned} T(k(a + bx + cx^2)) &= T((ka) + (kb)x + (kc)x^2) = \begin{bmatrix} -ka - kb & 0 \\ 3(kc) - (ka) & -2(kb) \end{bmatrix} \\ &= \begin{bmatrix} k(-a - b) & 0 \\ k(3c - a) & k(-2b) \end{bmatrix} \\ &= k \begin{bmatrix} -a - b & 0 \\ 3c - a & -2b \end{bmatrix} \\ &= kT(a + bx + cx^2). \end{aligned}$$

Next, observe that $a + bx + cx^2$ belongs to $\text{Ker}(T)$ if and only if $-a - b = 0$, $3c - a = 0$, and $-2b = 0$. These equations require that $b = 0$, $a = 0$, and $c = 0$. Thus, $\text{Ker}(T) = \{0\}$, which implies that $\text{Ker}(T)$ is 0-dimensional (with basis $\emptyset$), and that T is one-to-one. However, since $M_2(\mathbb{R})$ is 4-dimensional and P_2 is only 3-dimensional, we see immediately that T cannot be onto. By the Rank-Nullity Theorem, in fact, $\text{Rng}(T)$ must be 3-dimensional, and a basis is given by

$$\text{Basis for } \text{Rng}(T) = \{T(1), T(x), T(x^2)\} = \left\{ \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \right\}.$$

8. YES. We can verify that T respects addition and scalar multiplication as follows:

T respects addition: Let A, B belong to $M_2(\mathbb{R})$. Then

$$T(A + B) = (A + B) + (A + B)^T = A + B + A^T + B^T = (A + A^T) + (B + B^T) = T(A) + T(B).$$

T respects scalar multiplication: Let A belong to $M_2(\mathbb{R})$, and let k be a scalar. Then

$$T(kA) = (kA) + (kA)^T = kA + kA^T = k(A + A^T) = kT(A).$$

Thus, T is a linear transformation. Note that if A is any skew-symmetric matrix, then $T(A) = A + A^T = A + (-A) = 0$, so $\text{Ker}(T)$ consists precisely of the 2×2 skew-symmetric matrices. These matrices take the form $\begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}$, for a constant a , and thus a basis for $\text{Ker}(T)$ is given by $\left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$, and $\text{Ker}(T)$ is 1-dimensional. Consequently, T is not one-to-one.

Therefore, by Proposition 5.4.13, T also fails to be onto. In fact, by the Rank-Nullity Theorem, $\text{Rng}(T)$ must be 3-dimensional. A typical element of the range of T takes the form

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & b+c \\ c+b & 2d \end{bmatrix}.$$

The characterizing feature of this matrix is that it is symmetric. So $\text{Rng}(T)$ consists of all 2×2 symmetric matrices, and hence a basis for $\text{Rng}(T)$ is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

9. YES. We can verify that T respects addition and scalar multiplication as follows:

T respects addition: Let (a_1, b_1, c_1) and (a_2, b_2, c_2) belong to $\mathbb{R}^3$. Then

$$\begin{aligned} T((a_1, b_1, c_1) + (a_2, b_2, c_2)) &= T(a_1 + a_2, b_1 + b_2, c_1 + c_2) \\ &= (a_1 + a_2)x^2 + (2(b_1 + b_2) - (c_1 + c_2))x + (a_1 + a_2 - 2(b_1 + b_2) + (c_1 + c_2)) \\ &= [a_1x^2 + (2b_1 - c_1)x + (a_1 - 2b_1 + c_1)] + [a_2x^2 + (2b_2 - c_2)x + (a_2 - 2b_2 + c_2)] \\ &= T((a_1, b_1, c_1)) + T((a_2, b_2, c_2)). \end{aligned}$$

T respects scalar multiplication: Let (a, b, c) belong to $\mathbb{R}^3$ and let k be a scalar. Then

$$\begin{aligned} T(k(a, b, c)) &= T(ka, kb, kc) = (ka)x^2 + (2kb - kc)x + (ka - 2kb + kc) \\ &= k(ax^2 + (2b - c)x + (a - 2b + c)) \\ &= kT((a, b, c)). \end{aligned}$$

Thus, T is a linear transformation. Now, (a, b, c) belongs to $\text{Ker}(T)$ if and only if $a = 0$, $2b - c = 0$, and $a - 2b + c = 0$. These equations collectively require that $a = 0$ and $2b = c$. Setting $c = 2t$, we find that $b = t$. Hence, (a, b, c) belongs to $\text{Ker}(T)$ if and only if (a, b, c) has the form $(0, t, 2t) = t(0, 1, 2)$. Hence, $\{(0, 1, 2)\}$ is a basis for $\text{Ker}(T)$, which is therefore 1-dimensional. Hence, T is not one-to-one.

By Proposition 5.4.13, T is also not onto. In fact, the Rank-Nullity Theorem implies that $\text{Rng}(T)$ must be 2-dimensional. It is spanned by

$$\{T(1, 0, 0), T(0, 1, 0), T(0, 0, 1)\} = \{x^2 + 1, 2x - 2, -x + 1\},$$

but the last two polynomials are proportional to each other. Omitting the polynomial $2x - 2$ (this is an arbitrary choice; we could have omitted $-x + 1$ instead), we arrive at a basis for $\text{Rng}(T)$: $\{x^2 + 1, -x + 1\}$.

10. YES. We can verify that T respects addition and scalar multiplication as follows:

T respects addition: Let (x_1, x_2, x_3) and (y_1, y_2, y_3) be vectors in $\mathbb{R}^3$. Then

$$\begin{aligned} T((x_1, x_2, x_3) + (y_1, y_2, y_3)) &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= \begin{bmatrix} 0 & (x_1 + y_1) - (x_2 + y_2) + (x_3 + y_3) \\ -(x_1 + y_1) + (x_2 + y_2) - (x_3 + y_3) & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & x_1 - x_2 + x_3 \\ -x_1 + x_2 - x_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & y_1 - y_2 + y_3 \\ -y_1 + y_2 - y_3 & 0 \end{bmatrix} \\ &= T((x_1, x_2, x_3)) + T((y_1, y_2, y_3)). \end{aligned}$$

T respects scalar multiplication: Let (x_1, x_2, x_3) belong to $\mathbb{R}^3$ and let k be a scalar. Then

$$\begin{aligned} T(k(x_1, x_2, x_3)) &= T(kx_1, kx_2, kx_3) = \begin{bmatrix} 0 & (kx_1) - (kx_2) + (kx_3) \\ -(kx_1) + (kx_2) - (kx_3) & 0 \end{bmatrix} \\ &= k \begin{bmatrix} 0 & x_1 - x_2 + x_3 \\ -x_1 + x_2 - x_3 & 0 \end{bmatrix} \\ &= kT((x_1, x_2, x_3)). \end{aligned}$$

Thus, T is a linear transformation. Now, (x_1, x_2, x_3) belongs to $\text{Ker}(T)$ if and only if $x_1 - x_2 + x_3 = 0$ and $-x_1 + x_2 - x_3 = 0$. Of course, the latter equation is equivalent to the former, so the kernel of T consists simply of ordered triples (x_1, x_2, x_3) with $x_1 - x_2 + x_3 = 0$. Setting $x_3 = t$ and $x_2 = s$, we have $x_1 = s - t$, so a typical element of $\text{Ker}(T)$ takes the form $(s - t, s, t)$, where $s, t \in \mathbb{R}$. Extracting the free variables, we find a basis for $\text{Ker}(T)$: $\{(1, 1, 0), (-1, 0, 1)\}$. Hence, $\text{Ker}(T)$ is 2-dimensional.

By the Rank-Nullity Theorem, $\text{Rng}(T)$ must be 1-dimensional. In fact, $\text{Rng}(T)$ consists precisely of the set of 2×2 skew-symmetric matrices, with basis $\left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$. Since $M_2(\mathbb{R})$ is 4-dimensional, T fails to be onto.

11. We have

$$T(x, y, z) = (-x + 8y, 2x - 2y - 5z).$$

12. We have

$$T(x, y) = (-x + 4y, 2y, 3x - 3y, 3x - 3y, 2x - 6y).$$

13. We have

$$T(x) = \frac{x}{2}T(2) = \frac{x}{2}(-1, 5, 0, -2) = \left(-\frac{x}{2}, \frac{5x}{2}, 0, -x\right).$$

14. For an arbitrary 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, if we write

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + s \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + t \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + u \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

we can solve for r, s, t, u to find

$$r = d, \quad s = c, \quad t = a - b + c - d, \quad u = b - c.$$

Thus,

$$\begin{aligned} T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) &= T\left(d \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (a - b + c - d) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (b - c) \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) \\ &= dT\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] + cT\left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right] + (a - b + c - d)T\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right] + (b - c)T\left[\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right] \\ &= d(2, -5) + c(0, -3) + (a - b + c - d)(1, 1) + (b - c)(-6, 2) \\ &= (a - 7b + 7c + d, a + b - 4c - 6d). \end{aligned}$$

15. For an arbitrary element $ax^2 + bx + c$ in P_2 , if we write

$$ax^2 + bx + c = r(x^2 - x - 3) + s(2x + 5) + 6t = rx^2 + (-r + 2s)x + (-3r + 5s + 6t),$$

we can solve for r, s, t to find $r = a$, $s = \frac{1}{2}(a + b)$, and $t = \frac{1}{12}a - \frac{5}{12}b + \frac{1}{6}c$. Thus,

$$\begin{aligned} T(ax^2 + bx + c) &= T\left(a(x^2 - x - 3) + \frac{1}{2}(a + b)(2x + 5) + \frac{1}{2}a - \frac{5}{2}b + c\right) \\ &= aT(x^2 - x - 3) + \frac{1}{2}(a + b)T(2x + 5) + \left(\frac{1}{12}a - \frac{5}{12}b + \frac{c}{6}\right)T(6) \\ &= a \begin{bmatrix} -2 & 1 \\ -4 & -1 \end{bmatrix} + \frac{1}{2}(a + b) \begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix} + \left(\frac{1}{12}a - \frac{5}{12}b + \frac{c}{6}\right) \begin{bmatrix} 12 & 6 \\ 6 & 18 \end{bmatrix} \\ &= \begin{bmatrix} -a - 5b + 2c & 2a - 2b + c \\ -\frac{5}{2}a - \frac{3}{2}b + c & -\frac{1}{2}a - \frac{17}{2}b + 3c \end{bmatrix}. \end{aligned}$$

16. Since $\dim[P_5] = 6$ and $\dim[M_2(\mathbb{R})] = 4 = \dim[\text{Rng}(T)]$ (since T is onto), the Rank-Nullity Theorem gives

$$\dim[\text{Ker}(T)] = 6 - 4 = 2.$$

17. Since T is one-to-one, $\dim[\text{Ker}(T)] = 0$, so the Rank-Nullity Theorem gives

$$\dim[\text{Rng}(T)] = \dim[M_{2 \times 3}(\mathbb{R})] = 6.$$

18. Since A is lower triangular, its eigenvalues lie along the main diagonal, $\lambda_1 = 3$ and $\lambda_2 = -1$. Since A is 2×2 with two distinct eigenvalues, A is diagonalizable. To get an invertible matrix S such that $S^{-1}AS = D$, we need to find an eigenvector associated with each eigenvalue:

Eigenvalue $\lambda_1 = 3$: To get an eigenvector, we consider

$$\text{nullspace}(A - 3I) = \text{nullspace} \begin{bmatrix} 0 & 0 \\ 16 & -4 \end{bmatrix},$$

and we see that one possible eigenvector is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Eigenvalue $\lambda_2 = -1$: To get an eigenvector, we consider

$$\text{nullspace}(A + I) = \text{nullspace} \begin{bmatrix} 4 & 0 \\ 16 & 0 \end{bmatrix},$$

and we see that one possible eigenvector is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Putting the above results together, we form

$$S = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

19. To compute the eigenvalues, we find the characteristic equation

$$\det(A - \lambda I) = \det \begin{bmatrix} 13 - \lambda & -9 \\ 25 & -17 - \lambda \end{bmatrix} = (13 - \lambda)(-17 - \lambda) + 225 = \lambda^2 + 4\lambda + 4,$$

and the roots of this equation are $\lambda = -2, -2$.

Eigenvalue $\lambda = 2$: We compute

$$\text{nullspace}(A - 2I) = \text{nullspace} \begin{bmatrix} 15 & -9 \\ 25 & -15 \end{bmatrix},$$

but since there is only one linearly independent solution to the corresponding system (one free variable), the eigenvalue $\lambda = 2$ does not have two linearly independent solutions. Hence, A is not diagonalizable.

20. To compute the eigenvalues, we find the characteristic equation

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} -4 - \lambda & 3 & 0 \\ -6 & 5 - \lambda & 0 \\ 3 & -3 & -1 - \lambda \end{bmatrix} = (-1 - \lambda)[(-4 - \lambda)(5 - \lambda) + 18] \\ &= (-1 - \lambda)(\lambda^2 - \lambda - 2) \\ &= -(\lambda + 1)^2(\lambda - 2), \end{aligned}$$

so the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2$.

Eigenvalue $\lambda_1 = -1$: To get eigenvectors, we consider

$$\text{nullspace}(A + I) = \text{nullspace} \begin{bmatrix} -3 & 3 & 0 \\ -6 & 6 & 0 \\ 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

There are two free variables, $z = t$ and $y = s$. From the first equation $x = s$. Thus, two linearly independent eigenvectors can be obtained corresponding to $\lambda_1 = -1$:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Eigenvalue $\lambda_2 = 2$: To get an eigenvector, we consider

$$\text{nullspace}(A - 2I) = \text{nullspace} \begin{bmatrix} -6 & 3 & 0 \\ -6 & 3 & 0 \\ 3 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

We let $z = t$. Then $y = -2t$ from the middle line, and $x = -t$ from the top line. Thus, an eigenvector corresponding to $\lambda_2 = 2$ may be chosen as

$$\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}.$$

Putting the above results together, we form

$$S = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

21. To compute the eigenvalues, we find the characteristic equation

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1 - \lambda & 1 & 0 \\ -4 & 5 - \lambda & 0 \\ 17 & -11 & -2 - \lambda \end{bmatrix} = (-2 - \lambda) [(1 - \lambda)(5 - \lambda) + 4] \\ &= (-2 - \lambda)(\lambda^2 - 6\lambda + 9) \\ &= (-2 - \lambda)(\lambda - 3)^2, \end{aligned}$$

so the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$.

Eigenvalue $\lambda_1 = 3$: To get eigenvectors, we consider

$$\text{nullspace}(A - 3I) = \text{nullspace} \begin{bmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ 17 & -11 & -5 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 0 \\ 17 & -11 & -5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -5 \\ -2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The latter matrix contains only one unpivoted column, so that only one linearly independent eigenvector can be obtained. However, $\lambda_1 = 3$ occurs with multiplicity 2 as a root of the characteristic equation for the matrix. Therefore, the matrix is not diagonalizable.

22. We are given that the only eigenvalue of A is $\lambda = 2$.

Eigenvalue $\lambda = 2$: To get eigenvectors, we consider

$$\text{nullspace}(A - 2I) = \text{nullspace} \begin{bmatrix} -3 & -1 & 3 \\ 4 & 2 & -4 \\ -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ -3 & -1 & 3 \\ 4 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

We see that only one unpivoted column will occur in a row-echelon form of $A - 2I$, and thus, only one linearly independent eigenvector can be obtained. Since the eigenvalue $\lambda = 2$ occurs with multiplicity 3 as a root of the characteristic equation for the matrix, the matrix is not diagonalizable.

23. We are given that the eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = -1$.

Eigenvalue $\lambda_1 = 4$: We consider

$$\text{nullspace}(A - 4I) = \text{nullspace} \begin{bmatrix} 5 & 5 & -5 \\ 0 & -5 & 0 \\ 10 & 5 & -10 \end{bmatrix}.$$

The middle row tells us that nullspace vectors (x, y, z) must have $y = 0$. From this information, the first and last rows of the matrix tell us the same thing: $x = z$. Thus, an eigenvector corresponding to $\lambda_1 = 4$ may be chosen as

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Eigenvalue $\lambda_2 = -1$: We consider

$$\text{nullspace}(A + I) = \text{nullspace} \begin{bmatrix} 10 & 5 & -5 \\ 0 & 0 & 0 \\ 10 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 10 & 5 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

From the first row, a vector (x, y, z) in the nullspace must satisfy $10x + 5y - 5z = 0$. Setting $z = t$ and $y = s$, we get $x = \frac{1}{2}t - \frac{1}{2}s$. Hence, the eigenvectors corresponding to $\lambda_2 = -1$ take the form $(\frac{1}{2}t - \frac{1}{2}s, s, t)$, and so a basis for this eigenspace is

$$\left\{ \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Putting the above results together, we form

$$S = \begin{bmatrix} 1 & 1/2 & -1/2 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

28. We will compute the dimension of each of the two eigenspaces associated with the matrix A .

For $\lambda_1 = 1$, we compute as follows:

$$\text{nullspace}(A - I) = \text{nullspace} \begin{bmatrix} 4 & 8 & 16 \\ 4 & 0 & 8 \\ -4 & -4 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

which has only one unpivoted column. Thus, this eigenspace is 1-dimensional.

For $\lambda_2 = -3$, we compute as follows:

$$\text{nullspace}(A + 3I) = \text{nullspace} \begin{bmatrix} 8 & 8 & 16 \\ 4 & 4 & 8 \\ -4 & -4 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which has two unpivoted columns. Thus, this eigenspace is 2-dimensional. Between the two eigenspaces, we have a complete set of linearly independent eigenvectors. Hence, the matrix A in this case is diagonalizable. Therefore, A is diagonalizable, and we may take

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

(Of course, the eigenvalues of A may be listed in any order along the main diagonal of the Jordan canonical form, thus yielding other valid Jordan canonical forms for A .)

29. We will compute the dimension of each of the two eigenspaces associated with the matrix A .

For $\lambda_1 = -1$, we compute as follows:

$$\text{nullspace}(A + I) = \text{nullspace} \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & -2 \\ -1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix},$$

which has one unpivoted column. Thus, this eigenspace is 1-dimensional.

For $\lambda_2 = 3$, we compute as follows:

$$\text{nullspace}(A - 3I) = \text{nullspace} \begin{bmatrix} -1 & 1 & 1 \\ 2 & -2 & -2 \\ -1 & 0 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix},$$

which has one unpivoted column. Thus, this eigenspace is also one-dimensional.

Since we have only generated two linearly independent eigenvectors from the eigenvalues of A , we know that A is not diagonalizable, and hence, the Jordan canonical form of A is not a diagonal matrix. We must have one 1×1 Jordan block and one 2×2 Jordan block. To determine which eigenvalue corresponds to the 1×1 block and which corresponds to the 2×2 block, we must determine the multiplicity of the eigenvalues as roots of the characteristic equation of A .

A short calculation shows that $\lambda_1 = -1$ occurs with multiplicity 2, while $\lambda_2 = 3$ occurs with multiplicity 1. Thus, the Jordan canonical form of A is

$$J = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

30. There are 3 different possible Jordan canonical forms, up to a rearrangement of the Jordan blocks:

Case 1:

$$J = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

In this case, the matrix has four linearly independent eigenvectors, and because all Jordan blocks have size 1×1 , the maximum length of a cycle of generalized eigenvectors for this matrix is 1.

Case 2:

$$J = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

In this case, the matrix has three linearly independent eigenvectors (two corresponding to $\lambda = -1$ and one corresponding to $\lambda = 2$). There is a Jordan block of size 2×2 , and so a cycle of generalized eigenvectors can have a maximum length of 2 in this case.

Case 3:

$$J = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

In this case, the matrix has two linearly independent eigenvectors (one corresponding to $\lambda = -1$ and one corresponding to $\lambda = 2$). There is a Jordan block of size 3×3 , and so a cycle of generalized eigenvectors can have a maximum length of 3 in this case.

31. There are 7 different possible Jordan canonical forms, up to a rearrangement of the Jordan blocks:

Case 1:

$$J = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

In this case, the matrix has five linearly independent eigenvectors, and because all Jordan blocks have size 1×1 , the maximum length of a cycle of generalized eigenvectors for this matrix is 1.

Case 2:

$$J = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

In this case, the matrix has four linearly independent eigenvectors, and because the largest Jordan block is of size 2×2 , the maximum length of a cycle of generalized eigenvectors for this matrix is 2.

Case 3:

$$J = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

In this case, the matrix has three linearly independent eigenvectors, and because the largest Jordan block is of size 2×2 , the maximum length of a cycle of generalized eigenvectors for this matrix is 2.

Case 4:

$$J = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

In this case, the matrix has three linearly independent eigenvectors, and because the largest Jordan block is of size 3×3 , the maximum length of a cycle of generalized eigenvectors for this matrix is 3.

Case 5:

$$J = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

In this case, the matrix has two linearly independent eigenvectors, and because the largest Jordan block is of size 3×3 , the maximum length of a cycle of generalized eigenvectors for this matrix is 3.

Case 6:

$$J = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

In this case, the matrix has two linearly independent eigenvectors, and because the largest Jordan block is of size 4×4 , the maximum length of a cycle of generalized eigenvectors for this matrix is 4.

Case 7:

$$J = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

In this case, the matrix has only one linearly independent eigenvector, and because the largest Jordan block is of size 5×5 , the maximum length of a cycle of generalized eigenvectors for this matrix is 5.

32. There are 10 different possible Jordan canonical forms, up to a rearrangement of the Jordan blocks:

Case 1:

$$J = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

In this case, the matrix has six linearly independent eigenvectors, and because all Jordan blocks have size 1×1 , the maximum length of a cycle of generalized eigenvectors for this matrix is 1.

Case 2:

$$J = \begin{bmatrix} 6 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

In this case, the matrix has five linearly independent eigenvectors (three corresponding to $\lambda = 6$ and two corresponding to $\lambda = -3$), and because the largest Jordan block is of size 2×2 , the maximum length of a cycle of generalized eigenvectors for this matrix is 2.

Case 3:

$$J = \begin{bmatrix} 6 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

In this case, the matrix has four linearly independent eigenvectors (two corresponding to $\lambda = 6$ and two corresponding to $\lambda = -3$), and because the largest Jordan block is of size 2×2 , the maximum length of a cycle of generalized eigenvectors for this matrix is 2.

Case 4:

$$J = \begin{bmatrix} 6 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

In this case, the matrix has four linearly independent eigenvectors (two corresponding to $\lambda = 6$ and two corresponding to $\lambda = -3$), and because the largest Jordan block is of size 3×3 , the maximum length of a cycle of generalized eigenvectors for this matrix is 3.

Case 5:

$$J = \begin{bmatrix} 6 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

In this case, the matrix has three linearly independent eigenvectors (one corresponding to $\lambda = 6$ and two corresponding to $\lambda = -3$), and because the largest Jordan block is of size 4×4 , the maximum length of a cycle of generalized eigenvectors for this matrix is 4.

Case 6:

$$J = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

In this case, the matrix has five linearly independent eigenvectors (four corresponding to $\lambda = 6$ and one corresponding to $\lambda = -3$), and because the largest Jordan block is of size 2×2 , the maximum length of a cycle of generalized eigenvectors for this matrix is 2.

Case 7:

$$J = \begin{bmatrix} 6 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

In this case, the matrix has four linearly independent eigenvectors (three corresponding to $\lambda = 6$ and one corresponding to $\lambda = -3$), and because the largest Jordan block is of size 2×2 , the maximum length of a cycle of generalized eigenvectors for this matrix is 2.

Case 8:

$$J = \begin{bmatrix} 6 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

In this case, the matrix has three linearly independent eigenvectors (two corresponding to $\lambda = 6$ and one corresponding to $\lambda = -3$), and because the largest Jordan block is of size 2×2 , the maximum length of a cycle of generalized eigenvectors for this matrix is 2.

Case 9:

$$J = \begin{bmatrix} 6 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

In this case, the matrix has three linearly independent eigenvectors (two corresponding to $\lambda = 6$ and one corresponding to $\lambda = -3$), and because the largest Jordan block is of size 3×3 , the maximum length of a cycle of generalized eigenvectors for this matrix is 3.

Case 10:

$$J = \begin{bmatrix} 6 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

In this case, the matrix has two linearly independent eigenvectors (one corresponding to $\lambda = 6$ and one corresponding to $\lambda = -3$), and because the largest Jordan block is of size 4×4 , the maximum length of a cycle of generalized eigenvectors for this matrix is 4.

33. There are 15 different possible Jordan canonical forms, up to a rearrangement of Jordan blocks:

Case 1:

$$J = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$

In this case, the matrix has seven linearly independent eigenvectors (four corresponding to $\lambda = 2$ and three corresponding to $\lambda = -4$), and because all Jordan blocks are size 1×1 , the maximum length of a cycle of generalized eigenvectors for this matrix is 1.

Case 2:

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$

In this case, the matrix has six linearly independent eigenvectors (three corresponding to $\lambda = 2$ and three corresponding to $\lambda = -4$), and because the largest Jordan block is of size 2×2 , the maximum length of a cycle of generalized eigenvectors for this matrix is 2.

Case 3:

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$

In this case, the matrix has five linearly independent eigenvectors (two corresponding to $\lambda = 2$ and three corresponding to $\lambda = -4$), and because the largest Jordan block is of size 2×2 , the maximum length of a cycle of generalized eigenvectors for this matrix is 2.

Case 4:

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$

In this case, the matrix has five linearly independent eigenvectors (two corresponding to $\lambda = 2$ and three corresponding to $\lambda = -4$), and because the largest Jordan block is of size 3×3 , the maximum length of a cycle of generalized eigenvectors for this matrix is 3.

Case 5:

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$

In this case, the matrix has four linearly independent eigenvectors (one corresponding to $\lambda = 2$ and three corresponding to $\lambda = -4$), and because the largest Jordan block is of size 4×4 , the maximum length of a cycle of generalized eigenvectors for this matrix is 4.

Case 6:

$$J = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$

In this case, the matrix has six linearly independent eigenvectors (four corresponding to $\lambda = 2$ and two corresponding to $\lambda = -4$), and because the largest Jordan block is of size 2×2 , the maximum length of a cycle of generalized eigenvectors for this matrix is 2.

Case 7:

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$

In this case, the matrix has five linearly independent eigenvectors (three corresponding to $\lambda = 2$ and two corresponding to $\lambda = -4$), and because the largest Jordan block is of size 2×2 , the maximum length of a cycle of generalized eigenvectors for this matrix is 2.

Case 8:

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$

In this case, the matrix has four linearly independent eigenvectors (two corresponding to $\lambda = 2$ and two corresponding to $\lambda = -4$), and because the largest Jordan block is of size 2×2 , the maximum length of a cycle of generalized eigenvectors for this matrix is 2.

Case 9:

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$

In this case, the matrix has four linearly independent eigenvectors (two corresponding to $\lambda = 2$ and two corresponding to $\lambda = -4$), and because the largest Jordan block is of size 3×3 , the maximum length of a cycle of generalized eigenvectors for this matrix is 3.

Case 10:

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$

In this case, the matrix has three linearly independent eigenvectors (one corresponding to $\lambda = 2$ and two corresponding to $\lambda = -4$), and because the largest Jordan block is of size 4×4 , the maximum length of a cycle of generalized eigenvectors for this matrix is 4.

Case 11:

$$J = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$

In this case, the matrix has five linearly independent eigenvectors (four corresponding to $\lambda = 2$ and one corresponding to $\lambda = -4$), and because the largest Jordan block is of size 3×3 , the maximum length of a cycle of generalized eigenvectors for this matrix is 3.

Case 12:

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$

In this case, the matrix has four linearly independent eigenvectors (three corresponding to $\lambda = 2$ and one corresponding to $\lambda = -4$), and because the largest Jordan block is of size 3×3 , the maximum length of a cycle of generalized eigenvectors for this matrix is 3.

Case 13:

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$

In this case, the matrix has three linearly independent eigenvectors (two corresponding to $\lambda = 2$ and one corresponding to $\lambda = -4$), and because the largest Jordan block is of size 3×3 , the maximum length of a cycle of generalized eigenvectors for this matrix is 3.

Case 14:

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$

In this case, the matrix has three linearly independent eigenvectors (two corresponding to $\lambda = 2$ and one corresponding to $\lambda = -4$), and because the largest Jordan block is of size 3×3 , the maximum length of a cycle of generalized eigenvectors for this matrix is 3.

Case 15:

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix}.$$

In this case, the matrix has two linearly independent eigenvectors (one corresponding to $\lambda = 2$ and one corresponding to $\lambda = -4$), and because the largest Jordan block is of size 4×4 , the maximum length of a cycle of generalized eigenvectors for this matrix is 4.

34. FALSE. For instance, if $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, then we have eigenvalues $\lambda_A = \lambda_B = 1$, but the matrix $A - B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is invertible, and hence, zero is not an eigenvalue of $A - B$. This can also be verified directly.

35. FALSE. For instance, if $A = I_2$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $A^2 = B^2 = I_2$, but the matrices A and B are not similar. (Otherwise, there would exist an invertible matrix S such that $S^{-1}AS = B$. But since $A = I_2$ this reduces to $I_2 = B$, which is clearly not the case. Thus, no such invertible matrix S exists.)

36. To see that $T_1 + T_2$ is a linear transformation, we must verify that it respects addition and scalar multiplication:

$T_1 + T_2$ respects addition: Let $\mathbf{v}_1$ and $\mathbf{v}_2$ belong to V . Then we have

$$\begin{aligned}(T_1 + T_2)(\mathbf{v}_1 + \mathbf{v}_2) &= T_1(\mathbf{v}_1 + \mathbf{v}_2) + T_2(\mathbf{v}_1 + \mathbf{v}_2) \\ &= [T_1(\mathbf{v}_1) + T_1(\mathbf{v}_2)] + [T_2(\mathbf{v}_1) + T_2(\mathbf{v}_2)] \\ &= [T_1(\mathbf{v}_1) + T_2(\mathbf{v}_1)] + [T_1(\mathbf{v}_2) + T_2(\mathbf{v}_2)] \\ &= (T_1 + T_2)(\mathbf{v}_1) + (T_1 + T_2)(\mathbf{v}_2),\end{aligned}$$

where we have used the linearity of T_1 and T_2 individually in the second step.

$T_1 + T_2$ respects scalar multiplication: Let $\mathbf{v}$ belong to V and let k be a scalar. Then we have

$$\begin{aligned}(T_1 + T_2)(k\mathbf{v}) &= T_1(k\mathbf{v}) + T_2(k\mathbf{v}) \\ &= kT_1(\mathbf{v}) + kT_2(\mathbf{v}) \\ &= k[T_1(\mathbf{v}) + T_2(\mathbf{v})] \\ &= k(T_1 + T_2)(\mathbf{v}),\end{aligned}$$

as required.

There is no particular relationship between $\text{Ker}(T_1)$, $\text{Ker}(T_2)$, and $\text{Ker}(T_1 + T_2)$.

37. FALSE. For instance, consider $T_1 : \mathbb{R} \rightarrow \mathbb{R}$ defined by $T_1(x) = x$, and consider $T_2 : \mathbb{R} \rightarrow \mathbb{R}$ defined by $T_2(x) = -x$. Both T_1 and T_2 are linear transformations, and both of them are onto. However, $(T_1 + T_2)(x) = T_1(x) + T_2(x) = x + (-x) = 0$, so $\text{Rng}(T_1 + T_2) = \{0\}$, which implies that $T_1 + T_2$ is not onto.

38. FALSE. For instance, consider $T_1 : \mathbb{R} \rightarrow \mathbb{R}$ defined by $T_1(x) = x$, and consider $T_2 : \mathbb{R} \rightarrow \mathbb{R}$ defined by $T_2(x) = -x$. Both T_1 and T_2 are linear transformations, and both of them are one-to-one. However, $(T_1 + T_2)(x) = T_1(x) + T_2(x) = x + (-x) = 0$, so $\text{Ker}(T_1 + T_2) = \mathbb{R}$, which implies that $T_1 + T_2$ is not one-to-one.

39. Assume that

$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n) = \mathbf{0}.$$

We wish to show that

$$c_1 = c_2 = \cdots = c_n = 0.$$

To do this, use the linearity of T to rewrite the above equation as

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n) = \mathbf{0}.$$

Now, since $\text{Ker}(T) = \{\mathbf{0}\}$, we conclude that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a linearly independent set, we conclude that $c_1 = c_2 = \cdots = c_n = 0$, as required.

40. Assume that $V_1 \cong V_2$ and $V_2 \cong V_3$. Then there exist isomorphisms $T_1 : V_1 \rightarrow V_2$ and $T_2 : V_2 \rightarrow V_3$. Since the composition of two linear transformations is a linear transformation (Theorem 5.4.2), we have a linear transformation $T_2T_1 : V_1 \rightarrow V_3$. Moreover, since both T_1 and T_2 are one-to-one and onto, T_2T_1 is also one-to-one and onto (see Problem 39 in Section 5.4). Thus, $T_2T_1 : V_1 \rightarrow V_3$ is an isomorphism. Hence, $V_1 \cong V_3$, as required.

41. We have

$$A_i\mathbf{v} = \lambda_i\mathbf{v}$$

for each $i = 1, 2, \dots, k$. Thus,

$$\begin{aligned}
 (A_1 A_2 \dots A_k) \mathbf{v} &= (A_1 A_2 \dots A_{k-1})(A_k \mathbf{v}) = (A_1 A_2 \dots A_{k-1})(\lambda_k \mathbf{v}) = \lambda_k (A_1 A_2 \dots A_{k-1}) \mathbf{v} \\
 &= \lambda_k (A_1 A_2 \dots A_{k-2})(A_{k-1} \mathbf{v}) = \lambda_k (A_1 A_2 \dots A_{k-2})(\lambda_{k-1} \mathbf{v}) = \lambda_{k-1} \lambda_k (A_1 A_2 \dots A_{k-2}) \mathbf{v} \\
 &\vdots \\
 &= \lambda_2 \lambda_3 \dots \lambda_k (A_1 \mathbf{v}) = \lambda_2 \lambda_3 \dots \lambda_k (\lambda_1 \mathbf{v}) \\
 &= (\lambda_1 \lambda_2 \dots \lambda_k) \mathbf{v},
 \end{aligned}$$

which shows that $\mathbf{v}$ is an eigenvector of $A_1 A_2 \dots A_k$ with corresponding eigenvalue $\lambda_1 \lambda_2 \dots \lambda_k$.

42. We first show that T is a linear transformation:

T respects addition: Let A and B belong to $M_n(\mathbb{R})$. Then

$$T(A + B) = S^{-1}(A + B)S = S^{-1}AS + S^{-1}BS = T(A) + T(B),$$

and so T respects addition.

T respects scalar multiplication: Let A belong to $M_n(\mathbb{R})$, and let k be a scalar. Then

$$T(kA) = S^{-1}(kA)S = k(S^{-1}AS) = kT(A),$$

and so T respects scalar multiplication.

Next, we verify that T is both one-to-one and onto (of course, in view of Proposition 5.4.13, it is only necessary to confirm one of these two properties, but we will nonetheless verify them both):

T is one-to-one: Assume that $T(A) = 0_n$. That is, $S^{-1}AS = 0_n$. Left multiplying by S and right multiplying by S^{-1} on both sides of this equation yields $A = S0_n S^{-1} = 0_n$. Hence, $\text{Ker}(T) = \{0_n\}$, and so T is one-to-one.

T is onto: Let B be an arbitrary matrix in $M_n(\mathbb{R})$. Then

$$T(SBS^{-1}) = S^{-1}(SBS^{-1})S = (S^{-1}S)B(S^{-1}S) = I_n B I_n = B,$$

and hence, B belongs to $\text{Rng}(T)$. Since B was an arbitrary element of $M_n(\mathbb{R})$, we conclude that $\text{Rng}(T) = M_n(\mathbb{R})$. That is, T is onto.

Solutions to Section 6.1

True-False Review:

1. TRUE. This is essentially the statement of Theorem 6.1.3.

2. FALSE. As stated in Theorem 6.1.5, if there is any point x_0 in I such that $W[y_1, y_2, \dots, y_n](x_0) = 0$, then $\{y_1, y_2, \dots, y_n\}$ is linearly dependent on I .

3. FALSE. Many counterexamples are possible. Note that

$$(xD - Dx)(x) = xD(x) - D(x^2) = x - 2x = (-1)(x).$$

Therefore, $xD - Dx = -1$. Setting $L_1 = x$ and $L_2 = D$, we therefore see that $L_1 L_2 \neq L_2 L_1$ in this example.

4. TRUE. By assumption, $L_1(y_1 + y_2) = L_1(y_1) + L_1(y_2)$ and $L_1(cy) = cL_1(y)$ for all functions y, y_1, y_2 . Likewise, $L_2(y_1 + y_2) = L_2(y_1) + L_2(y_2)$ and $L_2(cy) = cL_2(y)$ for all functions y, y_1, y_2 . Therefore,

$$\begin{aligned}(L_1 + L_2)(y_1 + y_2) &= L_1(y_1 + y_2) + L_2(y_1 + y_2) \\ &= (L_1(y_1) + L_1(y_2)) + (L_2(y_1) + L_2(y_2)) \\ &= (L_1(y_1) + L_2(y_1)) + (L_1(y_2) + L_2(y_2)) \\ &= (L_1 + L_2)(y_1) + (L_1 + L_2)(y_2)\end{aligned}$$

and

$$\begin{aligned}(L_1 + L_2)(cy) &= L_1(cy) + L_2(cy) \\ &= cL_1(y) + cL_2(y) \\ &= c(L_1(y) + L_2(y)) \\ &= c(L_1 + L_2)(y).\end{aligned}$$

Therefore, $L_1 + L_2$ is a linear differential operator.

5. TRUE. By assumption $L(y_1 + y_2) = L(y_1) + L(y_2)$ and $L(ky) = kL(y)$ for all scalars k . Therefore, for all constants c , we have

$$(cL)(y_1 + y_2) = cL(y_1 + y_2) = c(L(y_1) + L(y_2)) = cL(y_1) + cL(y_2) = (cL)(y_1) + (cL)(y_2)$$

and

$$(cL)(ky) = c(L(ky)) = c(k(L(y))) = k(cL)(y).$$

Therefore, cL is a linear differential operator.

6. TRUE. We have

$$L(y_p + u) = L(y_p) + L(u) = F + 0 = F.$$

7. TRUE. We have

$$L(y_1 + y_2) = L(y_1) + L(y_2) = F_1 + F_2.$$

Problems:

1.

(a)

$$\begin{aligned}L(y(x)) &= (D - x)(2x - 3e^{2x}) = D(2x - 3e^{2x}) - x(2x - 3e^{2x}) \\ &= (2 - 6e^{2x}) - 2x^2 + 3xe^{2x} \\ &= 2(1 - x^2) + 3e^{2x}(x - 2).\end{aligned}$$

(b)

$$\begin{aligned}L(y(x)) &= (D - x)(3\sin^2 x) = D(3\sin^2 x) - x(3\sin^2 x) \\ &= 6\sin x \cos x - 3x\sin^2 x \\ &= 3\sin x(2x \cos x - x\sin x).\end{aligned}$$

2.

(a)

$$\begin{aligned}
L(y(x)) &= (D^2 - x^2D + x)(2x - 3e^{2x}) = D^2(2x - 3e^{2x}) - x^2D(2x - 3e^{2x}) + x(2x - 3e^{2x}) \\
&= D(2 - 6e^{2x}) - x^2(2 - 6e^{2x}) + x(2x - 3e^{2x}) \\
&= -12e^{2x} - 2x^2 + 6x^2e^{2x} + 2x^2 - 3xe^{2x} \\
&= 3e^{2x}(2x^2 - x - 4).
\end{aligned}$$

(b)

$$\begin{aligned}
L(y(x)) &= (D^2 - x^2D + x)(3\sin^2 x) = D^2(3\sin^2 x) - x^2D(3\sin^2 x) + x(3\sin^2 x) \\
&= D(6\sin x \cdot \cos x) - x^2(6\sin x \cdot \cos x) + 3x\sin^2 x \\
&= 12\cos^2 x - 6 - 6x^2\sin x \cdot \cos x + 3x\sin^2 x.
\end{aligned}$$

3.

(a)

$$\begin{aligned}
L(y(x)) &= (D^3 - 2xD^2)(2x - 3e^{2x}) = D^3(2x - 3e^{2x}) - 2xD^2(2x - 3e^{2x}) \\
&= D^3(-3e^{2x}) - 2xD^2(-3e^{2x}) \\
&= -24e^{2x} + 24xe^{2x} \\
&= 24e^{2x}(x - 1).
\end{aligned}$$

(b)

$$\begin{aligned}
L(y(x)) &= (D^3 - 2xD^2)(3\sin^2 x) = D^3(3\sin^2 x) - 2xD^2(3\sin^2 x) \\
&= D^2(6\sin x \cdot \cos x) - 2xD(6\sin x \cdot \cos x) \\
&= D(12\cos^2 x - 6) - 2x(12\cos^2 x - 6) \\
&= -24\sin x \cdot \cos x - 24x\cos^2 x + 12x.
\end{aligned}$$

4.

(a)

$$\begin{aligned}
L(y(x)) &= (D^3 - D + 4)(2x - 3e^{2x}) = D^3(2x - 3e^{2x}) - D(2x - 3e^{2x}) + 4(2x - 3e^{2x}) \\
&= D^2(2 - 6e^{2x}) - (2 - 6e^{2x}) + 8x - 12e^{2x} \\
&= D(-12e^{2x}) - (2 - 6e^{2x}) + 8x - 12e^{2x} \\
&= (-24e^{2x}) - (2 - 6e^{2x}) + 8x - 12e^{2x} \\
&= 8x - 30e^{2x} - 2.
\end{aligned}$$

(b)

$$\begin{aligned}
L(y(x)) &= (D^3 - D + 4)(3\sin^2 x) = D^3(3\sin^2 x) - D(3\sin^2 x) + 4(3\sin^2 x) \\
&= D^2(6\sin x \cdot \cos x) - (6\sin x \cdot \cos x) + 12\sin^2 x \\
&= D(12\cos^2 x - 6) - 6\sin x \cdot \cos x + 12\sin^2 x \\
&= -24\sin x \cdot \cos x - 6\sin x \cdot \cos x + 12\sin^2 x \\
&= 6\sin x(-3\cos x + 2\sin x).
\end{aligned}$$

5.

$$\begin{aligned}
L(y(x)) &= (x^2 D^2 + 2xD - 2)(x^{-2}) = x^2 D^2(x^{-2}) + 2xD(x^{-2}) - 2x^{-2} \\
&= x^2 D(-2x^{-3}) - 4x^{-2} - 2x^{-2} \\
&= 6x^{-2} - 6x^{-2} = 0.
\end{aligned}$$

Thus, $f \in \text{Ker}(L)$.

6.

$$\begin{aligned}
L(y(x)) &= (D^2 - x^{-1}D + 4x^2)(\sin(x^2)) = D^2(\sin(x^2)) - x^{-1}D(\sin(x^2)) + 4x^2(\sin(x^2)) \\
&= D(2x \cos(x^2)) - x^{-1}(2x \cos(x^2)) + 4x^2(\sin(x^2)) \\
&= 2[\cos(x^2) - 2x^2 \sin(x^2)] - 2 \cos(x^2) + 4x^2(\sin(x^2)) \\
&= 0.
\end{aligned}$$

Thus, $f \in \text{Ker}(L)$.

7.

$$\begin{aligned}
L(y(x)) &= (D^3 + D^2 + D + 1)(\sin x + \cos x) \\
&= D^3(\sin x + \cos x) + D^2(\sin x + \cos x) + D(\sin x + \cos x) + (\sin x + \cos x) \\
&= D^2(\cos x - \sin x) + D(\cos x - \sin x) + \cos x - \sin x + \sin x + \cos x \\
&= D(-\sin x - \cos x) + (-\sin x - \cos x) + 2 \cos x \\
&= -\cos x + \sin x - \sin x - \cos x + 2 \cos x \\
&= 0.
\end{aligned}$$

Thus, $f \in \text{Ker}(L)$.

8.

$$\begin{aligned}
L(y(x)) &= (-D^2 + 2D - 1)(xe^x) \\
&= -D^2(xe^x) + 2D(xe^x) - xe^x \\
&= -D(e^x + xe^x) + 2(e^x + xe^x) - xe^x \\
&= -(e^x + e^x + xe^x) + 2e^x + 2xe^x - xe^x \\
&= 0.
\end{aligned}$$

Thus, $f \in \text{Ker}(L)$.

9. $L(y(x)) = 0 \iff (D - 2x)y = 0 \iff y' - 2xy = 0$. This linear equation has integrating factor $I = e^{-\int 2x dx} = e^{-x^2}$, so that the differential equation can be written as $\frac{d}{dx}(e^{-x^2}y) = 0$, which has a general solution $e^{-x^2}y = c$, that is, $y(x) = ce^{x^2}$. Consequently,

$$\text{Ker}(L) = \{y \in C^1(\mathbb{R}) : y(x) = ce^{x^2}, c \in \mathbb{R}\}.$$

10. $L(y(x)) = 0 \iff (D^2 + 1)y = 0 \iff y'' + y = 0$. This is a second-order homogeneous linear differential equation, so by Theorem 6.1.3, the solution set to this equation forms a 2-dimensional vector space. A little thought shows that both $y_1 = \cos x$ and $y_2 = \sin x$ are solutions to the equation. Since they are linearly independent, these two functions form a basis for the solution set. Therefore,

$$\text{Ker}(L) = \{a \cos x + b \sin x : a, b \in \mathbb{R}\}.$$

11. $L(y(x)) = 0 \iff (D^2 + 2D - 15)y = 0 \iff y'' + 2y' - 15y = 0$. This is a second-order homogeneous linear differential equation, so by Theorem 6.1.3, the solution set to this equation forms a 2-dimensional vector space. Following the hint given in the text, we try for solutions of the form $y(x) = e^{rx}$. Substituting this into the differential equation, we get $r^2e^{rx} + 2re^{rx} - 15e^{rx} = 0$, or $e^{rx}(r^2 + 2r - 15) = 0$. It follows that $r^2 + 2r - 15 = 0$. That is $(r + 5)(r - 3) = 0$, and hence, $r = -5$ and $r = 3$ are the solutions. Therefore, we obtain the solutions $y_1 = e^{-5x}$ and $y_2 = e^{3x}$. Since they are linearly independent (by computing the Wronskian, for instance), these two functions form a basis for the solution set. Therefore,

$$\text{Ker}(L) = \{ae^{-5x} + be^{3x} : a, b \in \mathbb{R}\}.$$

12. $Ly = 0 \iff (x^2D + x)y = 0 \iff x^2y' + xy = 0$. Dividing by x^2 , we can express this as $y' + \frac{1}{x}y = 0$. This differential equation is separable: $\frac{1}{y}dy = -\frac{1}{x}dx$. Integrating both sides, we obtain $\ln|y| = -\ln|x| + c_1$. Therefore $|y| = e^{-\ln|x|+c_1} = c_2e^{-\ln|x|}$. Hence, $y(x) = c_3e^{-\ln|x|} = \frac{c_3}{|x|}$. Thus,

$$\text{Ker}(L) = \left\{ \frac{c}{|x|} : c \in \mathbb{R} \right\}.$$

13. We have

$$\begin{aligned} (L_1L_2)(f) &= L_1(f' - 2x^2f) \\ &= (D + 1)(f' - 2x^2f) \\ &= f'' + f' - 4xf - 2x^2f' - 2x^2f \\ &= f'' + (1 - 2x^2)f' - (4x + 2x^2)f, \end{aligned}$$

so that

$$L_1L_2 = D^2 + (1 - 2x^2)D - 2x(2 + x).$$

Furthermore,

$$\begin{aligned} (L_2L_1)(f) &= L_2(f' + f) \\ &= (D - 2x^2)(f' + f) \\ &= f'' + f' - 2x^2f' - 2x^2f, \end{aligned}$$

so that

$$L_2L_1 = D^2 + (1 - 2x^2)D - 2x^2.$$

Therefore, $L_1L_2 \neq L_2L_1$.

14. We have

$$\begin{aligned} (L_1L_2)(f) &= L_1(f' + (2x - 1)f) \\ &= (D + x)(f' + (2x - 1)f) \\ &= f'' + xf' + 2f + (2x - 1)f' + x(2x - 1)f \\ &= (D^2 + (3x - 1)D + (2x^2 - x + 2))(f), \end{aligned}$$

so that

$$L_1L_2 = D^2 + (3x - 1)D + (2x^2 - x + 2).$$

Furthermore,

$$\begin{aligned} (L_2L_1)(f) &= L_2(f' + xf) \\ &= (D + (2x - 1))(f' + xf) \\ &= f'' + (2x - 1)f' + f + xf' + (2x - 1)xf \\ &= (D^2 + (3x - 1)D + (2x^2 - x + 1))(f), \end{aligned}$$

so that

$$L_2L_1 = D^2 + (3x - 1)D + (2x^2 - x + 1).$$

Therefore, $L_1L_2 \neq L_2L_1$.

15. We have

$$\begin{aligned}(L_1L_2)(f) &= L_1(D + b_1)f \\ &= (D + a_1)(f' + b_1f) \\ &= f'' + [b_1 + a_1]f' + (b'_1 + a_1b_1)f.\end{aligned}$$

Thus

$$L_1L_2 = D^2 + (b_1 + a_1)D + (b'_1 + a_1b_1).$$

Similarly,

$$L_2L_1 = D^2 + (a_1 + b_1)D + (a'_1 + b_1a_1).$$

Thus $L_1L_2 - L_2L_1 = b'_1 - a'_1$, which is the zero operator if and only if $b'_1 = a'_1$ which can be integrated directly to obtain $b_1 = a_1 + c_2$, where c_2 is an arbitrary constant. Consequently we must have $L_2 = D + [a_1(x) + c_2]$.

16. $(D^3 + x^2D^2 - (\sin x)D + e^x)y = x^3$ and $y''' + x^2y'' - (\sin x)y' + e^xy = 0$.

17. $(D^2 + 4xD - 6x^2)y = x^2 \sin x$ and $y'' + 4xy' - 6x^2y = 0$.

18. x^2 and e^x are both continuous functions for all $x \in \mathbb{R}$ and in particular on any interval I containing $x_0 = 0$. Thus, $y(x) = 0$ is clearly a solution to the given differential equation and also satisfies the initial conditions. Thus, by the existence-uniqueness theorem, $y(x) = 0$ is the only solution to the initial-value problem.

19. Let $a_1, \dots, a_n$ be functions that are continuous on the interval I . Then, for any x_0 in I , the initial-value problem $y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_{n-1}(x)y' + a_n(x)y = 0$, $y(x_0) = 0$, $y'(x_0) = 0, \dots, y^{(n-1)}(x_0) = 0$, has only the trivial solution $y(x) = 0$.

Proof:

All of the conditions of the existence-uniqueness theorem are satisfied and $y(x) = 0$ is a solution; consequently, it is the only solution.

20. Given $y'' - 2y' - 3y = 0$ then $r^2 - 2r - 3 = 0 \implies r \in \{-1, 3\} \implies y(x) = c_1e^{-x} + c_2e^{3x}$.

21. Given $y'' + 7y' + 10y = 0$ then $r^2 + 7r + 10 = 0 \implies r \in \{-5, -2\} \implies y(x) = c_1e^{-5x} + c_2e^{-2x}$.

22. Given $y'' - 36y = 0$ then $r^2 - 36 = 0 \implies r \in \{-6, 6\} \implies y(x) = c_1e^{-6x} + c_2e^{6x}$.

23. Given $y'' + 4y' = 0$ then $r^2 + 4r = 0 \implies r \in \{-4, 0\} \implies y(x) = c_1e^{-4x} + c_2$.

24. Substituting $y(x) = e^{rx}$ into the given differential equation yields $e^{rx}(r^3 - 3r^2 - r + 3) = 0$, so that we will have a solution provided that r satisfies $r^3 - 3r^2 - r + 3 = 0$, that is, $(r - 1)(r + 1)(r - 3) = 0$. Consequently, three solutions to the given differential equation are $y_1(x) = e^x$, $y_2(x) = e^{-x}$, $y_3(x) = e^{3x}$. Further, the Wronskian of these solutions is

$$W[y_1, y_2, y_3] = \begin{vmatrix} e^x & e^{-x} & e^{3x} \\ e^x & -e^{-x} & 3e^{3x} \\ e^x & e^{-x} & 9e^{3x} \end{vmatrix} = -16e^{3x}.$$

Since the Wronskian is never zero, the solutions are linearly independent on any interval. Hence the general solution to the differential equation is $y(x) = c_1e^x + c_2e^{-x} + c_3e^{3x}$.

25. Substituting $y(x) = e^{rx}$ into the given differential equation yields $e^{rx}(r^3 + 3r^2 - 4r - 12) = 0$, so that we will have a solution provided that r satisfies $r^3 + 3r^2 - 4r - 12 = 0$, that is, $(r - 2)(r + 2)(r + 3) = 0$. Consequently, three solutions to the given differential equation are $y_1(x) = e^{2x}$, $y_2(x) = e^{-2x}$, $y_3(x) = e^{-3x}$. Further, the Wronskian of these solutions is

$$W[y_1, y_2, y_3] = \begin{vmatrix} e^{2x} & e^{-2x} & e^{-3x} \\ 2e^{2x} & -2e^{-2x} & -3e^{-3x} \\ 4e^{2x} & 4e^{-2x} & 9e^{-3x} \end{vmatrix} = -20e^{-3x}.$$

Since the Wronskian is never zero, the solutions are linearly independent on any interval. Hence the general solution to the differential equation is $y(x) = c_1e^{2x} + c_2e^{-2x} + c_3e^{-3x}$.

26. Substituting $y(x) = e^{rx}$ into the given differential equation yields $e^{rx}(r^3 + 3r^2 - 18r - 40) = 0$, so that we will have a solution provided r satisfies $r^3 + 3r^2 - 18r - 40 = 0$, that is, $(r + 5)(r + 2)(r - 4) = 0$. Consequently, three solutions to the differential equation are $y_1(x) = e^{-5x}$, $y_2(x) = e^{-2x}$, $y_3(x) = e^{4x}$. Further, the Wronskian of the solution is

$$W[y_1, y_2, y_3] = \begin{vmatrix} e^{-5x} & e^{-2x} & e^{4x} \\ -5e^{-5x} & -2e^{-2x} & 4e^{4x} \\ 25e^{-5x} & 4e^{-2x} & 16e^{4x} \end{vmatrix} = 162e^{-3x}.$$

Since the Wronskian is never zero, the solutions are linearly independent on any interval. Hence the general solution to the differential equation is $y(x) = c_1e^{-5x} + c_2e^{-2x} + c_3e^{4x}$.

27. Given $y''' - y'' - 2y' = 0$ then $r^3 - r^2 - 2r = 0 \implies r(r^2 - r - 2) = 0 \implies r \in \{-1, 0, 2\} \implies y(x) = c_1e^{-x} + c_2 + c_3e^{2x}$.

28. Given $y''' + y'' - 10y' + 8y = 0$ then $r^3 + r^2 - 10r + 8 = 0 \implies (r - 2)(r - 1)(r + 4) = 0 \implies r \in \{-4, 1, 2\} \implies y(x) = c_1e^{-4x} + c_2e^x + c_3e^{2x}$.

29. Given $y^{(iv)} - 2y''' - y'' + 2y' = 0$ then $r^4 - 2r^3 - r^2 - 2r = 0 \implies r(r^3 - 2r^2 - r - 2) = 0 \implies r(r - 2)(r - 1)(r + 1) = 0 \implies r \in \{-1, 0, 1, 2\} \implies y(x) = c_1e^{-x} + c_2 + c_3e^x + c_4e^{2x}$.

30. Given $y^{(iv)} - 13y'' + 36y = 0$ then $r^4 - 13r^2 + 36 = 0 \implies (r^2 - 9)(r^2 - 4) = 0 \implies r^2 \in \{4, 9\} \implies r \in \{-3, -2, 2, 3\} \implies y(x) = c_1e^{-3x} + c_2e^{-2x} + c_3e^{2x} + c_4e^{3x}$.

31. Given $x^2y'' + 3xy' - 8y = 0$, the trial solution gives $x^2r(r - 1)x^{r-2} + 3xrx^{r-1} - 8x^r = 0$, or $x^r[r(r - 1) + 3r - 8] = 0$. Therefore, $r^2 + 2r - 8 = 0$, which factors as $(r + 4)(r - 2) = 0$. Therefore, $r = -4$ or $r = 2$. Hence, we obtain the solutions $y_1(x) = x^{-4}$ and $y_2(x) = x^2$. Furthermore,

$$W[x^{-4}, x^2] = (x^{-4})(2x) - (-4x^{-5})(x^2) = 6x^{-3} \neq 0,$$

so that $\{x^{-4}, x^2\}$ is a linearly independent set of solutions to the given differential equation on $(0, \infty)$. Consequently, from Theorem 6.1.3, the general solution is given by $y(x) = c_1x^{-4} + c_2x^2$.

32. Given $2x^2y'' + 5xy' + y = 0$, the trial solution gives $2x^2r(r - 1)x^{r-2} + 5xrx^{r-1} + x^r = 0$, or $x^r[2r(r - 1) + 5r + 1] = 0$. Therefore, $2r^2 + 3r + 1 = 0$, which factors as $(2r + 1)(r + 1) = 0$. Therefore,

$r = -\frac{1}{2}$ and $r = -1$. Hence, we obtain the solutions $y_1(x) = x^{-\frac{1}{2}}$ and $y_2(x) = x^{-1}$. Furthermore,

$$W[x^{-\frac{1}{2}}, x^{-1}] = (x^{-\frac{1}{2}})(-\frac{1}{x^2}) - (-\frac{1}{2}x^{-\frac{3}{2}})(x^{-1}) = -\frac{1}{2}x^{-\frac{5}{2}} \neq 0,$$

so that $\{x^{-\frac{1}{2}}, x^{-1}\}$ is a linearly independent set of solutions to the given differential equation on $(0, \infty)$. Consequently, from Theorem 6.1.3, the general solution is given by $y(x) = c_1x^{-\frac{1}{2}} + c_2x^{-1}$.

33. Substituting $y(x) = x^r$ into the given differential equation yields $x^r[r(r-1)(r-2) + r(r-1) - 2r + 2] = 0$, so that r must satisfy $(r-1)(r-2)(r+1) = 0$. It follows that three solutions to the differential equation are $y_1(x) = x$, $y_2(x) = x^2$, $y_3(x) = x^{-1}$. Further, the Wronskian of these solutions is

$$W[y_1, y_2, y_3] = \begin{vmatrix} x & x^2 & x^{-1} \\ 1 & 2x & -x^{-2} \\ 0 & 2 & 2x^{-3} \end{vmatrix} = 6x^{-1}.$$

Since the Wronskian is nonzero on $(0, \infty)$, the solutions are linearly independent on this interval. Consequently, the general solution to the differential equation is $y(x) = c_1x + c_2x^2 + c_3x^{-1}$.

34. Given $x^3y''' + 3x^2y'' - 6xy' = 0$, the trial solution gives

$$x^3r(r-1)(r-2)x^{r-3} + 3x^2r(r-1)x^{r-2} - 6xr x^{r-1} = 0,$$

or

$$x^r[r(r-1)(r-2) + 3r(r-1) - 6r] = 0.$$

Therefore, $r(r-1)(r-2) + 3r(r-1) - 6r = 0$, or $r[(r-1)(r-2) + 3(r-1) - 6] = 0$. Therefore, $r[r^2 - 7] = 0$, so that $r = 0$ or $r = \pm\sqrt{7}$. Hence, we obtain the solutions $y_1(x) = 1$, $y_2(x) = x^{\sqrt{7}}$, and $y_3(x) = x^{-\sqrt{7}}$. Furthermore,

$$\begin{aligned} W[1, x^{\sqrt{7}}, x^{-\sqrt{7}}] &= \begin{vmatrix} 1 & x^{\sqrt{7}} & x^{-\sqrt{7}} \\ 0 & \sqrt{7}x^{\sqrt{7}-1} & -\sqrt{7}x^{-\sqrt{7}-1} \\ 0 & \sqrt{7}(\sqrt{7}-1)x^{\sqrt{7}-2} & -\sqrt{7}(-\sqrt{7}-1)x^{-\sqrt{7}-2} \end{vmatrix} \\ &= -7(-\sqrt{7}-1)x^{-3} + 7(\sqrt{7}-1)x^{-3} \\ &= 14\sqrt{7}x^{-3} \neq 0, \end{aligned}$$

so that $\{1, x^{\sqrt{7}}, x^{-\sqrt{7}}\}$ is a linearly independent set of solutions to the given differential equation on $(0, \infty)$. Consequently, from Theorem 6.1.3, the general solution is given by $y(x) = c_1 + c_2x^{\sqrt{7}} + c_3x^{-\sqrt{7}}$.

35. To determine a particular solution of the form $y_p(x) = A_0e^{3x}$, we substitute this solution into the differential equation:

$$(A_0e^{3x})'' + (A_0e^{3x})' - 6(A_0e^{3x}) = 18e^{3x}.$$

Therefore

$$9A_0e^{3x} + 3A_0e^{3x} - 6A_0e^{3x} = 18e^{3x},$$

which forces $A_0 = 3$. Therefore, $y_p(x) = 3e^{3x}$.

To obtain the general solution, we need to find the complementary function $y_c(x)$, the solution to the associated homogeneous differential equation: $y'' + y' - 6y = 0$. Seeking solutions of the form $y(x) = e^{rx}$, we obtain $r^2e^{rx} + re^{rx} - 6e^{rx} = 0$, or $e^{rx}(r^2 + r - 6) = 0$. Therefore, $r^2 + r - 6 = 0$, which factors as

$(r+3)(r-2) = 0$. Hence, $r = -3$ or $r = 2$. Therefore, we obtain the solutions $y_1(x) = e^{-3x}$ and $y_2(x) = e^{2x}$. Since

$$W[e^{-3x}, e^{2x}] = (e^{-3x})(2e^{2x}) - (-3e^{-3x})(e^{2x}) = 2e^{-x} + 3e^{-x} = 5e^{-x} \neq 0,$$

$\{e^{-3x}, e^{2x}\}$ is linearly independent. Therefore, the complementary function is $y_c(x) = c_1e^{-3x} + c_2e^{2x}$. By Theorem 6.1.7, the general solution to the differential equation is

$$y(x) = c_1e^{-3x} + c_2e^{2x} + 3e^{3x}.$$

36. Substituting $y(x) = A_0 + A_1x + A_2x^2$ into the differential equation yields

$$(A_0 + A_1x + A_2x^2)'' + (A_0 + A_1x + A_2x^2)' - 2(A_0 + A_1x + A_2x^2) = 4x^2$$

or

$$(2A_2 + A_1 - 2A_0) + (2A_2 - 2A_1)x - 2A_2x^2 = 4x^2.$$

Equating the powers of x on each side of this equation yields

$$2A_2 + A_1 - 2A_0 = 0, \quad 2A_2 - 2A_1 = 0, \quad -2A_2 = 4.$$

Solving for A_0, A_1 , and A_2 , we find that $A_0 = -3$, $A_1 = -2$, and $A_2 = -2$. Thus, $y_p(x) = -3 - 2x - 2x^2$.

To obtain the general solution, we need to find the complementary function $y_c(x)$, the solution to the associated homogeneous differential equation: $y'' + y' - 2y = 0$. Seeking solutions of the form $y(x) = e^{rx}$, we obtain $r^2e^{rx} + re^{rx} - 2e^{rx} = 0$, or $e^{rx}(r^2 + r - 2) = 0$. Therefore, $r^2 + r - 2 = 0$, which factors as $(r+2)(r-1) = 0$. Hence, $r = -2$ or $r = 1$. Therefore, we obtain the solutions $y_1(x) = e^{-2x}$ and $y_2(x) = e^x$. Since

$$W[e^{-2x}, e^x] = (e^{-2x})(e^x) - (-2e^{-2x})(e^x) = 3e^{-x} \neq 0,$$

$\{e^{-2x}, e^x\}$ is linearly independent. Therefore, the complementary function is $y_c(x) = c_1e^{-2x} + c_2e^x$. By Theorem 6.1.7, the general solution to the differential equation is

$$y(x) = c_1e^{-2x} + c_2e^x - 3 - 2x - 2x^2.$$

37. Substituting $y(x) = A_0e^{3x}$ into the given differential equation yields $A_0e^{3x}(27 + 18 - 3 - 2) = 4e^{3x}$, so that $A_0 = \frac{1}{10}e^{3x}$. Hence, a particular solution to the differential equation is $y_p(x) = \frac{1}{10}e^{3x}$. To determine the general solution we need to solve the associated homogeneous differential equation $y'' + 2y' - 2y = 0$. We try for solutions of the form $y(x) = e^{rx}$. Substituting into the homogeneous differential equation gives $e^{rx}(r^2 + 2r - 2) = 0$, so that we choose r to satisfy $r^2 + 2r - 2 = 0$, or equivalently, $(r+2)(r-1)(r+1) = 0$. It follows that three solutions to the differential equation are $y_1(x) = e^{-2x}$, $y_2(x) = e^x$, $y_3(x) = e^{-x}$. Further, the Wronskian of these solutions is

$$W[y_1, y_2, y_3] = \begin{vmatrix} e^{-2x} & e^x & e^{-x} \\ -2e^{-2x} & e^x & -e^{-x} \\ 4e^{-2x} & e^x & e^{-x} \end{vmatrix} = -6e^{-2x}.$$

Since the Wronskian is nonzero, the solutions are linearly independent on any interval. It follows that the complementary function for the given nonhomogeneous differential equation is $y_c(x) = c_1e^{-2x} + c_2e^x + c_3e^{-x}$, so the general solution to the differential equation is $y(x) = y_c(x) + y_p(x) = c_1e^{-2x} + c_2e^x + c_3e^{-x} + \frac{1}{10}e^{3x}$.

38. Substituting $y(x) = A_0 e^{-2x}$ into the differential equation yields

$$-8A_0 e^{-2x} + 4A_0 e^{-2x} + 20A_0 e^{-2x} + 8A_0 e^{-2x} = e^{-2x}.$$

Therefore, $24A_0 = 1$. Hence, $A_0 = \frac{1}{24}$. Thus, $y_p(x) = \frac{1}{24} e^{-2x}$.

To obtain the general solution, we need to find the complementary function $y_c(x)$, the solution to the associated homogeneous differential equation: $y''' + y'' - 10y' + 8y = 0$. Seeking solutions of the form $y(x) = e^{rx}$, we obtain $r^3 e^{rx} + r^2 e^{rx} - 10r e^{rx} + 8e^{rx} = 0$, or $e^{rx}(r^3 + r^2 - 10r + 8) = 0$. Therefore, $r^3 + r^2 - 10r + 8 = 0$, which has roots $r = 1$, $r = 2$, and $r = -4$. Therefore, we obtain the solutions $y_1(x) = e^x$, $y_2(x) = e^{2x}$, and $y_3(x) = e^{-4x}$. Since

$$W[e^x, e^{2x}, e^{-4x}] = \begin{vmatrix} e^x & e^{2x} & e^{-4x} \\ e^x & 2e^{2x} & -4e^{-4x} \\ e^x & 4e^{2x} & 16e^{-4x} \end{vmatrix} = 30e^{-x} \neq 0,$$

$\{e^x, e^{2x}, e^{-4x}\}$ is linearly independent. Thus, the complementary function is $y_c(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{-4x}$. By Theorem 6.1.7, the general solution to the differential equation is

$$y(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{-4x} + \frac{1}{24} e^{-2x}.$$

39. Substituting $y(x) = A_0 e^{4x}$ into the differential equation yields

$$64A_0 e^{4x} + 80A_0 e^{4x} + 24A_0 e^{4x} = -3e^{4x}.$$

Therefore, $168A_0 = -3$. Hence, $A_0 = -\frac{1}{56}$. Thus, $y_p(x) = -\frac{1}{56} e^{4x}$.

To obtain the general solution, we need to find the complementary function $y_c(x)$, the solution to the associated homogeneous differential equation: $y''' + 5y'' + 6y' = 0$. Seeking solutions of the form $y(x) = e^{rx}$, we obtain $r^3 e^{rx} + 5r^2 e^{rx} + 6r e^{rx} = 0$, or $e^{rx}(r^3 + 5r^2 + 6r) = 0$. Therefore, $r^3 + 5r^2 + 6r = 0$, which has roots $r = 0$, $r = -2$, and $r = -3$. Therefore, we obtain the solutions $y_1(x) = 1$, $y_2(x) = e^{-2x}$, and $y_3(x) = e^{-3x}$. Since

$$W[1, e^{-2x}, e^{-3x}] = \begin{vmatrix} 1 & e^{-2x} & e^{-3x} \\ 0 & -2e^{-2x} & -3e^{-3x} \\ 0 & 4e^{-2x} & 9e^{-3x} \end{vmatrix} = -6e^{-5x} \neq 0,$$

$\{1, e^{-2x}, e^{-3x}\}$ is linearly independent. Therefore, the complementary function is $y_c(x) = c_1 + c_2 e^{-2x} + c_3 e^{-3x}$. By Theorem 6.1.7, the general solution to the differential equation is

$$y(x) = c_1 + c_2 e^{-2x} + c_3 e^{-3x} - \frac{1}{56} e^{4x}.$$

40. Prior to the statement of Theorem 6.1.3 it was shown that the set of all solutions forms a vector space. We now show that the dimension of this solution space is n , by constructing a basis. Let $y_1, y_2, \dots, y_n$ be the unique solutions of the n initial-value problems:

$$a_0(x)y_i^{(n)} + a_1(x)y_i^{(n-1)} + \dots + a_{n-1}(x)y_i' + a_n(x)y_i = 0,$$

$$y_i^{(k-1)}(x_0) = \delta_{ik}, \quad k = 1, 2, \dots, n,$$

respectively. The Wronskian of these functions at x_0 is $W[y_1, y_2, \dots, y_n](x_0) = \det[I_n] = 1 \neq 0$ so that the solutions are linearly independent on I . We now show that they span the solution space.

Let $u(x)$ be any solution of the differential equation on I , and suppose that $u(x_0) = u_1$, $u'(x_0) = u_2, \dots, u^{(n-1)}(x_0) = u_n$, where $u_1, u_2, \dots, u_n$ are constants. It follows that $y = u(x)$ is the unique solution to the initial-value problem:

$$Ly = 0,$$

$$y(x_0) = u_1, \quad y'(x_0) = u_2, \dots, y^{(n-1)}(x_0) = u_n.$$

However, if we define $w(x) = u_1 y_1(x) + u_2 y_2(x) + \dots + u_n y_n(x)$, then $w(x)$ also satisfies this initial value problem. Thus, by uniqueness, we must have $u(x) = w(x)$, that is, $u(x) = u_1 y_1(x) + u_2 y_2(x) + \dots + u_n y_n(x)$. Thus we have shown that $\{y_1, y_2, \dots, y_n\}$ forms a basis for the solution space and hence the dimension of this solution space is n .

41. Consider the linear system

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) + \dots + c_n y_n(x_0) &= 0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) + \dots + c_n y_n'(x_0) &= 0 \\ &\vdots \\ c_1 y_1^{(n-1)}(x_0) + c_2 y_2^{(n-1)}(x_0) + \dots + c_n y_n^{(n-1)}(x_0) &= 0, \end{aligned}$$

where we are solving for $c_1, c_2, \dots, c_n$. The determinant of the matrix of coefficients of this system is $W[y_1, y_2, \dots, y_n](x_0) = 0$, so that the system has non-trivial solutions. Let $(\alpha_1, \alpha_2, \dots, \alpha_n)$ be one such non-trivial solution for $(c_1, c_2, \dots, c_n)$. Therefore, not all of the α_i are zero. Define the function $u(x)$ by $u(x) = \alpha_1 y_1(x) + \alpha_2 y_2(x) + \dots + \alpha_n y_n(x)$. It follows that $y = u(x)$ satisfies the initial-value problem:

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0$$

$$y(x_0) = 0, \quad y'(x_0) = 0, \dots, y^{(n-1)}(x_0) = 0.$$

However, $y(x) = 0$ also satisfies the above initial value problem and hence, by uniqueness, we must have $u(x) = 0$, that is, $\alpha_1 y_1(x) + \alpha_2 y_2(x) + \dots + \alpha_n y_n(x) = 0$, where not all of the α_i are zero. Thus, the functions $y_1, y_2, \dots, y_n$ are linearly dependent on I .

42. Let $\mathbf{v}_p$ be a particular solution to the equation (6.1.14), and let $\mathbf{v}$ be any solution to this equation. Then

$$T(\mathbf{v}_p) = \mathbf{w} \quad \text{and} \quad T(\mathbf{v}) = \mathbf{w}.$$

Subtracting these two equations gives

$$T(\mathbf{v}) - T(\mathbf{v}_p) = \mathbf{0},$$

or equivalently, since T is a linear transformation,

$$T(\mathbf{v} - \mathbf{v}_p) = \mathbf{0}.$$

But this latter equation implies that the vector $\mathbf{v} - \mathbf{v}_p$ is an element of $\text{Ker}(T)$ and therefore can be written as

$$\mathbf{v} - \mathbf{v}_p = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n,$$

for some scalars $c_1, c_2, \dots, c_n$. Hence,

$$\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n + \mathbf{v}_p.$$

43. Let y_1 and y_2 belong to $C^n(I)$, and let c be a scalar. We must show that

$$L(y_1 + y_2) = L(y_1) + L(y_2) \quad \text{and} \quad L(cy_1) = cL(y_1).$$

We have

$$\begin{aligned} L(y_1 + y_2) &= (D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)(y_1 + y_2) \\ &= D^n(y_1 + y_2) + a_1D^{n-1}(y_1 + y_2) + \cdots + a_{n-1}D(y_1 + y_2) + a_n(y_1 + y_2) \\ &= (D^n y_1 + a_1D^{n-1}y_1 + \cdots + a_{n-1}Dy_1 + a_n y_1) + (D^n y_2 + a_1D^{n-1}y_2 + \cdots + a_{n-1}Dy_2 + a_n y_2) \\ &= (D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)(y_1) + (D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)(y_2) \\ &= L(y_1) + L(y_2), \end{aligned}$$

and

$$\begin{aligned} L(cy_1) &= (D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)(cy_1) \\ &= D^n(cy_1) + a_1D^{n-1}(cy_1) + \cdots + a_{n-1}D(cy_1) + a_n(cy_1) \\ &= cD^n(y_1) + ca_1D^{n-1}(y_1) + \cdots + ca_{n-1}D(y_1) + ca_n(y_1) \\ &= c(D^n(y_1) + a_1D^{n-1}(y_1) + \cdots + a_{n-1}D(y_1) + a_n(y_1)) \\ &= c(D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)(y_1) \\ &= cL(y_1). \end{aligned}$$

Solutions to Section 6.2

True-False Review:

1. FALSE. Even if the auxiliary polynomial fails to have n distinct roots, the differential equation still has n linearly independent solutions. For example, if $L = D^2 + 2D + 1$, then the differential equation $Ly = 0$ has auxiliary polynomial with (repeated) roots $r = -1, -1$. Yet we do have two linearly independent solutions $y_1(x) = e^{-x}$ and $y_2(x) = xe^{-x}$ to the differential equation.

2. FALSE. Theorem 6.2.1 only applies to *polynomial* differential operators. However, in general, many counterexamples can be given. For example, note that

$$(xD - Dx)(x) = xD(x) - D(x^2) = x - 2x = (-1)(x).$$

Therefore, $xD - Dx = -1$. Setting $L_1 = x$ and $L_2 = D$, we therefore see that $L_1L_2 \neq L_2L_1$ in this example.

3. TRUE. This is really just the statement that a polynomial of degree n always has n roots, with multiplicities counted.

4. TRUE. Since 0 is a root of multiplicity four, each term of the polynomial differential operator must contain a factor of D^4 , so that any polynomial of degree three or less becomes zero after taking four (or more) derivatives. Therefore, for a *homogeneous* differential equation of this type, a polynomial of degree three or less must be a solution.

5. FALSE. Note that $r = 0$ is a root of the auxiliary polynomial, but only of multiplicity 1. The expression $c_1 + c_2x$ in the solution reflects $r = 0$ as a root of multiplicity 2.

6. TRUE. The roots of the auxiliary polynomial are $r = -3, -3, 5i, -5i$. The portion of the solution corresponding to the repeated root $r = -3$ is $c_1e^{-3x} + c_2xe^{-3x}$, and the portion of the solution corresponding to the complex conjugate pair $r = \pm 5i$ is $c_3 \cos 5x + c_4 \sin 5x$.

7. TRUE. The roots of the auxiliary polynomial are $r = 2 \pm i, 2 \pm i$. The terms corresponding to the first pair $2 \pm i$ are $c_1 e^{2x} \cos x$ and $c_2 e^{2x} \sin x$, and the repeated root gives two more terms: $c_3 x e^{2x} \cos x$ and $c_4 x e^{2x} \sin x$.

8. FALSE. Many counterexamples can be given. For instance, if $P(D) = D - 1$, then the general solution is $y(x) = ce^x$. However, the differential equation $(P(D))^2 y = (D - 1)^2 y$ has auxiliary equation with roots $r = 1, 1$ and general solution $z(x) = c_1 e^x + c_2 x e^x \neq xy(x)$.

Problems:

1. The auxiliary polynomial is $P(r) = r^2 + 2r - 3 = (r + 3)(r - 1)$. Therefore, the auxiliary equation has roots $r = -3$ and $r = 1$. Therefore, two linearly independent solutions to the given differential equation are

$$y_1(x) = e^{-3x} \quad \text{and} \quad y_2(x) = e^x.$$

By Theorem 6.1.3, the solution space to this differential equation is 2-dimensional, and hence $\{e^{-3x}, e^x\}$ forms a basis for the solution space.

2. The auxiliary polynomial is $P(r) = r^2 + 6r + 9 = (r + 3)^2$. Therefore, the auxiliary equation has roots $r = -3$ (with multiplicity 2). Therefore, two linearly independent solutions to the given differential equation are

$$y_1(x) = e^{-3x} \quad \text{and} \quad y_2(x) = x e^{-3x}.$$

By Theorem 6.1.3, the solution space to this differential equation is 2-dimensional, and hence $\{e^{-3x}, x e^{-3x}\}$ forms a basis for the solution space.

3. The auxiliary polynomial is $P(r) = r^2 - 6r + 25$. According to the quadratic equation, the auxiliary equation has roots $r = 3 \pm 4i$. Therefore, two linearly independent solutions to the given differential equation are

$$y_1(x) = e^{3x} \cos 4x \quad \text{and} \quad y_2(x) = e^{3x} \sin 4x.$$

By Theorem 6.1.3, the solution space to this differential equation is 2-dimensional, and hence $\{e^{3x} \cos 4x, e^{3x} \sin 4x\}$ forms a basis for the solution space.

4. The general vector in S takes the form $y(x) = c(\sin 4x + 5 \cos 4x)$. The full solution space for this differential equation is 2-dimensional by Theorem 6.1.3. Note that $\sin 4x$ belongs to the solution space and is linearly independent from $\sin 4x + 5 \cos 4x$ since

$$W[\sin 4x + 5 \cos 4x, \sin 4x] = 20 \neq 0.$$

Therefore, we can extend the basis for S to a basis for the entire solution space with $\{\sin 4x + 5 \cos 4x, \sin 4x\}$. Many other extensions are also possible, of course.

5. We have $r^2 - r - 2 = 0 \implies r \in \{-1, 2\} \implies y(x) = c_1 e^{-x} + c_2 e^{2x}$.

6. We have $r^2 - 6r + 9 = 0 \implies r \in \{3, 3\} \implies y(x) = c_1 e^{3x} + c_2 x e^{3x}$.

7. We have $r^2 + 6r + 25 = 0 \implies r \in \{-3 - 4i, -3 + 4i\} \implies y(x) = c_1 e^{-3x} \cos 4x + c_2 e^{-3x} \sin 4x$.

8. We have $(r + 1)(r - 5) = 0 \implies r \in \{-1, 5\} \implies y(x) = c_1 e^{-x} + c_2 e^{5x}$.

9. We have $(r + 2)^2 = 0 \implies r \in \{-2, -2\} \implies y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$.

10. We have $r^2 - 6r + 34 = 0 \implies r \in \{3 - 5i, 3 + 5i\} \implies y(x) = c_1 e^{3x} \cos 5x + c_2 e^{3x} \sin 5x$.
11. We have $r^2 + 10r + 25 = 0 \implies r \in \{-5, -5\} \implies y(x) = c_1 e^{-5x} + c_2 x e^{-5x}$.
12. We have $r^2 - 2 = 0 \implies r \in \{-\sqrt{2}, \sqrt{2}\} \implies y(x) = c_1 e^{-\sqrt{2}x} + c_2 e^{\sqrt{2}x}$.
13. We have $r^2 + 8r + 20 = 0 \implies r \in \{-4 - 2i, -4 + 2i\} \implies y(x) = c_1 e^{-4x} \cos 2x + c_2 e^{-4x} \sin 2x$.
14. We have $r^2 + 2r + 2 = 0 \implies r \in \{-1 - i, -1 + i\} \implies y(x) = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$.
15. We have $(r - 4)(r + 2) = 0 \implies r \in \{-2, 4\} \implies y(x) = c_1 e^{-2x} + c_2 e^{4x}$.
16. We have $r^2 - 14r + 58 = 0 \implies r \in \{7 - 3i, 7 + 3i\} \implies y(x) = c_1 e^{7x} \cos 3x + c_2 e^{7x} \sin 3x$.
17. We have $r^3 - r^2 + r - 1 = 0 \implies r \in \{1, i, -i\} \implies y(x) = c_1 e^x + c_2 \cos x + c_3 \sin x$.
18. We have $r^3 - 2r^2 - 4r + 8 = 0 \implies r \in \{-2, 2, 2\} \implies y(x) = c_1 e^{-2x} + (c_2 + c_3 x) e^{2x}$.
19. We have $(r - 2)(r^2 - 16) = 0 \implies r \in \{2, 4, -4\} \implies y(x) = c_1 e^{2x} + c_2 e^{4x} + c_3 e^{-4x}$.
20. We have $(r^2 + 2r + 10)^2 = 0 \implies r \in \{-1 + 3i, -1 + 3i, -1 - 3i, -1 - 3i\} \implies y(x) = e^{-x} [c_1 \cos 3x + c_2 \sin 3x + x(c_4 \cos 3x + c_5 \sin 3x)]$.
21. We have $(r^2 + 4)^2(r + 1) = 0 \implies r \in \{2i, 2i, -2i, -2i, -1\} \implies y(x) = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x + x(c_4 \cos 2x + c_5 \sin 2x)$.
22. We have $(r^2 + 3)(r + 1) = 0 \implies r \in \{-1, -1, -i\sqrt{3}, i\sqrt{3}\} \implies y(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 \cos(\sqrt{3}x) + c_4 \sin(\sqrt{3}x)$.
23. We have $r^2(r - 1) = 0 \implies r \in \{0, 0, 1\} \implies y(x) = c_1 + c_2 x + c_3 e^x$.
24. We have $r^4 - 8r^2 + 16 = (r - 2)^2(r + 2)^2 = 0 \implies r \in \{2, 2, -2, -2\} \implies y(x) = e^{2x}(c_1 + c_2 x) + e^{-2x}(c_3 + c_4 x)$.
25. We have $r^4 - 16 = 0 \implies r \in \{2, -2, 2i, -2i\} \implies y(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$.
26. We have $r^3 + 8r^2 + 22r + 20 = 0 \implies r \in \{-2, -3 + i, -3 - i\} \implies y(x) = c_1 e^{-2x} + e^{-3x}(c_2 \cos x + c_3 \sin x)$.
27. We have $r^4 - 16r^2 + 40r - 25 = 0 \implies r \in \{1, -5, 2 + i, 2 - i\} \implies y(x) = c_1 e^x + c_2 e^{-5x} + e^{2x}(c_3 \cos x + c_4 \sin x)$.
28. We have $(r - 1)^3(r^2 + 9) = 0 \implies r \in \{1, 1, 1, -3i, 3i\} \implies y(x) = e^x(c_1 + c_2 x + c_3 x^2) + c_4 \cos 3x + c_5 \sin 3x$.
29. We have $(r^2 - 2r + 2)^2(r^2 - 1) = 0 \implies r \in \{1 + i, 1 + i, 1 - i, 1 - i, 1, -1\} \implies y(x) = e^x(c_1 \cos x + c_2 \sin x) + x e^x(c_3 \cos x + c_4 \sin x) + c_5 e^{-x} + c_6 e^x$.
30. We have $(r + 3)(r - 1)(r + 5)^3 = 0 \implies r \in \{-3, 1, -5, -5, -5\} \implies y(x) = c_1 e^{-3x} + c_2 e^x + e^{-5x}(c_3 + c_4 x + c_5 x^2)$.
31. We have $(r^2 + 9)^3 = 0 \implies r \in \{3i, 3i, 3i, -3i, -3i, -3i\} \implies y(x) = c_1 \cos 3x + c_2 \sin 3x + x(c_3 \cos 3x + c_4 \sin 3x) + x^2(c_5 \cos 3x + c_6 \sin 3x)$.

32. We have $r^2 - 8r + 16 = 0 \implies r \in \{4, 4\} \implies y(x) = c_1 e^{4x} + c_2 x e^{4x}$. Now $2 = y(0) = c_1$ and $7 = y'(0) = 4c_1 + c_2$. Therefore, $c_1 = 2$ and $c_2 = -1$. Hence, the solution to this initial-value problem is $y(x) = 2e^{4x} - xe^{4x}$.

33. We have $r^2 - 4r + 5 = 0 \implies r \in \{2 + i, 2 - i\} \implies y(x) = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x$. Now $3 = y(0) = c_1$ and $5 = y'(0) = 2c_1 + c_2$. Therefore, $c_1 = 3$ and $c_2 = -1$. Hence, the solution to this initial-value problem is $y(x) = 3e^{2x} \cos x - e^{2x} \sin x$.

34. We have $r^3 - r^2 + r - 1 = 0 \implies r \in \{1, i, -i\} \implies y(x) = c_1 e^x + c_2 \cos x + c_3 \sin x$. Then since $y(0) = 0$, we have $c_1 + c_2 = 0$. Moreover, since $y'(0) = 1$, we have $c_1 + c_3 = 1$. Finally, $y''(0) = 2$ implies that $c_1 - c_2 = 2$. Solving these equations yields $c_1 = 1$, $c_2 = -1$, $c_3 = 0$. Hence, the solution to this initial-value problem is $y(x) = e^x - \cos x$.

35. We have $r^3 + 2r^2 - 4r - 8 = (r - 2)(r + 2)^2 = 0 \implies r \in \{2, -2, -2\} \implies y(x) = c_1 e^{2x} + c_2 e^{-2x} + c_3 x e^{-2x}$. Then since $y(0) = 0$, we have $c_1 + c_2 = 0$. Moreover, since $y'(0) = 6$, we have $2c_1 - 2c_2 + c_3 = 6$. Further, $y''(0) = 8$ implies that $4c_1 + 4c_2 - 4c_3 = 8$. Solving these equations yields $c_1 = 2$, $c_2 = c_3 = -2$. Hence, the solution to this initial-value problem is $y(x) = 2(e^{2x} - e^{-2x} - x e^{-2x})$.

36. Given $m > 0$ and $k > 0$, we have auxiliary polynomial $P(r) = r^2 - 2mr + (m^2 + k^2) = 0 \implies r = m \pm ki$. Therefore, $y(x) = e^{mx}(c_1 \cos kx + c_2 \sin kx)$. Differentiating, we obtain $y'(x) = m e^{mx}(c_1 \cos kx + c_2 \sin kx) + e^{mx}(-c_1 k \sin kx + c_2 k \cos kx)$. Now $0 = y(0) = c_1$ and $k = y'(0) = c_2 k$. Therefore, $c_1 = 0$ and $c_2 = 1$. Hence, the solution to this initial-value problem is $y(x) = e^{mx} \sin kx$.

37. Given $m > 0$ and $k > 0$, we have auxiliary polynomial $P(r) = r^2 - 2mr + (m^2 - k^2) = 0 \implies r = m \pm k$. Therefore, the general solution to this differential equation is $y(x) = a e^{(m+k)x} + b e^{(m-k)x} = e^{mx}(a e^{kx} + b e^{-kx})$. Letting $a = \frac{c_1 + c_2}{2}$ and $b = \frac{c_1 - c_2}{2}$ in the last equality gives

$$\begin{aligned} y(x) &= e^{mx} \left(\frac{c_1 + c_2}{2} e^{kx} + \frac{c_1 - c_2}{2} e^{-kx} \right) \\ &= e^{mx} \left(c_1 \frac{e^{kx} + e^{-kx}}{2} + c_2 \frac{e^{kx} - e^{-kx}}{2} \right) \\ &= e^{mx} (c_1 \cosh kx + c_2 \sinh kx). \end{aligned}$$

38. The auxiliary polynomial is $P(r) = r^2 + 2cr + k^2$. The quadratic formula supplies the roots of $P(r) = 0$: $r = -c \pm \sqrt{c^2 - k^2}$.

(a) If $c^2 < k^2$, then $c^2 - k^2 < 0$ and the roots above are complex. We can write $r = -c \pm \omega i$, where $\omega = \sqrt{k^2 - c^2}$. Thus,

$$y(t) = e^{-ct} (c_1 \cos \omega t + c_2 \sin \omega t)$$

and

$$y'(t) = e^{-ct} [(\omega c_2 - c c_1) \cos \omega t - (c c_2 + \omega c_1) \sin \omega t].$$

Using the initial conditions $y(0) = y_0$ and $y'(0) = 0$, we find that $c_1 = y_0$ and $\omega c_2 - y_0 c = 0$, or $c_2 = \frac{y_0 c}{\omega}$. Therefore,

$$y(t) = \frac{y_0}{\omega} e^{-ct} (\omega \cos \omega t + c \sin \omega t).$$

(b) Since $\omega = \sqrt{k^2 - c^2}$, we find $k = \sqrt{\omega^2 + c^2}$ (note k is assumed positive, so we take the positive square root). Therefore, continuing from part (a), we have

$$\begin{aligned} y(t) &= \frac{y_0}{\omega} e^{-ct} (\omega \cos \omega t + c \sin \omega t) \\ &= k \frac{y_0}{\omega} e^{-ct} \left(\frac{\omega}{k} \cos \omega t + \frac{c}{k} \sin \omega t \right) \\ &= k \frac{y_0}{\omega} e^{-ct} \left(\frac{\omega}{\sqrt{\omega^2 + c^2}} \cos \omega t + \frac{c}{\sqrt{\omega^2 + c^2}} \sin \omega t \right). \end{aligned}$$

Now since

$$\left(\frac{\omega}{\sqrt{\omega^2 + c^2}} \right)^2 + \left(\frac{c}{\sqrt{\omega^2 + c^2}} \right)^2 = 1,$$

there exists ϕ such that

$$\cos \phi = \frac{c}{\sqrt{\omega^2 + c^2}} \quad \text{and} \quad \sin \phi = \frac{\omega}{\sqrt{\omega^2 + c^2}};$$

that is, $\phi = \tan^{-1} \left(\frac{\omega}{c} \right)$. Hence, $y(t)$ can be written as

$$y(t) = \frac{ky_0}{\omega} e^{-ct} (\sin \phi \cos \omega t + \cos \phi \sin \omega t) = \left(\frac{ky_0}{\omega} \right) e^{-ct} \sin(\omega t + \phi).$$

The amplitude is not constant, but is given by $\left(\frac{ky_0}{\omega} \right) e^{-ct}$. Since this tends to zero as $t \rightarrow \infty$, the vibrations tend to die out as time goes on. The damping is exponential and the motion is reasonable.

39. Let $u(x, y) = e^{x/\alpha} f(\xi)$, where $\xi = \beta x - \alpha y$, and assume that $\alpha > 0$ and $\beta > 0$.

(a) Let us compute the partial derivatives of u :

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^{x/\alpha} \frac{df}{d\xi} + \frac{1}{\alpha} e^{x/\alpha} f \\ &= e^{x/\alpha} \frac{df}{d\xi} \frac{\partial \xi}{\partial x} + \frac{1}{\alpha} e^{x/\alpha} f \\ &= e^{x/\alpha} \frac{df}{d\xi} \beta + \frac{1}{\alpha} e^{x/\alpha} f \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= e^{x/\alpha} \frac{d^2 f}{d\xi^2} \frac{\partial \xi}{\partial x} \beta + \frac{\beta}{\alpha} e^{x/\alpha} \frac{df}{d\xi} \beta + \frac{1}{\alpha} e^{x/\alpha} \frac{df}{d\xi} \frac{\partial \xi}{\partial x} + \frac{1}{\alpha^2} e^{x/\alpha} f \\ &= \beta^2 e^{x/\alpha} \frac{d^2 f}{d\xi^2} + 2 \frac{\beta}{\alpha} e^{x/\alpha} \frac{df}{d\xi} + e^{x/\alpha} \frac{f}{\alpha^2}. \end{aligned}$$

Moreover,

$$\begin{aligned} \frac{\partial u}{\partial y} &= e^{x/\alpha} \frac{\partial f}{\partial \xi} \\ &= e^{x/\alpha} \frac{df}{d\xi} \frac{d\xi}{dy} \\ &= e^{x/\alpha} \frac{df}{d\xi} (-\alpha) \\ &= -\alpha e^{x/\alpha} \frac{df}{d\xi} \end{aligned}$$

and

$$\begin{aligned}
 \frac{\partial^2 u}{\partial y^2} &= -\alpha e^{x/\alpha} \frac{\partial}{\partial y} \left(\frac{df}{d\xi} \right) \\
 &= -\alpha e^{x/\alpha} \frac{d^2 f}{d\xi^2} \frac{\partial \xi}{\partial y} \\
 &= -\alpha e^{x/\alpha} \frac{d^2 f}{d\xi^2} (-\alpha) \\
 &= \alpha^2 e^{x/\alpha} \frac{d^2 f}{d\xi^2}.
 \end{aligned}$$

From the formulas for $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{x/\alpha} \left(\beta^2 \frac{d^2 f}{d\xi^2} + 2 \frac{\beta}{\alpha} \frac{df}{d\xi} + \frac{f}{\alpha^2} + \alpha^2 \frac{d^2 f}{d\xi^2} \right).$$

Now if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, then the last equation becomes

$$\beta^2 \frac{d^2 f}{d\xi^2} + 2 \frac{\beta}{\alpha} \frac{df}{d\xi} + \frac{f}{\alpha^2} + \alpha^2 \frac{d^2 f}{d\xi^2} = 0$$

or

$$(\alpha^2 + \beta^2) \frac{d^2 f}{d\xi^2} + 2 \frac{\beta}{\alpha} \frac{df}{d\xi} + \frac{f}{\alpha^2} = 0.$$

Therefore,

$$\frac{d^2 f}{d\xi^2} + \frac{2\beta}{\alpha(\alpha^2 + \beta^2)} \frac{df}{d\xi} + \frac{f}{\alpha^2(\alpha^2 + \beta^2)} = 0.$$

Letting $p = \frac{\beta}{\alpha(\alpha^2 + \beta^2)}$ and $q = \frac{1}{\alpha^2 + \beta^2}$, the last equation reduces to $\frac{d^2 f}{d\xi^2} + 2p \frac{df}{d\xi} + \frac{q}{\alpha^2} f = 0$.

(b) The auxiliary equation associated with Equation (6.2.10) is $r^2 + 2pr + \frac{q}{\alpha^2} = 0$. The quadratic formula yields the roots

$$r = -p \pm \sqrt{p^2 - \frac{q}{\alpha^2}}.$$

Therefore,

$$r = -\frac{\beta}{\alpha(\alpha^2 + \beta^2)} \pm \left[\frac{\beta^2}{\alpha^2(\alpha^2 + \beta^2)^2} - \frac{\alpha^2 + \beta^2}{\alpha^2(\alpha^2 + \beta^2)^2} \right]^{1/2}.$$

Hence,

$$r = \frac{-\beta \pm i\alpha}{\alpha(\alpha^2 + \beta^2)} = -p \pm iq.$$

Therefore,

$$f(\xi) = e^{-p\xi} [A \sin q\xi + B \cos q\xi].$$

Since $u(x, y) = e^{x/\alpha} f(\xi)$, we conclude that

$$u(x, y) = e^{x/\alpha} [e^{-p\xi} (A \sin q\xi + B \cos q\xi)] = e^{x/\alpha - p\xi} (A \sin q\xi + B \cos q\xi).$$

40. The auxiliary equation is $r^2 + a_1r + a_2 = 0$.

(a) Depending on whether $r_1 = r_2$, we know that the solution to the differential equation takes one of these two forms:

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x} \quad \text{or} \quad y(x) = e^{r_1x}(c_1 + c_2x).$$

In order for $\lim_{x \rightarrow +\infty} y(x) = 0$, we must have that the roots are negative: $r_1 < 0$ and $r_2 < 0$.

(b) Complex roots guarantee a solution of the form $y(x) = e^{ax}(c_1 \cos bx + c_2 \sin bx)$. In order for $\lim_{x \rightarrow +\infty} y(x) = 0$, we must have $a < 0$. Therefore, the complex conjugate roots must have negative real part.

(c) By the quadratic formula, the roots of the auxiliary equation are

$$r = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}.$$

Case 1: If $a_1^2 - 4a_2 = 0$, then $r = -\frac{a_1}{2} < 0$ is a double root and $y(x) = e^{rx}(c_1 + c_2x)$ goes to zero as $x \rightarrow \infty$ (see part (a)).

Case 2: If $a_1^2 - 4a_2 > 0$, then both roots $\frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$ are negative; thus, the solution $y(x) = c_1e^{r_1x} + c_2e^{r_2x}$ goes to zero as $x \rightarrow \infty$ (see part (a)).

Case 3: If $a_1^2 - 4a_2 < 0$, then the roots of the auxiliary polynomial are complex conjugates with negative real part. Therefore, the conclusion follows from part (b).

(d) In this case, the auxiliary equation is $r^2 + a_1r = 0$, with roots $r = 0$ and $r = -a_1$. The general solution to the differential equation in this case is $y(x) = c_1 + c_2e^{-a_1x}$. Thus, $\lim_{x \rightarrow \infty} y(x) = c_1$, a constant.

(e) In this case, the auxiliary equation is $r^2 + a_2 = 0$, with roots $r = \pm\sqrt{-a_2} = \pm\sqrt{a_2}i$. The general solution to the differential equation in this case is $y(x) = c_1 \cos(\sqrt{a_2}x) + c_2 \sin(\sqrt{a_2}x)$, which is clearly bounded for all $x \in \mathbb{R}$, since the sine and cosine functions are both bounded between ± 1 in value.

41. All the roots must have a negative real part.

42. Multiplication of polynomial differential operators is identical to the multiplication of polynomials in general, an operation that is widely known to be commutative: $P(D)Q(D) = Q(D)P(D)$.

43. This is the special case of Problem 44 for $m = 1$. Therefore, the solution to Problem 44 can be applied to solve this exercise as well.

44. The given equations are linearly independent if and only if

$$\sum_{k=0}^{m-1} c_k x^k e^{ax} \cos bx + \sum_{k=0}^m d_k x^k e^{ax} \sin bx = 0 \implies c_k = 0 \quad \text{and} \quad d_k = 0$$

for all $k \in \{0, 1, 2, \dots, m\}$. If we let

$$P(x) = \sum_{k=0}^m c_k x^k \quad \text{and} \quad Q(x) = \sum_{k=0}^m d_k x^k$$

then the first equation can be written as

$$e^{ax}[P(x) \cos bx + Q(x) \sin bx] = 0,$$

which is equivalent to

$$P(x) \cos bx + Q(x) \sin bx = 0.$$

Now $x = \frac{n\pi}{b}$ where n is an integer implies $\cos bx = \pm 1$ and $\sin bx = 0$ so $P\left(\frac{n\pi}{b}\right) = 0$ for all integers n . Also if $x = \frac{(2n+1)\pi}{b}$, where n is an integer, then $\cos bx = 0$ and $\sin bx = \pm 1$ which implies that $Q\left(\frac{(2n+1)\pi}{b}\right) = 0$ for all integers n . This in turn implies that $P(x)$ and $Q(x)$ are identically zero so $c_k = 0$ and $d_k = 0$ for all $k \in \{0, 1, 2, \dots, m\}$. Thus the given functions are linearly independent.

45. $y(x) = c_1 e^{-5x} + c_2 e^{-7x} + c_3 e^{19x}.$

46. $y(x) = c_1 e^{4x} + c_2 e^{-4x} + e^{-2x}(c_3 \cos 3x + c_4 \sin 3x).$

47. $y(x) = e^{-5x/2}[c_1 \cos(\sqrt{11}x/2) + c_2 \sin(\sqrt{11}x/2)] + e^{-3x/2}[c_3 \cos(\sqrt{7}x/2) + c_4 \sin(\sqrt{7}x/2)].$

48. $y(x) = c_1 e^{-4x} + c_2 \cos 5x + c_3 \sin 5x + x(c_4 \cos 5x + c_5 \sin 5x).$

49. $y(x) = c_1 e^{-3x} + c_2 \cos x + c_3 \sin x + x(c_4 \cos x + c_5 \sin x) + x^2(c_6 \cos x + c_7 \sin x).$

Solutions to Section 6.3

True-False Review:

1. FALSE. Under the given assumptions, we have $A_1(D)F_1(x) = 0$ and $A_2(D)F_2(x) = 0$. However, this means that

$$\begin{aligned}(A_1(D) + A_2(D))(F_1(x) + F_2(x)) &= A_1(D)F_1(x) + A_1(D)F_2(x) + A_2(D)F_1(x) + A_2(D)F_2(x) \\ &= A_1(D)F_2(x) + A_2(D)F_1(x),\end{aligned}$$

which is not necessarily zero. As a specific example, if $A_1(D) = D - 1$ and $A_2(D) = D - 2$, then $A_1(D)$ annihilates $F_1(x) = e^x$ and $A_2(D)$ annihilates $F_2(x) = e^{2x}$. However, $A_1(D) + A_2(D) = 2D - 3$ does not annihilate $e^x + e^{2x}$.

2. FALSE. The annihilator of $F(x)$ in this case is $A(D) = D^{k+1}$, since it takes $k + 1$ derivatives in order to annihilate x^k .

3. TRUE. We apply rule 1 in this section with $k = 1$, or we can compute directly that $(D - a)^2$ annihilates xe^{ax} .

4. FALSE. Some functions cannot be annihilated by a polynomial differential operator. Only those of the forms listed in 1-4 can be annihilated. For example, $F(x) = \ln x$ does not have an annihilator.

5. FALSE. For instance, if $F(x) = x$, $A_1(D) = A_2(D) = D$, then although $A_1(D)A_2(D)F(x) = 0$, neither $A_1(D)$ nor $A_2(D)$ annihilates $F(x) = x$.

6. FALSE. The annihilator of $F(x) = 3 - 5x$ is D^2 , but since $r = 0$ already occurs twice as a root of the auxiliary equation, the appropriate trial solution here is $y_p(x) = A_0 x^2 + A_0 x^3$.

7. FALSE. The annihilator of $F(x) = x^4$ is D^5 , but since $r = 0$ already occurs three times as a root of the auxiliary equation, the appropriate trial solution here is $y_p(x) = A_0x^3 + A_1x^4 + A_2x^5 + A_3x^6 + A_4x^7$.

8. TRUE. The annihilator of $F(x) = \cos x$ is $D^2 + 1$, but since $r = \pm i$ already occurs once as a complex conjugate pair of roots of the auxiliary equation, the appropriate trial solution is not $y_p(x) = A_0 \cos x + B_0 \sin x$; we must multiply by a factor of x to occur for the fact that $r = \pm i$ is a pair of roots of the auxiliary equation.

Problems:

Note: In Problems 1-16, we use the four boxed formulas on pages 473-474 of the text.

1. $A(D) = D^2(D - 1)$; $(D - 1)(2e^x) = 0$ and $D^2(3x) = 0 \implies D^2(D - 1)(2e^x - 3x) = 0$.

2. $A(D) = D + 3$; $(D + 3)(5e^{-3x}) = -15e^{-3x} + 15e^{-3x} = 0$.

3. $A(D) = (D - 7)^4(D^2 + 16)$; $(D - 7)^4(x^3e^{7x}) = 0$ and $(D^2 + 16)(5 \sin 4x) = 0 \implies (D - 7)^4(D^2 + 16)(x^3e^{7x} + 5 \sin 4x) = 0$.

4. $A(D) = (D^2 + 1)(D - 2)^2$; $(D^2 + 1)(\sin x) = 0$ and $(D - 2)^2(3xe^{2x}) = 0 \implies (D^2 + 1)(D - 2)^2(\sin x + 3xe^{2x}) = 0$.

5. $A(D) = (D^2 - 2D + 5)(D^2 + 4)$; In the expression $e^x \sin 2x$, $a = 1$ and $b = 2$, so that $D^2 - 2aD + (a^2 + b^2) = D^2 - 2D + 5 \implies (D^2 - 2D + 5)(e^x \sin 2x) = 0$. Moreover, in the expression $3 \cos 2x$, $a = 0$ and $b = 2$, so that $D^2 - 2aD + (a^2 + b^2) = D^2 + 4 \implies (D^2 + 4)(3 \cos 2x) = 0$. Thus we conclude that $(D^2 - 2D + 5)(D^2 + 4)(e^x \sin 2x + 3 \cos 2x) = 0$.

6. $A(D) = D^2 + 4D + 5$; In the expression $4e^{-2x} \sin x$, $a = -2$ and $b = 1$ so that $D^2 - 2aD + (a^2 + b^2) = D^2 + 4D + 5 \implies (D^2 + 4D + 5)(4e^{-2x} \sin x) = 0$.

7. $A(D) = (D^2 - 10D + 26)^3$; In the expression $e^{5x} \cos x$, $a = 5$ and $b = 1$ so that $D^2 - 2aD + (a^2 + b^2) = D^2 - 10D + 26 \implies (D^2 - 10D + 26)^3(e^{5x} \cos x) = 0$.

8. $A(D) = D^3(D - 4)^2$; Note that $D^3(2x^2) = 0$ and $(D - 4)^2[(1 - 3x)e^{4x}] = 0 \implies D^3(D - 4)^2((1 - 3x)e^{4x} + 2x^2) = 0$.

9. $A(D) = (D - 4)^2(D^2 - 8D + 41)D^2(D^2 + 4D + 5)^3$; Note that

$$(D - 4)^2(xe^{4x}) = 0, \quad (D^2 - 8D + 41)(-2e^{4x} \sin 5x) = 0, \quad D^2(3x) = 0, \quad \text{and } (D^2 + 4D + 5)^3(x^2e^{-2x} \cos x) = 0.$$

Therefore,

$$(D - 4)^2(D^2 - 8D + 41)D^2(D^2 + 4D + 5)^3(e^{4x}(x - 2 \sin 5x) + 3x - x^2e^{-2x} \cos x) = 0.$$

10. $A(D) = D^2 + 6D + 10$; In the expression $2e^{-3x} \sin x + 7e^{-3x} \cos x$, $a = -3$ and $b = 1$ so that $D^2 - 2aD + (a^2 + b^2) = D^2 + 6D + 10 \implies (D^2 + 6D + 10)(e^{-3x}(2 \sin x + 7 \cos x)) = 0$.

11. $A(D) = (D^2 + 9)^2$; Here we have applied Rule 3 (page 474) with $a = 0$, $b = 3$, and $k = 1$.

12. $A(D) = (D^2 + 1)^3$; Here we have applied Rule 3 (page 474) with $a = 0$, $b = 1$, and $k = 2$.

13. First we simplify

$$\sin^4 x = (\sin^2 x)^2 = \left(\frac{1 - \cos 2x}{2} \right)^2 = \frac{1}{4} \left[1 - 2 \cos 2x + \frac{1 + \cos 4x}{2} \right] = \frac{3 - 4 \cos 2x + \cos 4x}{8}.$$

We use $A_1(D) = D$ to annihilate $\frac{3}{8}$, $A_2(D) = D^2 + 4$ to annihilate $\frac{1}{2} \cos 2x$, and $A_3(D) = D^2 + 16$ to annihilate $\frac{1}{8} \cos 4x$. Hence, $A(D) = A_1(D)A_2(D)A_3(D) = D(D^2 + 4)(D^2 + 16) = D^5 + 20D^3 + 64D$.

14. Following the hint, we write $\cos^2 x = \frac{1 + \cos 2x}{2}$. We use $A_1(D) = D$ to annihilate $\frac{1}{2}$ and we use $A_2(D) = D^2 + 4$ to annihilate $\frac{1}{2}(\cos 2x)$. Hence, $A(D) = A_1(D)A_2(D) = D^3 + 4D$.

15. To find the annihilator of $F(x)$, let us use the identities

$$\cos^2 x = \frac{1 + \cos 2x}{2} \quad \text{and} \quad \sin^2 x = \frac{1 - \cos 2x}{2}$$

to rewrite the formula for $F(x)$:

$$\begin{aligned} F(x) &= \sin^2 x \cos^2 x \cos^2 2x \\ &= \frac{1 - \cos 2x}{2} \cdot \frac{1 + \cos 2x}{2} \cdot \frac{1 + \cos 4x}{2} \\ &= \frac{1}{8} [(1 - \cos^2 2x)(1 + \cos 4x)] \\ &= \frac{1}{8} \sin^2 2x (1 + \cos 4x) \\ &= \frac{1}{16} (1 - \cos 4x)(1 + \cos 4x) \\ &= \frac{1}{16} (1 - \cos^2 4x) \\ &= \frac{1}{16} \sin^2 4x = \frac{1}{32} (1 - \cos 8x) = \frac{1}{32} - \frac{1}{32} \cos 8x. \end{aligned}$$

The annihilator for $\frac{1}{32}$ is $A_1(D) = D$, and the annihilator for $\frac{1}{32} \cos 8x$ is $A_2(D) = D^2 + 64$. Therefore, the annihilator for $F(x)$ is $A(D) = A_1(D)A_2(D) = D(D^2 + 64) = D^3 + 64D$.

16. To find the annihilator of $F(x)$, let us use the identities

$$\cos^2 x = \frac{1 + \cos 2x}{2} \quad \text{and} \quad \sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \sin 2x = 2 \sin x \cos x$$

to rewrite the formula for $F(x)$:

$$\begin{aligned} F(x) &= \frac{1}{2} \sin x \cos x (1 + \cos 2x) \\ &= \frac{1}{4} (\sin 2x)(1 + \cos 2x) \\ &= \frac{1}{4} \sin 2x + \frac{1}{4} (\sin 2x)(\cos 2x) \\ &= \frac{1}{4} \sin 2x + \frac{1}{8} \sin 4x. \end{aligned}$$

The annihilator of $\frac{1}{4} \sin 2x$ is $A_1(D) = D^2 + 4$, and the annihilator of $\frac{1}{8} \sin 4x$ is $A_2(D) = D^2 + 16$. Therefore, the annihilator for $F(x)$ is $A(D) = A_1(D)A_2(D) = (D^2 + 4)(D^2 + 16) = D^4 + 20D^2 + 64$.

17. We have $P(r) = r^2 + 4r + 4 \implies y_c(x) = c_1 e^{-2x} + c_2 x e^{-2x}$. Now, $A(D) = (D+2)^2$. Operating on the given differential equation with $A(D)$ yields the homogeneous differential equation $(D+2)^4 y = 0$, with solution $y(x) = c_1 e^{-2x} + c_2 x e^{-2x} + A_0 x^2 e^{-2x} + A_1 x^3 e^{-2x}$. Therefore, $y_p(x) = A_0 x^2 e^{-2x} + A_1 x^3 e^{-2x}$. Differentiating y_p yields $y'_p = A_0 e^{-2x}(-2x^2 + 2x) + A_1 e^{-2x}(-2x^3 + 3x^2)$, $y''_p = A_0 e^{-2x}(4x^2 - 8x + 2) + A_1 e^{-2x}(4x^3 - 12x^2 + 6x)$. Substituting into the given differential equation and simplifying yields $2A_0 + 6A_1 x = 5x$, so that $A_0 = 0$, $A_1 = \frac{5}{6}$. Hence, $y_p(x) = \frac{5}{6} x^3 e^{-2x}$, and therefore $y(x) = c_1 e^{-2x} + c_2 x e^{-2x} + \frac{5}{6} x^3 e^{-2x}$.

18. We have $P(r) = r^2 + 1 \implies y_c(x) = c_1 \cos x + c_2 \sin x$. Now, $A(D) = D - 1$. Operating on the given differential equation with $A(D)$ yields $(D-1)(D^2+1)y = 0$, with solution $y(x) = c_1 \cos x + c_2 \sin x + A_0 e^x$. Hence, $y_p(x) = A_0 e^x$. Substitution into the given differential equation yields $A_0 = 3$, so that $y_p(x) = 3e^x$. Consequently, $y(x) = c_1 \cos x + c_2 \sin x + 3e^x$.

19. We have $P(r) = (r-2)(r+1) \implies y_c(x) = c_1 e^{2x} + c_2 e^{-x}$. Now, $A(D) = D - 2$. Operating on the given differential equation with $A(D)$ yields $(D-2)^2(D+1)y = 0$ with general solution $y(x) = c_1 e^{2x} + c_2 e^{-x} + A_0 x e^{2x}$. We therefore choose $y_p(x) = A_0 x e^{2x}$. Differentiating y_p yields $y'_p(x) = A_0 e^{2x}(2x+1)$, $y''_p(x) = A_0 e^{2x}(4x+4)$. Substituting into the given differential equation and simplifying we find $A_0 = \frac{5}{3}$. Hence, $y_p(x) = \frac{5}{3} x e^{2x}$, and so $y(x) = c_1 e^{2x} + c_2 e^{-x} + \frac{5}{3} x e^{2x}$.

20. We have $P(r) = r^2 + 16 \implies y_c(x) = c_1 \cos 4x + c_2 \sin 4x$. Now, $A(D) = D^2 + 1$. Operating on the given differential equation with $A(D)$ yields $(D^2+1)(D^2+16)y = 0$, so that $y(x) = c_1 \cos 4x + c_2 \sin 4x + A_0 \cos x + B_0 \sin x$. We therefore choose $y_p(x) = A_0 \cos x + B_0 \sin x$. Substitution into the given differential equation and simplification yields $15A_0 \cos x + 15B_0 \sin x = 4 \cos x$, so that $A_0 = \frac{4}{15}$, $B_0 = 0$. Hence, $y_p(x) = \frac{4}{15} \cos x$, so that $y(x) = c_1 \cos 4x + c_2 \sin 4x + \frac{4}{15} \cos x$.

21. We have $P(r) = (r-2)(r-3) \implies y_c(x) = c_1 e^{2x} + c_2 e^{3x}$. Moreover, the annihilator of $F(x) = 7e^{2x}$ is $A(D) = D - 2$. Operating on the differential equation with $A(D)$ gives

$$(D-2)^2(D-3)y = 0,$$

which has solution

$$y(x) = c_1 e^{2x} + c_2 e^{3x} + A_0 x e^{2x}.$$

Therefore, our trial solution is $y_p(x) = A_0 x e^{2x}$. We must solve for A_0 . Substituting the expression for $y_p(x)$ into the differential equation and simplification yields $A_0 = -7$, so $y_p(x) = -7x e^{2x}$. Hence, the general solution to the given differential equation is $y(x) = c_1 e^{2x} + c_2 e^{3x} - 7x e^{2x}$.

22. We have $P(r) = r^2 + 2r + 5 \implies y_c(x) = e^{-x}[c_1 \cos 2x + c_2 \sin 2x]$. Now, $A(D) = D^2 + 4$. Operating on the given differential equation with $A(D)$ yields $(D^2+4)(D^2+2D+5)y = 0$ with general solution $y(x) = e^{-x}(c_1 \cos 2x + c_2 \sin 2x) + A_0 \cos 2x + B_0 \sin 2x$. We therefore choose $y_p(x) = A_0 \cos 2x + B_0 \sin 2x$. Substitution into the given differential equation and simplification yields $(A_0 + 4B_0) \cos 2x + (-4A_0 + B_0) \sin 2x = 3 \sin 2x$. Consequently $A_0 + 4B_0 = 0$, $-4A_0 + B_0 = 3$. This system has a solution $A_0 = -\frac{12}{17}$, $B_0 = \frac{3}{17}$. Hence, $y_p(x) = -\frac{12}{17} \cos 2x + \frac{3}{17} \sin 2x$, and so $y(x) = e^{-x}(c_1 \cos 2x + c_2 \sin 2x) - \frac{12}{17} \cos 2x + \frac{3}{17} \sin 2x$.

23. We have $P(r) = r^2 + 6 \implies y_c(x) = c_1 \cos \sqrt{6}x + c_2 \sin \sqrt{6}x$. Next we must determine the annihilator of $F(x) = \sin^2 x \cos^2 x = \frac{1}{4}(1 - \cos 2x)(1 + \cos 2x) = \frac{1}{4}(1 - \cos^2 2x) = \frac{1}{4}\sin^2 2x = \frac{1}{8}(1 - \cos 4x) = \frac{1}{8} - \frac{1}{8}\cos 4x$, which is $A(D) = D(D^2 + 16)$. Operating on the given differential equation with $A(D)$ yields $D(D^2 + 16)(D^2 + 6)y = 0$, which has general solution $y(x) = c_1 \cos \sqrt{6}x + c_2 \sin \sqrt{6}x + A_0 + B_0 \cos 4x + C_0 \sin 4x$. We therefore choose $y_p(x) = A_0 + B_0 \cos 4x + C_0 \sin 4x$. Substitution into the given differential equation and simplification yields $-10B_0 \cos 4x - 10C_0 \sin 4x + 6A_0 = \frac{1}{8} - \frac{1}{8}\cos 4x$. Therefore, $A_0 = \frac{1}{48}$, $B_0 = \frac{1}{80}$, and $C_0 = 0$. Therefore, $y_p(x) = \frac{1}{48} + \frac{1}{80}\cos 4x$. Thus, the general solution to the given differential equation is

$$y(x) = c_1 \cos \sqrt{6}x + c_2 \sin \sqrt{6}x + \frac{1}{48} + \frac{1}{80}\cos 4x.$$

24. We have $P(r) = (r + 3)(r - 1) \implies y_c(x) = c_1 e^{-3x} + c_2 e^x$. Next we must determine the annihilator of $F(x) = \sin^2 x = \frac{1 - \cos 2x}{2}$, which is $A(D) = D(D^2 + 4) = D^3 + 4D$. Operating on the given differential equation with the annihilator $A(D)$ yields $D(D^2 + 4)(D + 3)(D - 1)y = 0$, which has the general solution $y(x) = c_1 e^{-3x} + c_2 e^x + A_0 + B_0 \cos 2x + C_0 \sin 2x$. We therefore choose $y_p(x) = A_0 + B_0 \cos 2x + C_0 \sin 2x$. Substitution into the given differential equation and simplification yields $A_0 = -\frac{1}{6}$, $B_0 = \frac{7}{130}$, and $C_0 = -\frac{2}{65}$. Therefore $y_p(x) = -\frac{1}{6} + \frac{7}{130}\cos 2x - \frac{2}{65}\sin 2x$. Thus, the general solution to the given differential equation is

$$y(x) = c_1 e^{-3x} + c_2 e^x - \frac{1}{6} + \frac{7}{130}\cos 2x - \frac{2}{65}\sin 2x.$$

25. We have $P(r) = r^3 + 2r^2 - 5r - 6 = (r - 2)(r + 3)(r + 1) \implies y_c(x) = c_1 e^{2x} + c_2 e^{-3x} + c_3 e^{-x}$. Now, $A(D) = D^3$. Operating on the given differential equation with $A(D)$ yields $D^3(D - 2)(D + 3)(D + 1)y = 0$, with general solution $y(x) = c_1 e^{2x} + c_2 e^{-3x} + c_3 e^{-x} + A_0 + A_1 x + A_2 x^2$. We therefore choose $y_p(x) = A_0 + A_1 x + A_2 x^2$. Substituting into the given differential equation and simplifying yields

$$4A_2 - 5(A_1 + 2A_2x) - 6(A_0 + A_1 x + A_2 x^2) = 4x^2,$$

so that $4A_2 - 5A_1 - 6A_0 = 0$, $-10A_2 - 6A_1 = 0$, $-6A_2 = 4$. This system has a solution $A_0 = -\frac{37}{27}$, $A_1 = \frac{10}{9}$, $A_2 = -\frac{2}{3}$. Hence, $y_p(x) = -\frac{37}{27} + \frac{10}{9}x - \frac{2}{3}x^2$, and so $y(x) = c_1 e^{2x} + c_2 e^{-3x} + c_3 e^{-x} - \frac{37}{27} + \frac{10}{9}x - \frac{2}{3}x^2$.

26. We have $P(r) = (r - 1)(r^2 + 1) \implies y_c(x) = c_1 e^x + c_2 \cos x + c_3 \sin x$. Now, $A(D) = D + 1$. Operating on the given differential equation with $A(D)$ yields $(D + 1)(D - 1)(D^2 + 1)y = 0$, with general solution $y(x) = c_1 e^x + c_2 \cos x + c_3 \sin x + A_0 e^{-x}$. We therefore choose $y_p(x) = A_0 e^{-x}$. Substitution into the given differential equation and simplification yields $A_0 = -\frac{9}{4}$. Hence, $y_p(x) = -\frac{9}{4}e^{-x}$, and so the general solution is $y(x) = c_1 e^x + c_2 \cos x + c_3 \sin x - \frac{9}{4}e^{-x}$.

27. We have $P(r) = (r + 1)^3 \implies y_c(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$. Now, $A(D) = (D - 2)(D + 1)$. Operating on the given differential equation with $A(D)$ yields $(D - 2)(D + 1)^4 y = 0$, with general solution $y(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} + A_0 x^3 e^{-x} + A_1 e^{2x}$. We therefore choose $y_p(x) = A_0 x^3 e^{-x} + A_1 e^{2x}$.

Differentiating y_p yields

$$y'_p = A_0 e^{-x}(-x^3 + 3x^2) + 2A_1 e^{2x}, \quad y''_p = A_0 e^{-x}(x^3 - 6x^2 + 6x) + 4A_1 e^{2x},$$

$$y'''_p = A_0 e^{-x}(-x^3 + 9x^2 - 18x + 6) + 8A_1 e^{2x}.$$

Substituting into the given differential equation and simplifying we find $6A_0 e^{-x} + 27A_1 e^{2x} = 2e^{-x} + 3e^{2x}$, so that $A_0 = \frac{1}{3}$, $A_1 = \frac{1}{9}$. Hence, $y_p(x) = \frac{1}{3}x^3 e^{-x} + \frac{1}{9}e^{2x}$, and therefore, the general solution to the differential equation is $y(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} + \frac{1}{3}x^3 e^{-x} + \frac{1}{9}e^{2x}$.

28. We have $P(r) = (r-1)(r-2)(r-3) \implies y_c(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$. An appropriate trial solution is $y_p(x) = A_0 e^{4x}$. Expanding the differential operators in the given differential equation gives $y''' - 6y'' + 11y' - 6y = 6e^{4x}$. Substituting for y_p into this differential equation and simplifying yields $A_0 = 1$, so that $y_p(x) = e^{4x}$. Hence, $y(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} + e^{4x}$. Imposing the given initial conditions leads to the following system:

$$c_1 + c_2 + c_3 = 3, \quad c_1 + 2c_2 + 3c_3 = 6, \quad c_1 + 4c_2 + 9c_3 = 14.$$

The reduced row-echelon form of the augmented matrix of this system is $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right]$, so that $c_1 = c_2 = c_3 = 1$. Consequently, the solution to the initial-value problem is $y(x) = e^x + e^{2x} + e^{3x} + e^{4x}$.

29. We have $P(r) = r^3 - 2r^2 - r + 2 = (r-1)(r+1)(r-2) \implies y_c(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{2x}$. We take a trial solution of the form $y_p(x) = A_0 e^{3x}$. Substitution into the differential equation yields $A_0(27 - 18 - 3 + 2) = 4$, so that $y_p(x) = \frac{1}{2}e^{3x}$. Consequently, $y(x) = c_1 e^x + c_2 e^{-x} + c_3 e^{2x} + \frac{1}{2}e^{3x}$.

30. We have $P(r) = (r+4)(r-4)(r-3) \implies y_c(x) = c_1 e^{-4x} + c_2 e^{4x} + c_3 e^{3x}$. We need the modified trial solution: $y_p(x) = A_0 x e^{3x}$. Differentiating y_p yields $y'_p = A_0 e^{3x} + 3A_0 x e^{3x}$, $y''_p(x) = 6A_0 e^{3x} + 9A_0 x e^{3x}$, $y'''_p = 27A_0 e^{3x} + 27A_0 x e^{3x}$. Substituting these expressions into the given differential equation and simplification leads to $A_0 = -\frac{6}{7}$. Hence, $y(x) = c_1 e^{-4x} + c_2 e^{4x} + c_3 e^{3x} - \frac{6}{7}x e^{3x}$.

31. We have $P(r) = (r-2)(r+3)(r+2) \implies y_c(x) = c_1 e^{2x} + c_2 e^{-3x} + c_3 e^{-2x}$. An appropriate trial solution is $y_p(x) = A_0 \cos x + B_0 \sin x$. Substitution into the given differential equation and simplification leads to $(5A_0 - 15B_0) \sin x - (15A_0 + 5B_0) \cos x = 4 \cos x$. Consequently, A_0 and B_0 must satisfy: $A_0 - 3B_0 = 0$, $15A_0 + 5B_0 = -4$. This system has a solution $A_0 = -\frac{6}{25}$, $B_0 = -\frac{2}{25}$. Hence, the general solution to the differential equation is $y(x) = c_1 e^{2x} + c_2 e^{-3x} + c_3 e^{-2x} - \frac{6}{25} \cos x - \frac{2}{25} \sin x$.

32. We have $P(r) = r^2(r^2 + 1) \implies y_c(x) = c_1 + c_2 x + c_3 \cos x + c_4 \sin x$. We must use the modified trial solution $y_p(x) = A_0 x^2 + A_1 x^3$. The differential equation is $y^{(iv)} + y'' = 6 - 2x$. Substituting the trial solution into this differential equation yields $2A_0 + 6A_1 x = 6 - 2x$, so that $A_0 = 3$ and $A_1 = \frac{1}{3}$. Hence, $y_p(x) = 3x^2 - \frac{1}{3}x^3$, and therefore $y(x) = c_1 + c_2 x + c_3 \cos x + c_4 \sin x + 3x^2 - \frac{1}{3}x^3$.

33. We have $P(r) = (r+1)^3 \implies y_c(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$. We must use the modified trial solution $y_p(x) = A_0 x^3 e^{-x}$. Differentiating this trial solution gives

$$y'_p = A_0 e^{-x}(-x^3 + 3x^2), \quad y''_p = A_0 e^{-x}(x^3 - 6x^2 + 6x), \quad y'''_p = A_0 e^{-x}(-x^3 + 9x^2 - 18x + 6).$$

The given differential equation is $y''' + 3y'' + 3y' + y = 0$. Substituting the trial solution into this differential equation and simplifying yields $A_0 = \frac{5}{6}$. Hence, $y_p(x) = \frac{5}{6}x^3e^{-x}$, and therefore the general solution to the differential equation is $y(x) = e^{-x}(c_1 + c_2x + c_3x^2 + \frac{5}{6}x^3)$.

34. We have $P(r) = (r+1)(r^2+1) \Rightarrow y_c(x) = c_1e^{-x} + c_2\cos x + c_3\sin x \Rightarrow y_p(x) = e^x(A_0 + A_1x)$.

35. We have $P(r) = (r^2 + 4r + 13)^2 \Rightarrow y_c(x) = e^{-2x}[c_1\cos 3x + c_2\sin 3x + x(c_3\cos 3x + c_4\sin 3x)] \Rightarrow y_p(x) = x^2e^{-2x}(A_0\cos 3x + B_0\sin 3x)$.

36. We have $P(r) = (r^2 + 4)(r-2)^3 \Rightarrow y_c(x) = c_1\cos 2x + c_2\sin 2x + e^{2x}(c_3 + c_4x + c_5x^2) \Rightarrow y_p(x) = A_0 + A_1x + x^3e^{2x}(A_2 + A_3x)$.

37. We have $P(r) = r^2(r-1)(r^2+4)^2 \Rightarrow y_c(x) = c_1 + c_2x + c_3e^x + c_4\cos 2x + c_5\sin 2x + x(c_6\cos 2x + c_7\sin 2x) \Rightarrow y_p(x) = A_0xe^x + x^2(A_1\cos 2x + A_2\sin 2x)$.

38. We have $P(r) = (r^2 - 2r + 2)^3(r-2)^2(r+4) \Rightarrow y_c(x) = e^x[c_1\cos x + c_2\sin x + x(c_3\cos x + c_4\sin x) + x^2(c_5\cos x + c_6\sin x)] + c_7e^{2x} + c_8xe^{2x} + c_9e^{-4x} \Rightarrow y_p(x) = x^3e^x(A_0\cos x + A_1\sin x) + A_2x^2e^{2x}$.

39. We have $P(r) = r(r^2 - 9)(r^2 - 4r + 5) \Rightarrow y_c(x) = c_1 + c_2e^{3x} + c_3e^{-3x} + e^{2x}(c_4\cos x + c_5\sin x) \Rightarrow y_p(x) = A_0xe^{3x} + xe^{2x}(A_1\cos x + A_2\sin x)$.

40. Consider

$$P(D)y = cx^ke^{ax}\cos bx \quad (40.1)$$

and let y_c denote the complementary function. The appropriate annihilator for (40.1) is

$$A(D) = [D^2 - 2aD + (a^2 + b^2)]^{k+1}$$

and so a trial solution for (40.1) can be determined from the general solution to

$$A(D)P(D)y = 0. \quad (40.2)$$

Two cases arise:

Case 1: If $r = a + ib$ is not a root of $P(r) = 0$, then the general solution of (40.2) will be of the form

$$y(x) = y_c(x) + e^{ax}[(A_0 + A_1x + \dots + A_kx^k)\cos bx + (B_0 + B_1x + \dots + B_kx^k)\sin bx]$$

so that an appropriate trial solution is

$$y_p(x) = e^{ax}[(A_0 + A_1x + \dots + A_kx^k)\cos bx + (B_0 + B_1x + \dots + B_kx^k)\sin bx].$$

Case 2: If $r = a + ib$ is a root of multiplicity m of $P(r) = 0$, then the complementary function y_c will contain the terms

$$e^{ax}[(c_0 + c_1x + \dots + c_{m-1}x^{m-1})\cos bx + (d_0 + d_1x + \dots + d_{m-1}x^{m-1})\sin bx].$$

The operator $A(D)P(D)$ will therefore contain the term $[D^2 - 2aD + (a^2 + b^2)]^{m+k+1}$, so that the terms in the general solution to (40.2) that do not arise in the complementary function are

$$y_p(x) = x^me^{ax}[(A_0 + A_1x + \dots + A_kx^k)\cos bx + (B_0 + B_1x + \dots + B_kx^k)\sin bx].$$

Solutions to Section 6.4

True-False Review:

1. **TRUE.**
2. **TRUE.**
3. **FALSE.** An appropriate complex-valued trial solution is $y_p(x) = A_0 x e^{ix}$.
4. **TRUE.**
5. **FALSE.** An appropriate complex-valued trial solution is $y_p(x) = A_0 x^2 e^{(2+5i)x}$.
6. **TRUE.**

Problems:

1. Consider $z'' + 2z' + z = 50e^{3ix}$. Let $z_p(x) = Ae^{3ix}$, $z'_p = 3Aie^{3ix}$, and $z''_p = -9Ae^{3ix}$. Substituting, we obtain

$$-9Ae^{3ix} + 2(3Aie^{3ix}) + Ae^{3ix} = 50e^{3ix} \implies A = -4 - 3i.$$

Hence,

$$z_p(x) = (-4 - 3i)e^{3ix} = (-4 - 3i)(\cos 3x + i \sin 3x) = (6 \sin 3x - 8 \cos 3x) + i(-4 \sin 3x - 3 \cos 3x).$$

Consequently,

$$y_p(x) = \text{Im}(z_p) = -3 \cos 3x - 4 \sin 3x.$$

2. Consider the complex equation $z'' - z = 10e^{(2+i)x}$. Let $z_p(x) = A_0 e^{(2+i)x}$. Substituting we get

$$A_0(3 + 4i)e^{(2+i)x} - A_0 e^{(2+i)x} = 10e^{(2+i)x} \implies A_0 = 1 - 2i.$$

Hence,

$$z_p(x) = (1 - 2i)e^{(2+i)x} = e^{2x}[(\cos x + 2 \sin x) + i(\sin x - 2 \cos x)]$$

so that

$$y_p(x) = \text{Re}(z_p) = e^{2x}(\cos x + 2 \sin x).$$

3. Consider $z'' + 4z' + 4z = 169e^{3ix}$. Let $z_p(x) = A_0 e^{3ix}$. Substituting we get

$$(-9 + 12i + 4)A_0 = 169 \implies A_0 = -5 - 12i.$$

Hence,

$$z_p = (-5 - 12i)(\cos 3x + i \sin 3x)$$

so that

$$y_p(x) = \text{Im}(z_p) = -12 \cos 3x - 5 \sin 3x.$$

4. Consider $z'' - z' - 2z = 20 - 20e^{2ix}$. Letting $z_p(x) = A + Be^{2ix}$, substituting back into the differential equation, and solving we get $A = -10$ and $B = 3 - i$. Hence,

$$z_p(x) = -10 + (3 - i)(\cos 2x + i \sin x)$$

so that

$$y_p(x) = \operatorname{Re}(z_p) = -10 + (3 \cos 2x + \sin 2x).$$

5. Consider $z'' + z = 3e^{(1+2i)x}$. Let $z_p(x) = A_0e^{(1+2i)x}$. Substituting into the differential equation we get $[(-3 + 4i) + 1]A_0 = 3 \implies A_0 = -\frac{3}{10}(1 + 2i)$. Hence,

$$z_p(x) = -\frac{3}{10}(1 + 2i)e^x(\cos 2x + i \sin 2x)$$

so that

$$y_p(x) = \operatorname{Re}(z_p) = \frac{3}{10}e^x(2 \sin 2x - \cos 2x).$$

6. Consider $z'' + 2z' + 2z = 2e^{(-1+i)x}$. Let $z_p(x) = A_0xe^{(-1+i)x}$. Substituting into the differential equation we get $A_0 = -i$. Hence,

$$z_p(x) = -ixe^{(-1+i)x} = xe^{-x} \sin x + i(-xe^{-x} \cos x),$$

so that

$$y_p(x) = \operatorname{Im}(z_p) = -xe^{-x} \cos x.$$

7. Consider $z'' - 4z = 100xe^{(1+i)x}$. Let $z_p(x) = (A_0 + A_1x)e^{(1+i)x}$. Substituting and solving the differential equation we get $A_0 = 2 - 14i$ and $A_1 = -20 - 10i$. Hence,

$$z_p(x) = e^x[(2 - 14i) + (-20 - 10i)x](\cos x + i \sin x)$$

so that

$$y_p(x) = \operatorname{Im}(z_p) = e^x[(-14 \cos x + 2 \sin x) - 10x(2 \sin x + \cos x)].$$

8. Consider $z'' + 2z' + 5z = 4e^{(-1+2i)x}$. Let $z_p(x) = A_0xe^{(-1+2i)x}$. Substituting into the differential equation and solving yields $A_0 = -i$. Hence,

$$z_p(x) = -ixe^{(-1+2i)x} = xe^{-x} \sin 2x - ixe^{-x} \cos 2x$$

so that

$$y_p(x) = \operatorname{Re}(z_p) = xe^{-x} \sin 2x.$$

9. Consider $z'' - 2z' + 10z = 24e^{(1+3i)x}$. Let $z_p(x) = Axe^{(1+3i)x}$. Substituting into the differential equation and solving we get $A = -4i$. Hence,

$$z_p(x) = -4ixe^{(1+3i)x}$$

so that

$$y_p(x) = \operatorname{Re}(z_p) = 4xe^x \sin 3x.$$

10. We can write $16 \cos 4x - 8 \sin 4x = 16 \left(\frac{e^{4ix} + e^{-4ix}}{2} \right) - 8 \left(\frac{e^{4ix} - e^{-4ix}}{2i} \right) = (8 + 4i)e^{4ix} + (8 - 4i)e^{-4ix}$. Now consider $z'' + 16z = 34e^x + (8 + 4i)e^{4ix} + (8 - 4i)e^{-4ix}$. Since $\pm 4i$ is a root of the auxiliary equation, a

reasonable choice for z_p is $z_p(x) = Ae^x + Bxe^{4ix} + Cxe^{-4ix}$. Substituting into the differential equation and solving, we get $A = 2$, $B = \frac{1}{2} - i$, and $C = \frac{1}{2} + i$. Hence,

$$z_p(x) = 2e^x + \left(\frac{1}{2} - i\right)xe^{4ix} + \left(\frac{1}{2} + i\right)xe^{-4ix} = 2e^x + x \cos 4x + 2x \sin 4x.$$

Therefore,

$$y_p(x) = 2e^x + x \cos 4x + 2x \sin 4x.$$

11. Consider $z'' + \omega_0^2 z = F_0 e^{\omega it}$.

Case 1: If $\omega \neq \omega_0$ let $z_p(t) = Ae^{\omega it}$. Substituting into the differential equation and solving we obtain $A = \frac{F_0}{\omega_0^2 - \omega^2}$. Hence, $z_p(t) = A(\cos \omega t + i \sin \omega t) = \frac{F_0}{\omega_0^2 - \omega^2} \cos \omega t + i \frac{F_0}{\omega_0^2 - \omega^2} \sin \omega t$. Now $y_p(t) = \text{Re}(z_p)$ so $y_p(t) = \frac{F_0}{\omega_0^2 - \omega^2} \cos \omega t$.

Case 2: If $\omega = \omega_0$ let $z_p(t) = Ate^{\omega_0 it}$. Substituting into the differential equation and solving we get $A = -\frac{F_0 i}{2\omega_0}$ so $z_p(t) = -\frac{F_0 i}{2\omega_0} te^{\omega_0 it} = \frac{F_0 t}{2\omega_0} \sin \omega_0 t - i \left(\frac{F_0 t}{2\omega_0} \cos \omega_0 t \right)$. Hence, $y_p(t) = \text{Re}(z_p)$ so $y_p(t) = \frac{F_0 t}{2\omega_0} \sin \omega_0 t$.

Solutions to Section 6.5

True-False Review:

1. TRUE. This is reflected in the negative sign appearing in Hooke's Law. The spring force F_s is given by $F_s = -kL_0$, where $k > 0$ and L_0 is the displacement of the spring from its equilibrium position. The spring force acts in a direction opposite to that of the displacement of the mass.

2. FALSE. The circular frequency is the *square root* of k/m :

$$\omega_0 = \sqrt{\frac{k}{m}}.$$

3. TRUE. The frequency of oscillation, denoted f in the text, and the period of oscillation, denoted T in the text, are inverses of one another: $fT = 1$. For instance, if a system undergoes $f = 3$ oscillations in one second, then each oscillation takes one-third of a second, so $T = \frac{1}{3}$.

4. TRUE. This is mathematically seen from Equation (6.5.14) in which the exponential factor $e^{-ct/2m}$ decays to zero for large t . Therefore, $y(t)$ becomes smaller and smaller as t increases. This is depicted in Figure 6.5.5.

5. FALSE. In the cases of critical damping and overdamping, the system cannot oscillate. In fact, the system passes through the equilibrium position at most once in these cases.

6. TRUE. The frequency of the driving term is denoted by ω , while the circular frequency of the spring-mass system is denoted by ω_0 . Case 1(b) in this section describes resonance, the situation in which $\omega = \omega_0$.

7. TRUE. Air resistance tends to dampen the motion of the spring-mass system, since it acts in a direction opposite to that of the motion of the spring. This is reflected in Equation (6.5.4) by the negative sign appearing in the formula for the damping force F_d .

8. FALSE. From the formula

$$T = 2\pi\sqrt{\frac{m}{k}}$$

for the period of oscillation, we see that for a larger mass, the period is larger (i.e. longer).

9. TRUE. We see in the section on free oscillations of a mechanical system, we see that the resulting motion of the mass is given by (6.5.11), (6.5.12), or (6.5.13), and in all of these cases, the amplitude of this system is bounded. Only in the case of forced oscillations with resonance can the amplitude increase without bound.

Problems:

1. We have $\frac{d^2y}{dt^2} + 4y = 0 \implies r^2 + 4 = 0 \implies r = \pm 2i \implies y(t) = A \cos 2t + B \sin 2t$ and $y'(t) = -2A \sin 2t + 2B \cos 2t$. Now $y(0) = 2 \implies A = 2$ and $y'(0) = 4 \implies B = 2$. Thus, $y(t) = 2 \cos 2t + 2 \sin 2t$.
Amplitude: $A_0 = \sqrt{2^2 + 2^2} = 2\sqrt{2}$,
Natural frequency: $\omega_0 = 2$,

Phase of the motion: $\phi = \tan^{-1}(1) = \frac{\pi}{4}$,

Period of oscillation: $T = \frac{2\pi}{\omega_0} = \pi$.

2. $\frac{d^2y}{dt^2} + \omega_0^2 y = 0 \implies r^2 + \omega_0^2 = 0 \implies r = \pm \omega_0 i \implies$

$$y(t) = A \cos \omega_0 t + B \sin \omega_0 t \quad \text{and} \quad y'(t) = -\omega_0 A \sin \omega_0 t + \omega_0 B \cos \omega_0 t.$$

Now $y(0) = y_0 \implies A = y_0$ and $y'(0) = v_0 \implies B = \frac{v_0}{\omega_0}$. Thus, $y(t) = y_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t$.

Amplitude: $A_0 = \sqrt{y_0^2 + \left(\frac{v_0}{\omega_0}\right)^2}$,

Natural frequency: $\omega_0 = \omega_0$,

Phase of the motion: $\sin \phi = \frac{y_0}{\sqrt{y_0^2 + \left(\frac{v_0}{\omega_0}\right)^2}}$, $\cos \phi = \frac{v_0}{\omega_0 \sqrt{y_0^2 + \left(\frac{v_0}{\omega_0}\right)^2}}$,

Period of oscillation: $T = \frac{2\pi}{\omega_0}$.

3.

(a) $3 \text{ N} = k (1\text{m}) \implies k = 3 \text{ N/m}$.

(b) $\omega_0 = \sqrt{k/m} = \sqrt{3/4} = \sqrt{3}/2$. $\frac{d^2y}{dt^2} + \frac{3}{4}y = 0 \implies y = A \cos(\sqrt{3}t/2) + B \sin(\sqrt{3}t/2)$. Since $y(0) = -1$,

we conclude that $A = -1$. Now $y'(t) = -A \frac{\sqrt{3}}{2} \sin(\sqrt{3}t/2) + B \frac{\sqrt{3}}{2} \cos(\sqrt{3}t/2)$ and $y'(0) = -\frac{1}{2}$, so that

$B = -\frac{\sqrt{3}}{3}$. Thus, $y(t) = -\cos(\sqrt{3}t/2) - \frac{\sqrt{3}}{3} \sin(\sqrt{3}t/2)$.

Amplitude: $A_0 = \sqrt{(-1)^2 + (-\sqrt{3}/3)^2} = 2\sqrt{3}/3$,

Phase of the motion: $\cos \phi = -\frac{\sqrt{3}}{2}$, $\sin \phi = \frac{1}{2} \implies \phi = \frac{5\pi}{6}$,

Period of oscillation: $T = \frac{2\pi}{\omega_0} = \frac{4\sqrt{3}\pi}{3} \text{ sec}$.

4. We have $P(r) = r^2 + 2r + 5 = 0 \implies r = -1 \pm 2i \implies$

$$y(t) = e^{-t}(A \cos 2t + B \sin 2t) \quad \text{and} \quad y'(t) = e^{-t}[(-2A - B) \sin 2t + (-A + 2B) \cos 2t].$$

Thus, $y(0) = 1 \implies A = 1$ and $y'(0) = 3 \implies B = 2$. Therefore, $y(t) = e^{-t}(\cos 2t + 2 \sin 2t)$. The motion is under-damped.

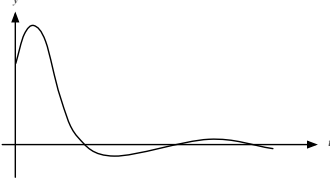


Figure 78: Figure for Problem 4

5. We have $P(r) = r^2 + 3r + 2 = 0 \implies r = -1$ or $r = -2 \implies$

$$y(t) = Ae^{-t} + Be^{-2t} \quad \text{and} \quad y'(t) = -Ae^{-t} - 2Be^{-2t}.$$

Thus $y(0) = 1$ and $y'(0) = 0 \implies A + B = 1$ and $-A - 2B = 0 \implies A = 2$ and $B = -1$. Therefore, $y(t) = 2e^{-t} - e^{-2t}$. The motion is over-damped.

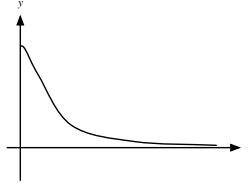


Figure 79: Figure for Problem 5

6. We have $P(r) = 4r^2 + 12r + 5 = 0 \implies r = -1/2$ or $r = -5/2 \implies$

$$y(t) = Ae^{-t/2} + Be^{-5t/2} \quad \text{and} \quad y'(t) = -\frac{A}{2}e^{-t/2} - \frac{5B}{2}e^{-5t/2}.$$

Since $y(0) = 1$ and $y'(0) = -3$, we have $A + B = 1$ and $-\frac{1}{2}A - \frac{5}{2}B = -3 \implies A = -\frac{1}{4}$ and $B = \frac{5}{4}$. Therefore, $y(t) = -\frac{1}{4}e^{-t/2} + \frac{5}{4}e^{-5t/2}$. The motion is over-damped.

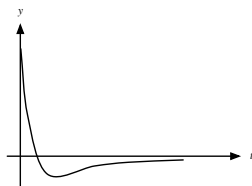


Figure 80: Figure for Problem 6

7. We have $P(r) = r^2 + 2r + 1 = 0 \implies r = -1$ (with multiplicity 2) $\implies$

$$y(t) = Ae^{-t} + Bte^{-t} \quad \text{and} \quad y'(t) = -Ae^{-t} + (1-t)Bte^{-t}.$$

Since $y(0) = -1$ and $y'(0) = 2$, we find that $A + 0 = -1$ and $-A + B = 2 \implies A = -1$ and $B = 1$. Therefore, $y(t) = -e^{-t} + te^{-t}$. The motion is critically-damped.

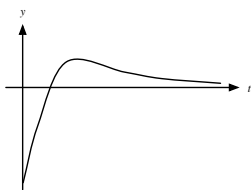


Figure 81: Figure for Problem 7

8. We have $P(r) = 4r^2 + 4r + 1 = 0 \implies r = -1/2$ (with multiplicity 2) $\implies$

$$y(t) = Ae^{-t/2} + Bte^{-t/2} \quad \text{and} \quad y'(t) = -\frac{A}{2}e^{-t/2} + B(1 - t/2)e^{-t/2}.$$

Since $y(0) = 4$ and $y'(0) = -1$, we find that $A = 4$ and $-\frac{1}{2}A + B = -1 \implies B = 1$. Therefore, $y(t) = 4e^{-t/2} + te^{-t/2}$. The motion is critically-damped.

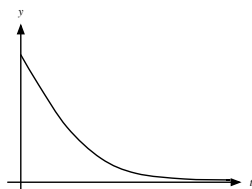


Figure 82: Figure for Problem 8

9. We have $P(r) = r^2 + 4r + 7 = 0 \implies -2 \pm \sqrt{3}i \implies$

$$y(t) = e^{-2t}(A \cos \sqrt{3}t + B \sin \sqrt{3}t) \quad \text{and} \quad y'(t) = e^{-2t}[(-\sqrt{3}A - 2B) \sin \sqrt{3}t + (-2A + \sqrt{3}B) \cos \sqrt{3}t].$$

Since $y(0) = 2$ and $y'(0) = -6$, we find that $A + 0 = 2$ and $-2A + \sqrt{3}B = 6 \implies A = 2$ and $B = \frac{10\sqrt{3}}{3}$ so $y(t) = e^{-2t}(2 \cos \sqrt{3}t + \frac{10\sqrt{3}}{3} \sin \sqrt{3}t)$. The motion is under-damped.

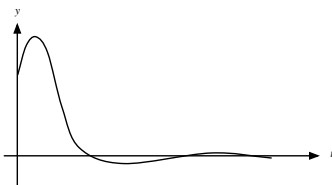


Figure 83: Figure for Problem 9

10. We have $P(r) = r^2 + 5r + 6 = 0 \implies r = -2$ or $r = -3 \implies$

$$y(t) = Ae^{-2t} + Be^{-3t} \quad \text{and} \quad y'(t) = -2Ae^{-2t} - 3Be^{-3t}.$$

Since $y(0) = -1$ and $y'(0) = 4$, we find that $A + B = -1$ and $-2A - 3B = 4 \implies A = 1$ and $B = -2$. Therefore, $y(t) = e^{-2t} - 2e^{-3t}$. The motion is over-damped.

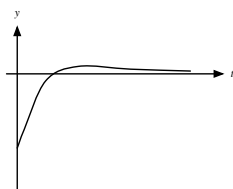


Figure 84: Figure for Problem 10

11. At equilibrium, $e^{-2t} - 2e^{-3t} = 0 \implies t = \ln 2$. For maximum displacement, $y'(t) = 0 \implies -2e^{-2t} + 6e^{-3t} = 0 \implies t = \ln 3$ so the maximum displacement is given by $y(\ln 3) = \frac{1}{27}$.

12. We have $r^2 + 2\alpha r + 1 = 0 \implies r = -\alpha \pm \sqrt{\alpha^2 - 1} \implies r = -\alpha \pm \mu$ where $\mu = \sqrt{\alpha^2 - 1}$ so that $y(t) = e^{-\alpha t}(c_1 e^{\mu t} + c_2 e^{-\mu t})$.

i.) System is under-damped $\implies \alpha^2 - 1 < 0 \implies |\alpha| < 1$, but $\alpha > 0$ so $0 < \alpha < 1$.

ii.) System is critically-damped $\implies |\alpha| = 1$, but $\alpha > 0$ so $\alpha = 1$.

iii.) System is over-damped $\implies |\alpha| > 1$, but $\alpha > 0$ so $\alpha > 1$. Since $y'(t) = e^{-\alpha t}[(\mu - \alpha)c_1 e^{\mu t} - (\mu + \alpha)c_2 e^{-\mu t}]$ and $y'(0) = 0$, we find that $(\mu - \alpha)c_1 - (\mu + \alpha)c_2 = 0 \implies \frac{c_1}{c_2} = \frac{\mu + \alpha}{\mu - \alpha} < 0$, which means that the system will pass through equilibrium.

13.

(a) We have $P(r) = r^2 + 3r + 2 = 0 \implies r = -1$ or $r = -2 \implies$

$$y(t) = Ae^{-t} + Be^{-2t} \quad \text{and} \quad y'(t) = -Ae^{-t} - 2Be^{-2t}.$$

Since $y(0) = 1$, we have $A + B = 1$, and since $y'(0) = -3$, we have $-A - 2B = -3$. Therefore, $A = -1$ and $B = 2$, so that $y(t) = -e^{-t} + 2e^{-2t}$.

(b) $y(t) = 0 \implies -e^{-t} + 2e^{-2t} = 0 \implies t = \ln 2$.

(c)

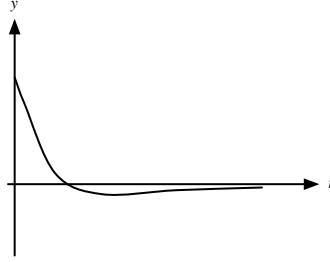


Figure 85: Figure for Problem 13

14.

(a) In the under-damped case, it has been shown in the text that the general solution is given by $y(t) = A_0 e^{-\frac{c}{2m}t} \cos(\mu t - \phi)$. Hence, $y'(t) = -\frac{cA_0}{2m} e^{-\frac{c}{2m}t} \cos(\mu t - \phi) - \mu A_0 e^{-\frac{c}{2m}t} \sin(\mu t - \phi)$. Setting $y'(t) = 0$ we obtain $\frac{c}{2m} \cos(\mu t - \phi) = -\mu \sin(\mu t - \phi)$, or equivalently, $\tan(\mu t - \phi) = -\frac{c}{c\mu m}$. Since the tangent function is periodic of period π , it follows that successive maxima (or minima) of $y(t)$ will occur when $T = \frac{2\pi}{\mu} = \frac{4\pi m}{\sqrt{4mk - c^2}}$.

(b) If $\frac{c^2}{4km} < 1$ then $T = \frac{4\pi m}{\sqrt{4mk - c^2}} \approx 2\pi \sqrt{\frac{m}{k}}$.

15. We have $P(r) = r^2 + \frac{c}{m}r + \frac{c^2}{4m^2} = 0 \implies r = -\frac{c}{2m}$ (with multiplicity 2). Thus,

$$y(t) = Ae^{rt} + Bte^{rt} \quad \text{and} \quad y'(t) = Are^{rt} + Brte^{rt} + Be^{rt}.$$

Since $y(0) = A$ and $y'(0) = Ar + B$, we find that $A = y_0$ and $Ar + B = v_0$. Therefore, $B = v_0 + \frac{y_0 c}{2m}$. Thus $y(t) = y_0 e^{rt} + \left(v_0 + \frac{y_0 c}{2m}\right) t e^{rt}$, or $y(t) = e^{rt} [y_0 + t(v_0 + \frac{c}{2m} y_0)]$. Since e^{rt} is never zero and $y_0 + t(v_0 + \frac{c}{2m} y_0)$ is linear in t , it follows that the product can have at most one zero.

16. Let r be the radius of the cylinder, ρ be the density of the cylinder, and ρ_0 be the density of the fluid. In equilibrium: $\rho \pi r^2 L g = \rho_0 \pi r^2 \frac{L}{4} g \implies \rho = \frac{\rho_0}{4}$.

In motion: $\rho \pi r^2 L \frac{d^2 y}{dt^2} = \rho \pi r^2 L g - \pi r^2 g \left(\frac{L}{4} + y\right) \rho_0 \implies L \rho \frac{d^2 y}{dt^2} = \rho_0 g y \implies \frac{d^2 y}{dt^2} + \frac{4g}{L} y = 0$. Hence,

$y(t) = c_1 \cos\left(2\sqrt{\frac{g}{L}}t\right) + c_2 \sin\left(2\sqrt{\frac{g}{L}}t\right)$. Thus the motion is simple harmonic with circular frequency

$\omega_0 = 2\sqrt{\frac{g}{L}}$ and period of motion $T = \pi\sqrt{\frac{L}{g}}$.

17. Let θ represent the angular displacement of the pendulum. From Equation (6.5.28), we have $\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$. Since $L = 0.5$, this becomes $\frac{d^2\theta}{dt^2} + 2g\theta = 0$. Therefore,

$$\theta(t) = c_1 \cos(\sqrt{2g}t) + c_2 \sin(\sqrt{2g}t).$$

Since $\theta(0) = \frac{1}{10}$, we have $c_1 = \frac{1}{10}$, and since $\frac{d\theta}{dt}(0) = 0$, we have $c_2 = 0$. Thus $\theta(t) = \frac{1}{10} \cos(\sqrt{2g}t)$.

18. Let θ represent the angular displacement of the pendulum. From Equation (6.5.28), we have $\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$, which implies

$$\theta(t) = A \cos\left(\sqrt{\frac{g}{L}}t\right) + B \sin\left(\sqrt{\frac{g}{L}}t\right).$$

Since $\theta(0) = \alpha$, we have $A = \alpha$, and since $\frac{d\theta}{dt}(0) = \beta$, we have $B = \sqrt{\frac{L}{g}}\beta$. Hence,

$$\theta(t) = \alpha \cos\left(\sqrt{\frac{g}{L}}t\right) + \sqrt{\frac{L}{g}}\beta \sin\left(\sqrt{\frac{g}{L}}t\right),$$

or $\theta(t) = A_0 \cos(\sqrt{\frac{g}{L}}t - \phi)$, where the amplitude is $A_0 = \sqrt{\frac{\alpha^2 g + \beta^2 L}{g}}$, the phase, ϕ , is determined by $\cos \phi = \frac{\alpha}{A_0}$ and $\sin \phi = \frac{\beta}{A_0} \sqrt{\frac{L}{g}}$, and the period is $T = 2\pi \sqrt{\frac{L}{g}}$.

19. In Problem 18, it was shown that the period of the simple pendulum is $T = 2\pi \sqrt{\frac{L}{g}}$. The time required for the pendulum to swing from its extreme position on one side to its extreme position on the other side is half the period, $T/2$. If $T/2 = 1$, then $\pi \sqrt{\frac{L}{g}} = 1$ and $L = \frac{g}{\pi^2}$, so that $L \approx 0.993$ meters.

20. We have $T = 2\pi \sqrt{\frac{L}{g}} \approx 2\pi \sqrt{\frac{0.9}{9.8}}$. The number of ticks per second is $\frac{2}{T} \approx \frac{1}{\pi} \sqrt{\frac{9.8}{0.9}}$. Thus the number of ticks per minute is approximately $\frac{60}{\pi} \sqrt{\frac{9.8}{0.9}} \approx 63$.

21. Since each side is stretched $2a$, it follows that the vertical component of each side is given by $2a \cos(\theta_0) = 2a \frac{4}{5} = \frac{8a}{5}$ where θ_0 is the angle the side of length $5a$ makes with the vertical. Consequently, in equilibrium, $2k \frac{8a}{5} = mg \implies k = \frac{5mg}{16a}$. Now when the mass is pulled down a small vertical distance, $y(t)$, from equilibrium the length $4a$ changes to $4a + y(t)$. Using the diagram below we see that the length of the hypotenuse is $\sqrt{y^2 + 8ay + 25a^2} = 5a \sqrt{1 + \frac{8}{25}y + \frac{y^2}{25a^2}} \approx 5a(1 + \frac{4}{25a}y + \dots)$.

Also, from the figure,

$$\cos \theta = \frac{y + 4a}{5a(1 + \frac{4}{25a}y + \dots)} \approx \frac{y + 4a}{5a} (1 - \frac{4}{25a}y + \dots) \approx \frac{4}{5} + y(\frac{1}{5a} - \frac{16}{125a}) + \dots \approx \frac{4}{5} + \frac{9}{125a}y.$$

Thus,

$$\begin{aligned}
 F &= -2k \left[5a \left(1 + \frac{4}{25a}y + \dots \right) - 3a \right] \cos \theta + mg \\
 &= -2k \left(2a + \frac{4}{5}y + \dots \right) \left(\frac{4}{5} + \frac{9}{125a}y + \dots \right) + mg \\
 &= -2k \left[\frac{8}{5}a + \left(\frac{16}{25} + \frac{18}{125} \right)y + \dots \right] + \frac{16ak}{5} \\
 &= -2k \frac{98}{125}y + \dots \\
 &= -2 \left(\frac{5mg}{16a} \right) \frac{98}{125}y + \dots \\
 &= -\frac{49mg}{100a}y + \dots \approx -\frac{49mg}{100a}y.
 \end{aligned}$$

Thus, $\frac{d^2y}{dt^2} + \frac{49g}{100a}y = 0$, and hence, the period is $T = 2\pi \sqrt{\frac{100a}{49g}} = \frac{20\pi}{7} \sqrt{\frac{a}{g}}$.

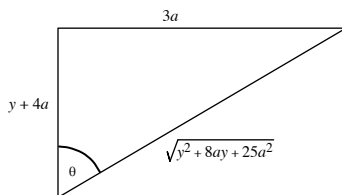


Figure 86: Figure for Problem 21

22. Since each side is stretched $b - a$, it follows that the vertical component of each side is given by $(L - L_0) \cos \theta_0$ where θ_0 is the angle the side of length L makes with the vertical. Consequently, in equilibrium,

$$2k(L - L_0) \cos \theta_0 = mg,$$

so that

$$k = \frac{mg}{2(L - L_0) \cos \theta_0} = \frac{mgL}{2(L - L_0) \sqrt{L^2 - L_0^2}}.$$

Now when the mass is pulled down a small vertical distance, $y(t)$, from equilibrium the length $\sqrt{L^2 - L_0^2}$ changes to $\sqrt{L^2 - L_0^2} + y(t)$. Using the diagram below we see that the length of the hypotenuse is

$$(y^2 + 2y\sqrt{L^2 - L_0^2} + L^2)^{1/2} = L \left(1 + \frac{2}{L^2}y\sqrt{L^2 - L_0^2} + \frac{y^2}{L^2} \right)^{1/2} \approx L \left(1 + y \frac{\sqrt{L^2 - L_0^2}}{L^2} \right)$$

for small y . Also from the figure,

$$\cos \theta \approx \frac{1}{L} (y + \sqrt{L^2 - L_0^2}) \left(\frac{L^2 - y\sqrt{L^2 - L_0^2}}{L^2} + \dots \right),$$

and so $\cos \theta \approx \cos \theta_0 + \frac{L_0^2}{L^3}y$. Consequently,

$$\begin{aligned}
 F &= -2k \left[L \left(1 + \frac{\sqrt{L^2 - L_0^2}}{L^2} y \right) - L_0 \right] \cos \theta + mg \\
 &\approx -2k \left[(L - L_0) + \frac{\sqrt{L^2 - L_0^2}}{L^2} y \right] \left(\cos \theta_0 + \frac{L_0^2}{L^3} y \right) + mg \\
 &= -2k \left\{ (L - L_0) \cos \theta_0 + y \left[\frac{L_0^2}{L^3} (L - L_0) + \frac{\sqrt{L^2 - L_0^2}}{L} \cos \theta_0 \right] \right\} + (L - L_0) \cos \theta_0 \\
 &= -2ky \left[\frac{L_0^2}{L^3} (L - L_0) + \left(\frac{\sqrt{L^2 - L_0^2}}{L} \right)^2 \right] \\
 &= -\frac{2ky}{L^3} (L^3 - L_0^2) \\
 &= -\frac{2y}{L^3} (L - L_0)(L^2 + L_0L + L_0^2) \frac{mgL}{2(L - L_0)\sqrt{L^2 - L_0^2}} \\
 &= -\frac{mg(L^2 + L_0L + L_0^2)}{L^2\sqrt{L^2 - L_0^2}} y.
 \end{aligned}$$

Hence, $m \frac{d^2 y}{dt^2} + \frac{mg(L^2 + L_0L + L_0^2)}{L^2\sqrt{L^2 - L_0^2}} y = 0$. Thus the period is $T = \frac{2\pi}{\omega}$ where $\omega^2 = \frac{g(L^2 + L_0L + L_0^2)}{L^2\sqrt{L^2 - L_0^2}}$.

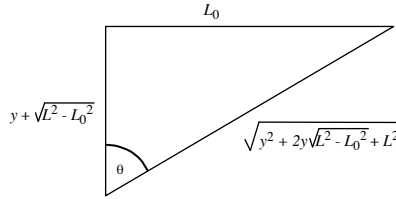


Figure 87: Figure for Problem 22

23.

(a) The motion is under-damped.

(b) We have $P(r) = r^2 + 2r + 5 = 0 \implies r = -1 \pm 2i$ so $y_h(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$. Letting $y_p(t) = C \cos 2t + D \sin 2t$, we obtain $y_p'(t) = -2C \sin 2t + 2D \cos 2t$ and $y_p''(t) = -4C \cos 2t - 4D \sin 2t$. Substituting these results into the original equation results in the system: $-4C + 4D + 5C = 0$ and $-4D - 4C + 5D = 17$. The solution is $C = -4$ and $D = 1$; consequently, $y_p(t) = -4 \cos 2t + \sin 2t$. The general solution is $y(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t) - 4 \cos 2t + \sin 2t$. Since $y(0) = -2$, we have $c_1 = 2$, and since $y'(0) = 0$ we have $2c_2 - c_1 + 2 = 0 \implies c_2 = 0$. Thus $y(t) = 2e^{-t} \cos 2t - 4 \cos 2t + \sin 2t$.

Transient part: $2e^{-t} \cos 2t$.

Steady state part: $-4 \cos 2t + \sin 2t$.

24. Since the system is resonating, $\omega = \omega_0$. Thus $r^2 + \omega_0^2 = 0 \implies r = \pm \omega_0 i \implies y_c(t) = A \cos \omega_0 t + B \sin \omega_0 t$. Let $y_p(t) = t(C \cos \omega_0 t + D \sin \omega_0 t)$ so that $y_p'(t) = (\omega_0 D t + C) \cos \omega_0 t + (D - \omega_0 C t) \sin \omega_0 t$ and $y_p''(t) =$

$\omega_0 D \cos \omega_0 t - (\omega_0^2 D + C \omega_0) \sin \omega_0 t - \omega_0 C \sin \omega_0 t + (D \omega_0 - \omega_0^2 C t) \cos \omega_0 t$. Substituting these results into $y'' + \omega_0 y = F_0 \sin \omega_0 t$ and equating like terms we obtain $D = 0$ and $C = -\frac{F_0}{2\omega_0}$ so $y_p(t) = -\frac{F_0}{2\omega_0} t \cos \omega_0 t$.

From this last equation we have $y(t) = A \cos \omega_0 t + B \sin \omega_0 t - \frac{F_0}{2\omega_0} t \cos \omega_0 t$. Since $y(0) = 0$, we have $A = 0$, and since $y'(0) = 0$, we have $\omega_0 B - \frac{F_0}{2\omega_0} = 0 \implies B = \frac{F_0}{2\omega_0^2}$. Thus, $y(t) = \frac{F_0}{2\omega_0} \sin \omega_0 t - \frac{F_0}{2\omega_0} t \cos \omega_0 t$.

25. Let $y_p(t) = A \sin t + B \cos t \implies y_p'(t) = A \cos t - B \sin t \implies y_p''(t) = -A \sin t - B \cos t$. Substituting these results into $y''(t) + 3y'(t) + 2y(t) = 10 \sin t$, we obtain the system $A - 3B = 10$ and $3A + B = 0$ so $A = 1$ and $B = -3$. Hence, $y_p(t) = \sin t - 3 \cos t = \sqrt{A^2 + B^2} \sin[t + \tan^{-1}(B/A)] = \sqrt{10} \sin(t - \tan^{-1} 3)$.

26. Case 1: $\omega \neq \omega_0$.

Let $y_p = A \cos \omega t + B \sin \omega t$ so $y_p'' = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t$. From $\frac{d^2 y}{dt^2} + \omega_0^2 y = \frac{F_0}{m} \cos \omega t$, we obtain $(-\omega^2 + \omega_0^2)(A \cos \omega t + B \sin \omega t) = \frac{F_0}{m} \cos \omega t \implies A \cos \omega t + B \sin \omega t = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \implies A = \frac{F_0}{m(\omega_0^2 - \omega^2)}$ and $B = 0$. Hence, $y_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$.

Case 2: $\omega_0 = \omega$.

Let $y_p(t) = t(A \cos \omega_0 t + B \sin \omega_0 t)$ so that $y_p'(t) = A \cos \omega_0 t + B \sin \omega_0 t + t(-A\omega_0 \sin \omega_0 t + B\omega_0 \cos \omega_0 t)$, and

$$\begin{aligned} y_p''(t) &= 2(-A\omega_0 \sin \omega_0 t + B\omega_0 \cos \omega_0 t) - t(A\omega_0^2 \cos \omega_0 t + B\omega_0^2 \sin \omega_0 t) \\ &= \sin \omega_0 t(-2A\omega_0 - Bt\omega_0^2) + \cos \omega_0 t(2B\omega_0 - tA\omega_0^2). \end{aligned}$$

Hence,

$$\begin{aligned} y_p'' + \omega_0 y &= \sin \omega_0 t(-2A\omega_0 - Bt\omega_0^2 + Bt\omega_0^2) + \cos \omega_0 t(2B\omega_0 - At\omega_0^2 + At\omega_0^2) \\ &= \sin \omega_0 t(-2A\omega_0) + \cos \omega_0 t(2B\omega_0) = \frac{F_0}{m} \cos \omega_0 t. \end{aligned}$$

Hence, $A = 0$ and $B = \frac{F_0}{2m\omega_0}$. Thus, $y_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$.

27. If $\frac{\omega}{\omega_0} = \frac{m}{n}$, where m and n are integers, then from the given equation

$$\begin{aligned} y\left(t + \frac{2\pi n}{\omega_0}\right) &= A_0 \cos\left[\omega_0\left(t + \frac{2\pi n}{\omega_0}\right) - \phi\right] + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos\left[\omega\left(t + \frac{2\pi n}{\omega_0}\right)\right] \\ &= A_0 \cos(\omega_0 t + 2\pi n - \phi) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos\left(\omega t + \frac{2\pi \omega n}{\omega_0}\right) \\ &= A_0 \cos(\omega_0 t - \phi) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t + 2\pi m) \\ &= A_0 \cos(\omega_0 t - \phi) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t \\ &= y(t). \end{aligned}$$

Thus the motion is periodic with period $T = \frac{2\pi n}{\omega_0}$.

28. From Problem 27, we have $y_c(t) = A_0 \cos(\frac{3}{4}t - \phi)$. Let $y_p(t) = A \cos 2t + B \sin 2t$. Taking the first and second derivatives of y_p and substituting the results into the differential equation gives

$$-4(A \cos 2t + B \sin 2t) + \frac{9}{16}(A \cos 2t + B \sin 2t) = 55 \cos 2t.$$

Equating like terms and solving for the constants we obtain $A = -16$ and $B = 0$. Consequently, $y(t) = A_0 \cos(\frac{3}{4}t - \phi) - 16 \cos 2t$. Using the result of Problem 27, $\frac{\omega}{\omega_0} = \frac{8}{3}$ implies that the motion is periodic with period $T = \frac{2\pi(3)}{3/4} = 8\pi$.

29. Let $y_p = A \cos \omega t + B \sin \omega t$ so that $y'_p = -\omega A \sin \omega t + B \omega \cos \omega t$ and $y''_p = -\omega^2 A \cos \omega t - B \omega^2 \sin \omega t$. Substituting these results into the original equation yields

$$-\omega^2 A \cos \omega t - B \omega^2 \sin \omega t + \frac{c}{m}(-\omega A \sin \omega t + B \omega \cos \omega t) + \frac{k}{m}(A \cos \omega t + B \sin \omega t) = \frac{F_0}{m} \cos \omega t,$$

which implies that

$$\cos \omega t(-\omega^2 A + \frac{c\omega}{m}B + \frac{k}{m}A) + \sin \omega t(-\omega^2 B - \frac{c\omega}{m}A + \frac{k}{m}B) = \frac{F_0}{m} \cos \omega t.$$

Therefore,

$$-\omega^2 B - \frac{c\omega}{m}A + \frac{k}{m}B = 0 \quad \text{and} \quad -\omega^2 A + \frac{c\omega}{m}B + \frac{k}{m}A = \frac{F_0}{m}.$$

Thus, $(\frac{k}{m} - \omega^2)A + \frac{m\omega}{c}B = \frac{F_0}{m}$. Solving this system for A and B yields

$$B = \frac{F_0 c \omega}{(k - m\omega^2)^2 + c^2 \omega^2} \quad \text{and} \quad A = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + c^2 \omega^2}.$$

Thus, the particular solution is $y_p(t) = \frac{F_0}{(k - m\omega^2)^2 + c^2 \omega^2}[(k - m\omega^2) \cos \omega t + c \omega \sin \omega t]$.

30. To minimize $H = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2 \omega^2} = \sqrt{m^2 \omega_0^4 - 2m^2 \omega_0^2 \omega^2 + m^2 \omega^4 + c^2 \omega^2}$, we set

$$0 = \frac{dH}{d\omega} = \frac{1}{2}[m^2 \omega^4 + \omega^2(c^2 - 2m^2 \omega_0^2) + m^2 \omega_0^4]^{-1/2}[4m^2 \omega^3 + 2\omega(c^2 - 2m^2 \omega_0^2)].$$

Therefore, $4m^2 \omega^3 + 2\omega(c^2 - 2m^2 \omega_0^2) = 0$, so that $\omega = 0$ or $2m^2 \omega^2 = 2m^2 \omega_0^2 - c^2$. Since $\omega \neq 0$, the latter case holds, and implies that $\omega = \pm \sqrt{\omega_0^2 - \frac{c^2}{2m^2}}$. When $\omega = \sqrt{\omega_0^2 - \frac{c^2}{2m^2}}$ the amplitude of the steady-state solution is a maximum.

31.

(a) From $\frac{d^2 y}{dt^2} + \frac{c}{m} \frac{dy}{dt} + \frac{k}{m} y = \frac{F(t)}{m}$, we substitute the given information: $y''(t) + 2y'(t) + 5y(t) = 8 \cos \omega t$. Then $r^2 + 2r + 5 = 0 \implies r = -1 \pm 2i$ so let $y_c(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$ and $y_p(t) = A \cos \omega t + B \sin \omega t$. Thus we obtain the system $-\omega^2 B - 2\omega A + 5B = 0$ and $-\omega^2 A + 2\omega B + 5A = 8$. Solving this system yields:

$$A = \frac{40 - 8\omega^2}{\omega^4 - 6\omega^2 + 25} \quad \text{and} \quad B = \frac{16\omega}{\omega^4 - 6\omega^2 + 25}.$$

Transient solution: $y_T(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$.

Steady state solution: $y_S(t) = \frac{40 - 8\omega^2}{\omega^4 - 6\omega^2 + 25} \cos \omega t + \frac{16\omega}{\omega^4 - 6\omega^2 + 25} \sin \omega t$.

(b) Since $m = 1$, $k = 5$, and $c = 2$, we have $\omega = \sqrt{\omega_0^2 - \frac{c^2}{2m^2}} = \sqrt{5 - 2} = \sqrt{3}$ maximizes the amplitude of the steady-state solution (see Problem 30). Therefore, using (a), we have $A = 1$ and $B = \sqrt{3}$, so that

$$\begin{aligned} y_p(t) &= \cos(\sqrt{3}t) + \sqrt{3} \sin(\sqrt{3}t) \\ &= 2 \left[\frac{1}{2} \cos(\sqrt{3}t) + \frac{\sqrt{3}}{2} \sin(\sqrt{3}t) \right] \\ &= 2 \cos(\sqrt{3}t - \pi/3). \end{aligned}$$

32.

(a) Since $F(t) = 4e^{-t} \cos 2t$, as $t \rightarrow \infty$, $F(t)$ exhibits oscillatory behavior with diminishing amplitude. In fact, $F(t) \rightarrow 0$.

(b) The differential equation is $y''(t) + 2y'(t) + 5y(t) = 4e^{-t} \cos 2t$. Then $P(r) = r^2 + 2r + 5 = 0 \implies r = -1 \pm 2i$, so we have

$$y_c(t) = e^{-t}(A \cos 2t + B \sin 2t) \quad \text{and} \quad y_p(t) = te^{-t}(C \cos 2t + D \sin 2t).$$

Now from $y_p''(t) + 2y_p'(t) + 5y_p(t) = 4e^{-t} \cos 2t$, we obtain $e^{-t}(4D \cos 2t - 4C \sin 2t) + te^{-t}(0) = 4e^{-t} \cos 2t$ which implies that $C = 0$ and $D = 1$. Thus, $y_p(t) = te^{-t} \sin 2t$ and $y(t) = e^{-t}(A \cos 2t + B \sin 2t) + te^{-t} \sin 2t$. As $t \rightarrow \infty$, $y(t) \rightarrow 0$.

33. If $\frac{d^2 y}{dt^2} + 16y = 0$ then $r^2 + 16 = 0 \implies r = \pm 4i \implies y_c(t) = A_0 \cos(4t - \phi)$. Now let a particular solution, $y_p(t)$, take the form $y_p(t) = e^{-t}(A \sin t + B \cos t)$. Substituting this expression into the differential equation, we obtain

$$2e^{-t}(B \sin t - A \cos t) + 16e^{-t}(A \sin t + B \cos t) = 130e^{-t} \cos t,$$

which implies that

$$(2B + 16A) \sin t + (16B - 2A) \cos t = 130 \cos t.$$

Thus, $B + 8A = 0$ and $8B - A = 65$. Hence, $A = -1$ and $B = 8$. Consequently, $y_p(t) = e^{-t}(8 \cos t - \sin t)$.

Transient part: $e^{-t}(8 \cos t - \sin t)$.

Steady-state part: $A_0 \cos(4t - \phi)$.

Solutions to Section 6.6

True-False Review:

1. TRUE. The differential equation governing the situation in which no driving electromotive force is present is (6.6.3), and its solutions are given in (6.6.4). In all cases, $q(t) \rightarrow 0$ as $t \rightarrow \infty$. Since the charge decays to zero, the rate of change of charge, which is the current in the circuit, eventually decreases to zero as well.

2. TRUE. For the given constants, it is true that $R^2 < 4L/C$, since $R^2 = 16$ and $4L/C = 272$.

3. TRUE. The amplitude of the steady-state current is given by Equation (6.6.6), which is maximum when $\omega^2 = \omega$. Substituting $\omega^2 = \frac{1}{LC}$, it follows that the amplitude of the steady-state current will be a maximum when

$$\omega = \omega_{max} = \frac{1}{\sqrt{LC}}.$$

4. TRUE. From the form of the amplitude of the steady-state current given in Equation (6.6.6), we see that the amplitude A is directly proportional to the amplitude E_0 of the external driving force.

5. FALSE. The current $i(t)$ in the circuit is the derivative of the charge $q(t)$ on the capacitor, given in the solution to Example 6.6.1:

$$q(t) = A_0 e^{-Rt/2L} \cos(\mu t - \phi) + \frac{E_0}{H} \cos(\omega t - \eta).$$

The derivative of this is not inversely proportional to R .

6. FALSE. The charge on the capacitor decays over time if there is no driving force.

Problems:

1. We are given that $R = \frac{3}{2}$, $L = \frac{1}{2}$, $C = \frac{2}{3}$, $E_0 = 13$, and $\omega = 3$. So $\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(1/2)(2/3)}} = \sqrt{3}$,
 $H = \sqrt{L^2(\omega_0^2 - \omega^2) + R^2\omega^2} = \sqrt{\frac{1}{4}(3 - 9)^2 + \frac{9}{4}(9)} = \frac{3\sqrt{13}}{2}$, $A = \frac{\omega E_0}{H} = \frac{3 \cdot 13}{3\sqrt{13}/2} = 2\sqrt{13}$, $\cos \eta = \frac{L(\omega_0^2 - \omega^2)}{H} = -\frac{2}{\sqrt{13}}$, and $\sin \eta = \frac{R\omega}{H} = \frac{3}{\sqrt{13}}$. Thus

$$\begin{aligned} i_S(t) &= -A \sin(\omega t - \eta) \\ &= -2\sqrt{13} \sin(3t - \eta) \\ &= -2\sqrt{13}(\sin 3t \cos \eta - \cos 3t \sin \eta) \\ &= -2\sqrt{13} \left[\sin 3t \left(-\frac{2}{\sqrt{13}} \right) - \cos 3t \left(\frac{3}{\sqrt{13}} \right) \right] \\ &= 2(2 \sin 3t + 3 \cos 3t). \end{aligned}$$

2. Given the RLC circuit with $L = 4$, $C = \frac{1}{17}$, $E = E_0$ and $R = 4$, we obtain $\mu = \frac{\sqrt{\frac{4L}{C} - R^2}}{2L} = 2$. Thus $q_c(t) = e^{-t/2}(c_1 \cos \mu t + c_2 \sin \mu t)$. Since the differential equation takes the form $\frac{d^2 q}{dt^2} + \frac{dq}{dt} + \frac{17}{4}q = \frac{E_0}{4}$, we can let $q_p(t) = K$ so that $q_p'' = q_p' = 0$. Substituting these results into the differential equation and solving for K yields $K = \frac{E_0}{17}$; hence, $q_p(t) = E_0/17$ and $q(t) = e^{-t/2}(c_1 \cos \mu t + c_2 \sin \mu t) + \frac{E_0}{17}$. As $t \rightarrow \infty$, $q(t) \rightarrow \frac{E_0}{17}$. Since q_c tends to zero as t tends to infinity, for large t that particular solution, $q_p(t)$, will be the dominant part of $q(t)$. The steady-state current is given by $i_S(t) = \frac{dq_p}{dt} = 0$.

3. When $R = 0$, the differential equation is $\frac{d^2 q}{dt^2} + \frac{1}{LC}q = E_0 \cos \omega t$, and so $q_c(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$, where $\omega_0 = \sqrt{\frac{1}{LC}}$.

Case 1: If $\omega = \sqrt{\frac{1}{LC}}$, then let $q_p(t) = t(A \cos \omega t + B \sin \omega t)$. Hence,

$$q_p'' + \omega^2 q_p = E_0 \cos \omega t,$$

and so

$$2\omega B \cos \omega t - 2\omega A \sin \omega t = E_0 \cos \omega t.$$

Therefore, $A = 0$ and $B = \frac{E_0}{2\omega}$, and therefore $q_p(t) = \frac{tE_0}{2\omega} \sin \omega t$ so that

$$q(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{tE_0}{2\omega} \sin \omega t.$$

Thus, as $t \rightarrow \infty$, $q(t) \rightarrow \infty$ when $\omega = \sqrt{\frac{1}{LC}}$.

Case 2: If $\omega \neq \sqrt{\frac{1}{LC}}$, then we take $q_p(t) = A \cos \omega t$ so that

$$q_p'' + \frac{1}{LC} q_p = E_0 \cos \omega t,$$

which implies that

$$-\omega^2 A \cos \omega t + \frac{A}{LC} \cos \omega t = E_0 \cos \omega t \implies \left(\frac{A}{LC} - \omega^2 A \right) \cos \omega t = E_0 \cos \omega t.$$

Therefore, $A = \frac{E_0 LC}{1 - LC\omega^2}$, and so $q_p(t) = \frac{E_0 LC}{1 - LC\omega^2} \cos \omega t$. Hence,

$$q(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{E_0 LC}{1 - LC\omega^2} \cos \omega t,$$

which is a bounded expression. If $\omega = \sqrt{\frac{1}{LC}}$, then $i(t) = \frac{dq}{dt} = (\omega c_2 + \frac{tE_0}{2}) \cos \omega t + (\frac{E_0}{2\omega} - \omega c_1) \sin \omega t$ which implies that $i(t) \rightarrow \infty$ as $t \rightarrow \infty$. If $\omega \neq \sqrt{\frac{1}{LC}}$, then

$$i(t) = \frac{dq}{dt} = \frac{CL\omega E_0}{CL\omega^2 - L} \sin \omega t + \omega_0 c_2 \cos \omega t - \omega_0 c_1 \sin \omega t,$$

which is bounded.

4. Given the RLC circuit with $R = 16$, $L = 8$, $C = \frac{1}{40}$, $E(t) = 17 \cos 2t$, $E_0 = 17$, and $\omega = 2$,

we have $\mu = \frac{\sqrt{\frac{4L}{C} - R^2}}{2L} = 2$, $\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{8(1/40)}} = \sqrt{5}$, and $H = \sqrt{L^2(\omega_0^2 - \omega^2) + R^2\omega^2} = \sqrt{8(5 - 2^2)^2 + 16^2 \cdot 2^2} = 8\sqrt{17}$. Then $\cos \eta = \frac{L(\omega_0^2 - \omega^2)}{H} = \frac{1}{\sqrt{17}}$ and $\sin \eta = \frac{R\omega}{H} = \frac{4}{\sqrt{17}}$ so that

$$q(t) = e^{-\frac{Rt}{2L}} (c_1 \cos \mu t + c_2 \sin \mu t) + \frac{E_0}{H} (\cos \omega t \cos \eta + \sin \omega t \sin \eta) = e^{-t} (c_1 \cos 2t + c_2 \sin 2t) + \frac{1}{8} (\cos 2t + 4 \sin 2t).$$

Now $q(0) = 0 \implies c_1 = -\frac{1}{8}$. Thus,

$$i(t) = \frac{dq}{dt} = e^{-t}[(2c_2 - c_1) \cos 2t + (-2c_1 - c_2) \sin 2t] + \cos 2t - \frac{1}{4} \sin 2t.$$

Since $i(0) = 0$, we have $c_2 = -\frac{9}{16}$. Thus,

$$i(t) = e^{-t}(-\cos 2t + \frac{13}{16} \sin 2t) + \cos 2t - \frac{1}{4} \sin 2t.$$

5. Given the RLC circuit with $R = 3$, $L = \frac{1}{2}$, $C = \frac{1}{5}$, $E(t) = 2 \cos \omega t$, and $E_0 = 2$. So $\mu = \frac{\sqrt{\frac{4L}{C} - R^2}}{2L} = 1$, $\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{10}$, and $H = \sqrt{L^2(\omega_0^2 - \omega^2) + R^2\omega^2} = \sqrt{(1/2)^2(10 - \omega^2)^2 + 3^2\omega^2} = \frac{\sqrt{(10 - \omega^2)^2 + 36\omega^2}}{2}$. Then $\cos \eta = \frac{L(\omega_0^2 - \omega^2)}{H} = \frac{10 - \omega^2}{2H}$ and $\sin \eta = \frac{R\omega}{H} = \frac{3\omega}{H}$ so that

$$q(t) = A_0 e^{-\frac{Rt}{2L}} \cos(\mu t - \phi) + \frac{E_0}{H} \cos(\omega t - \eta),$$

where $c_1 = A_0 \cos \phi$ and $c_2 = A_0 \sin \phi$. Therefore,

$$q(t) = A_0 e^{-3t} \cos(t - \phi) + \frac{2}{H} \cos(\omega t - \eta)$$

and

$$i(t) = \frac{dq}{dt} = -A_0 e^{-3t} [3 \cos(t - \phi) + \sin(t - \phi)] - \frac{2\omega}{H} \sin(\omega t - \eta).$$

Thus $\omega_{\max} = \sqrt{\frac{1}{LC}} = \sqrt{10}$.

6. Differentiating the equation $\frac{d^2 q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{1}{LC} q = \frac{1}{L} E(t)$ with respect to t yields $\frac{d^3 q}{dt^3} + \frac{R}{L} \frac{d^2 q}{dt^2} + \frac{1}{LC} \frac{dq}{dt} = \frac{1}{L} \frac{dE}{dt}$, but $i(t) = \frac{dq}{dt}$ so it follows that $\frac{d^2 i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{1}{LC} i = \frac{1}{L} \frac{dE}{dt}$.

7. Since $E(t) = E_0 e^{-at}$, $\frac{dE}{dt} = -aE_0 e^{-at}$. The equation for current is identical to the equation governing charge, except it has a different nonhomogeneous term.

$$i_c(t) = A_0 e^{-\frac{Rt}{2L}} \cos(\mu t - \phi),$$

and substituting $i_p(t) = Ae^{-at}$ into $\frac{d^2 i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{1}{LC} i = \frac{1}{L} \frac{dE}{dt}$, we have

$$a^2 A e^{-at} + \frac{R}{L} (-a A e^{-at}) + \frac{1}{LC} (A e^{-at}) = \frac{1}{L} (-a E_0 e^{-at}).$$

Therefore,

$$A e^{-at} (a^2 - \frac{aR}{L} + \frac{1}{LC}) = -\frac{aE_0}{L} e^{-at},$$

and so $A = -\frac{aCE_0}{a^2LC - aCR + 1}$. Hence,

$$i_p(t) = -\frac{aCE_0}{a^2LC - aCR + 1}e^{-at} \text{ and } i(t) = A_0e^{-\frac{Rt}{2L}} \cos(\mu t - \phi) - \frac{aCE_0}{a^2LC - aCR + 1}e^{-at},$$

where $\mu = \frac{\sqrt{\frac{4L}{C} - R^2}}{2L}$.

8. Given $R = 2$, $L = \frac{1}{2}$, $C = \frac{2}{5}$ and $E(t) = \begin{cases} 50t, & 0 \leq t < \pi, \\ 50\pi, & t \geq \pi, \end{cases}$, we have $\mu = \frac{\sqrt{\frac{4L}{C} - R^2}}{2L} = 1$. Thus

$$q_c(t) = e^{-\frac{Rt}{2L}}(c_1 \cos(\mu t) + c_2 \sin(\mu t)) = e^{-2t}(c_1 \cos t + c_2 \sin t). \text{ From (6.6.2), } \frac{d^2q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{1}{LC}q = \frac{E(t)}{L}, \text{ and}$$

on the interval $0 < t < \pi$ this becomes $\frac{d^2q}{dt^2} + 4\frac{dq}{dt} + 5q = 100t$. Using the method of undetermined coefficients, if $q_p(t) = At + B$ then from the differential equation we obtain $A = 20$ and $B = -16$ so $q_p(t) = 20t - 16$ which in turn yields $i_p(t) = 20$. On the interval $t \geq \pi$ the differential equation becomes $\frac{d^2q}{dt^2} + 4\frac{dq}{dt} + 5q = 100\pi$. Again using the method of undetermined coefficients the time with $q_p(t) = K$ we obtain after substitution and equating like terms that $K = 20\pi$ so $q_p(t) = 20\pi$ and $i_p(t) = 0$.

On $[0, \pi]$, $q(t) = e^{-2t}(c_1 \cos t + c_2 \sin t) + 20t - 16$. Given $q(0) = 0 \implies c_1 = 16$ and $\frac{dq}{dt}(0) = i(0) = 0 \implies c_2 = 12$ and so $q(t) = e^{-2t}(16 \cos t + 12 \sin t) + 20t - 16$. Therefore,

$$\lim_{t \rightarrow \pi^-} q(t) = 20\pi - 16(e^{-2\pi} + 1).$$

Further, $i(t) = q'(t) = -e^{-2t}(20 \cos t + 40 \sin t) + 20$, so that

$$\lim_{t \rightarrow \pi^-} i(t) = 20(e^{-2\pi} + 1).$$

On $[\pi, \infty)$, $q(t) = e^{-2t}(c_3 \cos t + c_4 \sin t) + 20\pi$. Continuity of the solution at $t = \pi$ requires that

$$q(\pi) = \lim_{t \rightarrow \pi^-} q(t),$$

so that

$$20\pi - c_3e^{-2\pi} = 20\pi - 16(e^{-2\pi} + 1).$$

Consequently, $c_3 = 16(e^{2\pi} + 1)$. Now $i(t) = \frac{dq}{dt} = i(t) = e^{-2t}[(c_4 - 2c_3) \cos t - (c_3 + 2c_4) \sin t]$, and continuity at $t = \pi$ requires

$$i(\pi) = \lim_{t \rightarrow \pi^-} i(t).$$

Therefore

$$20(e^{-2\pi} + 1) = -e^{-2\pi}(c_4 - 2c_3)$$

so that $c_4 = 12(e^{2\pi} + 1)$.

Consequently, $i(t) = e^{-2t}[-20(e^{2\pi} + 1) \cos t - 40(e^{2\pi} + 1) \sin t] = -20e^{-2t}(e^{-2\pi} + 1)(\cos t + 2 \sin t)$ for $t \geq \pi$. Thus

$$i(t) = \begin{cases} 20[1 - e^{-2t}(\cos t + 2 \sin t)], & 0 \leq t < \pi, \\ -20e^{-2t}(e^{-2\pi} + 1)(\cos t + 2 \sin t), & t \geq \pi. \end{cases}$$

Solutions to Section 6.7

True-False Review:

1. **TRUE.** This is essentially the statement of the variation-of-parameters method (Theorem 6.7.1).
2. **FALSE.** The solutions $y_1, y_2, \dots, y_n$ must form a *linearly independent* set of solutions to the associated homogeneous differential equation (6.7.18).
3. **FALSE.** The requirement on the functions $u_1, u_2, \dots, u_n$, according to Theorem 6.7.6, is that they satisfy Equations (6.7.22). Because these equations involve the *derivatives* of $u_1, u_2, \dots, u_n$ only, constants of integration can be arbitrarily chosen in solving (6.7.22), and therefore, addition of constants to the functions $u_1, u_2, \dots, u_n$ again yields valid solutions to (6.7.22).

Problems:

1. Setting $y'' + 6y' + 9y = 0 \implies r^2 + 6r + 9 = 0 \implies r \in \{-3, -3\} \implies y_c(x) = e^{-3x}(c_1 + c_2x)$. Let $y_1(x) = e^{-3x}$ and $y_2(x) = xe^{-3x}$. We have $W[y_1, y_2](x) = e^{-6x}$. Then a particular solution to the given differential equation is $y_p(x) = u_1y_1 + u_2y_2$, where $e^{-3x}u_1' + xe^{-3x}u_2' = 0$ and $-3e^{-3x}u_1' + e^{-3x}(1-3x)u_2' = \frac{2e^{-3x}}{x^2+1}$. That is, $u_1' + xu_2' = 0$, $-3u_1' + (1-3x)u_2' = \frac{2}{x^2+1}$. Solving for u_1 and u_2 (and setting integration constants to zero), we have

$$u_1 = - \int^x \frac{2te^{-3t}(e^{-3t})}{e^{-6t}(t^2+1)} dt = - \int^x \frac{2t}{t^2+1} dt = -\ln(x^2+1)$$

and

$$u_2 = \int^x \frac{2e^{-3t}(e^{-3t})}{e^{-6t}(t^2+1)} dt = \int^x \frac{2}{t^2+1} dt = 2 \tan^{-1}(x).$$

Thus $y_p(x) = -e^{-3x} \ln(x^2+1) + 2xe^{-3x} \tan^{-1}(x)$. Therefore, $y(x) = e^{-3x}[c_1 + c_2x + 2x \tan^{-1}(x) - \ln(x^2+1)]$.

2. Setting $y'' - 4y = 0 \implies r^2 - 4 = 0 \implies r \in \{2, -2\} \implies y_c(x) = c_1e^{2x} + c_2e^{-2x}$. Let $y_1(x) = e^{2x}$ and $y_2(x) = e^{-2x}$. We have $W[y_1, y_2](x) = -4$. Then a particular solution to the given differential equation is $y_p(x) = u_1y_1 + u_2y_2$, where $e^{2x}u_1' + e^{-2x}u_2' = 0$, $2e^{2x}u_1' - 2e^{-2x}u_2' = \frac{8}{e^{2x}+1}$. Solving for u_1 and u_2 (and setting integration constants to zero), we have

$$u_1 = - \int^x \frac{8e^{-2t}}{(e^{2t}+1)W[y_1, y_2]} dt = \int^x \frac{2e^{-2t}}{e^{2t}+1} dt = \ln(e^{2x}+1) - 2x - e^{-2x}$$

and

$$u_2 = \int^x \frac{8e^{2t}}{(e^{2t}+1)W[y_1, y_2]} dt = -\ln(e^{2x}+1),$$

where we have used a calculator to evaluate the integral required for u_1 . Thus,

$$y_p(x) = [\ln(e^{2x}+1) - 2x - e^{-2x}]e^{2x} - \ln(e^{2x}+1)e^{-2x}$$

and the general solution is

$$y(x) = c_1e^{2x} + c_2e^{-2x} + [\ln(e^{2x}+1) - 2x - e^{-2x}]e^{2x} - \ln(e^{2x}+1)e^{-2x}.$$

3. Setting $y'' - 4y' + 5y = 0 \implies r^2 - 4r + 5 = 0 \implies r \in \{2 + i, 2 - i\} \implies y_c(x) = e^{2x}(c_1 \cos x + c_2 \sin x)$. Let $y_1(x) = e^{2x} \cos x$ and $y_2(x) = e^{2x} \sin x$. Then $W[y_1, y_2](x) = e^{4x}$. A particular solution to the given differential equation is therefore $y_p = u_1 y_1 + u_2 y_2$, where

$$e^{2x} \cos x u'_1 + e^{2x} \sin x u'_2 = 0$$

and

$$e^{2x}(2 \cos x - \sin x)u'_1 + e^{2x}(2 \sin x + \cos x)u'_2 = e^{2x} \tan x.$$

Solving for u_1 and u_2 (and setting integration constants to zero), we have

$$u_1 = - \int^x \frac{e^{2t} \sin t (e^{2t} \tan t)}{e^{4t}} dt = \int^x \sin t \tan t dt = \sin x - \ln |\sec x + \tan x|$$

and

$$u_2 = \int^x \frac{e^{2t} \cos t (e^{2t} \tan t)}{e^{4t}} dt = \int^x \sin t dt = -\cos x.$$

Consequently, $y_p(x) = e^{2x} \cos x (\sin x - \ln |\sec x + \tan x|) - e^{2x} \sin x \cos x = -e^{2x} \cos x \ln |\sec x + \tan x|$. Thus, $y(x) = e^{2x} [\cos x (c_1 - \ln |\sec x + \tan x|) + c_2 \sin x]$.

4. Given $y'' - 6y' + 9y = 4e^{3x} \ln x \implies r^2 - 6r + 9 = (r - 3)^2 = 0 \implies r \in \{3, 3\} \implies y_c(x) = c_1 e^{3x} + c_2 x e^{3x}$. Let $y_1(x) = e^{3x}$ and $y_2(x) = x e^{3x}$. We have $W[y_1, y_2](x) = e^{6x}$. Then a particular solution to the given differential equation is $y_p(x) = y_1 u_1 + y_2 u_2$, where

$$e^{3x} u'_1 + x e^{3x} u'_2 = 0, \quad 3e^{3x} u'_1 + e^{3x} (3x + 1) u'_2 = 4e^{3x} \ln x.$$

Solving for u_1 and u_2 (and setting integration constants to zero), we have

$$u_1 = - \int^x \frac{(te^{3t})(4e^{3t} \ln t)}{e^{6t}} dt = - \int^x 4t \ln t dt = x^2(1 - 2 \ln x)$$

and

$$u_2 = \int^x \frac{(e^{3t})(4e^{3t} \ln t)}{e^{6t}} dt = \int^x 4 \ln t dt = 4x(\ln x - 1).$$

Consequently,

$$y_p(x) = x^2 e^{3x} (1 - 2 \ln x) + 4x^2 e^{3x} (\ln x - 1) = x^2 e^{3x} (2 \ln x - 3),$$

so that

$$y(x) = c_1 e^{3x} + c_2 x e^{3x} + x^2 e^{3x} (2 \ln x - 3).$$

5. Setting $y'' + 4y' + 4y = 0 \implies r^2 + 4r + 4 = (r + 2)^2 = 0 \implies r \in \{-2, -2\} \implies y_c(x) = c_1 e^{-2x} + c_2 x e^{-2x}$. Let $y_1(x) = e^{-2x}$ and $y_2(x) = x e^{-2x}$. We have $W[y_1, y_2](x) = e^{-4x}$. Then a particular solution to the given differential equation is $y_p(x) = y_1 u_1 + y_2 u_2$, where

$$u'_1 + x u'_2 = 0, \quad -2u'_1 + (1 - 2x)u'_2 = \frac{1}{x^2}.$$

Solving for u_1 and u_2 (and setting integration constants to zero), we have

$$u_1 = - \int^x \frac{(te^{-2t})e^{-2t}}{t^2 e^{-4t}} dt = - \int^x \frac{1}{t} dt = -\ln x$$

and

$$u_2 = \int^x \frac{(e^{-2t})e^{-2t}}{t^2 e^{-4t}} dt = -\frac{1}{x}.$$

Consequently,

$$y_p(x) = -e^{-2x} \ln x - e^{-2x}.$$

Hence,

$$y(x) = e^{-2x}(c_1 + c_2 x - \ln x - 1).$$

6. Setting $y'' + 9y = 0 \implies r^2 + 9 = 0 \implies r \in \{3i, -3i\} \implies y_c(x) = c_1 \cos 3x + c_2 \sin 3x$. Let $y_1(x) = \cos 3x$ and $y_2(x) = \sin 3x$. Then,

$$W[y_1, y_2](x) = (\cos 3x)(3 \cos 3x) - (\sin 3x)(-3 \sin 3x) = 3 \neq 0$$

so that $y_p(x) = y_1 u_1 + y_2 u_2$, where

$$u_1 = - \int^x \frac{(\sin 3t)(18 \sec^3 3t)}{3} dt = -6 \int^x (\sin 3t)(\sec^3 3t) dt = -\tan^2 3x$$

and

$$u_2 = \int^x \frac{(\cos 3t)(18 \sec^3 3t)}{3} dt = 6 \int^x \sec^2 3t dt = 2 \tan 3x.$$

Consequently,

$$y_p(x) = -\cos 3x \tan^2 3x + 2 \sin 3x \tan 3x.$$

Hence,

$$y(x) = c_1 \cos 3x + c_2 \sin 3x - \cos 3x \tan^2 3x + 2 \sin 3x \tan 3x.$$

7. Setting $y'' - y = 0 \implies r^2 - 1 = 0 \implies r \in \{-1, 1\} \implies y_c(x) = c_1 e^x + c_2 e^{-x}$. Let $y_1(x) = e^x$ and $y_2(x) = e^{-x}$. Then,

$$W[y_1, y_2](x) = (e^x)(-e^{-x}) - (e^x)(e^{-x}) = -2 \neq 0$$

so that

$$\begin{aligned} y_p(x) &= -y_1 \int \frac{y_2 F}{W} dx + y_2 \int \frac{y_1 F}{W} dx \\ &= e^x \int \frac{1 - e^{-2x}}{e^x + e^{-x}} dx - e^{-x} \int \frac{e^{2x} - 1}{e^x + e^{-x}} dx \\ &= e^x [2 \tan^{-1}(e^x) + e^{-x}] - e^{-x} [e^x - 2 \tan^{-1}(e^x)] \\ &= 2(e^x + e^{-x}) \tan^{-1}(e^x). \end{aligned}$$

Hence, $y(x) = c_1 e^x + c_2 e^{-x} + 4 \cosh x \tan^{-1}(e^x)$.

8. Setting $y'' - 2my' + m^2 y = 0 \implies r^2 - 2mr + m^2 = 0 \implies r \in \{m, m\} \implies y_c(x) = e^{mx}(c_1 + c_2 x)$. Let $y_1(x) = e^{mx}$ and $y_2(x) = x e^{mx}$. Then,

$$W[y_1, y_2](x) = (e^{mx})(e^{mx}[mx + 1]) - (m e^{mx})(x e^{mx}) = e^{2mx} \neq 0$$

so that

$$\begin{aligned} y_p(x) &= -y_1 \int \frac{y_2 F}{W} dx + y_2 \int \frac{y_1 F}{W} dx \\ &= -e^{mx} \int \frac{x}{1 + x^2} dx + x e^{mx} \int \frac{1}{1 + x^2} dx \\ &= -\frac{e^{mx}}{2} \ln(1 + x^2) + x e^{mx} \tan^{-1}(x). \end{aligned}$$

Thus, $y(x) = e^{mx}(c_1 + c_2x + x \tan^{-1}(x) - \ln \sqrt{1+x^2})$.

9. Setting $y'' - 2y' + y = 0 \implies r^2 - 2r + 1 = 0 \implies r \in \{1, 1\} \implies y_c(x) = c_1e^x + c_2xe^x$. Let $y_1(x) = e^x$ and $y_2(x) = xe^x$. Then,

$$W[y_1, y_2](x) = (e^x)(e^x[x+1]) - (e^x)(xe^x) = e^{2x} \neq 0$$

so that

$$\begin{aligned} y_p(x) &= -y_1 \int \frac{y_2 F}{W} dx + y_2 \int \frac{y_1 F}{W} dx \\ &= -4e^x \int x^{-2} \ln x dx + 4xe^x \int x^{-3} \ln x dx \\ &= \frac{e^x}{x} (2 \ln x + 3). \end{aligned}$$

Consequently, $y(x) = e^x[c_1 + c_2x + x^{-1}(2 \ln x + 3)]$.

10. Setting $y'' + 2y' + y = 0 \implies r^2 + 2r + 1 = 0 \implies r \in \{-1, -1\} \implies y_c(x) = e^{-x}(c_1 + c_2x)$. Let $y_1(x) = e^{-x}$ and $y_2(x) = xe^{-x}$. Then,

$$W[y_1, y_2](x) = (e^{-x})(e^{-x}[1-x]) + (xe^{-x})(-e^{-x}) = e^{-2x} \neq 0$$

so that

$$\begin{aligned} y_p(x) &= -y_1 \int \frac{y_2 F}{W} dx + y_2 \int \frac{y_1 F}{W} dx \\ &= -e^{-x} \int \frac{x}{\sqrt{4-x^2}} dx + xe^{-x} \int \frac{1}{\sqrt{4-x^2}} dx \\ &= -e^{-x}(-\sqrt{4-x^2}) + xe^{-x} \sin^{-1}\left(\frac{x}{2}\right). \end{aligned}$$

Consequently, $y(x) = e^{-x}(c_1 + c_2x + x \sin^{-1}\left(\frac{x}{2}\right) + \sqrt{4-x^2})$.

11. Setting $y'' + 2y' + 17 = 0 \implies r^2 + 2r + 17 = 0 \implies r \in \{1+4i, 1-4i\} \implies y_c(x) = e^{-x}(c_1 \cos 4x + c_2 \sin 4x)$. Let $y_1(x) = e^{-x} \cos 4x$ and $y_2(x) = e^{-x} \sin 4x$. Then

$$W[y_1, y_2](x) = (e^{-x} \cos 4x)(e^{-x}[4 \cos 4x - \sin 4x]) - (e^{-x} \sin 4x)[-e^{-x}(4 \sin 4x + \cos 4x)] = 4e^{-2x}$$

so that

$$\begin{aligned} y_p(x) &= -y_1 \int \frac{y_2 F}{W} dx + y_2 \int \frac{y_1 F}{W} dx \\ &= -16e^{-x} \cos 4x \int \frac{\sin 4x}{3 + \sin^2 4x} dx + 16e^{-x} \sin 4x \int \frac{\cos 4x}{3 + \sin^2 4x} dx \\ &= -16e^{-x} \cos 4x \left[-\frac{1}{8} \tanh^{-1}(\cos(4x/2)) \right] + 16e^{-x} \sin 4x \left[\frac{\sqrt{3}}{12} \tan^{-1}(\sin(4x/\sqrt{3})) \right] \\ &= 2e^{-x} \cos 4x \tanh^{-1}(\cos(4x/2)) + \frac{4\sqrt{3}}{3} e^{-x} \sin 4x \tan^{-1}(\sin(4x/\sqrt{3})) \end{aligned}$$

Therefore, $y(x) = e^{-x} \left[c_1 \cos 4x + c_2 \sin 4x + \cos 4x \ln \left(\frac{\cos 4x + 2}{\cos 4x - 2} \right) + \frac{4\sqrt{3}}{3} \sin 4x \tan^{-1}(\sin 4x/\sqrt{3}) \right]$.

12. Setting $y'' + 9y = 0 \implies r^2 + 9 = 0 \implies r \in \{-3i, 3i\} \implies y_c(x) = c_1 \cos 3x + c_2 \sin 3x$. Let $y_1(x) = \cos 3x$ and $y_2(x) = \sin 3x$. Then,

$$W[y_1, y_2](x) = (\cos 3x)(3 \cos 3x) - (\sin 3x)(-3 \cos 3x) = 3 \neq 0$$

so that

$$\begin{aligned} y_p(x) &= -y_1 \int \frac{y_2 F}{W} dx + y_2 \int \frac{y_1 F}{W} dx \\ &= -12 \cos 3x \int \frac{\sin 3x}{4 - \cos^2 3x} dx + 12 \sin 3x \int \frac{\cos 3x}{4 - \cos^2 3x} dx \\ &= -12 \cos 3x \left[-\frac{1}{6} \tanh^{-1}(\cos 3x/2) \right] + 12 \sin 3x \left[\frac{\sqrt{3}}{9} \tan^{-1}(\sin 3x/\sqrt{3}) \right]. \end{aligned}$$

Thus, $y(x) = [c_1 + 2 \tanh^{-1}(\cos 3x/2)] \cos 3x + [c_2 + \frac{4\sqrt{3}}{3} \tan^{-1}(\sin 3x/\sqrt{3})] \sin 3x$.

13. Setting $y'' - 10y' + 25y = 0 \implies r^2 - 10r + 25 = 0 \implies r \in \{5, 5\} \implies y_c(x) = e^{5x}(c_1 + c_2x)$. Let $y_1(x) = e^{5x}$ and $y_2(x) = xe^{5x}$. Then,

$$W[y_1, y_2](x) = (e^{5x})[e^{5x}(5x + 1)] - (xe^{5x})(5e^{5x}) = e^{10x} \neq 0$$

so that

$$\begin{aligned} y_p(x) &= -y_1 \int \frac{y_2 F}{W} dx + y_2 \int \frac{y_1 F}{W} dx \\ &= -2e^{5x} \int \frac{x}{4 + x^2} dx + 2xe^{5x} \int \frac{1}{4 + x^2} dx \\ &= -e^{5x} \ln(4 + x^2) + xe^{5x} \tan^{-1}(x/2) \end{aligned}$$

Hence, $y(x) = e^{5x}[c_1 + c_2x - \ln(4 + x^2) + x \tan^{-1}(x/2)]$.

14. Setting $y'' - 6y' + 13y = 0 \implies r^2 - 6r + 13 = 0 \implies r \in \{3 + 2i, 3 - 2i\} \implies y_c(x) = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$. Let $y_1(x) = e^{3x} \cos 2x$ and $y_2(x) = e^{3x} \sin 2x$. Then,

$$W[y_1, y_2](x) = (e^{3x} \cos 2x)(e^{3x}[2 \cos 2x + 3 \sin 2x]) - (e^{3x} \sin 2x)(e^{3x}[3 \cos 2x - 2 \sin 2x]) = 2e^{6x}$$

so that

$$\begin{aligned} y_p(x) &= -y_1 \int \frac{y_2 F}{W} dx + y_2 \int \frac{y_1 F}{W} dx \\ &= -2e^{3x} \cos 2x \int \sin 2x \sec^2 2x dx + 2e^{3x} \sin 2x \int \sec 2x dx \\ &= e^{3x}(\sin 2x \ln |\sec 2x + \tan 2x| - 1). \end{aligned}$$

Hence, $y(x) = e^{3x}(c_1 \cos 2x + c_2 \sin 2x + \sin 2x \ln |\sec 2x + \tan 2x| - 1)$.

15. The complementary function for the differential equation is $y_c(x) = c_1 \cos x + c_2 \sin x$. To determine a particular solution to the given differential equation we first find particular solutions to each of the differential equations $y'' + y = \sec x$, and $y'' + y = 4e^x$, and then add the two solutions together. Using variation-of-parameters, a particular solution to $y'' + y = \sec x$ is $y_{p1}(x) = u_1 \cos x + u_2 \sin x$, where

$$\cos x u'_1 + \sin x u'_2 = 0, \quad -\sin x u'_1 + \cos x u'_2 = \sec x.$$

This pair of equations has a solution $u'_1 = -\tan x$ and $u'_2 = 1$. Consequently, we can choose

$$u_1(x) = \ln(\cos x) \quad \text{and} \quad u_2(x) = x,$$

so that

$$y_{p1}(x) = \cos x \ln(\cos x) + x \sin x.$$

From the method of undetermined coefficients, the differential equation $y'' + y = 4e^x$ has a particular solution of the form $y_{p_2}(x) = 2e^x$. A particular solution to the given differential equation is therefore

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x) = \cos x \ln(\cos x) + x \sin x + 2e^x,$$

and the general solution is

$$y(x) = c_1 \cos x + c_2 \sin x + \cos x \ln(\cos x) + x \sin x + 2e^x.$$

16. The complementary function for the differential equation is $y_c(x) = c_1 \cos x + c_2 \sin x$. To determine a particular solution to the given differential equation we first find particular solutions to the differential equations $y'' + y = \csc x$ and $y'' + y = 2x^2 + 5x + 1$, and then add the two solutions together. Using variation-of-parameters, a particular solution to $y'' + y = \csc x$ is $y_{p_1}(x) = u_1 \cos x + u_2 \sin x$, where

$$\cos x u'_1 + \sin x u'_2 = 0, \quad -\sin x u'_1 + \cos x u'_2 = \csc x.$$

This pair of equations has a solution

$$u'_1 = -1 \quad \text{and} \quad u'_2 = \cot x.$$

Consequently, we can choose $u_1(x) = -x$ and $u_2(x) = \ln(\sin x)$, so that

$$y_{p_1}(x) = -x \cos x + \ln(\sin x) \sin x.$$

From the method of undetermined coefficients, the differential equation $y'' + y = 2x^2 + 5x + 1$ has a particular solution of the form $y_{p_2}(x) = A_0 + A_1x + A_2x^2$. Substituting into the preceding differential equation and equating coefficients yields $A_0 = -3$, $A_1 = 5$ and $A_2 = 2$. Hence, $y_{p_2}(x) = -3 + 5x + 2x^2$. A particular solution to the given differential equation is therefore

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x) = -x \cos x + \ln(\sin x) \sin x - 3 + 5x + 2x^2$$

and the general solution to the differential equation is

$$y(x) = c_1 \cos x + c_2 \sin x - x \cos x + \ln(\sin x) \sin x - 3 + 5x + 2x^2.$$

17. The complementary function for the differential equation is $y_c(x) = c_1 e^{-2x} + c_2 x e^{-2x}$. To determine a particular solution to the given differential equation, we first find particular solutions to each of the differential equations $y'' + 4y' + 4y = 15e^{-2x} \ln x$ and $y'' + 4y' + 4y = 25 \cos x$ separately, and then we add the two solutions together. Using variation-of-parameters, a particular solution to $y'' + 4y' + 4y = 15e^{-2x} \ln x$ is $y_{p_1}(x) = u_1 e^{-2x} + u_2 x e^{-2x}$, where

$$e^{-2x} u'_1 + x e^{-2x} u'_2 = 0, \quad -2e^{-2x} u'_1 + e^{-2x} (1 - 2x) u'_2 = 15e^{-2x} \ln x.$$

This pair of equations has a solution

$$u'_1 = -15x \ln x, \quad u'_2 = 15 \ln x.$$

Consequently, we can choose $u_1(x) = -\frac{15}{4}x^2(2 \ln x - 1)$ and $u_2(x) = 15x(\ln x - 1)$ so that

$$y_{p_1}(x) = -\frac{15}{4}x^2 e^{-2x} (2 \ln x - 1) + 15x^2 e^{-2x} (\ln x - 1) = \frac{15}{4}x^2 e^{-2x} (2 \ln x - 3).$$

According to the method of undetermined coefficients, the differential equation $y'' + 4y' + 4y = 25 \cos x$ has a particular solution of the form $y_{p_2}(x) = A_0 \cos x + A_1 \sin x$. Substituting into the preceding differential equation and equating the coefficients yields $3A_0 + 4A_1 = 25$ and $-4A_0 + 3A_1 = 0$, with a solution $A_0 = 3$, $A_1 = 4$. Consequently,

$$y_{p_2}(x) = 3 \cos x + 4 \sin x.$$

A particular solution to the given differential equation is therefore

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x) = \frac{15}{4}x^2e^{-2x}(2 \ln x - 3) + 3 \cos x + 4 \sin x,$$

and the general solution is

$$y(x) = c_1e^{-2x} + c_2xe^{-2x} + \frac{15}{4}x^2e^{-2x}(2 \ln x - 3) + 3 \cos x + 4 \sin x.$$

18. The complementary function for the differential equation is $y_c(x) = c_1e^{-2x} + c_2xe^{-2x}$. To determine a particular solution to the given differential equation we first find a particular solution to each of the differential equations $y'' + 4y' + 4y = \frac{4e^{-2x}}{1+x^2}$ and $y'' + 4y' + 4y = 2x^2 - 1$ separately, and then we add the two solutions together. Using variation-of-parameters, a particular solution to $y'' + 4y' + 4y = \frac{4e^{-2x}}{1+x^2}$ is $y_{p_1}(x) = u_1e^{-2x} + u_2xe^{-2x}$, where

$$e^{-2x}u'_1 + xe^{-2x}u'_2 = 0, \quad -2e^{-2x}u'_1 + (1-2x)e^{-2x}u'_2 = \frac{4e^{-2x}}{1+x^2}.$$

This pair of equations has a solution $u'_1 = -\frac{4x}{1+x^2}$ and $u'_2 = \frac{4}{1+x^2}$, so that we can choose $u_1(x) = -2 \ln(1+x^2)$ and $u_2(x) = 4 \tan^{-1} x$. Consequently,

$$y_{p_1}(x) = -2e^{-2x} \ln(1+x^2) + 4xe^{-2x} \tan^{-1} x.$$

According to the method of undetermined coefficients, a particular solution to $y'' + 4y' + 4y = 2x^2 - 1$ is of the form $y_{p_2}(x) = A_0 + A_1x + A_2x^2$. Substitution into the preceding differential equation and equating coefficients yields $4A_2 = 2$, $8A_2 + 4A_1 = 0$ and $2A_2 + 4A_1 + 4A_0 = -1$. These equations have solution $A_0 = \frac{1}{2}$, $A_1 = -1$ and $A_2 = \frac{1}{2}$. Hence,

$$y_{p_2}(x) = \frac{1}{2} - x + \frac{1}{2}x^2 = \frac{1}{2}(x-1)^2.$$

A particular solution to the given differential equation is therefore,

$$y_p(x) = y_{p_1}(x) + y_{p_2}(x) = -2e^{-2x} \ln(1+x^2) + 4xe^{-2x} \tan^{-1} x + \frac{1}{2}(x-1)^2,$$

and the general solution is

$$y(x) = c_1e^{-2x} + c_2xe^{-2x} - 2e^{-2x} \ln(1+x^2) + 4xe^{-2x} \tan^{-1} x + \frac{1}{2}(x-1)^2.$$

19. The complementary function for the differential equation is $y_c(x) = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x}$. Let $y_1(x) = e^{2x}$, $y_2(x) = x e^{2x}$ and $y_3(x) = x^2 e^{2x}$. The system (6.7.22) from the text assumes the form

$$\begin{aligned} u'_1 + x u'_2 + x^2 u'_3 &= 0, \\ 2u'_1 + (1 + 2x)u'_2 + (2x + 2x^2)u'_3 &= 0, \\ 4u'_1 + (4 + 4x)u'_2 + (2 + 8x + 4x^2)u'_3 &= 36 \ln x, \end{aligned}$$

where we have divided each equation by e^{2x} . Solving the system we obtain $u'_1 = 18x^2 \ln x$, $u'_2 = -36x \ln x$, and $u'_3 = 18 \ln x$ from which it follows that $u_1 = 2x^3(3 \ln x - 1)$, $u_2 = 9x^2(1 - 2 \ln x)$ and $u_3 = 18x(\ln x - 1)$. Hence,

$$\begin{aligned} y_p(x) &= u_1 y_1 + u_2 y_2 + u_3 y_3 \\ &= [2x^3(3 \ln x - 1)]e^{2x} + [9x^2(1 - 2 \ln x)]x e^{2x} + [18x(\ln x - 1)]x^2 e^{2x} \\ &= x^3 e^{2x}(6 \ln x - 11). \end{aligned}$$

Therefore, the general solution is

$$y(x) = c_1 e^{2x} + c_2 x e^{2x} + c_3 x^2 e^{2x} + x^3 e^{2x}(6 \ln x - 11).$$

20. The complementary function to the given differential equation is $y_c(x) = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$. Let $y_1(x) = e^x$, $y_2(x) = x e^x$ and $y_3(x) = x^2 e^x$. The system (6.7.22) from the text assumes the form

$$\begin{aligned} u'_1 + x u'_2 + x^2 u'_3 &= 0, \\ u'_1 + (1 + x)u'_2 + (2x + x^2)u'_3 &= 0, \\ u'_1 + (2 + x)u'_2 + (2 + 4x + x^2)u'_3 &= \frac{2}{x^2}, \end{aligned}$$

where we have divided each equation by e^x . Solving the system we obtain $u'_1 = 1$, $u'_2 = -2x^{-1}$, and $u'_3 = x^{-2}$, from which it follows that $u_1 = x$, $u_2 = -2 \ln x$, and $u_3 = -x^{-1}$. Hence, $y_p(x) = e^x(x - 2x \ln x - x) = -2x e^x \ln x$, and therefore the general solution to the differential equation is

$$y(x) = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + e^x(x - 2x \ln x - x) = -2x e^x \ln x.$$

21. The complementary function for the differential equation is $y_c(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$. The system (6.7.22) from the text assumes the form

$$\begin{aligned} u'_1 + x u'_2 + x^2 u'_3 &= 0, \\ -u'_1 + (1 - x)u'_2 + (2x - x^2)u'_3 &= 0, \\ u'_1 + (x - 2)u'_2 + (2 - 4x + x^2)u'_3 &= \frac{2}{1 + x^2}, \end{aligned}$$

where we have divided each equation by e^{-x} . Solving the system we obtain $u'_1 = \frac{x^2}{1 + x^2}$, $u'_2 = -\frac{2x}{1 + x^2}$, and $u'_3 = \frac{1}{1 + x^2}$ from which it follows that $u_1 = x - \tan^{-1} x$, $u_2 = -\ln(x^2 + 1)$, and $u_3 = \tan^{-1} x$. Thus,

$$\begin{aligned} y_p(x) &= u_1 y_1 + u_2 y_2 + u_3 y_3 \\ &= (x - \tan^{-1} x)e^{-x} - x e^{-x} \ln(x^2 + 1) + x^2 e^{-x} \tan^{-1} x. \end{aligned}$$

Therefore, the general solution to the differential equation is

$$y(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} + (x - \tan^{-1} x)e^{-x} - x e^{-x} \ln(x^2 + 1) + x^2 e^{-x} \tan^{-1} x.$$

22. The complementary function for the differential equation is $y_c(x) = c_1 + c_2e^{3x} + c_3xe^{3x}$. Let $z = y'$. The given differential equation becomes $z'' - 6z' + 9z = 12e^{3x}$. Then $r^2 - 6r + 9 = 0 \implies r \in \{3, 3\} \implies z_1(x) = e^{3x}$ and $z_2(x) = xe^{3x}$ are two linearly independent solutions of the homogeneous differential equation $z'' - 6z' + 9z = 0$. The system (6.7.22) in the text therefore assumes the form

$$\begin{aligned} u'_1 + xu'_2 &= 0, \\ 3u'_1 + (3x+1)u'_2 &= 12. \end{aligned}$$

Solving the system we obtain $u'_1 = -12x$ and $u_2 = 12$ so $u_1 = -6x^2$ and $u_2 = 12x$. Hence, $z_p(x) = u_1z_1 + u_2z_2 = (-6x^2)e^{3x} + (12x)(xe^{3x}) = 6x^2e^{3x}$. Therefore,

$$y_p(x) = \int z_p dx = 6 \int x^2 e^{3x} dx = \frac{2}{9}(9x^2 - 6x + 2)e^{3x},$$

and so the general solution to the differential equation is

$$y(x) = c_1 + c_2e^{3x} + c_3xe^{3x} + \frac{2}{9}(9x^2 - 6x + 2)e^{3x}.$$

The method of undetermined coefficients would be an easier way to solve this problem.

23. Given $y'' - y = F(x) \implies y_c(x) = c_1e^x + c_2e^{-x}$. Choose $y_1(x) = e^x$ and $y_2(x) = e^{-x}$. Then $W[y_1, y_2](x) = (e^x)(-e^{-x}) - (e^{-x})(e^x) = -2$. Hence, $K(x, t) = \frac{e^te^{-x} - e^{-t}e^x}{-2} = \frac{1}{2}(e^{x-t} - e^{t-x}) = \sinh(x-t)$, so that

$$y_p(x) = \int_{x_0}^x \sinh(x-t)F(t)dt.$$

24. Given $y'' + y' - 2y = F(x) \implies y_c(x) = c_1e^{-2x} + c_2e^x$. Choose $y_1(x) = e^{-2x}$ and $y_2(x) = e^x$. Then $W[y_1, y_2](x) = (e^{-2x})(e^x) - (-2e^{-2x})(e^x) = 3e^{-x}$. Hence, $K(x, t) = \frac{e^{-2t}e^x - e^te^{-2x}}{3e^{-x}} = \frac{1}{3}[e^{x-t} - e^{2(t-x)}]$, so that

$$y_p(x) = \frac{1}{3} \int_{x_0}^x [e^{x-t} - e^{2(t-x)}]F(t)dt.$$

25. Given $y'' + 5y' + 4y = F(x) \implies y_c(x) = c_1e^{-4x} + c_2e^{-x}$. Choose $y_1(x) = e^{-4x}$ and $y_2(x) = e^{-x}$. Then $W[y_1, y_2](x) = (e^{-4x})(-e^{-x}) - (e^{-x})(-4e^{-4x}) = 3e^{-5x}$. Hence, $K(x, t) = \frac{e^{-4t}e^{-x} - e^{-t}e^{-4x}}{3e^{-5t}} = \frac{1}{3}[e^{t-x} - e^{4(t-x)}]$, so that

$$y_p(x) = \frac{1}{3} \int_{x_0}^x [e^{t-x} - e^{4(t-x)}]F(t)dt.$$

26. Given $y'' + 4y' - 12y = F(x) \implies y_c(x) = c_1e^{2x} + c_2e^{-6x}$. Choose $y_1(x) = e^{2x}$ and $y_2(x) = e^{-6x}$. Then $W[y_1, y_2](t) = (e^{2t})(-6e^{-6t}) - (2e^{2t})(e^{-6t}) = -8e^{-4t}$. Hence, $K(x, t) = \frac{e^{2t}e^{-6x} - e^{-6t}e^{2x}}{-8e^{-4t}}$, so that

$$y_p(x) = -\frac{1}{8} \int_{x_0}^x [e^{6(t-x)} - e^{-2(t-x)}]F(t)dt.$$

27. Given $y'' + y = \sec x \implies y_c(x) = c_1 \cos x + c_2 \sin x$. Choose $y_1(x) = \cos x$ and $y_2(x) = \sin x$. Then $W[y_1, y_2](x) = (\cos x)(\cos x) - (\sin x)(-\sin x) = 1$. Hence, $K(x, t) = \cos t \sin x - \sin t \cos x$, so that

$$y_p(x) = \int_0^x (\cos t \sin x - \sin t \cos x)(\sec t) dt = \int_0^x (\sin x - \tan t \cos x) dt.$$

Thus $y_p(x) = x \sin x + \ln(\cos x) \cos x$. Consequently, $y(x) = c_1 \cos x + c_2 \sin x + x \sin x + \ln(\cos x) \cos x$. Then $y(0) = 0$ yields $c_1 = 0$ and $y'(0) = 1$ yields $c_2 = 1$. Therefore,

$$y(x) = \sin x + x \sin x + \ln(\cos x) \cos x.$$

28. Given $y'' - 4y' + 4y = 5xe^{2x} \implies y_c(x) = c_1 e^{2x} + c_2 x e^{2x}$. Choose $y_1(x) = e^{2x}$ and $y_2(x) = x e^{2x}$. Then $W[y_1, y_2](x) = (e^{2x})(e^{2x}[2x + 1]) - (2e^{2x})(x e^{2x}) = e^{4x}$. Hence, $K(x, t) = \frac{e^{2t} x e^{2x} - t e^{2t} e^{2x}}{e^{4t}} = e^{2(x-t)}(x - t)$, so that

$$y_p(x) = \int_0^x 5e^{2(x-t)}(x - t)t e^{2t} dt = 5e^{2x} \int_0^x (xt - t^2) dt = \frac{5}{6} x^3 e^{2x}.$$

Consequently,

$$y(x) = c_1 e^{2x} + c_2 x e^{2x} + \frac{5}{6} x^3 e^{2x}.$$

Then since $y(0) = 1$, we have $c_1 = 1$, and since $y'(0) = 0$, we have $c_2 = -2$. Therefore,

$$y(x) = e^{2x} \left(1 - 2x + \frac{5}{6} x^3 \right).$$

29. Given $y'' - 2ay' + a^2 y = F(x) \implies y_c(x) = c_1 e^{ax} + c_2 x e^{ax}$. Choose $y_1(x) = e^{ax}$ and $y_2(x) = x e^{ax}$. Then $W[y_1, y_2](x) = e^{2ax}$. Hence, $K(x, t) = \frac{e^{at} x e^{ax} - t e^{at} e^{ax}}{e^{2at}} = e^{a(x-t)}(x - t)$.

(a)

$$y_p(x) = \alpha \int_0^x e^{a(x-t)}(x - t) \frac{e^{at}}{t^2 + \beta^2} dt = \alpha e^{ax} \int_0^x \left(\frac{x}{t^2 + \beta^2} - \frac{t}{t^2 + \beta^2} \right) dt.$$

$$\text{Thus } y_p(x) = \alpha e^{ax} \left[\frac{x}{\beta} \tan^{-1} \left(\frac{x}{\beta} \right) - \frac{1}{2} \ln \left(\frac{x^2 + \beta^2}{\beta^2} \right) \right].$$

(b)

$$y_p(x) = \alpha \int_0^x e^{a(x-t)}(x - t) \frac{e^{at}}{(\beta^2 - t^2)^{1/2}} dt = \alpha e^{ax} \int_0^x \left[\frac{x}{(\beta^2 - t^2)^{1/2}} - \frac{t}{(\beta^2 - t^2)^{1/2}} \right] dt.$$

$$\text{Thus } y_p(x) = \alpha e^{ax} \left[x \sin^{-1} \left(\frac{x}{\beta} \right) + (\beta^2 - x^2)^{1/2} - \beta^2 \right].$$

(c)

$$y_p(x) = \int_0^x e^{a(x-t)}(x - t) e^{at} t^\alpha \ln t dt = -e^{ax} \int_0^x t^{\alpha+1} \ln t dt + x e^{at} \int_0^x t^\alpha \ln t dt.$$

$$\text{If } \alpha = -1 \text{ then } y_p(x) = \frac{x e^{ax}}{2} [(\ln x)^2 - 2 \ln x - 2].$$

$$\text{If } \alpha = -2 \text{ then } y_p(x) = -e^{ax} \left[\frac{(\ln x)^2}{2} + \ln x + 1 \right].$$

$$\text{If } \alpha \neq -1 \text{ and } \alpha \neq -2 \text{ then } y_p(x) = \frac{e^{ax} x^{\alpha+2}}{(\alpha+1)(\alpha+2)} \left[\ln x - \frac{2\alpha+3}{(\alpha+1)(\alpha+2)} \right].$$

30. Given $y'' + y = F(x) \implies y_c(x) = c_1 \cos x + c_2 \sin x$. Choose $y_1(x) = \cos x$ and $y_2(x) = \sin x$. Then $W[y_1, y_2](x) = 1$, and so $K(x, t) = \cos t \sin x - \sin t \cos x = \sin(x - t)$. Hence,

$$y_p(x) = \int_{x_0}^x \sin(x - t)F(t)dt$$

so that

$$y(x) = c_1 \cos x + c_2 \sin x + \int_{x_0}^x \sin(x - t)dt.$$

Imposing the initial conditions yields the two equations

$$c_1 \cos x_0 + c_2 \sin x_0 = y_0 \quad \text{and} \quad -c_1 \sin x_0 + c_2 \cos x_0 = y_1.$$

Solving these systems for c_1 and c_2 , we obtain

$$c_1 = y_0 \cos x_0 - y_1 \sin x_0 \quad \text{and} \quad c_2 = y_1 \cos x_0 + y_0 \sin x_0,$$

so that

$$\begin{aligned} y(x) &= (y_0 \cos x_0 - y_1 \sin x_0) \cos x + (y_1 \cos x_0 + y_0 \sin x_0) \sin x + \int_{x_0}^x F(t) \sin(x - t)dt \\ &= y_0(\cos x \cos x_0 + \sin x \sin x_0) + y_1(\sin x \cos x_0 - \cos x \sin x_0) + \int_{x_0}^x F(t) \sin(x - t)dt \\ &= y_0 \cos(x - x_0) + y_1 \sin(x - x_0) + \int_{x_0}^x F(t) \sin(x - t)dt. \end{aligned}$$

31. The complementary equation for the differential equation is $y_c(x) = c_1 e^{rx} + c_2 x e^{rx} + c_3 x^2 e^{rx}$. Let $y_1(x) = e^{rx}$, $y_2(x) = x e^{rx}$, and $y_3(x) = x^2 e^{rx}$. The system (6.7.22) in the text therefore assumes the form

$$\begin{aligned} u'_1 + t u'_2 + t^2 u'_3 &= 0, \\ r u'_1 + (rt + 1) u'_2 + (rt^2 + 2t) u'_3 &= 0, \\ r^2 e^{rt} u'_1 + (r^2 t + 2r) e^{rt} u'_2 + (r^2 t^2 + 4rt + 2) e^{rt} u'_3 &= F(x). \end{aligned}$$

Row reducing the augmented matrix,

$$\left[\begin{array}{cccc} 1 & t & t^2 & 0 \\ r & rt + 1 & rt^2 + 2t & 0 \\ r^2 e^{rt} & (r^2 t + 2r) e^{rt} & (r^2 t^2 + 4rt + 2) e^{rt} & F(x) \end{array} \right],$$

we obtain

$$\left[\begin{array}{cccc} 1 & t & t^2 & 0 \\ 0 & 1 & 2t & 0 \\ 0 & 0 & 1 & \frac{F(x)}{2e^{rt}} \end{array} \right].$$

By back-substitution, we find that $u'_3 = \frac{F(x)}{2e^{rt}}$, $u'_2 = \frac{tF(x)}{e^{rt}}$ and $u'_1 = \frac{t^2 F(x)}{2e^{rt}}$. Thus,

$$u_1(x) = \frac{1}{2} \int_a^x \frac{t^2 F(x)}{e^{rt}} dt, \quad u_2(x) = -\frac{1}{2} \int_a^x \frac{2tF(x)}{e^{rt}} dt \quad \text{and} \quad u_3(x) = \frac{1}{2} \int_a^x \frac{F(x)}{2e^{rt}} dt.$$

Hence,

$$\begin{aligned}
 y_p(x) &= e^{rx}u_1 + xe^{rx}u_2 + x^2e^{rx}u_3 \\
 &= \frac{e^{rx}}{2} \int_a^x \frac{t^2 F(t)}{e^{rt}} dt - \frac{xe^{rx}}{2} \int_a^x \frac{2tF(t)}{e^{rt}} dt + \frac{x^2e^{rx}}{2} \int_a^x \frac{F(t)}{e^{rt}} dt \\
 &= \frac{1}{2} \int_a^x F(t)[t^2 - 2tx + x^2]e^{r(x-t)} dt \\
 &= \frac{1}{2} \int_a^x F(t)(x-t)^2 e^{r(x-t)} dt.
 \end{aligned}$$

32.

(a) Note that $F(x) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ is precisely the right-hand side vector in the system (6.7.22), and therefore, the solutions for the unknowns $u'_1, u'_2, \dots, u'_k$ are precisely given by

$$u'_k = \frac{F(x)W_k(x)}{W[y_1, y_2, \dots, y_n](x)}, \quad k = 1, 2, \dots, n,$$

by a direct application of Cramer's Rule.

(b) Using part (a), we have

$$\begin{aligned}
 y_p(x) &= y_1(x)u_1(x) + y_2(x)u_2(x) + \dots + y_n(x)u_n(x) \\
 &= y_1(x) \int_{x_0}^x \frac{W_1(t)}{W(t)} F(t) dt + y_2(x) \int_{x_0}^x \frac{W_2(t)}{W(t)} F(t) dt + \dots + y_n(x) \int_{x_0}^x \frac{W_n(t)}{W(t)} F(t) dt.
 \end{aligned}$$

Since the $y_i(x)$ depend on x , whereas the integration is performed with respect to t , we can combine the integrals in the preceding equation to obtain

$$y_p(x) = \int_{x_0}^x \left[\frac{y_1(x)W_1(t) + y_2(x)W_2(t) + \dots + y_n(x)W_n(t)}{W[y_1, y_2, \dots, y_n](t)} \right] F(t) dt = \int_{x_0}^x K(x, t) F(t) dt,$$

which is the desired expression.

33. Three linearly independent solutions to the associated homogeneous problem are $y_1(x) = e^{-3x}$, $y_2(x) = e^{3x}$, and $y_3(x) = e^{-5x}$. Then

$$\begin{aligned}
 W[y_1, y_2, y_3](t) &= \begin{vmatrix} e^{-3t} & e^{3t} & e^{-5t} \\ -3e^{-3t} & 3e^{3t} & -5e^{-5t} \\ 9e^{-3t} & 9e^{3t} & 25e^{-5t} \end{vmatrix} = 96e^{-5t}, \\
 \widetilde{W}_1 &= \begin{vmatrix} 0 & e^{3t} & e^{-5t} \\ 0 & 3e^{3t} & -5e^{-5t} \\ 1 & 9e^{3t} & 25e^{-5t} \end{vmatrix} = -8e^{-2t}, \quad \widetilde{W}_2 = \begin{vmatrix} e^{-3t} & 0 & e^{-5t} \\ -3e^{-3t} & 0 & -5e^{-5t} \\ 9e^{-3t} & 1 & 25e^{-5t} \end{vmatrix} = 2e^{-8t}, \quad \widetilde{W}_3 = \begin{vmatrix} e^{-3t} & e^{3t} & 0 \\ -3e^{-3t} & 3e^{3t} & 0 \\ 9e^{-3t} & 9e^{3t} & 1 \end{vmatrix} = 6.
 \end{aligned}$$

So, $K(x, t) = \frac{e^{-3x}(-8e^{-2t}) + e^{3x}(2e^{-8t}) + e^{-5x}(6)}{96e^{-5t}} = \frac{1}{48}[e^{3(x-t)} + 3e^{-5(x-t)} - 4e^{-3(x-t)}]$. Hence,

$$y_p(x) = \frac{1}{48} \int_{x_0}^x [e^{3(x-t)} + 3e^{-5(x-t)} - 4e^{-3(x-t)}] F(t) dt.$$

34. Three linearly independent solutions to the associated homogeneous problem are $y_1(x) = e^{-x}$, $y_2(x) = \cos 3x$, and $y_3(x) = \sin 3x$. Then

$$W[y_1, y_2, y_3](t) = \begin{vmatrix} e^{-t} & \cos 3t & \sin 3t \\ -e^{-t} & -3 \sin 3t & 3 \cos 3t \\ e^{-t} & -9 \cos 3t & -9 \sin 3t \end{vmatrix} = 30e^{-t},$$

$$\widetilde{W}_1 = \begin{vmatrix} 0 & \cos 3t & \sin 3t \\ 0 & -3 \sin 3t & 3 \cos 3t \\ 1 & -9 \cos 3t & -9 \sin 3t \end{vmatrix} = 3, \quad \widetilde{W}_2 = \begin{vmatrix} e^{-t} & 0 & \sin 3t \\ -e^{-t} & 0 & 3 \cos 3t \\ e^{-t} & 1 & -9 \sin 3t \end{vmatrix} = e^{-t}[\sin 3t + 3 \cos 3t],$$

$$\widetilde{W}_3 = \begin{vmatrix} e^{-t} & \cos 3t & 0 \\ -e^{-t} & -3 \sin 3t & 0 \\ e^{-t} & -9 \cos 3t & 1 \end{vmatrix} = e^{-t}[\cos 3t - 3 \sin 3t].$$

So,

$$\begin{aligned} K(x, t) &= \frac{e^{-x}(3) - \cos 3xe^{-t}[\sin 3t + 3 \cos 3t] + \sin 3xe^{-t}[\cos 3t - 3 \sin 3t]}{30e^{-t}} \\ &= \frac{1}{30}[3e^{t-x} + (\sin 3x \cos 3t - \cos 3x \sin 3t) - 3(\cos 3x \cos 3t + \sin 3x \sin 3t)] \\ &= \frac{1}{30}[3e^{t-x} + \sin(3(x-t)) - 3 \cos(3(x-t))]. \end{aligned}$$

Hence,

$$y_p(x) = \frac{1}{30} \int_{x_0}^x [3e^{t-x} + \sin(3(x-t)) - 3 \cos(3(x-t))] F(t) dt.$$

35. Three linearly independent solutions to the associated homogeneous problem are $y_1(x) = e^{-4x}$, $y_2(x) = xe^{-4x}$, and $y_3(x) = e^{2x}$. Then

$$W[y_1, y_2, y_3](t) = \begin{vmatrix} e^{-4t} & te^{-4t} & e^{2t} \\ -4e^{-4t} & (1-4t)e^{-4t} & 2e^{2t} \\ 16e^{-4t} & (-8+16t)e^{-4t} & 4e^{2t} \end{vmatrix} = 36e^{-6t},$$

$$\widetilde{W}_1 = \begin{vmatrix} 0 & te^{-4t} & e^{2t} \\ 0 & (1-4t)e^{-4t} & 2e^{2t} \\ 1 & (-8+16t)e^{-4t} & 4e^{2t} \end{vmatrix} = e^{-2t}(6t-1), \quad \widetilde{W}_2 = \begin{vmatrix} e^{-4t} & 0 & e^{2t} \\ -4e^{-4t} & 0 & 2e^{2t} \\ 16e^{-4t} & 1 & 4e^{2t} \end{vmatrix} = -6e^{-2t},$$

$$\widetilde{W}_3 = \begin{vmatrix} e^{-4t} & te^{-4t} & 0 \\ -4e^{-4t} & (1-4t)e^{-4t} & 0 \\ 16e^{-4t} & (-8+16t)e^{-4t} & 1 \end{vmatrix} = e^{-8t}.$$

Hence, $K(x, t) = \frac{e^{-2t}(6t-1)e^{-4x} - 6e^{-2t}(xe^{-4x}) + e^{-8t}(e^{2x})}{36e^{-6t}} = \frac{1}{36}[e^{4(t-x)}(6(t-x)-1) + e^{2(x-t)}]$. Consequently,

$$y_p(x) = \frac{1}{36} \int_{x_0}^x [e^{4(t-x)}(6(t-x)-1) + e^{2(x-t)}] F(t) dt.$$

36. Three linearly independent solutions to the associated homogeneous problem are $y_1(x) = e^{2x} \cos 3x$, $y_2(x) = e^{2x} \sin 3x$, and $y_3(x) = e^{3x}$. Then

$$W[y_1, y_2, y_3](t) = \begin{vmatrix} e^{2t} \cos 3t & e^{2t} \sin 3t & e^{3t} \\ e^{2t}(2 \cos 3t - 3 \sin 3t) & e^{2t}(2 \sin 3t + 3 \cos 3t) & 3e^{3t} \\ e^{2t}(-5 \cos 3t - 12 \sin 3t) & e^{2t}(-5 \sin 3t + 12 \cos 3t) & 9e^{3t} \end{vmatrix} = 30e^{7t}.$$

Furthermore,

$$\begin{aligned}\widetilde{W}_1(t) &= \begin{vmatrix} 0 & e^{2t} \sin 3t & e^{3t} \\ 0 & e^{2t}(2 \sin 3t + 3 \cos 3t) & 3e^{3t} \\ 1 & e^{2t}(-5 \sin 3t + 12 \cos 3t) & 9e^{3t} \end{vmatrix} = e^{5t}(\sin 3t - 3 \cos 3t), \\ \widetilde{W}_2(t) &= \begin{vmatrix} e^{2t} \cos 3t & 0 & e^{3t} \\ e^{2t}(2 \cos 3t - 3 \sin 3t) & 0 & 3e^{3t} \\ e^{2t}(-5 \cos 3t - 12 \sin 3t) & 1 & 9e^{3t} \end{vmatrix} = -e^{5t}(\cos 3t + 3 \sin 3t), \\ \widetilde{W}_3 &= \begin{vmatrix} e^{2t} \cos 3t & e^{2t} \sin 3t & 0 \\ e^{2t}(2 \cos 3t - 3 \sin 3t) & e^{2t}(2 \sin 3t + 3 \cos 3t) & 0 \\ e^{2t}(-5 \cos 3t - 12 \sin 3t) & e^{2t}(-5 \sin 3t + 12 \cos 3t) & 1 \end{vmatrix} = 3e^{4t}.\end{aligned}$$

Hence,

$$\begin{aligned}K(x, t) &= \frac{e^{5t}(\sin 3t - 3 \cos 3t)e^{2x} \cos 3x - e^{5t}(\cos 3t + 3 \sin 3t)e^{2x} \sin 3x + 3e^{4t}e^{3x}}{30e^{7t}} \\ &= K(x, t) = \frac{1}{30}e^{2(x-t)}[\sin(3(t-x)) - 3 \cos(3(t-x))] + \frac{1}{10}e^{3(x-t)}.\end{aligned}$$

Consequently,

$$y_p(x) = \int_{x_0}^x \left[\frac{1}{30}e^{2(x-t)}[\sin(3(t-x)) - 3 \cos(3(t-x))] + \frac{1}{10}e^{3(x-t)} \right] F(t) dt.$$

37. Three linearly independent solutions to the associated homogeneous problem are $y_1(x) = e^x$, $y_2(x) = e^{2x}$, and $y_3(x) = e^{-4x}$. Then

$$W[y_1, y_2, y_3](t) = \begin{vmatrix} e^t & e^{2t} & e^{-4t} \\ e^t & 2e^{2t} & -4e^{-4t} \\ e^t & 4e^{2t} & 16e^{-4t} \end{vmatrix}.$$

Furthermore,

$$\widetilde{W}_1(t) = \begin{vmatrix} 0 & e^{2t} & e^{-4t} \\ 0 & 2e^{2t} & -4e^{-4t} \\ 1 & 4e^{2t} & 16e^{-4t} \end{vmatrix} = -6e^{-2t}, \quad \widetilde{W}_2(t) = \begin{vmatrix} e^t & 0 & e^{-4t} \\ e^t & 0 & -4e^{-4t} \\ e^t & 1 & 16e^{-4t} \end{vmatrix} = 5e^{-3t}, \quad \widetilde{W}_3(t) = \begin{vmatrix} e^t & e^{2t} & 0 \\ e^t & 2e^{2t} & 0 \\ e^t & 4e^{2t} & 1 \end{vmatrix} = e^{3t}.$$

Consequently,

$$\begin{aligned}K(x, t) &= \frac{1}{30e^{-t}} [e^x(-6e^{-2t}) + e^{2x}(5e^{-3t}) + e^{-4x}(e^{3t})] \\ &= \frac{1}{30} [e^{-4(x-t)} + 5e^{2(x-t)} - 6e^{x-t}],\end{aligned}$$

and a particular solution to the given differential equation is

$$y_p(x) = \frac{1}{30} \int_{x_0}^x [e^{-4(x-t)} + 5e^{2(x-t)} - 6e^{x-t}] F(t) dt.$$

Solutions to Section 6.8

True-False Review:

1. FALSE. First of all, the given equation only addresses the case of a second-order Cauchy-Euler equation. Secondly, the term containing y' must contain a factor of x (see Equation 6.8.1):

$$x^2 y'' + a_1 x y' + a_2 y = 0.$$

2. TRUE. The indicial equation (6.8.2) in this case is

$$r(r-1) - 2r - 18 = r^2 - 3r - 18 = 0,$$

with real and distinct roots $r = 6$ and $r = -3$. Hence, $y_1(x) = x^6$ and $y_2(x) = x^{-3}$ are two linearly independent solutions to this Cauchy-Euler equation.

3. FALSE. The indicial equation (6.8.2) in this case is

$$r(r-1) + 9r + 16 = r^2 + 8r + 16 = (r+4)^2 = 0,$$

and so we have only one real root, $r = 4$. Therefore, the only solutions to this Cauchy-Euler equation of the form $y = x^r$ take the form $y = cx^{-4}$. Therefore, only one linearly independent solution of the form $y = x^r$ to the Cauchy-Euler equation has been obtained.

4. TRUE. The indicial equation (6.8.2) in this case is

$$r(r-1) + 6r + 6 = r^2 + 5r + 6 = (r+2)(r+3) = 0,$$

with roots $r = -2$ and $r = -3$. Therefore, the general solution to this Cauchy-Euler equation is

$$y(x) = c_1 x^{-2} + c_2 x^{-3} = \frac{c_1}{x^2} + \frac{c_2}{x^3}.$$

Therefore, as $x \rightarrow +\infty$, $y(x)$ for all values of the constants c_1 and c_2 .

5. TRUE. A solution obtained by the method in this section containing the function $\ln x$ implies a repeated root to the indicial equation. In fact, such a solution takes the form $y_2(x) = x^{r_1} \ln x$. Therefore, if $y(x) = \ln x/x$ is a solution, we conclude that $r_1 = r_2 = -1$ is the only root of the indicial equation $r(r-1) + a_1 r + a_2 = 0$. From Case 2 in this section, we see that $-1 = -\frac{a_1-1}{2}$, and so $a_1 = 3$. Moreover, the discriminant of Equation (6.8.3) must be zero:

$$(a_1 - 1)^2 - 4a_2 = 0.$$

That is, $2^2 - 4a_2 = 0$. Therefore, $a_2 = 1$. Therefore, the Cauchy-Euler equation in question, with $a_1 = 3$ and $a_2 = 1$, must be as given.

6. FALSE. The indicial equation (6.8.2) in this case is

$$r(r-1) - 5r - 7 = r^2 - 6r - 7 = (r-7)(r+1) = 0,$$

with roots $r = 7$ and $r = -1$. Therefore, the general solution to this Cauchy-Euler equation is

$$y(x) = c_1 x^7 + c_2 x^{-1},$$

and this is not an oscillatory function of x for any choice of constants c_1 and c_2 .

Problems:

1. We are given that $x^2 y'' - xy' + 5y = 0$. If $y(x) = x^r$ then the indicial equation is $r^2 - 2r + 5 = 0 \implies r = 1 \pm 2i \implies y_1(x) = x \sin(2 \ln x)$, $y_2(x) = x \cos(2 \ln x) \implies y(x) = x[c_1 \sin(2 \ln x) + c_2 \cos(2 \ln x)]$.

2. We are given that $x^2y'' - 6y = 0$. If $y(x) = x^r$ then the indicial equation is $r^2 - r - 6 = 0 \implies r \in \{-2, 3\} \implies y(x) = c_1x^{-2} + c_2x^3$.

3. We are given that $x^2y'' - 3xy' + 4y = 0$. If $y(x) = x^r$ then the indicial equation is $r^2 - 4r + 4 = 0 \implies r \in \{2, 2\} \implies y_1(x) = x^2$, $y_2(x) = x^2 \ln x \implies y(x) = x^2(c_1 + c_2 \ln x)$.

4. We are given that $x^2y'' - 4xy' + 4y = 0$. If $y(x) = x^r$ then the indicial equation is $r^2 - 5r + 4 = 0 \implies r \in \{1, 4\} \implies y(x) = c_1x + c_2x^4$.

5. We are given that $x^2y'' + 3xy' + y = 0$. If $y(x) = x^r$ then the indicial equation is $r^2 + 2r + 1 = 0 \implies r \in \{-1, -1\} \implies y_1(x) = x^{-1}$, $y_2(x) = x^{-1} \ln x \implies y(x) = x^{-1}(c_1 + c_2 \ln x)$.

6. We are given that $x^2y'' + 5xy' + 13y = 0$. If $y(x) = x^r$ then the indicial equation is $r^2 + 4r + 13 = 0 \implies r = -2 \pm 3i \implies y(x) = x^{-2}[c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x)]$.

7. We are given that $x^2y'' - xy' - 35y = 0$. If $y(x) = x^r$ then the indicial equation is $r^2 - 2r - 35 = 0 \implies r \in \{-5, 7\} \implies y_1(x) = x^7$, $y_2(x) = x^{-5} \implies y(x) = c_1x^7 + c_2x^{-5}$.

8. We are given that $x^2y'' + xy' + 16y = 0$. If $y(x) = x^r$ then the indicial equation is $r^2 + 16 = 0 \implies r \in \{-4i, 4i\} \implies y(x) = c_1 \cos(4 \ln x) + c_2 \sin(4 \ln x)$.

9. If $y(x) = x^r$ then the indicial equation is $r^2 - m^2 = 0 \implies r \in \{-m, m\} \implies y(x) = c_1x^m + c_2x^{-m}$.

10. If $y(x) = x^r$ then the indicial equation is $r^2 - 2mr + m^2 = 0 \implies r \in \{m, m\} \implies y_1(x) = x^m$, $y_2(x) = x^m \ln x \implies y(x) = x^m(c_1 + c_2 \ln x)$.

11. If $y(x) = x^r$ then the indicial equation is $r^2 - 2mr + (m^2 + k^2) = 0 \implies r \in \{m + ki, m - ki\} \implies y(x) = x^m[c_1 \cos(k \ln x) + c_2 \sin(k \ln x)]$.

12.

(a) Let $x = e^z$. Then $z = \ln x$ and $\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}$ and $\frac{d^2y}{dx^2} = \frac{1}{x^2} \frac{d^2y}{dz^2} - \frac{1}{x^2} \frac{dy}{dz}$. Hence, substituting into the given differential equation we obtain

$$x^2 \left(\frac{1}{x^2} - \frac{1}{x^2} \frac{dy}{dz} \right) + a_1 x \left(\frac{1}{x} \frac{dy}{dz} \right) + a_2 y = 0,$$

so that

$$\frac{d^2y}{dz^2} + (a_1 - 1) \frac{dy}{dz} + a_2 y = 0.$$

(b) Let $x = e^z$. Then

$$\begin{aligned}
 W[y_1, y_2](x) &= y_1(x) \frac{dy_2(x)}{dx} - \frac{dy_1(x)}{dx} y_2(x) \\
 &= y_1(z) \frac{dy_2(z)}{dx} - \frac{dy_1(z)}{dx} y_2(z) \\
 &= y_1(z) \frac{dy_2(z)}{dz} \frac{dz}{dx} - y_2(z) \frac{dy_1(z)}{dz} \frac{dz}{dx} \\
 &= \frac{dz}{dx} \left[y_1(z) \frac{dy_2(z)}{dz} - \frac{dy_1(z)}{dz} y_2(z) \right] \\
 &= \frac{dz}{dx} W[y_1, y_2](z).
 \end{aligned}$$

Thus, if $y_1(z)$ and $y_2(z)$ are linearly independent solutions of (6.8.24), then $y_1(\ln x)$ and $y_2(\ln x)$ are linearly independent solutions of (6.8.23).

13. If $y(x) = (-x)^r$, $y' = r(-x)^{r-1}$, $y'' = r(r-1)(-x)^{r-2}$. Substituting these results into the original equation and simplifying yields $x^2 r(r-1)(-x)^{r-2} + ax(r)(-x)^{r-1} + b(-x)^r = 0 \implies r^2 - r + ar + b = 0 \implies r^2 + r(a-1) + b = 0$.

14. Consider $x^2 y'' + 4xy' + 2y = 0$. For $y(x) = x^r$, the indicial equation is given by $r^2 + 3r + 2 = 0 \implies r \in \{-1, -2\} \implies y_c(x) = c_1 x^{-1} + c_2 x^{-2}$. Let $y_1(x) = x^{-1}$ and $y_2(x) = x^{-2}$. Then $W[y_1, y_2](x) = (x^{-1})(-2x^{-3}) - (x^{-2})(-x^{-2}) = -x^{-4} \neq 0$ for $x > 0$, so that

$$y_p(x) = -y_1 \int \frac{y_2 F}{W} dx + y_2 \int \frac{y_1 F}{W} dx = \frac{4}{x} \int \ln x dx - \frac{4}{x^2} \int x \ln x dx = 2 \ln x - 3.$$

Thus, $y(x) = c_1 x^{-1} + c_2 x^{-2} + 2 \ln x - 3$.

15. Consider $x^2 y'' - 4xy' + 6y = 0$. For $y(x) = x^r$, the indicial equation is given by $r^2 - 5r + 6 = 0 \implies r \in \{2, 3\} \implies y_c(x) = c_1 x^2 + c_2 x^3$. Let $y_1(x) = x^2$ and $y_2(x) = x^3$. Then $W[y_1, y_2](x) = (x^2)(3x^2) - (x^3)(2x) = x^4 \neq 0$ for $x > 0$, so that

$$\begin{aligned}
 y_p(x) &= -y_1 \int \frac{y_2 F}{W} dx + y_2 \int \frac{y_1 F}{W} dx = -x^2 \int \frac{x^3(x^4 \sin x)}{x^2 x^4} dx + x^3 \int \frac{x^2(x^4 \sin x)}{x^2 x^4} dx \\
 &= -x^2 \int x \sin x dx + x^3 \int \sin x dx = -x^2 \sin x.
 \end{aligned}$$

Thus, $y(x) = c_1 x^2 + c_2 x^3 - x^2 \sin x$.

16. Consider $x^2 y'' + 6xy' + 6y = 0$. For $y(x) = x^r$, the indicial equation is given by $r^2 + 5r + 6 = 0 \implies r \in \{-3, -2\} \implies y_c(x) = c_1 x^{-2} + c_2 x^{-3}$. Let $y_1(x) = x^{-2}$ and $y_2(x) = x^{-3}$. Then $W[y_1, y_2](x) = (x^{-2})(-3x^{-4}) - (x^{-3})(-2x^{-3}) = -x^{-6} \neq 0$ so that

$$\begin{aligned}
 y_p(x) &= -y_1 \int \frac{y_2 F}{W} dx + y_2 \int \frac{y_1 F}{W} dx = -\frac{1}{x^2} \int \frac{(1/x^3)4e^{2x}}{x^2(-1/x^6)} dx + \frac{1}{x^3} \int \frac{(1/x^2)(4e^{2x})}{x^2(-1/x^6)} dx \\
 &= \frac{4}{x^2} \int x e^{2x} dx - \frac{4}{x^3} \int x^2 e^{2x} dx = \frac{e^{2x}(x-1)}{x^3}.
 \end{aligned}$$

Thus, $y(x) = c_1 x^{-2} + c_2 x^{-3} + \frac{e^{2x}(x-1)}{x^3}$.

17. Consider $x^2y'' - 3xy' + 4y = 0$. For $y(x) = x^r$, the indicial equation is given by $r^2 - 4r + 4 = 0 \implies r \in \{2, 2\} \implies y_c(x) = c_1x^2 + c_2x^2 \ln x$. Let $y_1(x) = x^2$ and $y_2(x) = x^2 \ln x$. Then $W[y_1, y_2](x) = (x^2)(2x \ln x) - (x^2 \ln x)(2x) = x^3 \neq 0$ for $x > 0$, so that

$$\begin{aligned} y_p(x) &= -y_1 \int \frac{y_2 F}{W} dx + y_2 \int \frac{y_1 F}{W} dx = -x^2 \int \frac{x^2 \ln x (x^2 / \ln x)}{x^2 x^3} dx + x^2 \ln x \int \frac{x^2 (x^2 / \ln x)}{x^2 x^3} dx \\ &= -x^2 \ln x + x^2 \ln x \ln |\ln x| \end{aligned}$$

for $x > 0$. Hence, $y(x) = x^2[c_1 + c_2 \ln x + \ln x(\ln |\ln x| - 1)]$.

18. Consider $x^2y'' + 4xy' + 2y = 0$. For $y(x) = x^r$, the indicial equation is given by $r^2 + 3r + 2 = 0 \implies r \in \{-1, -2\} \implies y_c(x) = c_1x^{-1} + c_2x^{-2}$. Let $y_1(x) = x^{-2}$ and $y_2(x) = x^{-1}$. Then $W[y_1, y_2](x) = (x^{-2})(-x^{-2}) - (x^{-1})(-2x^{-3}) = x^{-4} \neq 0$ for $x > 0$, so that

$$\begin{aligned} y_p(x) &= -y_1 \int \frac{y_2 F}{W} dx + y_2 \int \frac{y_1 F}{W} dx = -\frac{1}{x^2} \int \frac{(1/x) \cos x}{x^2(1/x^4)} dx + \frac{1}{x} \int \frac{(1/x^2) \cos x}{x^2(1/x^4)} dx \\ &= -\frac{1}{x^2} \int x \cos x dx + \frac{1}{x} \int \cos x dx = -\frac{1}{x^2} \cos x. \end{aligned}$$

Therefore, $y(x) = c_1x^{-1} + c_2x^{-2} - \frac{1}{x^2} \cos x$.

19. Consider $x^2y'' + xy' + 9y = 0$. For $y(x) = x^r$, the indicial equation is given by $r^2 + 9 = 0 \implies r \in \{-3i, 3i\} \implies y_c(x) = c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x)$. Let $y_1(x) = \cos(3 \ln x)$ and $y_2(x) = \sin(3 \ln x)$. Then

$$W[y_1, y_2](x) = [\cos(3 \ln x)] \left[\frac{3 \cos(3 \ln x)}{x} \right] - [\sin(3 \ln x)] \left[-\frac{3 \sin(3 \ln x)}{x} \right] = \frac{3}{x} \neq 0$$

for $x > 0$, so that

$$y_p(x) = -y_1 \int \frac{y_2 F}{W} dx + y_2 \int \frac{y_1 F}{W} dx = -\cos(3 \ln x) \int \frac{\sin(3 \ln x) 9 \ln x}{x^2(3/x)} dx + \sin(3 \ln x) \int \frac{\cos(3 \ln x) (9 \ln x)}{x^2(3/x)} dx.$$

Making the change of variables $u = 3 \ln x$ in the preceding integrals yields

$$\begin{aligned} y_p(x) &= -\frac{1}{3} \cos(3 \ln x) \int u \sin u du + \frac{1}{3} \sin(3 \ln x) \int u \cos u du \\ &= -\frac{1}{3} \cos(3 \ln x) (-u \cos u + \sin u) + \frac{1}{3} \sin(3 \ln x) (u \sin u + \cos u) = \frac{1}{3} \ln x. \end{aligned}$$

Consequently, $y(x) = c_1 \cos(3 \ln x) + c_2 \sin(3 \ln x) + \frac{1}{3} \ln x$.

20. Consider $x^2y'' - xy' + 5y = 0$. For $y(x) = x^r$, the indicial equation is given by $r^2 - 2r + 5 = 0 \implies r \in \{1-2i, 1+2i\} \implies y_c(x) = x[c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)]$. Let $y_1(x) = x \cos(2 \ln x)$ and $y_2(x) = x \sin(2 \ln x)$. Then

$$W[y_1, y_2](x) = [x \cos(2 \ln x)][2 \cos(2 \ln x) + \sin(2 \ln x)] - [x \sin(2 \ln x)][\cos(2 \ln x) - 2 \sin(2 \ln x)] = 2x \neq 0$$

for $x > 0$, so that

$$\begin{aligned} y_p(x) &= -y_1 \int \frac{y_2 F}{W} dx + y_2 \int \frac{y_1 F}{W} dx \\ &= -x \cos(2 \ln x) \int \frac{x \sin(2 \ln x)[8x(\ln x)^2]}{x^2(2x)} dx + x \sin(2 \ln x) \int \frac{x \cos(2 \ln x)[8x(\ln x)^2]}{x^2(2x)} dx \\ &= -4x \cos(2 \ln x) \int \frac{\sin(2 \ln x) \cdot (\ln x)^2}{x} dx + 4x \sin(2 \ln x) \int \frac{\cos(2 \ln x) \cdot (\ln x)^2}{x} dx. \end{aligned}$$

Making the change of variables $u = 2 \ln x$ in the preceding integrals yields

$$\begin{aligned} y_p(x) &= -\frac{x \cos(2 \ln x)}{2} \int u^2 \sin u du + \frac{x \sin(2 \ln x)}{2} \int u^2 \cos u du \\ &= x[2(\ln x)^2 - 1]. \end{aligned}$$

Thus, $y(x) = x[c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x) + 2(\ln x)^2 - 1]$.

21. Consider $x^2 y'' - (2m-1)xy' + m^2 y = 0$. For $y(x) = x^r$, the indicial equation is given by $r^2 - 2mr + m^2 = 0 \implies r \in \{m, m\} \implies y_c(x) = c_1 x^m + c_2 x^m \ln x$. Let $y_1(x) = x^m$ and $y_2(x) = x^m \ln x$. Then

$$W[y_1, y_2](x) = (x^m)[x^{m-1}(1 + m \ln x)] - (x^m \ln x)(mx^{m-1}) = x^{2m-1} \neq 0$$

for $x > 0$, so that

$$\begin{aligned} y_p(x) &= -y_1 \int \frac{y_2 F}{W} dx + y_2 \int \frac{y_1 F}{W} dx = -x^m \int \frac{x^m \ln x [x^m (\ln x)^k]}{x^2 (x^2)^{m-1}} dx + x^m \ln x \int \frac{x^m [x^m (\ln x)^k]}{x^2 (x^2)^{m-1}} dx \\ &= -x^m \int \frac{(\ln x)^{k+1}}{x} dx + x^m \ln x \int \frac{(\ln x)^k}{x} dx. \end{aligned} \quad (21.1)$$

Case 1: If $k \neq -1$ and $k \neq -2$, then Equation (21.1) becomes

$$y_p(x) = -\frac{x^m (\ln x)^{k+2}}{k+2} + x^m \ln x \frac{(\ln x)^{k+1}}{k+1} = \frac{x^m (\ln x)^{k+2}}{(k+1)(k+2)}.$$

Case 2: If $k = -1$, then Equation (21.1) becomes

$$y_p(x) = -x^m \int \frac{1}{x} dx + x^m \ln x \int \frac{1}{x \ln x} dx = (\ln |\ln x| - 1)x^m \ln x.$$

Case 3: If $k = -2$ then Equation (21.1) becomes

$$y_p(x) = -x^m \int \frac{1}{x \ln x} dx + x^m \ln x \int \frac{1}{x (\ln x)^2} dx = -x^m (1 + \ln |\ln x|).$$

22.

(a) The indicial equation is $r(r-1) - r + 5 = 0 \implies r \in \{1+2i, 1-2i\}$ so the general solution to the differential equation is $y(x) = x[c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)]$. Given $y(1) = \sqrt{2} \implies c_1 = \sqrt{2}$. Further, $y'(x) = \sqrt{2} \cos(2 \ln x) + c_2 \sin(2 \ln x) + x \left[-\frac{2}{x} \sqrt{2} \sin(2 \ln x) + \frac{2}{x} c_2 \cos(2 \ln x) \right]$, so that $y'(1) =$

$3\sqrt{2} \implies c_2 = \sqrt{2}$. Hence, $y(x) = \sqrt{2}x[\cos(2 \ln x) + \sin(2 \ln x)] = 2x \left[\frac{\sqrt{2}}{2} \cos(2 \ln x) + \frac{\sqrt{2}}{2} \sin(2 \ln x) \right] = 2x[\cos(2 \ln x) \cos(\pi/4) + \sin(2 \ln x) \sin(\pi/4)] = 2x \cos(2 \ln x - \pi/4)$.

(b) The zeros of $y(x)$ occur when

$$2 \ln x - \pi/4 = (2n + 1)\pi/2, \quad n = 0, \pm 1, \pm 2, \dots$$

Solving for x gives:

$$x = e^{\pi(4n+3)/8}, \quad n = 0, \pm 1, \pm 2, \dots$$

The zeros corresponding to $n = -2, -1, 0, 1, 2$ are, respectively, 0.1403669226, 0.6752319066, 3.248187814, 15.62533401, 75.16531584.

(c)

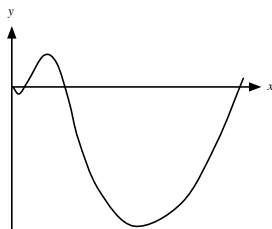


Figure 88: Figure for Problem 22

23.

(a) The indicial equation is $r(r - 1) + r + 25 = 0 \implies r \in \{5i, -5i\}$. The general solution to the differential equation is therefore $y(t) = c_1 \cos(5 \ln t) + c_2 \sin(5 \ln t)$. Given $y(1) = \frac{3\sqrt{3}}{2}$ then $c_1 = \frac{3\sqrt{3}}{2}$ and $y'(1) = \frac{15}{2}$ then $c_2 = \frac{3}{2}$. Hence, the solution to the initial-value problem is

$$y(t) = 3 \left[\frac{\sqrt{3}}{2} \cos(5 \ln t) + \frac{1}{2} \sin(5 \ln t) \right] = 3 \cos(5 \ln t - \pi/6).$$

(b) The zeros of $y(t)$ occur when $5 \ln t - \pi/6 = (2n + 1)\pi/2, \quad n = 0, \pm 1, \pm 2, \dots$
Solving for t yields

$$t = e^{\pi(6n+5)/30}.$$

(c)

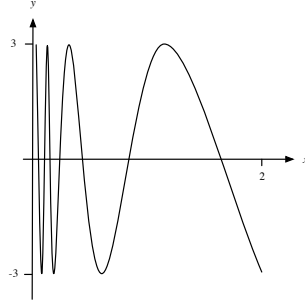


Figure 89: Figure for Problem 23(c)

(d) The motion is oscillatory but not periodic. Consequently the system is not performing simple harmonic motion.

24.

(a) We have

$$\begin{aligned} Ax^a [c_1 \cos(b \ln x) + c_2 \sin(b \ln x)] &= \frac{Ax^a}{\sqrt{c_1^2 + c_2^2}} \left[\frac{c_1}{\sqrt{c_1^2 + c_2^2}} \cos(b \ln x) + \frac{c_2}{\sqrt{c_1^2 + c_2^2}} \sin(b \ln x) \right] \\ &= \frac{Ax^a}{\sqrt{c_1^2 + c_2^2}} [\cos(b \ln x) \sin \phi + \sin(b \ln x) \cos \phi], \end{aligned}$$

where $\cos \phi = \frac{c_1}{\sqrt{c_1^2 + c_2^2}}$ and $\sin \phi = \frac{c_2}{\sqrt{c_1^2 + c_2^2}}$. Hence,

$$y(x) = Ax^a \cos(b \ln x - \phi).$$

(b) The zeros of the solution arise when

$$b \ln x - \phi = \frac{(2k+1)\pi}{2}, \quad k = 0, \pm 1, \pm 2, \dots;$$

that is, when

$$x_k = e^{[(2k+1)\pi/2+\phi]/b}, \quad k = 0, \pm 1, \pm 2, \dots$$

Hence there are an infinite number of zeros. As $k \rightarrow -\infty$, the zeros of the function approach $x = 0$, whereas if $k \rightarrow +\infty$, the zeros of the function also tend to $+\infty$. The distance between successive zeros is

$$\begin{aligned} \Delta x_k &= e^{[(2k+3)\pi/2+\phi]/b} - e^{[(2k+1)\pi/2+\phi]/b} = e^{(2k\pi+\phi)/b} (e^{3\pi/b} - e^{\pi/b}) \\ &= e^{[2\pi(k+1)+\phi]/b} (e^{\pi/b} - e^{-\pi/b}) \\ &= 2e^{[2\pi(k+1)+\phi]/b} \sinh(\pi/b). \end{aligned}$$

We see that $\lim_{k \rightarrow -\infty} \Delta x_k = 0$. Therefore, as $x \rightarrow 0^+$, the distance between successive zeros also approaches zero. On the other hand, $\lim_{k \rightarrow \infty} \Delta x_k = \infty$ so as $x \rightarrow \infty$, the distance between successive zeros increases without bound.

(c) We have $y(x) = Ax^a \cos(b \ln x - \phi)$.

If $a > 0$: The general behavior is oscillatory. The amplitude increases as $x \rightarrow \infty$, whereas the amplitude approaches zero as $x \rightarrow 0^+$.

If $a < 0$: The general behavior is oscillatory. The amplitude approaches zero as $x \rightarrow \infty$, whereas the amplitude increases without bound as $x \rightarrow 0^+$.

If $a = 0$: The general behavior is oscillatory with constant amplitude.

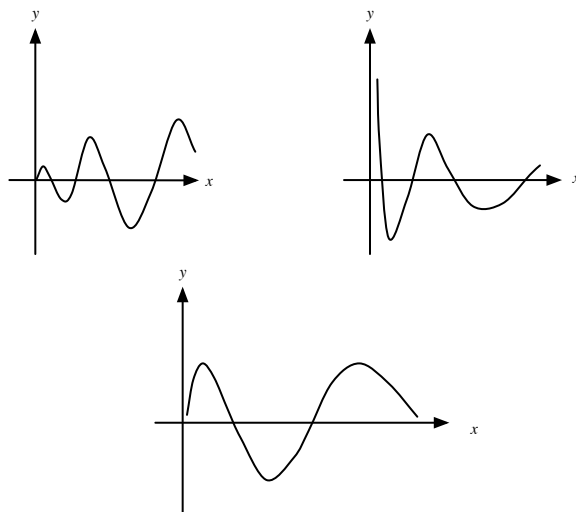


Figure 90: Figure for Problem 24

Solutions to Section 6.9

Problems:

1. We are given that $y_1(x) = x^2$. Let $y_2(x) = x^2 u$ so $y_2' = 2xu + x^2 u'$, and $y_2'' = x^2 u'' + 4xu' + 2u$. Substituting these results into the given differential equation and simplifying we obtain $xu'' + u' = 0 \Rightarrow \frac{u''}{u'} = -\frac{1}{x} \Rightarrow u(x) = c_1 \ln x + c_2$. Letting $c_1 = 1$ and $c_2 = 0$, we obtain a second linearly independent solution, $y_2(x) = x^2 \ln x$.

2. We are given that $y_1(x) = x \sin x$. Let $y_2(x) = (x \sin x)u$ so that $y_2' = xu \cos x + (x \sin x)u' + u \sin x$, and $y_2'' = -xu \sin x + 2((x \cos x)u' + (x \sin x)u'') + 2u \cos x + 2(\sin x)u'$. Substituting these results into the original equation and simplifying we obtain $\sin x u'' + 2 \cos x u' = 0 \Rightarrow \frac{u''}{u'} = -\frac{2 \cos x}{\sin x} \Rightarrow u' = c_1 \csc^2 x \Rightarrow u(x) = -c_1 \cot x + c_2$. Letting $c_1 = -1$ and $c_2 = 0$, we obtain a second linearly independent solution, $y_2(x) = x \cos x$.

3. We are given $y_1(x) = e^x$. Let $y_2(x) = ue^x$ so that $y_2' = e^x u + e^x u'$, and $y_2'' = e^x u + 2e^x u' + e^x u''$. Substituting these results into the given differential equation and simplifying we obtain $xu'' + u' = 0 \Rightarrow \frac{u''}{u'} = -\frac{1}{x} \Rightarrow u' = \frac{c_1}{x} \Rightarrow u(x) = c_1 \ln x + c_2$. Letting $c_1 = 1$ and $c_2 = 0$, we obtain a second linearly independent solution, $y_2(x) = e^x \ln x$.

4. We are given $y_1(x) = \sin x^2$. Let $y_2(x) = u \sin x^2$ so that $y'_2 = u' \sin x^2 + 2xu \cos x^2$, and $y''_2 = u'' \sin x^2 + 4xu' \cos x^2 + 2u \cos x^2 - 4x^2 u \sin x^2$. Substituting these results into the given differential equation and simplifying we obtain $\frac{u''}{u'} = \frac{1}{x} - 4x \cot x^2 \Rightarrow u' = c_1 x \csc^2(x^2) \Rightarrow u(x) = -\frac{1}{2}c_1 \cot x^2 + c_2$. Letting $c_1 = 1$ and $c_2 = 0$, we obtain a second linearly independent solution, $y_2(x) = \cos(x^2)$.

5. We are given $y_1(x) = x$ and $-1 < x < 1$. Let $y_2(x) = ux$ so that $y'_2 = u'x + u$ and $y''_2 = u''x + 2u'$. Substituting these results into the original equation and simplifying we obtain $u''(x - x^3) + u'(2 - 4x^2) = 0 \Rightarrow \frac{u''}{u'} = \frac{4x^2 - 2}{x - x^3} \Rightarrow \frac{u''}{u'} = -\frac{2}{x} - \frac{1}{1+x} + \frac{1}{1-x} \Rightarrow \ln|u'| = -\ln|x| - \ln|1+x| + \ln|1-x| + c_1 \Rightarrow u'(x) = \frac{c_1}{(1-x^2)x^2}$, $-1 < x < 1 \Rightarrow u(x) = c_2 \left[\frac{1}{2} \ln \left(\frac{x+1}{1-x} \right) - \frac{1}{x} \right] + c_3$. Letting $c_2 = 1$ and $c_3 = 0$ we obtain a second linearly independent solution, $y_2(x) = \frac{1}{2}x \ln \left(\frac{x+1}{1-x} \right) - 1$.

6. We are given $y_1(x) = x^{-1/2} \sin x$. Letting $y_2(x) = ux^{-1/2} \sin x$, we have

$$y'_2 = ux^{-1/2} \cos x + (u'x^{-1/2} - \frac{1}{2}ux^{-3/2}) \sin x$$

and

$$y''_2 = u'x^{-1/2} \cos x + u \left(-\frac{1}{2}x^{-3/2} \cos x - x^{-1/2} \sin x \right) + u''x^{-1/2} \sin x + u' \left(x^{-1/2} \cos x - \frac{1}{2}x^{-3/2} \sin x \right) - \frac{1}{2}u'x^{-3/2} \sin x - \frac{1}{2}u \left(x^{-3/2} \cos x - \frac{3}{2}x^{-5/2} \sin x \right).$$

Substituting these results into the given differential equation and simplifying we obtain $4u''x^{3/2} \sin x + 8u'x^{3/2} \cos x = 0 \Rightarrow \frac{u''}{u'} = -2 \cot x \Rightarrow \ln|u'| = -2 \ln|\sin x| + \ln|c_1| \Rightarrow u'(x) = c_2 \csc^2(x) \Rightarrow u = -c_2 \cot(x) + c_3$. Letting $c_2 = 1$ and $c_3 = 0$ we obtain a second linearly differential solution, $y_2(x) = x^{-1/2} \cos x$.

7.

(a) If $y_1(x) = x^\lambda$, then the corresponding indicial equation is given by $\lambda^2 - 2m\lambda + m^2 = 0 \Rightarrow \lambda \in \{m, m\}$ so one particular solution of the given differential equation is $y_1(x) = x^m$.

(b) Let $y_2(x) = x^m u$ so $y'_2 = x^m u' + mx^{m-1}u$ and $y''_2 = x^m u'' + mux^{m-2} + m^2 x^{m-2}u + 2mx^{m-1}u'$.

Substituting these results into the original equation and simplifying yields $x^{m+1}u' + x^{m+2}u'' = 0 \Rightarrow \frac{u''}{u'} = -\frac{1}{x} \Rightarrow u'(x) = \frac{c}{x} \Rightarrow u(x) = c \ln x + c_1$. Letting $c = 1$ and $c_1 = 0$ we obtain a second linearly independent solution, $y_2(x) = x^m \ln x$.

8. If $y(x) = a_0 + a_1x + a_2x^2$, then $y'(x) = a_1 + 2a_2x$ and $y''(x) = 2a_2$. Substituting these results into the original equation and simplifying yields $-2a_1x + (8a_2 - 2a_0) = 0 \Rightarrow a_1 = 0$ and $a_0 = 4a_2$. Hence, $y(x) = c_1(4+x^2)$ is a solution to the differential equation where c_1 is an arbitrary constant. Let $y_1(x) = 4+x^2$. If $y(x) = (4+x^2)u$, then $y'(x) = (4+x^2)u' + 2xu$ and $y''(x) = (4+x^2)u'' + 4xu' + 2u$. Substituting these results into the original equation and simplifying yields $(4+x^2)u'' + 4xu' = 0$. Substituting $v = u'$ into the last equation gives $(4+x^2)v' + 4xv = 0$. Separating the variables gives $\frac{dv}{v} = -\frac{4x}{4+x^2}dx \Rightarrow \ln|v| =$

$-2 \ln(4+x^2) + 1_3 \implies v = \frac{c_2}{(4+x^2)^2}$. Hence, $\frac{du}{dx} = \frac{c_2}{(4+x^2)^2} \implies u(x) = c_2 \left[\tan^{-1}\left(\frac{x}{2}\right) + \frac{2x}{x^2+4} \right] + c_3$. Thus, $y(x) = (4+x^2)u$ becomes

$$y(x) = c_2[(4+x^2) \tan^{-1}(x/2) + 2x] + c_3(4+x^2),$$

which is the general solution to the given differential equation. A second linearly independent solution is

$$y_2(x) = (4+x^2) \tan^{-1}(x/2) + 2x.$$

9.

(a) If $y_1(x) = e^{\alpha x}$ then $y_1'(x) = \alpha e^{\alpha x}$ and $y_1''(x) = \alpha^2 e^{\alpha x}$ so from the original equation we have $xy_1'' + (\alpha x + \beta)y_1' + \alpha\beta y_1 = x(\alpha^2 e^{\alpha x}) - (\alpha x + \beta)(\alpha e^{\alpha x}) + \alpha e^{\alpha x} = \alpha^2 x - (\alpha x + \beta)\alpha + \alpha\beta = 0$. Hence, $y_1(x) = e^{\alpha x}$ is a solution to the differential equation.

(b) If $y(x) = e^{\alpha x}u$, then $y' = (u' + \alpha u)e^{\alpha x}$ and $y'' = [u'' + \alpha(2u' + \alpha u)]e^{\alpha x}$. Substituting these results into the original equation and simplifying yields $xu'' + (\alpha x - \beta)u' = 0 \implies \frac{u''}{u'} = -\alpha + \beta x^{-1} \implies u' = c_1 x^{\beta} e^{-\alpha x} \implies u(x) = c_1 \int x^{\beta} e^{-\alpha x} dx + c_2$. Letting $c_1 = 1$ and $c_2 = 0$ we obtain $y_2(x) = \int x^{\beta} e^{-\alpha x} dx$ as a second linearly independent solution.

(c) If $\alpha = 1$ and β is a non-negative integer then $y_2(x) = e^x \int x^{\beta} e^{-x} dx$. Use repeated integration by parts:

$$\begin{aligned} y_2(x) &= e^x(-x^{\beta} e^{-x} + \beta \int x^{\beta-1} e^{-x} dx) \\ &= -\beta! \left[\frac{x^{\beta}}{\beta!} - \frac{e^x}{(\beta-1)!} \int x^{\beta-1} e^{-x} dx \right] \\ &= -\beta! \left\{ \frac{x^{\beta}}{\beta!} - \frac{e^x}{(\beta-1)!} \left[-x^{\beta-1} e^{-x} + (\beta-1) \int x^{\beta-2} e^{-x} dx \right] \right\} \\ &= -\beta! \left[\frac{x^{\beta}}{\beta!} + \frac{x^{\beta-1}}{(\beta-1)!} - \frac{e^x}{(\beta-2)!} \int x^{\beta-2} e^{-x} dx \right] \\ &= -\beta! \left[\frac{x^{\beta}}{\beta!} + \frac{x^{\beta-1}}{(\beta-1)!} + \cdots + \frac{x^2}{2!} - \frac{e^x}{1!} \left(\int x e^{-x} dx \right) \right] \\ &= -\beta! \left[\frac{x^{\beta}}{\beta!} + \frac{x^{\beta-1}}{(\beta-1)!} + \cdots + \frac{x^2}{2!} + x + 1 \right]. \end{aligned}$$

10. We are given that $y_1(x) = e^{3x}$. If $y_2(x) = ue^{3x}$, then $y_2' = (u' + 3u)e^{3x}$ and $y_2'' = (u'' + 6u' + 9u)e^{3x}$. Substituting these results into the original equation and simplifying we obtain $u'' = 15\sqrt{x}$, which implies that $u' = 10x^{3/2} + c_1$ and $u = 4x^{5/2} + c_1x + c_2$. Thus, the general solution is

$$y(x) = e^{3x}(4x^{5/2} + c_1x + c_2).$$

11. We are given that $y_1(x) = e^{2x}$. If $y_2(x) = ue^{2x}$, then $y_2' = (u' + 2u)e^{2x}$ and $y_2'' = (u'' + 4u' + 4u)e^{2x}$. Substituting these results into the original equation and simplifying yields $u'' = 4 \ln x$, which implies that $u' = 4x \ln x - 4x + c_2$ and $u = c_1 + c_2x + x^2(2 \ln x - 3)$. Thus the general solution is

$$y(x) = e^{2x}[c_1 + c_2x + x^2(2 \ln x - 3)].$$

12. We are given that $y_1(x) = \sqrt{x}$. If $y_2(x) = u\sqrt{x}$, then we compute that $y'_2 = u'\sqrt{x} + \frac{1}{2}ux^{-1/2}$ and $y''_2 = u''\sqrt{x} + u'x^{-1/2} - \frac{1}{4}ux^{-3/2}$. Substituting these results into the original equation and simplifying, we obtain $4x^2u'' + 4xu' = \ln x \implies xu'' + u' = \frac{\ln x}{4x} \implies \frac{d(xu')}{dx} = \frac{\ln x}{4x} \implies xu' = \frac{1}{8}(\ln x)^2 + c_1 \implies u(x) = \frac{1}{24}(\ln x)^3 + c_1 \ln x + c_2$. Consequently, the general solution is

$$y(x) = \sqrt{x} \left[\frac{1}{24}(\ln x)^3 + c_1 \ln x + c_2 \right].$$

13. We are given that $y_1(x) = \sin x$ where $0 < x < \pi$. If $y_2(x) = u \sin x$, then $y'_2 = u' \sin x + u \cos x$ and $y''_2 = u'' \sin x + 2u' \cos x - u \sin x$. Substituting these results into the original equation and simplifying yields $u'' \sin x + 2u' \cos x = \csc x \implies u'' \sin^2 x + 2u' \sin x \cos x = 1 \implies u' = \frac{x}{\sin^2 x} + \frac{c_3}{\sin^2 x} \implies u(x) = -x \cot x + \ln(\sin x) - c_3 \cot x + c_4$. Choosing $c_3 = c_4 = 0$, we obtain $u(x) = -x \cot x + \ln(\sin x)$. Hence, the general solution is

$$y(x) = c_1 \sin x + c_2 \sin x [\ln(\sin x) - x \cot x].$$

14. We are given that $y_1(x) = e^{2x}$ where $x > 0$. If $y_2(x) = ue^{2x}$, then we compute $y'_2 = (u' + 2u)e^{2x}$ and $y''_2 = (u'' + 4u' + 4u)e^{2x}$. Substituting these results into the original equation and simplifying we obtain $u''x + u'(2x - 1) = 8x^2$, or equivalently, $u'' + \frac{2x-1}{x}u' = 8x$. An integrating factor for the last equation is $\frac{e^{2x}}{x^2}$ so that, $\frac{d}{dx} \left(\frac{e^{2x}}{x^2} u' \right) = 8e^{2x} \implies u' = 4x + c_3xe^{-2x}$. Hence, $u(x) = 2x^2 + c_3 \left(-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} \right) + c_2$. Thus, the general solution is

$$y(x) = 2x^2e^{2x} + c_1(2x + 1) + c_2e^{2x}.$$

15. We are given that $y_1(x) = x^2$, where $x > 0$. If $y_2(x) = ux^2$ then $y'_2 = u'x^2 + 2xu$ and $y''_2 = u''x^2 + 4u'x + 2u$. Substituting these results into the original differential equation and simplifying we obtain $u''x + u' = 8x \implies xu' = 4x^2 + c_2 \implies u(x) = 2x^2 + c_2 \ln x + c_1$. Hence, the general solution is

$$y(x) = x^2(c_1 + c_2 \ln x + 2x^2).$$

16. Since $y_1(x)$ is a solution of the associated homogeneous differential equation we have $y''_1 + py'_1 + qy_1 = 0$. Let $y_2(x) = u(x)y_1(x)$ so that $y'_2 = u'y_1 + uy'_1$ and $y''_2 = u''y_1 + 2u'y'_1 + uy''_1$. Substituting these results into the original equation and simplifying we obtain the following equalities:

$$\begin{aligned} y''_2 + py'_2 + qy_2 &= u''y_1 + 2u'y'_1 + uy''_1 + p(u'y_1 + uy'_1) + quy_1 \\ &= u(y''_1 + py'_1 + qy_1) + u''y_1 + 2u'y'_1 + pu'y_1 \\ &= y_1u'' + (2y'_1 + py_1)u' \\ &= y_1 \left[u'' + \left(2\frac{y'_1}{y_1} + p \right) u' \right] \\ &= y_1 \left[v' + \left(2\frac{y'_1}{y_1} + p \right) v \right] = y_1 \left(\frac{r}{y_1} \right) = r \end{aligned}$$

Therefore, $y_2(x) = u(x)y_1(x)$ is a solution to the given differential equation provided $v = u'$ is a solution of the equation $v' + \left(2\frac{y'_1}{y_1} + p \right) v = \frac{r}{y_1}$. The last differential equation has integrating factor

$$I(x) = e^{\int^x \left(2\frac{y'_1}{y_1} + p \right) dt} = e^{2 \int^x \frac{y'_1}{y_1} dt} e^{\int^x p dt} = y_1^2 e^{\int^x p dt},$$

so

$$\frac{d}{dt}[I(t)v] = I(t)\frac{r}{y_1} \implies I(x)v = \int^x I(t)(r/y_1)dt + c_2 \implies v(x) = I^{-1}(x) \int^x I(t)(r/y_1)dt + c_2 I^{-1}(x).$$

Since $\frac{du}{dx} = v$, we have

$$u(x) = \int [I^{-1}(x) \int^x I(t)(r/y_1)dt]dx + c_1 + c_2 \int^x I^{-1}(t)dt.$$

Hence,

$$y(x) = y_1(x) \left\{ \int [I^{-1}(x) \int^x I(t)(r/y_1)dt]dx + c_1 + c_2 \int^x I^{-1}(t)dt \right\}.$$

Linearly independent solutions: $y_1(x)$ and $y_1(x) \int^x I^{-1}(t)dt$.

Particular solution: $y_1 \int [I^{-1}(x) \int^x I(t)(r/y_1)dt]dx$.

Solutions to Section 6.10

Problems:

$$\begin{aligned} 1. \quad Ly &= (D^2 + 3)(e^{x^3}) = D^2(e^{x^3}) + 3e^{x^3} = D(3x^2e^{x^3}) + 3e^{x^3} = e^{x^3}(9x^4 + 6x) + 3e^{x^3} \\ &= 3e^{x^3}(3x^4 + 2x + 1). \end{aligned}$$

$$2. \quad Ly = 5 \cdot \left(\frac{1}{1+x^2} \right) = \frac{5}{1+x^2}.$$

$$\begin{aligned} 3. \quad Ly &= \left(\frac{1}{x}D^2 + xD - 2 \right) (4 \sin x) = \frac{1}{x}D^2(4 \sin x) + xD(4 \sin x) - 2(4 \sin x) \\ &= -\frac{4}{x} \sin x + 4x \cos x - 8 \sin x. \end{aligned}$$

$$\begin{aligned} 4. \quad Ly &= [x^2D^3 - (\sin x)D](e^{2x} + \cos x) = x^2D^3(e^{2x} + \cos x) - (\sin x)D(e^{2x} + \cos x) \\ &= x^2(8e^{2x} + \sin x) - \sin x(2e^{2x} - \sin x). \end{aligned}$$

5.

$$\begin{aligned} Ly &= [(x^2 + 1)D^3 - (\cos x)D + 5x^2](\ln x + 8x^5) \\ &= (x^2 + 1)D^3(\ln x + 8x^5) - (\cos x)D(\ln x + 8x^5) + 5x^2(\ln x + 8x^5) \\ &= (x^2 + 1) \left(\frac{2}{x^3} + 480x^4 \right) - (\cos x) \left(\frac{1}{x} + 40x^4 \right) + 5x^2(\ln x + 8x^5) \\ &= \frac{2}{x} + \frac{2}{x^3} + 480x^6 + 480x^4 - \frac{\cos x}{x} - 40x^4 \cos x + 5x^2 \ln x + 40x^7. \end{aligned}$$

$$6. \quad Ly = 4x^2D[\sin^2(x^2 + 1)] = 16x^3 \sin(x^2 + 1) \cos(x^2 + 1).$$

$$7. \quad P(r) = r^3 + 3r^2 - 4 = (r - 1)(r + 2)^2 = 0 \implies r = 1, r = -2 \text{ (multiplicity 2)}.$$

General solution:

$$y(x) = c_1e^x + c_2e^{-2x} + c_3xe^{-2x}.$$

$$8. \quad P(r) = r^3 + 11r^2 + 36r + 26 = (r + 1)(r^2 + 10r + 26) = 0 \implies r = -1, r = -5 + i.$$

General solution:

$$y(x) = c_1e^{-x} + c_2e^{-5x}(c_2 \cos x + c_3 \sin x).$$

9. $P(r) = r^4 + 13r^2 + 36 = (r^2 + 4)(r^2 + 9) \implies r = \pm 2i, r = \pm 3i.$

General solution:

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos 3x + c_4 \sin 3x.$$

10. $P(r) = r^3 + 10r^2 + 25r = r(r + 5)^2 \implies r = 0, r = -5$ (multiplicity 2).

General solution:

$$y(x) = c_1 + c_2 e^{-5x} + c_3 x e^{-5x}.$$

11. $P(r) = (r + 3)^2(r^2 - 4r + 13) \implies r = -3$ (multiplicity 2), $r = 2 \pm 3i.$

General solution:

$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x} + c_3 x^2 e^{-3x} + e^{2x}(c_4 \cos 3x + c_5 \sin 3x).$$

12. $P(r) = (r^2 - 2r + 2)^3 \implies r = 1 \pm i$ (multiplicity 3).

General solution:

$$y(x) = e^x [c_1 \cos x + c_2 \sin x + x(c_3 \cos x + c_4 \sin x) + x^2(c_5 \cos x + c_6 \sin x)].$$

13. $P(r) = (r^2 + 4r + 4)(r - 3) = (r + 2)^2(r - 3) \implies r = -2$ (multiplicity 2), $r = 3.$

General solution:

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x} + c_3 e^{3x}.$$

14. $A(D) = D^2(D + 1).$

15. $A(D) = D^2 - 6D + 10.$

16. $A(D) = (D^2 + 16)^6.$

17. $A(D) = (D^2 + 1)^2(D + 2).$

18. In operator form the given differential equation is

$$(D^2 + 6D + 9)y = 4e^{-3x},$$

that is

$$(D + 3)^2 y = 4e^{-3x}. \quad (0.0.22)$$

Therefore the complementary function is

$$y_c(x) = c_1 e^{-3x} + c_2 x e^{-3x}.$$

The annihilator of $F(x) = 4e^{-3x}$ is $A(D) = D + 3$. Operating on (0.0.22) with $D + 3$ yields

$$(D + 3)^3 y = 0$$

which has general solution

$$y(x) = y_c(x) + A_0 x^2 e^{-3x}.$$

Consequently, an appropriate trial solution for (0.0.22) is

$$y_p(x) = A_0 x^2 e^{-3x}.$$

19. In operator form the given differential equation is

$$(D^2 + 6D + 9)y = 4e^{-2x},$$

that is

$$(D + 3)^2 y = 4e^{-2x}. \quad (0.0.23)$$

Therefore the complementary function is

$$y_c(x) = c_1 e^{-3x} + c_2 x e^{-3x}.$$

The annihilator of $F(x) = e^{-2x}$ is $A(D) = D + 2$. Operating on (0.0.23) with $D + 2$ yields

$$(D + 2)(D + 3)^2 y = 0$$

which has general solution

$$y(x) = y_c(x) + A_0 e^{-2x}.$$

Consequently, an appropriate trial solution for (0.0.23) is

$$y_p(x) = A_0 e^{-2x}.$$

20. In operator form the given differential equation is

$$(D^3 - 6D^2 + 25D)y = x^2,$$

that is

$$D(D^2 - 6D + 25)y = x^2. \quad (0.0.24)$$

Therefore the complementary function is

$$y_c(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + c_3.$$

The annihilator of $F(x) = x^2$ is $A(D) = D^3$. Operating on (0.0.24) with D^3 yields

$$D^4(D^2 - 6D + 25)y = 0$$

which has general solution

$$y(x) = y_c(x) + A_0 x + A_1 x^2 + A_2 x^3.$$

Consequently, an appropriate trial solution for (0.0.24) is

$$y_p(x) = A_0 x + A_1 x^2 + A_2 x^3.$$

21. In operator form the given differential equation is

$$(D^3 - 6D^2 + 25D)y = \sin 4x,$$

that is

$$D(D^2 - 6D + 25)y = \sin 4x. \quad (0.0.25)$$

Therefore the complementary function is

$$y_c(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + c_3.$$

The annihilator of $F(x) = \sin 4x$ is $A(D) = D^2 + 16$. Operating on (0.0.25) with $D^2 + 16$ yields

$$(D^2 + 16)D(D^2 - 6D + 25)y = 0$$

which has general solution

$$y(x) = y_c(x) + A_0 \cos 4x + A_1 \sin 4x.$$

Consequently, an appropriate trial solution for (0.0.25) is

$$y_p(x) = A_0 \cos 4x + A_1 \sin 4x.$$

22. In operator form the given differential equation is

$$(D^3 + 9D^2 + 24D + 16)y = 8e^{-x} + 1,$$

that is

$$(D + 1)(D + 4)^2y = 8e^{-x} + 1. \quad (0.0.26)$$

Therefore the complementary function is

$$y_c(x) = c_1e^{-4x} + c_2xe^{-4x} + c_3e^{-x}.$$

The annihilator of $F(x) = 8e^{-x} + 1$ is $A(D) = D(D + 1)$. Operating on (0.0.26) with $D(D + 1)$ yields

$$D(D + 1)^2(D + 4)^2y = 0$$

which has general solution

$$y(x) = y_c(x) + A_0 + A_1xe^{-x}.$$

Consequently, an appropriate trial solution for (0.0.26) is

$$y_p(x) = A_0 + A_1xe^{-x}.$$

23. In operator form the given differential equation is

$$(D^6 + 3D^4 + 3D^2 + 1)y = 2 \sin x,$$

that is

$$(D^2 + 1)^3y = 2 \sin x. \quad (0.0.27)$$

Therefore the complementary function is

$$y_c(x) = c_1 \cos x + c_2 \sin x + x(c_3 \cos x + c_4 \sin x) + x^2(c_5 \cos x + c_6 \sin x).$$

The annihilator of $F(x) = 2 \sin x$ is $A(D) = D^2 + 1$. Operating on (0.0.27) with $D^2 + 1$ yields

$$(D^2 + 1)^4y = 0$$

which has general solution

$$y(x) = y_c(x) + x^3(A_0 \cos x + A_1 \sin x).$$

Consequently, an appropriate trial solution for (0.0.27) is

$$y_p(x) = x^3(A_0 \cos x + A_1 \sin x).$$

24.

(a) From Problem 19 the complementary function for the given differential equation is

$$y_c(x) = c_1 e^{-3x} + c_2 x e^{-3x},$$

and an appropriate trial solution is

$$y_p(x) = A_0 e^{-2x}.$$

Inserting this expression for $y_p(x)$ into the given differential equation yields

$$e^{-2x}(4A_0 - 12A_0 + 9A_0) = 4e^{-2x},$$

so that $y_p(x) = 4e^{-2x}$. Consequently, the general solution to the given differential equation is

$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x} + 4e^{-2x}.$$

(b) Choosing $y_1(x) = e^{-3x}$, $y_2(x) = x e^{-3x}$, we have $W[y_1, y_2](x) = e^{-6x}$. Hence a particular solution to the given differential equation is

$$y_p(x) = -e^{-3x} \int \frac{x e^{-3x} \cdot 4e^{-2x}}{e^{-6x}} dx + x e^{-3x} \int \frac{e^{-3x} \cdot 4e^{-2x}}{e^{-6x}} dx = 4e^{-2x}.$$

Therefore the differential equation has general solution

$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x} + 4e^{-2x}.$$

25.

(a) From Problem 20 the complementary function for the given differential equation is

$$y_c(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + c_3,$$

and an appropriate trial solution is

$$y_p(x) = A_0 x + A_1 x^2 + A_2 x^3.$$

Inserting this expression for $y_p(x)$ into the given differential equation yields

$$6A_2 - 6(2A_1 + 6A_2 x) + 25(A_0 + 2A_1 x + 3A_2 x^2) = x^2,$$

so that A_0, A_1, A_2 must satisfy

$$25A_0 - 12A_1 + 6A_2 = 0, \quad 50A_1 - 36A_2 = 0, \quad 75A_2 = 1.$$

Hence,

$$A_0 = \frac{22}{15625}, \quad A_1 = \frac{6}{625}, \quad A_2 = \frac{1}{75},$$

so that

$$y_p(x) = \frac{22}{15625}x + \frac{6}{625}x^2 + \frac{1}{75}x^3.$$

Consequently, the general solution to the given differential equation is

$$y(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + c_3 + \frac{22}{15625}x + \frac{6}{625}x^2 + \frac{1}{75}x^3.$$

(b) Choosing

$$y_1(x) = 1, \quad y_2(x) = e^{3x} \cos 4x, \quad y_3(x) = e^{3x} \sin 4x,$$

we have

$$W[y_1, y_2, y_3](x) = \begin{vmatrix} 1 & e^{3x} \cos 4x & e^{3x} \sin 4x \\ 0 & e^{3x}(3 \cos 4x - 4 \sin 4x) & e^{3x}(3 \sin 4x + 4 \cos 4x) \\ 0 & e^{3x}(-7 \cos 4x - 24 \sin 4x) & e^{3x}(-7 \sin 4x + 24 \cos 4x) \end{vmatrix} = 100e^{6x}.$$

Then a particular solution to the given differential equation is

$$y_p(x) = u_1 + u_2 e^{3x} \cos 4x + u_3 e^{3x} \sin 4x$$

where

$$\begin{aligned} u_1' + e^{3x} \cos 4x u_2' &+ e^{3x} \sin 4x u_3' &= 0, \\ e^{3x}(3 \cos 4x - 4 \sin 4x)u_2' &+ e^{3x}(3 \sin 4x + 4 \cos 4x)u_3' &= 0, \\ e^{3x}(-7 \cos 4x - 24 \sin 4x)u_2' &+ e^{3x}(-7 \sin 4x + 24 \cos 4x)u_3' &= x^2. \end{aligned}$$

Solving this system yields:

$$u_1' = \frac{1}{25}x^2, \quad u_2' = -\frac{1}{100}e^{-3x}x^2(3 \sin 4x + 4 \cos 4x), \quad u_3' = \frac{1}{100}e^{-3x}x^2(3 \cos 4x - 4 \sin 4x).$$

Hence,

$$\begin{aligned} y_p(x) &= \frac{1}{75}x^3 - \frac{1}{100}e^{3x} \cos 4x \int e^{-3x}x^2(3 \sin 4x + 4 \cos 4x)dx \\ &\quad + \frac{1}{100}e^{3x} \sin 4x \int e^{-3x}x^2(3 \cos 4x - 4 \sin 4x)dx \\ &= \frac{1}{75}x^3 + \frac{6}{625}x^2 + \frac{22}{15625}x - \frac{168}{390625}, \end{aligned}$$

and the general solution to the given differential equation is

$$y(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + c_3 + \frac{1}{75}x^3 + \frac{6}{625}x^2 + \frac{22}{15625}x.$$

26.

(a) From Problem 21 the complementary function for the given differential equation is

$$y_c(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + c_3,$$

and an appropriate trial solution is

$$y_p(x) = A_0 \cos 4x + B_0 \sin 4x.$$

Inserting this expression for $y_p(x)$ into the given differential equation yields

$$64A_0 \sin 4x - 64B_0 \cos 4x - 6(-16A_0 \cos 4x - 16B_0 \sin 4x) + 25(-4A_0 \sin 4x + 4B_0 \cos 4x) = \sin 4x,$$

that is

$$(-36A_0 + 96B_0) \sin 4x + (96A_0 + 36B_0) \cos 4x = \sin 4x$$

so that A_0 , and B_0 must satisfy

$$-36A_0 + 96B_0 = 1, \quad 96A_0 + 36B_0 = 0.$$

Hence,

$$A_0 = -\frac{1}{292}, \quad B_0 = \frac{2}{219},$$

so that

$$y_p(x) = -\frac{1}{292} \cos 4x + \frac{2}{219} \sin 4x.$$

Consequently, the general solution to the given differential equation is

$$y(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + c_3 - \frac{1}{292} \cos 4x + \frac{2}{219} \sin 4x.$$

(b) Choosing $y_1(x) = e^{3x} \cos 4x$, $y_2(x) = e^{3x} \sin 4x$, $y_3(x) = 1$, a particular solution to the given differential equation is

$$y_p(x) = e^{3x} \cos 4x \cdot u_1 + e^{3x} \sin 4x \cdot u_2 + u_3 \quad (0.0.28)$$

where

$$\begin{aligned} e^{3x} \cos 4x \cdot u'_1 + e^{3x} \sin 4x \cdot u'_2 &+ u'_3 = 0 \\ e^{3x}(3 \cos 4x - 4 \sin 4x)u'_1 + e^{3x}(3 \sin 4x + 4 \cos 4x)u'_2 &= 0 \\ e^{3x}(-7 \cos 4x - 24 \sin 4x)u'_1 + e^{3x}(-7 \sin 4x + 24 \cos 4x)u'_2 &= \sin 4x. \end{aligned}$$

Solving this system using Cramer's rule yields:

$$\begin{aligned} u'_1 &= \frac{\begin{vmatrix} 0 & e^{3x} \sin 4x & 1 \\ 0 & e^{3x}(3 \sin 4x + 4 \cos 4x) & 0 \\ \sin 4x & e^{3x}(-7 \sin 4x + 24 \cos 4x) & 0 \end{vmatrix}}{\begin{vmatrix} e^{3x} \cos 4x & e^{3x} \sin 4x & 1 \\ e^{3x}(3 \cos 4x - 4 \sin 4x) & e^{3x}(3 \sin 4x + 4 \cos 4x) & 0 \\ e^{3x}(-7 \cos 4x - 24 \sin 4x) & e^{3x}(-7 \sin 4x + 24 \cos 4x) & 0 \end{vmatrix}} \\ &= -\frac{1}{100} e^{-3x} (3 \sin^2 4x + 4 \cos 4x \sin 4x) = -\frac{1}{100} e^{-3x} \left[\frac{3}{2} (1 - \cos 8x) + 2 \sin 8x \right], \\ u'_2 &= \frac{\begin{vmatrix} e^{3x} \cos 4x & 0 & 1 \\ e^{3x}(3 \cos 4x - 4 \sin 4x) & 0 & 0 \\ e^{3x}(-7 \cos 4x - 24 \sin 4x) & \sin 4x & 0 \end{vmatrix}}{\begin{vmatrix} e^{3x} \cos 4x & e^{3x} \sin 4x & 1 \\ e^{3x}(3 \cos 4x - 4 \sin 4x) & e^{3x}(3 \sin 4x + 4 \cos 4x) & 0 \\ e^{3x}(-7 \cos 4x - 24 \sin 4x) & e^{3x}(-7 \sin 4x + 24 \cos 4x) & 0 \end{vmatrix}} \\ &= \frac{1}{100} e^{-3x} (3 \sin 4x \cos 4x - 4 \sin^2 4x) = \frac{1}{100} e^{-3x} \left[\frac{3}{2} \sin 8x + 2(\cos 8x - 1) \right], \\ u'_3 &= \frac{\begin{vmatrix} e^{3x} \cos 4x & e^{3x} \sin 4x & 0 \\ e^{3x}(3 \cos 4x - 4 \sin 4x) & e^{3x}(3 \sin 4x + 4 \cos 4x) & 0 \\ e^{3x}(-7 \cos 4x - 24 \sin 4x) & e^{3x}(-7 \sin 4x + 24 \cos 4x) & \sin 4x \end{vmatrix}}{\begin{vmatrix} e^{3x} \cos 4x & e^{3x} \sin 4x & 1 \\ e^{3x}(3 \cos 4x - 4 \sin 4x) & e^{3x}(3 \sin 4x + 4 \cos 4x) & 0 \\ e^{3x}(-7 \cos 4x - 24 \sin 4x) & e^{3x}(-7 \sin 4x + 24 \cos 4x) & 0 \end{vmatrix}} = \frac{1}{25} \sin 4x. \end{aligned}$$

Integrating these expressions and substituting into (0.0.28) yields

$$y_p(x) = \frac{2}{219} \sin 4x - \frac{1}{292} \cos 4x,$$

so that the general solution to the given differential equation is

$$y(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + c_3 + \frac{2}{219} \sin 4x - \frac{1}{292} \cos 4x.$$

27.

(a) In operator form the given differential equation is

$$(D^2 - 4)y = 5e^x$$

that is

$$(D - 2)(D + 2)y = 5e^x. \quad (0.0.29)$$

Therefore the complementary function is

$$y_c(x) = c_1 e^{2x} + c_2 e^{-2x}.$$

The annihilator of $F(x) = 5e^x$ is $A(D) = D - 1$. Operating on (0.0.29) with $D - 1$ yields

$$(D - 1)(D - 2)(D + 2)y = 0$$

which has general solution

$$y(x) = y_c(x) + A_0 e^x.$$

Consequently, an appropriate trial solution for (0.0.29) is

$$y_p(x) = A_0 e^x.$$

Inserting this expression for $y_p(x)$ into the given differential equation yields

$$A_0 e^x (1 - 4) = 5e^x$$

so that $A_0 = -\frac{5}{3}$. Hence,

$$y_p(x) = -\frac{5}{3} e^x,$$

and the general solution to the given differential equation is

$$y(x) = c_1 e^{2x} + c_2 e^{-2x} - \frac{5}{3} e^x.$$

(b) Choosing $y_1(x) = e^{2x}$, $y_2(x) = e^{-2x}$, we have $W[y_1, y_2](x) = -4$. Hence a particular solution to the given differential equation is

$$y_p(x) = -e^{2x} \int \frac{e^{-2x} \cdot 5e^x}{-4} dx + e^{-2x} \int \frac{e^{2x} \cdot 5e^x}{-4} dx = -\frac{5}{3} e^x.$$

Therefore the differential equation has general solution

$$y(x) = c_1 e^{2x} + c_2 e^{-2x} - \frac{5}{3} e^x.$$

28.

(a) In operator form the given differential equation is

$$(D^2 + 2D + 1)y = 2xe^{-x}$$

that is

$$(D + 1)^2 y = 2xe^{-x}. \quad (0.0.30)$$

Therefore the complementary function is

$$y_c(x) = c_1 e^{-x} + c_2 x e^{-x}.$$

The annihilator of $F(x) = 2xe^{-x}$ is $A(D) = (D + 1)^2$. Operating on (0.0.30) with $(D + 1)^2$ yields

$$(D + 1)^4 y = 0$$

which has general solution

$$y(x) = y_c(x) + A_0 x^2 e^{-x} + A_1 x^3 e^{-x}.$$

Consequently, an appropriate trial solution for (0.0.30) is

$$y_p(x) = e^{-x}(A_0 x^2 + A_1 x^3).$$

Differentiating this trial solution with respect to x yields

$$\begin{aligned} y'_p(x) &= e^{-x}(-A_0 x^2 - A_1 x^3 + 2A_0 x + 3A_1 x^2), \\ y''_p(x) &= e^{-x}(A_0 x^2 + A_1 x^3 - 4A_0 x - 6A_1 x^2 - 2A_0 + 6A_1 x). \end{aligned}$$

Inserting these results into the given differential equation yields

$$\begin{aligned} e^{-x}(A_0 x^2 + A_1 x^3 - 4A_0 x - 6A_1 x^2 + 2A_0 + 6A_1 x) + 2e^{-x}(-A_0 x^2 - A_1 x^3 + 2A_0 x + 3A_1 x^2) \\ + e^{-x}(A_0 x^2 + A_1 x^3) = 2xe^{-x}, \end{aligned}$$

that is,

$$2A_0 + 6A_1 x = 2x.$$

Consequently,

$$A_0 = 0, \quad A_1 = \frac{1}{3}.$$

Hence,

$$y_p(x) = \frac{1}{3} x^3 e^{-x},$$

and the general solution to the given differential equation is

$$y(x) = c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{3} x^3 e^{-x}.$$

(b) Choosing $y_1(x) = e^{-x}$, $y_2(x) = x e^{-x}$, we have $W[y_1, y_2](x) = e^{-2x}$. Hence a particular solution to the given differential equation is

$$y_p(x) = -e^{-x} \int \frac{x e^{-x} \cdot 2x e^{-x}}{e^{-2x}} dx + x e^{-x} \int \frac{e^{-x} \cdot 2x e^{-x}}{e^{-2x}} dx = \frac{1}{3} x^3 e^{-x}.$$

Therefore the given differential equation has general solution

$$y(x) = c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{3} x^3 e^{-x}.$$

29.

(a) In operator form the given differential equation is

$$(D^2 - 1)y = 4e^x$$

that is

$$(D - 1)(D + 1)y = 4e^x. \quad (0.0.31)$$

Therefore the complementary function is

$$y_c(x) = c_1 e^x + c_2 e^{-x}.$$

The annihilator of $F(x) = 4e^x$ is $A(D) = D - 1$. Operating on (0.0.31) with $D - 1$ yields

$$(D - 1)^2(D + 1)y = 0$$

which has general solution

$$y(x) = y_c(x) + A_0 x e^x.$$

Consequently, an appropriate trial solution for (0.0.31) is

$$y_p(x) = A_0 x e^x.$$

Differentiating this trial solution with respect to x yields

$$y'_p(x) = A_0 e^x(x + 1), \quad y''_p(x) = A_0 e^x(x + 2).$$

Inserting these expressions into the given differential equation yields

$$A_0 e^x(x + 2) - A_0 x e^x = 4e^x,$$

so that $A_0 = 2$. Hence,

$$y_p(x) = 2x e^x,$$

and the general solution to the given differential equation is

$$y(x) = c_1 e^x + c_2 e^{-x} + 2x e^x.$$

(b) Choosing $y_1(x) = e^x$, $y_2(x) = e^{-x}$, we have $W[y_1, y_2](x) = -2$. Hence a particular solution to the given differential equation is

$$y_p(x) = -e^x \int \frac{e^{-x} \cdot 4e^x}{-2} dx + e^{-x} \int \frac{e^x \cdot 4e^x}{-2} dx = 2x e^x - e^x.$$

The last term in this expression for $y_p(x)$ can be omitted since it is part of the complementary function. Consequently, the given differential equation has general solution

$$y(x) = c_1 e^x + c_2 e^{-x} + 2x e^x.$$

30. The given differential equation does not have constant coefficients, and therefore the annihilator method cannot be applied.

31. The nonhomogeneous term $F(x) = \ln x$ cannot be annihilated, and therefore the annihilator method cannot be applied.

32. In operator form the given differential equation is

$$(D^2 + 2D - 3)y = 5e^x$$

that is

$$(D + 3)(D - 1)y = 5e^x. \quad (0.0.32)$$

Therefore the complementary function is

$$y_c(x) = c_1 e^x + c_2 e^{-3x}.$$

The annihilator of $F(x) = 5e^x$ is $A(D) = D - 1$. Operating on (0.0.32) with $D - 1$ yields

$$(D - 1)^2(D + 3)y = 0$$

which has general solution

$$y(x) = y_c(x) + A_0 x e^x.$$

Consequently, an appropriate trial solution for (0.0.32) is

$$y_p(x) = A_0 x e^x.$$

33. The nonhomogeneous term $F(x) = \tan x$ cannot be annihilated, and therefore the annihilator method cannot be applied.

34. In operator form the given differential equation is

$$(D^2 + 1)y = 4 \cos 2x + 3e^x. \quad (0.0.33)$$

Therefore the complementary function is

$$y_c(x) = c_1 \cos x + c_2 \sin x.$$

The annihilator of $F(x) = 4 \cos 2x + 3e^x$ is $A(D) = (D^2 + 4)(D - 1)$. Operating on (0.0.33) with $(D^2 + 4)(D - 1)$ yields

$$(D^2 + 4)(D - 1)(D^2 + 1)y = 0$$

which has general solution

$$y(x) = y_c(x) + A_0 \cos 2x + A_1 \sin 2x + A_2 e^x.$$

Consequently, an appropriate trial solution for (0.0.33) is

$$y_p(x) = A_0 \cos 2x + A_1 \sin 2x + A_2 e^x.$$

35. In operator form the given differential equation is

$$(D^2 - 8D + 16)y = 7e^{4x},$$

that is

$$(D - 4)^2 y = 7e^{4x}. \quad (0.0.34)$$

Therefore the complementary function is

$$y_c(x) = c_1 e^{4x} + c_2 x e^{4x}.$$

The annihilator of $F(x) = 7e^{4x}$ is $A(D) = D - 4$. Operating on (0.0.34) with $D - 4$ yields

$$(D - 4)^3 y = 0$$

which has general solution

$$y(x) = y_c(x) + A_0 x^2 e^{4x}.$$

Consequently, an appropriate trial solution for (0.0.34) is

$$y_p(x) = A_0 x^2 e^{4x}.$$

36. The differential equation does not have constant coefficients and therefore the annihilator method cannot be applied.

37. In operator form the given differential equation is

$$(D^2 - 2D + 5)y = 7e^x \cos x + \sin x. \quad (0.0.35)$$

Therefore the complementary function is

$$y_c(x) = c_1 e^x (c_1 \cos 2x + c_2 \sin 2x).$$

The annihilator of $F(x) = 7e^x \cos x + \sin x$ is $A(D) = (D^2 - 2D + 2)(D^2 + 1)$. Operating on (0.0.35) with $(D^2 - 2D + 2)(D^2 + 1)$ yields

$$(D^2 - 2D + 2)(D^2 + 1)(D^2 - 2D + 5)y = 0$$

which has general solution

$$y(x) = y_c(x) + e^x (A_0 \cos x + A_1 \sin x) + A_2 \cos x + A_3 \sin x.$$

Consequently, an appropriate trial solution for (0.0.35) is

$$y_p(x) = e^x (A_0 \cos x + A_1 \sin x) + A_2 \cos x + A_3 \sin x.$$

38. In operator form the given differential equation is

$$(D^2 + 4)y = 7 \cos^2 x$$

that is, using the trigonometric identity $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$

$$(D^2 + 4)y = \frac{7}{2}(\cos 2x + 1). \quad (0.0.36)$$

Therefore the complementary function is

$$y_c(x) = c_1 \cos 2x + c_2 \sin 2x.$$

The annihilator of $F(x) = \frac{7}{2}(\cos 2x + 1)$ is $A(D) = D(D^2 + 4)$. Operating on (0.0.36) with $D(D^2 + 4)$ yields

$$D(D^2 + 4)^2 y = 0$$

which has general solution

$$y(x) = y_c(x) + A_0 + x(A_1 \cos 2x + A_2 \sin 2x).$$

Consequently, an appropriate trial solution for (0.0.36) is

$$y_p(x) = A_0 + x(A_1 \cos 2x + A_2 \sin 2x).$$

39. In operator form the given differential equation is

$$(D^2 - 2aD + a^2 + b^2)y = e^{at}(4t + \cos bt). \quad (0.0.37)$$

Therefore the complementary function is

$$y_c(t) = e^{at}(c_1 \cos bt + c_2 \sin bt).$$

The annihilator of $F(t) = e^{at}(4t + \cos bt)$ is $A(D) = (D - a)^2(D^2 - 2aD + a^2 + b^2)$. Operating on (0.0.37) with $(D - a)^2(D^2 - 2aD + a^2 + b^2)$ yields

$$(D - a)^2(D^2 - 2aD + a^2 + b^2)^2 y = 0$$

which has general solution

$$y(t) = y_c(t) + e^{at}(A_0 + A_1 t) + te^{at}(A_2 \cos bt + A_3 \sin bt).$$

Consequently, an appropriate trial solution for (0.0.37) is

$$y_p(t) = e^{at}(A_0 + A_1 t) + te^{at}(A_2 \cos bt + A_3 \sin bt).$$

40. In operator form the given differential equation is

$$(D^2 + 4)y = 7e^x. \quad (0.0.38)$$

Therefore the complementary function is

$$y_c(x) = c_1 \cos 2x + c_2 \sin 2x.$$

The annihilator of $F(x) = 7e^x$ is $A(D) = D - 1$. Operating on (0.0.38) with $D - 1$ yields

$$(D - 1)(D^2 + 4)y = 0$$

which has general solution

$$y(x) = y_c(x) + A_0 e^x.$$

Consequently, an appropriate trial solution for (0.0.38) is

$$y_p(x) = A_0 e^x.$$

Inserting this expression for y_p into the given differential equation yields

$$5A_0e^x = 7e^x,$$

so that $A_0 = \frac{7}{5}$. Hence,

$$y_p(x) = \frac{7}{5}e^x,$$

and the general solution to the given differential equation is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{7}{5}e^x.$$

41. In operator form the given differential equation is

$$(D^2 + 2D - 3)y = 2xe^{-3x}$$

that is

$$(D + 3)(D - 1)y = 2xe^{-3x}. \quad (0.0.39)$$

Therefore the complementary function is

$$y_c(x) = c_1e^{-3x} + c_2e^x.$$

The annihilator of $F(x) = 2xe^{-3x}$ is $A(D) = (D + 3)^2$. Operating on (0.0.39) with $(D + 3)^2$ yields

$$(D + 3)^3(D - 1)y = 0$$

which has general solution

$$y(x) = y_c(x) + e^{-3x}(A_0x + A_1x^2).$$

Consequently, an appropriate trial solution for (0.0.39) is

$$y_p(x) = e^{-3x}(A_0x + A_1x^2).$$

Differentiating this trial solution with respect to x yields

$$\begin{aligned} y'_p(x) &= e^{-3x}(-3A_0x - 3A_1x^2 + A_0 + 2A_1x), \\ y''_p(x) &= e^{-3x}(9A_0x + 9A_1x^2 - 3A_0 - 6A_1x - 3A_0 - 6A_1x + 2A_1). \end{aligned}$$

Inserting these expressions into the given differential equation yields

$$\begin{aligned} e^{-3x}(9A_0x + 9A_1x^2 - 6A_0 - 12A_1x + 2A_1) + 2e^{-3x}(-3A_0x - 3A_1x^2 + A_0 + 2A_1x) \\ - 3e^{-3x}(A_0x + A_1x^2) = 2xe^{-3x}, \end{aligned}$$

which simplifies to

$$-4A_0 + 2A_1 - 8A_1x = 2x.$$

Therefore A_0 and A_1 must satisfy

$$-4A_0 + 2A_1 = 0, \quad -8A_1 = 2,$$

so that $A_0 = -\frac{1}{8}$ and $A_1 = -\frac{1}{4}$. Hence,

$$y_p(x) = e^{-3x} \left(-\frac{1}{8}x - \frac{1}{4}x^2 \right) = -\frac{1}{8}xe^{-3x}(2x + 1),$$

and the general solution to the given differential equation is

$$y(x) = c_1 e^{-3x} + c_2 e^x - \frac{1}{8} x e^{-3x} (2x + 1).$$

42. In operator form the given differential equation is

$$(D^2 + 4D)y = 4x^2$$

that is

$$D(D + 4)y = 4x^2. \quad (0.0.40)$$

Therefore the complementary function is

$$y_c(x) = c_1 + c_2 e^{-4x}.$$

The annihilator of $F(x) = 4x^2$ is $A(D) = D^3$. Operating on (0.0.40) with D^3 yields

$$D^4(D + 4)y = 0$$

which has general solution

$$y(x) = y_c(x) + A_1 x + A_2 x^2 + A_3 x^3.$$

Consequently, an appropriate trial solution for (0.0.40) is

$$y_p(x) = A_1 x + A_2 x^2 + A_3 x^3.$$

Differentiating this trial solution with respect to x yields

$$y'_p(x) = A_1 + 2A_2 x + 3A_3 x^2, \quad y''_p(x) = 2A_2 + 6A_3 x.$$

Inserting these expressions into the given differential equation yields

$$2A_2 + 6A_3 x + 4(A_1 + 2A_2 x + 3A_3 x^2) = 4x^2,$$

that is,

$$A_2 + 2A_1 + x(3A_3 + 4A_2) + 6A_3 x^2 = 2x^2.$$

Therefore A_1, A_2 , and A_3 must satisfy

$$A_2 + 2A_1 = 0, \quad 3A_3 + 4A_2 = 0, \quad 6A_3 = 2,$$

so that $A_1 = \frac{1}{8}$, $A_2 = -\frac{1}{4}$, and $A_3 = \frac{1}{3}$. Hence,

$$y_p(x) = \frac{1}{8}x - \frac{1}{4}x^2 + \frac{1}{3}x^3 = \frac{1}{24}x(3 - 6x + 8x^2),$$

and the general solution to the given differential equation is

$$y(x) = c_1 + c_2 e^{-4x} + \frac{1}{24}x(3 - 6x + 8x^2).$$

43. In operator form the given differential equation is

$$(D^2 + 4)y = 8 \cos 2x. \quad (0.0.41)$$

Therefore the complementary function is

$$y_c(x) = c_1 \cos 2x + c_2 \sin 2x.$$

The annihilator of $F(x) = 8 \cos 2x$ is $A(D) = D^2 + 4$. Operating on (0.0.41) with $D^2 + 4$ yields

$$(D^2 + 4)^2 y = 0$$

which has general solution

$$y(x) = y_c(x) + x(A_0 \cos 2x + A_1 \sin 2x).$$

Consequently, an appropriate trial solution for (0.0.41) is

$$y_p(x) = x(A_0 \cos 2x + A_1 \sin 2x).$$

Differentiating this trial solution with respect to x yields

$$\begin{aligned} y'_p(x) &= A_0 \cos 2x + A_1 \sin 2x + x(-2A_0 \sin 2x + A_1 \cos 2x), \\ y''_p(x) &= -4A_0 \sin 2x + 4A_1 \cos 2x + x(-4A_0 \cos 2x - A_1 \sin 2x). \end{aligned}$$

Inserting these expressions into the given differential equation yields

$$-4A_0 \sin 2x + 4A_1 \cos 2x + x(-4A_0 \cos 2x - A_1 \sin 2x) + 4x(A_0 \cos 2x + A_1 \sin 2x) = 8 \cos 2x,$$

that is,

$$-4A_0 \sin 2x + 4A_1 \cos 2x = 8 \cos 2x.$$

Consequently, $A_0 = 0$, and $A_1 = 2$. Therefore,

$$y_p(x) = 2x \sin 2x,$$

and the general solution to the given differential equation is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + 2x \sin 2x.$$

44. In operator form the given differential equation is

$$(D^2 - 8D + 16)y = 5e^{4x},$$

that is

$$(D - 4)^2 y = 5e^{4x}. \quad (0.0.42)$$

Therefore the complementary function is

$$y_c(x) = c_1 e^{4x} + c_2 x e^{4x}.$$

The annihilator of $F(x) = 5e^{4x}$ is $A(D) = D - 4$. Operating on (0.0.42) with $D - 4$ yields

$$(D - 4)^3 y = 0$$

which has general solution

$$y(x) = y_c(x) + A_0 x^2 e^{4x}.$$

Consequently, an appropriate trial solution for (0.0.42) is

$$y_p(x) = A_0 x^2 e^{4x}.$$

Differentiating this trial solution with respect to x yields

$$y'_p(x) = A_0 e^{4x}(4x^2 + 2x), \quad y''_p(x) = A_0 e^{4x}(16x^2 + 16x + 2).$$

Inserting these expressions into the given differential equation yields

$$A_0 e^{4x}(16x^2 + 16x + 2) - 8A_0 e^{4x}(4x^2 + 2x) - 16A_0 x^2 e^{4x} = 5e^{4x},$$

which reduces to $2A_0 = 5$, so that $A_0 = \frac{5}{2}$. Therefore,

$$y_p(x) = \frac{5}{2} x^2 e^{4x},$$

and the general solution to the given differential equation is

$$y(x) = c_1 e^{4x} + c_2 x e^{4x} + \frac{5}{2} x^2 e^{4x}.$$

45. In operator form the given differential equation is

$$(D^2 - 1)y = 3e^{2x} + \sin x,$$

that is

$$(D - 1)(D + 1)y = 3e^{2x} + \sin x. \quad (0.0.43)$$

Therefore the complementary function is

$$y_c(x) = c_1 e^x + c_2 e^{-x}.$$

The annihilator of $F(x) = 3e^{2x} + \sin x$ is $A(D) = (D^2 + 1)(D - 2)$. Operating on (0.0.43) with $(D^2 + 1)(D - 2)$ yields

$$(D^2 + 1)(D - 2)(D - 1)(D + 1)y = 0$$

which has general solution

$$y(x) = y_c(x) + A_0 e^{2x} + A_1 \cos x + A_2 \sin x.$$

Consequently, an appropriate trial solution for (0.0.43) is

$$y_p(x) = A_0 e^{2x} + A_1 \cos x + A_2 \sin x.$$

Differentiating this trial solution with respect to x yields

$$y'_p(x) = 2A_0 e^{2x} - A_1 \sin x + A_2 \cos x, \quad y''_p(x) = 4A_0 e^{2x} - A_1 \cos x - A_2 \sin x.$$

Inserting these expressions into the given differential equation yields

$$4A_0 e^{2x} - A_1 \cos x - A_2 \sin x - (A_0 e^{2x} + A_1 \cos x + A_2 \sin x) = 3e^{2x} + \sin x,$$

that is,

$$3A_0 e^{2x} - 2A_1 \cos x - 2A_2 \sin x = 3e^{2x} + \sin x.$$

Therefore, $A_0 = 1$, $A_1 = 0$, $A_2 = -\frac{1}{2}$, so that

$$y_p(x) = e^{2x} - \frac{1}{2} \sin x,$$

and the general solution to the given differential equation is

$$y(x) = c_1 e^x + c_2 e^{-x} + e^{2x} - \frac{1}{2} \sin x.$$

46. In operator form the given differential equation is

$$(D^2 - D - 2)y = 15e^{2x},$$

that is

$$(D - 2)(D + 1)y = 15e^{2x}. \quad (0.0.44)$$

Therefore the complementary function is

$$y_c(x) = c_1 e^{2x} + c_2 e^{-x}.$$

The annihilator of $F(x) = 15e^{2x}$ is $A(D) = D - 2$. Operating on (0.0.44) with $D - 2$ yields

$$(D - 2)^2(D + 1)y = 0$$

which has general solution

$$y(x) = y_c(x) + A_0 x e^{2x}.$$

Consequently, an appropriate trial solution for (0.0.44) is

$$y_p(x) = A_0 x e^{2x}.$$

Differentiating this trial solution with respect to x yields

$$y_p'(x) = A_0 e^{2x}(2x + 1), \quad y_p''(x) = A_0 e^{2x}(4x + 4).$$

Inserting these expressions into the given differential equation yields

$$A_0 e^{2x}[4x + 4 - (2x + 1) - 2x] = 15e^{2x},$$

that is,

$$3A_0 = 15.$$

Therefore, $A_0 = 5$, so that

$$y_p(x) = 5x e^{2x},$$

and the general solution to the given differential equation is

$$y(x) = c_1 e^{2x} + c_2 e^{-x} + 5x e^{2x}.$$

The initial conditions $y(0) = 0$, $y'(0) = 8$ require, respectively,

$$c_1 + c_2 = 0, \quad 2c_1 - c_2 + 5 = 8$$

so that $c_1 = 1, c_2 = -1$. Therefore the solution to the given initial-value problem is

$$y(x) = e^{2x} - e^{-x} + 5xe^{2x}$$

47. The complementary function is

$$y_c(x) = c_1 \cos x + c_2 \sin x.$$

Choosing

$$y_1(x) = \cos x, \quad y_2(x) = \sin x$$

we have

$$W[y_1, y_2](x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

so that

$$y_p(x) = -\cos x \int \sin x \cdot \frac{1}{\sin x} dx + \sin x \int \cos x \cdot \frac{1}{\sin x} dx = -x \cos x + \sin x \ln |\sin x|.$$

Hence,

$$y(x) = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \ln |\sin x|.$$

48. The complementary function is

$$y_c(x) = c_1 \cos x + c_2 \sin x.$$

Choosing

$$y_1(x) = \cos x, \quad y_2(x) = \sin x$$

we have

$$W[y_1, y_2](x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

so that

$$y_p(x) = -\cos x \int \sin x \cdot \frac{\sin x}{\cos x} dx + \sin x \int \cos x \cdot \frac{\sin x}{\cos x} dx = -\cos x \ln \left| \frac{1 + \sin x}{\cos x} \right|.$$

Hence,

$$y(x) = c_1 \cos x + c_2 \sin x - \cos x \ln \left| \frac{1 + \sin x}{\cos x} \right|.$$

49. The complementary function is

$$y_c(x) = c_1 e^{mx} + c_2 x e^{mx}.$$

Choosing

$$y_1(x) = e^{mx}, \quad y_2(x) = x e^{mx}$$

we have

$$W[y_1, y_2](x) = \begin{vmatrix} e^{mx} & x e^{mx} \\ m e^{mx} & e^{mx}(mx + 1) \end{vmatrix} = e^{2mx}$$

so that

$$y_p(x) = -e^{mx} \int \frac{x e^{mx} \cdot e^{mx} \ln x}{e^{2mx}} dx + x e^{mx} \int \frac{e^{mx} \cdot e^{mx} \ln x}{e^{2mx}} dx = \frac{1}{4} x^2 e^{mx} (2 \ln x - 3).$$

Hence,

$$y(x) = c_1 e^{mx} + c_2 x e^{mx} + \frac{1}{4} x^2 e^{mx} (2 \ln x - 3).$$

50. The given differential equation should read

$$y'' + 2y' + y = x^{-1} e^{-x}$$

The complementary function is

$$y_c(x) = c_1 e^{-x} + c_2 x e^{-x}.$$

Choosing

$$y_1(x) = e^{-x}, \quad y_2(x) = x e^{-x}$$

we have

$$W[y_1, y_2](x) = \begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x}(1-x) \end{vmatrix} = e^{-2x}$$

so that

$$y_p(x) = -e^{-x} \int \frac{x e^{-x} \cdot x^{-1} e^{-x}}{e^{-2x}} dx + x e^{-x} \int \frac{e^{-x} \cdot x^{-1} e^{-x}}{e^{-2x}} dx = -x e^{-x} + x e^{-x} \ln |x|.$$

Hence,

$$y(x) = c_1 e^{-x} + c_2 x e^{-x} + x e^{-x} \ln |x|.$$

51. The complementary function is

$$y_c(x) = c_1 e^x + c_2 x e^x.$$

Choosing

$$y_1(x) = e^x, \quad y_2(x) = x e^x$$

we have

$$W[y_1, y_2](x) = \begin{vmatrix} e^x & x e^x \\ e^x & e^x(x+1) \end{vmatrix} = e^{2x}$$

so that

$$y_p(x) = -e^x \int \frac{x e^x \cdot e^x \ln x}{e^{2x}} dx + x e^x \int \frac{e^x \cdot e^x \ln x}{e^{2x}} dx = \frac{1}{4} x^2 e^x (2 \ln x - 3).$$

Hence,

$$y(x) = c_1 e^x + c_2 x e^x + \frac{1}{4} x^2 e^x (2 \ln x - 3)$$

52. Let $y(x) = u \cdot e^{mx}$. Then

$$y'(x) = e^{mx}(u' + mu), \quad y'' = e^{mx}(u'' + 2mu' + m^2u).$$

Inserting these results into the given differential equation yields

$$e^{mx}(u'' + 2mu' + m^2u) - 2me^{mx}(u' + mu) + m^2e^{mx}u = e^{mx} \ln x,$$

that is

$$u'' = \ln x.$$

Integrating twice gives

$$u(x) = \frac{1}{4}x^2(2\ln x - 3) + c_1x + c_2,$$

so that

$$y(x) = e^{mx} \left[\frac{1}{4}x^2(2\ln x - 3) + c_1x + c_2 \right].$$

53. The indicial equation is

$$r(r-1) + 9r + 6 = 0 \implies r^2 + 8r + 16 = 0 \implies (r+4)^2 = 0,$$

with root $r = -4$ (multiplicity 2). Consequently, the given differential equation has general solution

$$y(x) = c_1x^{-4} + c_2x^{-4} \ln x.$$

54. The indicial equation is

$$r(r-1) + 9r + 15 = 0 \implies r^2 + 8r + 15 = 0 \implies (r+3)(r+5) = 0,$$

with roots $r = -3, r = -5$. Consequently, the given differential equation has general solution

$$y(x) = c_1x^{-3} + c_2x^{-5}.$$

55. The indicial equation is

$$r(r-1) - 11r + 37 = 0 \implies r^2 - 12r + 37 = 0,$$

with roots $r = 6 \pm i$. Consequently, the given differential equation has general solution

$$y(x) = c_1x^6 \cos(\ln x) + c_2x^6 \sin(\ln x).$$

56. The indicial equation is

$$r(r-1) + r + 25 = 0 \implies r^2 + 25 = 0,$$

with roots $r = \pm 5i$. Consequently, the given differential equation has general solution

$$y(x) = c_1 \cos(5 \ln x) + c_2 \sin(5 \ln x).$$

57. The indicial equation is

$$r(r-1) - 2r - 18 = 0 \implies r^2 - 3r - 18 = 0 \implies (r+3)(r-6) = 0,$$

with roots $r = -3, r = 6$. Consequently, the given differential equation has general solution

$$y(x) = c_1x^{-3} + c_2x^6.$$

58. The indicial equation is

$$r(r-1) - r = 0 \implies r(r-2) = 0,$$

with roots $r = 0, r = 2$. Consequently, the given differential equation has general solution

$$y(x) = c_1 + c_2 x^2.$$

59. The indicial equation is

$$r(r-1) + 9r + 6 = 0 \implies r^2 + 8r + 16 = 0 \implies (r+4)^2 = 0,$$

with root $r = -4$ (multiplicity 2). Consequently, the complementary function is

$$y_c(x) = c_1 x^{-4} + c_2 x^{-4} \ln x.$$

Choosing

$$y_1(x) = x^{-4}, \quad y_2(x) = x^{-4} \ln x$$

we have

$$W[y_1, y_2](x) = \begin{vmatrix} x^{-4} & x^{-4} \ln x \\ -4x^{-5} & x^{-5}(1 - 4 \ln x) \end{vmatrix} = x^{-9}$$

so that

$$y_p(x) = -x^{-4} \int \frac{x^{-4} \ln x \cdot x^{-5}}{x^{-9}} dx + x^{-4} \ln x \int \frac{x^{-4} \cdot x^{-5}}{x^{-9}} dx = x^{-3}.$$

Hence,

$$y(x) = c_1 x^{-4} + c_2 x^{-4} \ln x + x^{-3}.$$

60. The indicial equation is

$$r(r-1) - 3r - 12 = 0 \implies r^2 - 4r - 12 = 0 \implies (r-6)(r+2) = 0,$$

with roots $r = 6, r = -2$. Consequently, the complementary function is

$$y_c(x) = c_1 x^6 + c_2 x^{-2}.$$

Choosing

$$y_1(x) = x^6, \quad y_2(x) = x^{-2}$$

we have

$$W[y_1, y_2](x) = \begin{vmatrix} x^6 & x^{-2} \\ 6x^5 & -2x^{-3} \end{vmatrix} = -8x^3$$

so that

$$y_p(x) = -x^6 \int \frac{x^{-2}(x^2 + 5)}{-8x^3} dx + x^{-2} \int \frac{x^6(x^2 + 5)}{-8x^3} dx = -\frac{1}{48} x^2 (4x^2 + 15).$$

Hence,

$$y(x) = c_1 x^6 + c_2 x^{-2} - \frac{1}{48} x^2 (4x^2 + 15).$$

61. The indicial equation is

$$r(r-1) - 5r + 10 = 0 \implies r^2 - 6r + 10 = 0,$$

with roots $r = 3 \pm i$. Consequently, the complementary function is

$$y_c(x) = c_1 x^3 \cos(\ln x) + c_2 x^3 \sin(\ln x).$$

Choosing

$$y_1(x) = x^3 \cos(\ln x), \quad y_2(x) = x^3 \sin(\ln x)$$

we have

$$W[y_1, y_2](x) = \begin{vmatrix} x^3 \cos(\ln x) & x^3 \sin(\ln x) \\ x^2[3 \cos(\ln x) - \sin(\ln x)] & x^2[3 \sin(\ln x) + \cos(\ln x)] \end{vmatrix} = x^5$$

so that

$$y_p(x) = -x^3 \cos(\ln x) \int \frac{x^3 \sin(\ln x) \cdot x}{x^5} dx + x^3 \sin(\ln x) \int \frac{x^3 \cos(\ln x) \cdot x}{x^5} dx = x^3.$$

Hence,

$$y(x) = c_1 x^3 \cos(\ln x) + c_2 x^3 \sin(\ln x) + x^3.$$

62. In operator form the given differential equation is

$$(D^2 - 4D - 5)y = e^{3x} \sin 2x$$

that is

$$(D - 5)(D + 1)y = e^{3x} \sin 2x. \quad (0.0.45)$$

Therefore the complementary function is

$$y_c(x) = c_1 e^{5x} + c_2 e^{-x}.$$

The annihilator of $F(x) = e^{3x} \sin 2x$ is $A(D) = D^2 - 6D + 13$. Operating on (0.0.45) with $D^2 - 6D + 13$ yields

$$(D^2 - 6D + 13)(D - 5)(D + 1)y = 0$$

which has general solution

$$y(x) = y_c(x) + A_0 e^{3x} \cos 2x + A_1 e^{3x} \sin 2x.$$

Consequently, an appropriate trial solution for (0.0.45) is

$$y_p(x) = A_0 e^{3x} \cos 2x + A_1 e^{3x} \sin 2x.$$

Differentiating this trial solution with respect to x yields

$$\begin{aligned} y'_p(x) &= A_0 e^{3x} (3 \cos 2x - 2 \sin 2x) + A_1 e^{3x} (3 \sin 2x + 2 \cos 2x), \\ y''_p(x) &= A_0 e^{3x} (5 \cos 2x - 12 \sin 2x) + A_1 e^{3x} (5 \sin 2x + 12 \cos 2x). \end{aligned}$$

Inserting these expressions into the given differential equation yields

$$\begin{aligned} A_0 e^{3x} (5 \cos 2x - 12 \sin 2x) + A_1 e^{3x} (5 \sin 2x + 12 \cos 2x) - 4[A_0 e^{3x} (3 \cos 2x - 2 \sin 2x) \\ + A_1 e^{3x} (3 \sin 2x + 2 \cos 2x)] - 5(A_0 e^{3x} \cos 2x + A_1 e^{3x} \sin 2x) = e^{3x} \sin 2x, \end{aligned}$$

which simplifies to

$$4(-3A_0 + A_1) \cos 2x - 4(A_0 + 3A_1) \sin 2x = \sin 2x.$$

Therefore A_0 and A_1 must satisfy

$$-3A_0 + A_1 = 0, \quad A_0 + 3A_1 = -\frac{1}{4},$$

so that $A_0 = -\frac{1}{40}$ and $A_1 = -\frac{3}{40}$. Hence,

$$y_p(x) = -\frac{1}{40}e^{3x}(\cos 2x + 3 \sin 2x),$$

and the general solution to the given differential equation is

$$y(x) = c_1e^{5x} + c_2e^{-x} - \frac{1}{40}e^{3x}(\cos 2x + 3 \sin 2x).$$

63. The complementary function is

$$y_c(x) = c_1e^{-2x} + c_2xe^{-2x}.$$

Choosing

$$y_1(x) = e^{-2x}, \quad y_2(x) = xe^{-2x}$$

we have

$$W[y_1, y_2](x) = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x}(1-2x) \end{vmatrix} = e^{-4x}.$$

so that

$$y_p(x) = -e^{-2x} \int \frac{xe^{-2x}}{e^{-4x}} \cdot \frac{\ln x}{xe^{2x}} dx + xe^{-2x} \int \frac{e^{-2x}}{e^{-4x}} \cdot \frac{\ln x}{xe^{2x}} dx = \frac{1}{2}xe^{-2x}[(\ln x)^2 - 2 \ln x + 2].$$

Hence,

$$y(x) = c_1e^{-2x} + c_2xe^{-2x} + \frac{1}{2}xe^{-2x}[(\ln x)^2 - 2 \ln x + 2].$$

64. In operator form the given differential equation is

$$(D^2 - 2D + 26)y = e^x \cos 5x. \quad (0.0.46)$$

Therefore the complementary function is

$$y_c(x) = c_1e^x \cos 5x + c_2e^x \sin 5x.$$

The annihilator of $F(x) = e^x \cos 5x$ is $A(D) = D^2 - 2D + 26$. Operating on (0.0.46) with $D^2 - 2D + 26$ yields

$$(D^2 - 2D + 26)^2y = 0$$

which has general solution

$$y(x) = y_c(x) + xe^x(A_0 \cos 5x + A_1 \sin 5x).$$

Consequently, an appropriate trial solution for (0.0.46) is

$$y_p(x) = xe^x(A_0 \cos 5x + A_1 \sin 5x).$$

Differentiating this trial solution with respect to x yields

$$\begin{aligned}y_p'(x) &= e^x[(A_0x + A_0 + 5A_1x) \cos 5x + (A_1x + A_1 - 5A_0x) \sin 5x], \\y_p''(x) &= e^x[(-24A_0x + 2A_0 + 10A_1x + 10A_1) \cos 5x + (-24A_1x + 2A_1 - 10A_0x - 10A_0) \sin 5x].\end{aligned}$$

Inserting these expressions into the given differential equation yields

$$\begin{aligned}(-24A_0x + 2A_0 + 10A_1x + 10A_1) \cos 5x + (-24A_1x + 2A_1 - 10A_0x - 10A_0) \sin 5x \\- 2[(A_0x + A_0 + 5A_1x) \cos 5x + (A_1x + A_1 - 5A_0x) \sin 5x] \\+ 26x(A_0 \cos 5x + A_1 \sin 5x) = \cos 5x,\end{aligned}$$

which simplifies to

$$-10A_0 \sin 5x + 10A_2 \cos 5x = \cos 5x.$$

Therefore $A_0 = 0$ and $A_1 = \frac{1}{10}$. Hence,

$$y_p(x) = \frac{1}{10}xe^x \sin 5x,$$

and the general solution to the given differential equation is

$$y(x) = c_1e^x \cos 5x + c_2e^x \sin 5x + \frac{1}{10}xe^x \sin 5x.$$

65. In operator form the given differential equation is

$$(D^2 - 9D + 20)y = x^3e^{5x},$$

that is,

$$(D - 4)(D - 5)y = x^3e^{5x}. \quad (0.0.47)$$

Therefore the complementary function is

$$y_c(x) = c_1e^{4x} + c_2e^{5x}.$$

The annihilator of $F(x) = x^3e^{5x}$ is $A(D) = (D - 5)^4$. Operating on (0.0.47) with $(D - 5)^4$ yields

$$(D - 5)^5(D - 4)y = 0$$

which has general solution

$$y(x) = y_c(x) + e^{5x}(A_0x + A_1x^2 + A_2x^3 + A_3x^4).$$

Consequently, an appropriate trial solution for (0.0.47) is

$$y_p(x) = e^{5x}(A_0x + A_1x^2 + A_2x^3 + A_3x^4).$$

Differentiating this trial solution with respect to x yields

$$\begin{aligned}y_p'(x) &= e^{5x}(5A_0x + 5A_1x^2 + 5A_2x^3 + 5A_3x^4 + A_0 + 2A_1x + 3A_2x^2 + 4A_3x^3), \\y_p''(x) &= e^{5x}(25A_0x + 25A_1x^2 + 25A_2x^3 + 10A_0 + 20A_1x + 30A_2x^2 + 40A_3x^3 \\&\quad + 25A_3x^4 + 12A_3x^2 + 2A_1 + 6A_2x).\end{aligned}$$

Inserting these expressions into the given differential equation yields

$$\begin{aligned} & (25A_0x + 25A_1x^2 + 25A_2x^3 + 10A_0 + 20A_1x + 30A_2x^2 + 40A_3x^3 + 25A_3x^4 + 12A_3x^2 \\ & + 2A_1 + 6A_2x) - 9(5A_0x + 5A_1x^2 + 5A_2x^3 + 5A_3x^4 + A_0 + 2A_1x + 3A_2x^2 + 4A_3x^3) \\ & + 20(A_0x + A_1x^2 + A_2x^3 + A_3x^4) = x^3. \end{aligned}$$

which simplifies to

$$4A_3x^3 + 3(A_2 + 4A_3)x^2 + 2(A_1 + 3A_2)x + A_0 + 2A_1 = x^3.$$

Therefore A_0, A_1, A_2, A_3 must satisfy

$$4A_3 = 1, \quad A_2 + 4A_3 = 0, \quad A_1 + 3A_2 = 0, \quad A_0 + 2A_1 = 0,$$

so that $A_0 = -6, A_1 = 3, A_2 = -1, A_3 = \frac{1}{4}$. Consequently,

$$y_p(x) = \frac{1}{4}xe^{5x}(x^3 - 4x^2 + 12x - 24),$$

and the general solution to the given differential equation is

$$y(x) = c_1e^{4x} + c_2e^{5x} + \frac{1}{4}xe^{5x}(x^3 - 4x^2 + 12x - 24).$$

66. The complementary function is

$$y_c(x) = c_1e^{4x} \cos x + c_2e^{4x} \sin x.$$

Choosing

$$y_1(x) = e^{4x} \cos x, \quad y_2(x) = e^{4x} \sin x$$

we have

$$W[y_1, y_2](x) = \begin{vmatrix} e^{4x} \cos x & e^{4x} \sin x \\ e^{4x}(4 \cos x - \sin x) & e^{4x}(4 \sin x + \cos x) \end{vmatrix} = e^{8x}$$

so that

$$\begin{aligned} y_p(x) &= -e^{4x} \cos x \int \frac{e^{4x} \sin x \cdot e^{4x} \csc x}{e^{8x}} dx + e^{4x} \sin x \int \frac{e^{4x} \cos x \cdot e^{4x} \csc x}{e^{8x}} dx \\ &= e^{4x}(\sin x \ln |\sin x| - x \cos x). \end{aligned}$$

Hence,

$$y(x) = c_1e^{4x} \cos x + c_2e^{4x} \sin x + e^{4x}(\sin x \ln |\sin x| - x \cos x).$$

Solutions to Section 7.1

True-False Review:

1. TRUE. If we make the substitution $x \rightarrow x_1$ and $y \rightarrow x_2$, then in terms of Definition 7.1.1, we have two equations with $a_{11}(t) = t^2, a_{12}(t) = -t, a_{21}(t) = \sin t, a_{22}(t) = 5$, and $b_1(t) = b_2(t) = 0$.

2. TRUE. If we make the substitution $x \rightarrow x_1$ and $y \rightarrow x_2$, then in terms of Definition 7.1.1, we have two equations with $a_{11}(t) = t^4, a_{12}(t) = -e^t, a_{21}(t) = t^2 + 3, a_{22}(t) = 0$, and $b_1(t) = 4$ and $b_2(t) = -t^2$.

3. FALSE. The term txy on the right-hand side of the formula for x' is non-linear because the unknown functions x and y are multiplied together, and this is prohibited in the form of a first-order linear system of differential equations.

4. FALSE. The term $e^y x$ on the right-hand side of the formula for y' is non-linear because the known function y appears as an exponent. This is prohibited in the form of a first-order linear system of differential equations.

5. TRUE. If we consider Equation (7.1.16) with $n = 3$, then the text describes how the substitution $x_1 = x$, $x_2 = x'$, $x_3 = x''$ enables us to replace (7.1.16) with the equivalent first-order linear system

$$x'_1 = x_2, \quad x'_2 = x_3, \quad x'_3 = -a_3(t)x_1 - a_2(t)x_2 - a_1(t)x_1 + F(t).$$

6. FALSE. The technique for converting a first-order linear system of two differential equations to a second-order linear differential equation only applies in the case where the coefficients $a_{ij}(t)$ of the system (7.1.1) are constants.

7. FALSE. As indicated in Definition 7.1.6, an initial-value problem consists of auxiliary conditions all of which are applied at the *same time* t_0 . In this system, the value of x is specified at $t = 0$ while the value of y is specified at $t = 1$.

8. TRUE. This has the required form in Definition 7.1.6. Note that both x and y have initial-values specified at $t_0 = 0$.

9. FALSE. There is no initial-value specified for the function $y(t)$. The value $y(2)$ would be required in order for the to be an initial-value problem.

10. FALSE. As indicated in Definition 7.1.6, an initial-value problem consists of auxiliary conditions all of which are applied at the *same time* t_0 . In this system, the value of x is specified at $t = 3$ while the value of y is specified at $t = -3$.

Problems:

1. Writing the given system of differential equations in operator form gives us the equations

$$(D - 2)x_1 + 3x_2 = 0 \quad \text{and} \quad -x_1 + (D + 2)x_2 = 0.$$

Operating on the first equation with $D + 2$, we obtain $(D + 2)(D - 2)x_1 + 3(D + 2)x_2 = 0$. Combining this with the second equation above yields $(D + 2)(D - 2)x_1 + 3x_1 = 0$, or $(D^2 - 1)x_1 = 0$. Therefore, since the characteristic equation for this differential equation has roots $r = \pm 1$, we obtain

$$x_1(t) = c_1 e^t + c_2 e^{-t}.$$

Hence,

$$\begin{aligned} x_2(t) &= -\frac{1}{3}(D - 2)x_1 \\ &= -\frac{1}{3}(c_1 e^t - c_2 e^{-t} - 2c_1 e^t - 2c_2 e^{-t}) \\ &= \frac{1}{3}c_1 e^t + c_2 e^{-t}. \end{aligned}$$

To summarize, the solution to the system of differential equations is

$$x_1(t) = c_1 e^t + c_2 e^{-t} \quad \text{and} \quad x_2(t) = \frac{1}{3}c_1 e^t + c_2 e^{-t}.$$

2. Writing the given system of differential equations in operator form gives us the equations

$$(D - 4)x_1 - 2x_2 = 0 \quad \text{and} \quad x_1 + (D - 1)x_2 = 0.$$

Operating on the first equation with $D - 1$, we obtain $(D - 1)(D - 4)x_1 - 2(D - 1)x_2 = 0$. Combining this with the second equation above yields $(D - 1)(D - 4)x_1 + 2x_1 = 0$, or $(D^2 - 5D + 6)x_1 = 0$. Therefore, since the characteristic equation for this differential equation has roots $r = 2$ and $r = 3$, we obtain

$$x_1(t) = c_1e^{2t} + c_2e^{3t}.$$

Hence,

$$\begin{aligned} x_2(t) &= \frac{1}{2}(D - 4)x_1 \\ &= \frac{1}{2}(2c_1e^{2t} + 3c_2e^{3t} - 4c_1e^{2t} - 4c_2e^{3t}) \\ &= -c_1e^{2t} - \frac{1}{2}c_2e^{3t}. \end{aligned}$$

To summarize, the solution to the given system of differential equations is

$$x_1(t) = c_1e^{2t} + c_2e^{3t} \quad \text{and} \quad x_2(t) = -c_1e^{2t} - \frac{1}{2}c_2e^{3t}.$$

3. Writing the given system of differential equations in operator form gives us the equations

$$(D - 2)x_1 - 4x_2 = 0 \quad \text{and} \quad 4x_1 + (D + 6)x_2 = 0.$$

Operating on the first equation with $D + 6$, we obtain $(D + 6)(D - 2)x_1 - 4(D + 6)x_2 = 0$. Combining this with the second equation above yields $(D + 6)(D - 2)x_1 + 16x_1 = 0$, or $(D^2 + 4D + 4)x_1 = 0$. Since the characteristic equation for this differential equation has root $r = -2$ with multiplicity 2, we obtain

$$x_1(t) = c_1e^{-2t} + c_2te^{-2t}.$$

Hence,

$$\begin{aligned} x_2(t) &= \frac{1}{4}(D - 2)x_1 \\ &= \frac{1}{4}(-2c_1e^{-2t} + c_2e^{-2t} - 2c_2te^{-2t} - 2c_1e^{-2t} - 2c_2te^{-2t}) \\ &= \frac{1}{4}(-4c_1e^{-2t} + c_2(1 - 4t)e^{-2t}) \\ &= -c_1e^{-2t} + \frac{1}{4}c_2(1 - 4t)e^{-2t}. \end{aligned}$$

To summarize, the solution to the given system of differential equations is

$$x_1(t) = c_1e^{-2t} + c_2te^{-2t} \quad \text{and} \quad x_2(t) = -c_1e^{-2t} + \frac{1}{4}c_2(1 - 4t)e^{-2t}.$$

4. Writing the given system of differential equations in operator form gives us the equations

$$Dx_1 - 2x_2 = 0 \quad \text{and} \quad 2x_1 + Dx_2 = 0.$$

Operating on the first equation with D , we obtain $D^2x_1 - 2Dx_2 = 0$. Combining this with the second equation above yields $D^2x_1 + 4x_1 = 0$, or $(D^2 + 4)x_1 = 0$. Since the characteristic equation for this differential equation has roots $r = \pm 2i$, we obtain

$$x_1(t) = c_1 \cos 2t + c_2 \sin 2t.$$

Hence,

$$\begin{aligned} x_2(t) &= \frac{1}{2}Dx_1 \\ &= \frac{1}{2}(-2c_1 \sin 2t + 2c_2 \cos 2t) \\ &= -c_1 \sin 2t + c_2 \cos 2t. \end{aligned}$$

To summarize, the solution to the given system of differential equations is

$$x_1(t) = c_1 \cos 2t + c_2 \sin 2t \quad \text{and} \quad x_2(t) = -c_1 \sin 2t + c_2 \cos 2t.$$

5. Writing the given system of differential equations in operator form gives us the equations

$$(D - 1)x_1 + 3x_2 = 0 \quad \text{and} \quad -3x_1 + (D - 1)x_2 = 0.$$

Operating on the first equation with $D - 1$, we obtain $(D - 1)^2x_1 + 3(D - 1)x_2 = 0$. Combining this with the second equation above yields $(D - 1)^2x_1 + 9x_1 = 0$, or $(D^2 - 2D + 10)x_1 = 0$. Since the characteristic equation for this differential equation has roots $r = 1 \pm 3i$, we obtain

$$x_1(t) = c_1 e^t \cos 3t + c_2 e^t \sin 3t.$$

Hence,

$$\begin{aligned} x_2(t) &= -\frac{1}{3}(D - 1)x_1 \\ &= -\frac{1}{3}(c_1 e^t \cos 3t - 3c_1 e^t \sin 3t + 3c_2 e^t \cos 3t + c_2 e^t \sin 3t - c_1 e^t \cos 3t - c_2 e^t \sin 3t) \\ &= -\frac{1}{3}(-3c_1 e^t \sin 3t + 3c_2 e^t \cos 3t) \\ &= c_1 e^t \sin 3t - c_2 e^t \cos 3t. \end{aligned}$$

To summarize, the solution to the given system of differential equations is

$$x_1(t) = c_1 e^t \cos 3t + c_2 e^t \sin 3t \quad \text{and} \quad x_2(t) = c_1 e^t \sin 3t - c_2 e^t \cos 3t.$$

6. Writing the given system of differential equations in operator form gives us the equations

$$(D - 2)x_1 = 0, \quad (D - 1)x_2 + x_3 = 0, \quad -x_2 + (D - 1)x_3 = 0.$$

From the first equation, we have

$$x_1(t) = c_1 e^{2t}.$$

Substituting for x_2 from the last equation into the middle equation, we obtain $(D - 1)^2x_3 + x_3 = 0$, or $(D^2 - 2D + 2)x_3 = 0$. Since the characteristic equation for this differential equation has roots $r = 1 \pm i$, we obtain

$$x_3(t) = c_2 e^t \cos t + c_3 e^t \sin t.$$

Hence,

$$\begin{aligned}x_2(t) &= (D-1)x_3 \\&= c_2 e^t \cos t - c_2 e^t \sin t + c_3 e^t \sin t + c_3 e^t \cos t - c_2 e^t \cos t - c_3 e^t \sin t \\&= -c_2 e^t \sin t + c_3 e^t \cos t.\end{aligned}$$

To summarize, the solution to the given system of differential equations is

$$x_1(t) = c_1 e^{2t}, \quad x_2(t) = -c_2 e^t \sin t + c_3 e^t \cos t, \quad x_3(t) = c_2 e^t \cos t + c_3 e^t \sin t.$$

7. Writing the given system of differential equations in operator form gives us the equations

$$(D+2)x_1 - x_2 - x_3 = 0, \quad -x_1 + (D+1)x_2 - 3x_3 = 0, \quad x_2 + (D+3)x_3 = 0.$$

Operating on the middle equation with $D+2$, we obtain $-(D+2)x_1 + (D+2)(D+1)x_2 - 3(D+2)x_3 = 0$. Combining this with the first equation to eliminate x_1 yields $(D^2+3D+1)x_2 - (3D+7)x_3 = 0$. Operating on the third equation with D^2+3D+1 yields $(D^2+3D+1)x_2 + (D^2+3D+1)(D+3)x_3 = 0$. Combining this with previous result, we obtain $[(D^2+3D+1)(D+3) + (3D+7)]x_3 = 0$. That is, $(D^3+6D^2+13D+10)x_3 = 0$, or $(D+2)(D^2+4D+5)x_3 = 0$. Since the characteristic equation for this differential equation has roots $r = -2$ and $r = -2 \pm i$, we obtain

$$x_3(t) = c_1 e^{-2t} + c_2 e^{-2t} \cos t + c_3 e^{-2t} \sin t.$$

From the third equation in the original system, we have

$$x_2(t) = -(D+3)x_3 = -e^{-2t} [c_1 + c_2(\cos t - \sin t) + c_3(\cos t + \sin t)],$$

and from the second equation in the original system, we have

$$x_1(t) = (D+1)x_2 - 3x_3 = -e^{-2t} [2c_1 + c_2 \cos t + c_3 \sin t].$$

To summarize, the solution to the given system of differential equations is

$$\begin{aligned}x_1(t) &= -e^{-2t} [2c_1 + c_2 \cos t + c_3 \sin t], \\x_2(t) &= -e^{-2t} [c_1 + c_2(\cos t - \sin t) + c_3(\cos t + \sin t)], \\x_3(t) &= c_1 e^{-2t} + c_2 e^{-2t} \cos t + c_3 e^{-2t} \sin t.\end{aligned}$$

8. Writing the given system of differential equations in operator form gives us the equations

$$Dx_1 - 2x_2 = 0 \quad \text{and} \quad -x_1 + (D-1)x_2 = 0.$$

Operating on the first equation with $D-1$, we get $(D-1)Dx_1 - 2(D-1)x_2 = 0$. Combining this with the second equation above, we obtain $(D-1)Dx_1 - 2x_1 = 0$, or $(D^2-D-2)x_1 = 0$. Since the characteristic equation for this differential equation has roots $r = 2$ and $r = -1$, we obtain

$$x_1(t) = c_1 e^{2t} + c_2 e^{-t}.$$

Hence,

$$\begin{aligned}x_2(t) &= \frac{1}{2} Dx_1 \\&= \frac{1}{2} (2c_1 e^{2t} - c_2 e^{-t}) \\&= c_1 e^{2t} - \frac{1}{2} c_2 e^{-t}.\end{aligned}$$

Imposing the initial conditions $x_1(0) = 3$ and $x_2(0) = 0$, we find that $c_1 = 1$ and $c_2 = 2$. Therefore,

$$x_1(t) = e^{2t} + 2e^{-t} \quad \text{and} \quad x_2(t) = e^{2t} - e^{-t}.$$

9. Writing the given system of differential equations in operator form gives us the equations

$$(D - 2)x_1 - 5x_2 = 0 \quad \text{and} \quad x_1 + (D + 2)x_2 = 0.$$

Operating on the first equation with $D + 2$, we obtain $(D + 2)(D - 2)x_1 - 5(D + 2)x_2 = 0$. Combining this with the second equation above, we obtain $(D + 2)(D - 2)x_1 + 5x_1 = 0$, or $(D^2 + 1)x_1 = 0$. Since the characteristic equation for this differential equation has roots $r = \pm i$, we obtain

$$x_1(t) = c_1 \cos t + c_2 \sin t.$$

Hence,

$$\begin{aligned} x_2(t) &= \frac{1}{5}(D - 2)x_1 \\ &= \frac{1}{5}(-c_1 \sin t + c_2 \cos t - 2c_1 \cos t - 2c_2 \sin t) \\ &= \frac{1}{5}(-c_1(\sin t + 2 \cos t) + c_2(\cos t - 2 \sin t)). \end{aligned}$$

Imposing the initial conditions $x_1(0) = 0$ and $x_2(0) = 1$, we find that $c_1 = 0$ and $c_2 = 5$. Therefore,

$$x_1(t) = 5 \sin t \quad \text{and} \quad x_2(t) = \cos t - 2 \sin t.$$

10. Writing the given system of differential equations in operator form gives us the equations

$$(D - 2)x_1 - x_2 = 0 \quad \text{and} \quad x_1 + (D - 4)x_2 = 0.$$

Operating on the first equation with $D - 4$, we get $(D - 4)(D - 2)x_1 - (D - 4)x_2 = 0$. Combining this with the second equation above, we obtain $(D - 4)(D - 2)x_1 + x_1 = 0$, or $(D^2 - 6D + 9)x_1 = 0$. Since the characteristic equation for this differential equation has roots $r = 3$ (with multiplicity 2), we obtain

$$x_1(t) = c_1 e^{3t} + c_2 t e^{3t}.$$

Hence,

$$\begin{aligned} x_2(t) &= (D - 2)x_1 \\ &= 3c_1 e^{3t} + c_2 e^{3t} + 3c_2 t e^{3t} - 2c_1 e^{3t} - 2c_2 t e^{3t} \\ &= e^{3t}(c_1 + c_2 + c_2 t). \end{aligned}$$

Imposing the initial conditions $x_1(0) = 1$ and $x_2(0) = 3$, we obtain $c_1 = 1$ and $c_2 = 2$. Therefore,

$$x_1(t) = e^{3t} + 2t e^{3t} \quad \text{and} \quad x_2(t) = e^{3t}(3 + 2t).$$

11. Writing the given system of differential equations in operator form gives us the equations

$$(D - 1)x_1 - 2x_2 = 5e^{4t} \quad \text{and} \quad -2x_1 + (D - 1)x_2 = 0.$$

Operating of the first equation with $D - 1$ yields

$$(D - 1)^2 x_1 - 2(D - 1)x_2 = 5(D - 1)e^{4t},$$

or

$$(D - 1)^2 x_1 - 2(D - 1)x_2 = 15e^{4t}.$$

Combining this with the second equation above, we obtain $(D - 1)^2 x_1 - 4x_1 = 15e^{4t}$, or $(D^2 - 2D - 3)x_1 = 15e^{4t}$. The complementary function for x_1 is

$$x_{1c}(t) = c_1 e^{-t} + c_2 e^{3t}.$$

Using the trial solution $x_{1p}(t) = Ae^{4t}$, we find that $A = 3$, so that

$$x_1(t) = c_1 e^{-t} + c_2 e^{3t} + 3e^{4t}.$$

Hence,

$$\begin{aligned} x_2(t) &= \frac{1}{2} [(D - 1)x_1 - 5e^{4t}] \\ &= \frac{1}{2} [-c_1 e^{-t} + 3c_2 e^{3t} + 12e^{4t} - c_1 e^{-t} - c_2 e^{3t} - 3e^{4t} - 5e^{4t}] \\ &= \frac{1}{2} [-2c_1 e^{-t} + 2c_2 e^{3t} + 4e^{4t}] \\ &= -c_1 e^{-t} + c_2 e^{3t} + 2e^{4t}. \end{aligned}$$

Therefore,

$$x_1(t) = c_1 e^{-t} + c_2 e^{3t} + 3e^{4t} \quad \text{and} \quad x_2(t) = -c_1 e^{-t} + c_2 e^{3t} + 2e^{4t}.$$

12. Writing the given system of differential equations in operator form gives us the equations

$$(D + 2)x_1 - x_2 = t \quad \text{and} \quad 2x_1 + (D - 1)x_2 = 1.$$

Operating on the first equation with $D - 1$ yields

$$(D - 1)(D + 2)x_1 - (D - 1)x_2 = (D - 1)t$$

or

$$(D - 1)(D + 2)x_1 - (D - 1)x_2 = 1 - t.$$

Combining this with the second equation above yields

$$(D - 1)(D + 2)x_1 + (2x_1 - 1) = 1 - t$$

or

$$(D^2 + D)x_1 = 2 - t.$$

The complementary function for x_1 is

$$x_{1c}(t) = c_1 + c_2 e^{-t}.$$

Using the trial solution $x_{1p}(t) = (A_0 + A_1 t)t$, we find that $A_0 = 3$ and $A_1 = -\frac{1}{2}$, so that

$$x_1(t) = c_1 + c_2 e^{-t} + (3 - \frac{1}{2}t)t.$$

Hence,

$$\begin{aligned}x_2(t) &= (D+2)x_1 - t \\ &= 2c_1 + c_2e^{-t} - t^2 + 4t + 3.\end{aligned}$$

Therefore,

$$x_1(t) = c_1 + c_2e^{-t} + \left(3 - \frac{1}{2}t\right)t \quad \text{and} \quad x_2(t) = 2c_1 + c_2e^{-t} - t^2 + 4t + 3.$$

13. Writing the given system of differential equations in operator form gives us the equations

$$(D-1)x_1 - x_2 = e^{2t} \quad \text{and} \quad -3x_1 + (D+1)x_2 = 5e^{2t}.$$

Operating on the first equation with $D+1$ yields $(D+1)(D-1)x_1 - (D+1)x_2 = (D+1)e^{2t}$. Combining this with the second equation above yields

$$[(D+1)(D-1) - 3]x_1 = 5e^{2t} + (D+1)e^{2t} = 8e^{2t},$$

or

$$(D^2 - 4)x_1 = 8e^{2t}.$$

The complementary function for x_1 is

$$x_{1c}(t) = c_1e^{2t} + c_2e^{-2t}.$$

Using the trial solution $x_{1p}(t) = Ate^{2t}$, we find that $A = 2$. Therefore,

$$x_1(t) = c_1e^{2t} + c_2e^{-2t} + 2te^{2t}.$$

Hence,

$$\begin{aligned}x_2(t) &= x_1'(t) - x_1(t) - e^{2t} \\ &= 2c_1e^{2t} - 2c_2e^{-2t} + 2e^{2t} + 4te^{2t} - c_1e^{2t} - c_2e^{-2t} - 2te^{2t} - e^{2t} \\ &= (c_1 + 2t + 1)e^{2t} - 3c_2e^{-2t}.\end{aligned}$$

Therefore,

$$x_1(t) = c_1e^{2t} + c_2e^{-2t} + 2te^{2t} \quad \text{and} \quad x_2(t) = (c_1 + 2t + 1)e^{2t} - 3c_2e^{-2t}.$$

14. Letting $x_1 = x$, $x_2 = y$, and $x_3 = \frac{dy}{dt}$ yields the first-order system

$$\frac{dx_1}{dt} = tx_2 + \cos t, \quad \frac{dx_2}{dt} = x_3, \quad \frac{dx_3}{dt} = -x_1 + tx_2 + e^t + \cos t.$$

15. Letting $x_1 = x$, $x_2 = \frac{dx}{dt}$, $x_3 = y$, and $x_4 = \frac{dy}{dt}$ yields the first-order system

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = -x_1 + 3x_4 + \sin t, \quad \frac{dx_3}{dt} = x_4, \quad \frac{dx_4}{dt} = tx_2 + e^tx_3 + t^2.$$

16. Letting $y_1 = y$ and $y_2 = \frac{dy}{dt}$, we obtain the system

$$\frac{dy_1}{dt} = y_2 \quad \text{and} \quad \frac{dy_2}{dt} = -2ty_2 - y_1 + \cos t.$$

17. Letting $y_1 = y$ and $y_2 = \frac{dy}{dt}$, we obtain the system

$$\frac{dy_1}{dt} = y_2 \quad \text{and} \quad \frac{dy_2}{dt} = -by_1 - ay_2 + F(t).$$

18. Letting $y_1 = y$, $y_2 = \frac{dy}{dt}$, and $y_3 = \frac{d^2y}{dt^2}$, we obtain the system

$$\frac{dy_1}{dt} = y_2, \quad \frac{dy_2}{dt} = y_3, \quad \text{and} \quad \frac{dy_3}{dt} = e^t y_1 - t^2 y_2 + t.$$

19. Letting $u_1 = x$, $u_2 = \frac{dx}{dt}$, $u_3 = y$, and $u_4 = \frac{dy}{dt}$, we obtain the system

$$\frac{du_1}{dt} = u_2, \quad \frac{du_2}{dt} = \frac{1}{m_1} [-k_1 u_1 + k_2 (u_3 - u_1)], \quad \frac{du_3}{dt} = u_4, \quad \frac{du_4}{dt} = -\frac{k_2}{m_2} (u_3 - u_1),$$

with initial conditions

$$u_1(0) = \alpha_1, \quad u_2(0) = \alpha_2, \quad u_3(0) = \alpha_3, \quad u_4(0) = \alpha_4.$$

20. Rewriting the differential equations, we have

$$x'_1 + (\tan t)x_1 = 3 \cos^2 t \quad \text{and} \quad -x_1 + x'_2 - (\tan t)x_2 = 2 \sin t.$$

The first equation has integrating factor $I(t) = e^{\int \tan t \, dt} = \sec t$. Thus, $\frac{d}{dt}(x_1 \sec t) = 3 \cos t$. Integrating, we obtain

$$x_1(t) = \cos t(3 \sin t + c_1).$$

Consequently, x_2 must satisfy $x'_2 - (\tan t)x_2 = 2 \sin t + x_1 = 2 \sin t + 3 \sin t \cos t + c_1 \cos t$. This is a first-order linear differential equation with integrating factor $I(t) = e^{-\int \tan t \, dt} = \cos t$. Thus, $\frac{d}{dt}(x_2 \cos t) = 2 \sin t \cos t + 3 \sin t \cos^2 t + c_1 \cos^2 t$, so that

$$x_2(t) = \sec t \left[\sin^2 t - \cos^3 t + \frac{1}{2} c_1 \left(\frac{1}{2} \sin 2t + t \right) + c_2 \right].$$

Imposing the initial conditions $x_1(0) = 4$ and $x_2(0) = 0$ yields $c_1 = 4$ and $c_2 = 1$. Therefore, we have

$$x_1(t) = \cos t(3 \sin t + 4) \quad \text{and} \quad x_2(t) = \sec t \left[\sin^2 t - \cos^3 t + 2 \left(\frac{1}{2} \sin 2t + t \right) + 1 \right].$$

Solutions to Section 7.2

True-False Review:

1. **TRUE.** If the unknown function $\mathbf{x}(t)$ is a column n -vector function, then $A(t)$ must contain n columns in order to be able to compute $A(t)\mathbf{x}(t)$. And since $\mathbf{x}'(t)$ is also a column n -vector function, then $A(t)\mathbf{x}(t)$ must have n rows, and therefore, $A(t)$ must also have n rows. Therefore, $A(t)$ contains the same number of rows and columns.

2. **FALSE.** If two columns of a matrix are interchanged, the determinant changes sign:

$$\det([\mathbf{x}_1(t), \mathbf{x}_2(t)]) = -\det([\mathbf{x}_2(t), \mathbf{x}_1(t)]).$$

3. FALSE. Many counterexamples are possible. For instance, if

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ t \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ t^2 \end{bmatrix},$$

then if we set

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t) = \mathbf{0},$$

then we arrive at the equations

$$c_1 + c_2 + c_3 = 0 \quad \text{and} \quad c_1 + tc_2 + t^2c_3 = 0.$$

Setting $t = 0$, we conclude that $c_1 = 0$. Next, setting $t = -1$, we have $-c_2 + c_3 = 0 = c_2 + c_3$, and this requires $c_2 = c_3 = 0$. Therefore $c_1 = c_2 = c_3 = 0$. Hence, $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent.

4. FALSE. Theorem 7.2.4 asserts that the Wronskian of these vector functions must be nonzero *for some* $t_0 \in I$, not for all values in I .

5. FALSE. To the contrary, an $n \times n$ linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$ *always* has n linearly independent solutions, regardless of any condition on the determinant of the matrix A . This is explained in the paragraph preceding Example 7.2.7 and is proved in Theorem 7.3.2 in the next section.

6. TRUE. This is actually described in the preceding section with Equation (7.1.16) and following. We can replace the fourth-order linear differential equation for $x(t)$ by making the change of variables $x_1 = x$, $x_2 = x'$, $x_3 = x''$, and $x_4 = x'''$. In so doing, we obtain the equivalent first-order linear system

$$x'_1 = x_2, x'_2 = x_3, x'_3 = x_4, x'_4 = -a_4(t)x_1 - a_3(t)x_2 - a_2(t)x_3 - a_1(t)x_4 + F(t).$$

7. FALSE. We have

$$(\mathbf{x}_0(t) + \mathbf{b}(t))' = \mathbf{x}'_0(t) + \mathbf{b}'(t) = A(t)\mathbf{x}_0(t) + \mathbf{b}'(t),$$

and this in general is not the same as

$$A(t)(\mathbf{x}_0(t) + \mathbf{b}(t)) + \mathbf{b}(t).$$

Problems:

1. We compute the Wronskian:

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} e^t & e^t \\ -e^t & e^t \end{vmatrix} = 2e^{2t} \neq 0,$$

so that by Theorem 7.2.4 the vector functions are linearly independent on $(-\infty, \infty)$.

2. We compute the Wronskian:

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} t & t \\ t & t^2 \end{vmatrix} = t^2(t - 1),$$

so that since this is not identically zero on $(-\infty, \infty)$, the vector functions are linearly independent on $(-\infty, \infty)$ by Theorem 7.2.4.

3. We compute the Wronskian:

$$\begin{aligned} W[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3](t) &= \begin{vmatrix} t+1 & e^t & 1 \\ t-1 & e^{2t} & \sin t \\ 2t & e^{3t} & \cos t \end{vmatrix} \\ &= (t+1)e^{2t} \cos t + 2te^t \sin t + (t-1)e^{3t} - 2te^{2t} - (t+1)e^{3t} \sin t - (t-1)e^t \cos t. \end{aligned}$$

Substituting $t = 0$, we find that $W[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3](0) = 1$, so that the Wronskian is not identically zero on $(-\infty, \infty)$. Therefore, by Theorem 7.2.4, the vector functions are linearly independent on $(-\infty, \infty)$.

4. We compute the Wronskian:

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} t & |t| \\ t & t \end{vmatrix} = t^2 - t|t|.$$

This is not identically zero on $(-\infty, \infty)$. For instance, $W[\mathbf{x}_1, \mathbf{x}_2](-1) = 2 \neq 0$, so by Theorem 7.2.4, the given vector functions are linearly independent on $(-\infty, \infty)$. Note that on any interval contained in $[0, \infty)$ consisting of nonnegative values of t only, $\mathbf{x}_1$ and $\mathbf{x}_2$ become the same vector function and therefore they are linearly *dependent* on any such interval.

5. We compute the Wronskian:

$$\begin{aligned} W[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3](t) &= \begin{vmatrix} \sin t & t & \sinh t \\ \cos t & 1-t & \cosh t \\ 1 & 1 & 1 \end{vmatrix} \\ &= (1-t)\sin t + t\cosh t + \cos t\sinh t - (1-t)\sinh t - \sin t\cosh t - t\cos t. \end{aligned}$$

A quick computation with, say $t = 1$, shows that this Wronskian is not identically zero on $(-\infty, \infty)$. Therefore, by Theorem 7.2.4, the given vector functions are linearly independent on $(-\infty, \infty)$.

6. By inspection we see that $3\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{0}$, so that $\{\mathbf{x}_1, \mathbf{x}_2\}$ is linearly dependent.

7. By inspection we see that $4\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{0}$ so that $\{\mathbf{x}_1, \mathbf{x}_2\}$ is linearly dependent.

8. We see that $2\mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_3 = \mathbf{0}$ so that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly dependent.

9. We see that $3\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3 = \mathbf{0}$ so that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly dependent.

10. We must verify the ten axioms appearing in Definition 4.2.1. All of these steps are routine. For example, clearly if we add two elements in $V_n(I)$, the result is another vector in $V_n(I)$, and similarly, if we scalar multiply an element of $V_n(I)$ by a scalar, we obtain another vector in $V_n(I)$. Hence, $V_n(I)$ is closed under addition and scalar multiplication. The axioms A3 and A4 of Definition 4.2.1 follow directly since they hold for functions defined on an interval I and the addition and scalar multiplication operations in $V_n(I)$ are defined componentwise. The zero vector in $V_n(I)$ is the column n -vector function whose elements

are all zero (for all t in I). Next, the additive inverse of the column n -vector function $\mathbf{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_n(t) \end{bmatrix}$

is $\begin{bmatrix} -v_1(t) \\ -v_2(t) \\ \vdots \\ -v_n(t) \end{bmatrix}$. The last four axioms A7-A10 of a vector space are all satisfied for functions defined on an

interval I , so the componentwise nature of the addition and scalar multiplication in $V_n(I)$ now implies that these same axioms carry over to vectors in $V_n(I)$.

11. Let

$$S = \{\mathbf{x} \in V_n(I) : \mathbf{x}' = A\mathbf{x}\}.$$

Note that $\mathbf{x} = \mathbf{0}$ is a solution to $\mathbf{x}' = A\mathbf{x}$, so that $\mathbf{0}$ belongs to S . Therefore, S is nonempty. Next, we must verify that S is closed under addition and scalar multiplication:

Closure under addition: Suppose that $\mathbf{u}$ and $\mathbf{v}$ belong to S . This means that $\mathbf{u}' = A\mathbf{u}$ and $\mathbf{v}' = A\mathbf{v}$. Therefore $(\mathbf{u} + \mathbf{v})' = \mathbf{u}' + \mathbf{v}' = A\mathbf{u} + A\mathbf{v} = A(\mathbf{u} + \mathbf{v})$, which shows that $\mathbf{u} + \mathbf{v}$ belongs to S . Hence, S is closed under addition.

Closure under scalar multiplication: Suppose that $\mathbf{u}$ belongs to S and c is a scalar. From the fact that $\mathbf{u}' = A\mathbf{u}$, we deduce that $(c\mathbf{u})' = c\mathbf{u}' = c(A\mathbf{u}) = A(c\mathbf{u})$, which shows that $c\mathbf{u}$ belongs to S . Hence, S is closed under scalar multiplication.

12. The system $\mathbf{x}' = A\mathbf{x}$ can be written as two separate differential equations:

$$x_1' = 2x_1 - 4x_2 \quad \text{and} \quad x_2' = x_1 - 3x_2.$$

These equations in turn can be written in operator form as

$$(D - 2)x_1 + 4x_2 = 0 \quad \text{and} \quad -x_1 + (D + 3)x_2 = 0,$$

respectively. Operating on the second equation with $D - 2$ and adding the result to the first equation gives $(D^2 + D - 2)x_2 = 0$. That is, $(D + 2)(D - 1)x_2 = 0$. Therefore, we obtain the solution

$$x_2(t) = c_1 e^t + c_2 e^{-2t}.$$

Hence,

$$\begin{aligned} x_1(t) &= (D + 3)x_2 \\ &= 4c_1 e^t + c_2 e^{-2t}. \end{aligned}$$

Therefore, the general solution is

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 4c_1 e^t + c_2 e^{-2t} \\ c_1 e^t + c_2 e^{-2t} \end{bmatrix},$$

which can be written as

$$\mathbf{x}(t) = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2,$$

where $\mathbf{x}_1(t) = \begin{bmatrix} 4e^t \\ e^t \end{bmatrix}$ and $\mathbf{x}_2(t) = \begin{bmatrix} e^{-2t} \\ e^{-2t} \end{bmatrix}$. Since

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} 4e^t & e^{-2t} \\ e^t & e^{-2t} \end{vmatrix} = 3e^{-t} \neq 0,$$

the solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent on $(-\infty, \infty)$.

13.

(a) Letting $x_1 = y$ and $x_2 = y'$, the given differential equation can be replaced by the first-order system

$$\mathbf{x}_1' = \mathbf{x}_2 \quad \text{and} \quad \mathbf{x}_2' = -bx_1 - ax_2.$$

That is $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}$.

(b) Supposing that $y_1 = f_1(t)$ and $y_2 = f_2(t)$ are solutions to (7.2.4), we have that

$$\frac{d^2 y_1}{dt^2} + a \frac{dy_1}{dt} + by_1 = 0 \quad \text{and} \quad \frac{d^2 y_2}{dt^2} + a \frac{dy_2}{dt} + by_2 = 0,$$

or

$$f_1''(t) + af_1'(t) + bf_1(t) = 0 \quad \text{and} \quad f_2''(t) + af_2'(t) + bf_2(t) = 0.$$

Substituting $\mathbf{x}_1(t) = \begin{bmatrix} f_1(t) \\ f_1'(t) \end{bmatrix}$ into $\mathbf{x}' = A\mathbf{x}$, we obtain

$$\begin{bmatrix} f_1'(t) \\ f_1''(t) \end{bmatrix} = \begin{bmatrix} f_1'(t) \\ -bf_1(t) - af_1'(t) \end{bmatrix},$$

which corresponds precisely to the differential equation for f_1 above. Likewise, substituting $\mathbf{x}_2(t) = \begin{bmatrix} f_2(t) \\ f_2'(t) \end{bmatrix}$ into $\mathbf{x}' = A\mathbf{x}$, we obtain

$$\begin{bmatrix} f_2'(t) \\ f_2''(t) \end{bmatrix} = \begin{bmatrix} f_2'(t) \\ -bf_2(t) - af_2'(t) \end{bmatrix},$$

which corresponds precisely to the differential equation for f_2 above.

(c) Using the Wronskian in $V_2(I)$ we have $W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$. Similarly, using the Wronskian in $C^1(I)$,

$$W[y_1, y_2](t) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}.$$

Solutions to Section 7.3

True-False Review:

1. **FALSE.** With $\mathbf{b}(t) \neq \mathbf{0}$, note that $\mathbf{x}(t) = \mathbf{0}$ is not a solution to the system $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$, and lacking the zero vector, the solution set to the differential equation cannot form a vector space.

2. **TRUE.** Since any fundamental matrix $X(t) = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n]$ is comprised of linearly independent columns, the Invertible Matrix Theorem guarantees that the fundamental matrix is invertible under all circumstances.

3. **TRUE.** More than this, a fundamental solution set for $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ is in fact a *basis* for the space of solutions to the linear system. In particular, it spans the space of all solutions to $\mathbf{x}' = A\mathbf{x}$.

4. **TRUE.** This is essentially the content of Theorem 7.3.6. The general solution to the homogeneous vector differential equation $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ is $\mathbf{x}_c(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \dots + c_n\mathbf{x}_n$, and therefore the general solution to $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{b}(t)$ is of the form $\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t)$.

Problems:

1. We first show that $\mathbf{x}_1$ is a solution to $\mathbf{x}' = A\mathbf{x}$:

$$A\mathbf{x}_1 = \begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} e^{4t} \\ 2e^{4t} \end{bmatrix} = \begin{bmatrix} 4e^{4t} \\ 8e^{4t} \end{bmatrix} = \mathbf{x}_1'.$$

Next we show that $\mathbf{x}_2$ is a solution to $\mathbf{x}' = A\mathbf{x}$:

$$A\mathbf{x}_2 = \begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 3e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} -3e^{-t} \\ -e^{-t} \end{bmatrix} = \mathbf{x}_2'.$$

To check $\{\mathbf{x}_1, \mathbf{x}_2\}$ for linear independence, we compute the Wronskian:

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} e^{4t} & 3e^{-t} \\ 2e^{4t} & e^{-t} \end{vmatrix} = -5e^{3t} \neq 0,$$

so $\{\mathbf{x}_1, \mathbf{x}_2\}$ is linearly independent. Therefore, the general solution to the system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} e^{4t} \\ 2e^{4t} \end{bmatrix} + c_2 \begin{bmatrix} 3e^{-t} \\ e^{-t} \end{bmatrix}.$$

Finally, we use the initial condition $\mathbf{x}(0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ to determine c_1 and c_2 : We have

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

which implies that

$$c_1 + 3c_2 = -2 \quad \text{and} \quad 2c_1 + c_2 = 1.$$

Solving this system, we obtain $c_1 = 1$ and $c_2 = -1$. Thus, the particular solution to this initial value problem is

$$\mathbf{x}(t) = \begin{bmatrix} e^{4t} \\ 2e^{4t} \end{bmatrix} - \begin{bmatrix} 3e^{-t} \\ e^{-t} \end{bmatrix}.$$

2. We first show that $\mathbf{x}_1$ is a solution to $\mathbf{x}' = A\mathbf{x}$:

$$A\mathbf{x}_1 = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ -2e^{2t} \end{bmatrix} = bf x'_1.$$

Next we show that $\mathbf{x}_2$ is a solution to $\mathbf{x}' = A\mathbf{x}$:

$$A\mathbf{x}_2 = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t}(1+t) \\ -te^{2t} \end{bmatrix} = \begin{bmatrix} 3e^{2t} + 2te^{2t} \\ -e^{2t} - 2te^{2t} \end{bmatrix} = \mathbf{x}'_2.$$

To check $\{\mathbf{x}_1, \mathbf{x}_2\}$ for linear independence, we compute the Wronskian:

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} e^{2t} & e^{2t}(1+t) \\ -e^{2t} & -te^{2t} \end{vmatrix} = e^{4t} \neq 0,$$

so $\{\mathbf{x}_1, \mathbf{x}_2\}$ is linearly independent. Therefore, the general solution to the system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{2t}(1+t) \\ -te^{2t} \end{bmatrix}.$$

3. We first show that $\mathbf{x}_1$ is a solution to $\mathbf{x}' = A\mathbf{x}$:

$$A\mathbf{x}_1 = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 9 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{x}'_1.$$

Next we show that $\mathbf{x}_2$ is a solution to $\mathbf{x}' = A\mathbf{x}$:

$$A\mathbf{x}_2 = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} e^{2t} \\ 3e^{2t} \\ e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 6e^{2t} \\ 2e^{2t} \end{bmatrix} = \mathbf{x}'_2.$$

Next we show that $\mathbf{x}_3$ is a solution to $\mathbf{x}' = A\mathbf{x}$:

$$A\mathbf{x}_3 = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} e^{4t} \\ e^{4t} \\ e^{4t} \end{bmatrix} = \begin{bmatrix} 4e^{4t} \\ 4e^{4t} \\ 4e^{4t} \end{bmatrix} = \mathbf{x}'_3.$$

To check $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ for linear independence, we use the Wronskian:

$$W[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3](t) = \begin{vmatrix} -3 & e^{2t} & e^{4t} \\ 9 & 3e^{2t} & e^{4t} \\ 5 & e^{2t} & e^{4t} \end{vmatrix} = -16e^{6t} \neq 0,$$

so $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent. Therefore, the general solution to the system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -3 \\ 9 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ 3e^{2t} \\ e^{2t} \end{bmatrix} + c_3 \begin{bmatrix} e^{4t} \\ e^{4t} \\ e^{4t} \end{bmatrix}.$$

4. We first show that $\mathbf{x}_1$ is a solution to $\mathbf{x}' = A\mathbf{x}$:

$$A\mathbf{x}_1 = \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2t \\ e^t \end{bmatrix} = \begin{bmatrix} 2 \\ e^t \end{bmatrix} = \mathbf{x}'_1.$$

Next we show that $\mathbf{x}_2$ is a solution to $\mathbf{x}' = A\mathbf{x}$:

$$A\mathbf{x}_2 = \begin{bmatrix} \frac{1}{t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3e^t \end{bmatrix} = \begin{bmatrix} 0 \\ 3e^t \end{bmatrix} = \mathbf{x}'_2.$$

To check $\{\mathbf{x}_1, \mathbf{x}_2\}$ for linear independence, we use the Wronskian:

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} 2t & 0 \\ e^t & 3e^t \end{vmatrix} = 6te^t,$$

which is nonzero provided that $t \neq 0$. Therefore, $\{\mathbf{x}_1, \mathbf{x}_2\}$ is linearly independent. Therefore, the general solution to the system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3e^t \end{bmatrix}.$$

5. We first show that $\mathbf{x}_1$ is a solution to $\mathbf{x}' = A\mathbf{x}$:

$$A\mathbf{x}_1 = \begin{bmatrix} 1/t & t \\ -1/t & 0 \end{bmatrix} \begin{bmatrix} t \sin t \\ \cos t \end{bmatrix} = \begin{bmatrix} \sin t + t \cos t \\ -\sin t \end{bmatrix} = \mathbf{x}'_1.$$

Next we show that $\mathbf{x}_2$ is a solution to $\mathbf{x}' = A\mathbf{x}$:

$$A\mathbf{x}_2 = \begin{bmatrix} 1/t & t \\ -1/t & 0 \end{bmatrix} \begin{bmatrix} -t \cos t \\ \sin t \end{bmatrix} = \begin{bmatrix} -\cos t + t \sin t \\ \cos t \end{bmatrix} = \mathbf{x}'_2.$$

To check $\{\mathbf{x}_1, \mathbf{x}_2\}$ for linear independence, we use the Wronskian:

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} t \sin t & -t \cos t \\ \cos t & \sin t \end{vmatrix} = t,$$

which is nonzero for all choices of $t \neq 0$. Therefore, $\{\mathbf{x}_1, \mathbf{x}_2\}$ is linearly independent. Therefore, the general solution to the system is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} t \sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} -t \cos t \\ \sin t \end{bmatrix}.$$

6. We have

$$A(t)X = A(t)[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n] = [A\mathbf{x}_1, A\mathbf{x}_2, \dots, A\mathbf{x}_n] = [\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_n] = X'.$$

Thus, $X' = A(t)X$ as required.

7.

(a) Write the fundamental matrix as $X(t) = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n]$, where the set of vector functions $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ is linearly independent. Therefore, by Theorem 7.3.2, the general solution to the system $\mathbf{x}' = A(t)\mathbf{x}$ is

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \dots + c_n\mathbf{x}_n = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = X(t)\mathbf{c},$$

where $\mathbf{c}$ is a vector of constants.

(b) Note that $\mathbf{x} = X(t)X^{-1}(t_0)\mathbf{x}_0$ is a solution to $\mathbf{x}' = A\mathbf{x}$ by part (a). Moreover, if we evaluate $\mathbf{x}(t_0)$ we get $\mathbf{x}(t_0) = X(t_0)X^{-1}(t_0)\mathbf{x}_0 = \mathbf{x}_0$, so that $\mathbf{x}$ also satisfies the initial condition in this problem. Since solutions to such an initial-value problem are unique (Theorem 7.3.1), $\mathbf{x}$ is *the* solution to this initial-value problem.

Solutions to Section 7.4

True-False Review:

1. **TRUE.** If $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, then

$$\mathbf{x}'(t) = \lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}(\lambda\mathbf{v}) = e^{\lambda t}(A\mathbf{v}) = A(e^{\lambda t}\mathbf{v}) = A\mathbf{x}(t).$$

This calculation appears prior to Theorem 7.4.1.

2. **TRUE.** The two real-valued solutions are derived in the work preceding Example 7.4.6. They are

$$\mathbf{x}_1(t) = e^{at}(\cos btr - \sin bts) \quad \text{and} \quad \mathbf{x}_2(t) = e^{at}(\sin btr + \cos bts),$$

where $\mathbf{r}$ and $\mathbf{s}$ are the real and imaginary parts of an eigenvector $\mathbf{v} = \mathbf{r} + i\mathbf{s}$ corresponding to $\lambda = a + bi$.

3. **FALSE.** Many counterexamples can be given. For instance, if $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then the matrices A and B have the same characteristic equation: $\lambda^2 = 0$. However, whereas any constant vector function $\mathbf{x}(t) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is a solution to $\mathbf{x}' = A\mathbf{x}$, we have $B\mathbf{x} = \begin{bmatrix} c_2 \\ 0 \end{bmatrix}$ and $\mathbf{x}' = \mathbf{0}$. Therefore, unless $c_2 = 0$,

$\mathbf{x}$ is not a solution to the system $\mathbf{x}' = B\mathbf{x}$. Hence, the systems $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{x}' = B\mathbf{x}$ have differing solution sets.

4. TRUE. Assume that the eigenvalue/eigenvector pairs are $(\lambda_1, \mathbf{v}_1), (\lambda_2, \mathbf{v}_2), \dots, (\lambda_n, \mathbf{v}_n)$. In this case, the general solution to both linear systems is the same:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2 + \dots + c_n e^{\lambda_n t} \mathbf{v}_n.$$

5. TRUE. The two real-valued solutions forming a basis for the set of solutions to $\mathbf{x}' = A\mathbf{x}$ in this case are

$$\mathbf{x}_1(t) = e^{at}(\cos bt \mathbf{r} - \sin bt \mathbf{s}) \quad \text{and} \quad \mathbf{x}_2(t) = e^{at}(\sin bt \mathbf{r} + \cos bt \mathbf{s}).$$

The general solution is a linear combination of these, and since $a > 0$, $e^{at} \rightarrow \infty$ as $t \rightarrow \infty$. Since $\mathbf{x}_1$ and $\mathbf{x}_2$ cannot cancel out in a linear combination (since they are linearly independent), e^{at} remains as a factor in any particular solution to the vector differential equation, and as this factor tends to ∞ , $\|\mathbf{x}(t)\| \rightarrow \infty$ as $t \rightarrow \infty$.

6. FALSE. If $\mathbf{x}(0)$ is parallel to $\mathbf{v}_2$, then since $\lambda_2 > 0$, the solution $\mathbf{x}(t)$ has the form $\mathbf{x}(t) = ce^{\lambda_2 t} \mathbf{v}_2$ and will tend to move *away* from the origin as $t \rightarrow \infty$.

Problems:

1. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} -2 - \lambda & -7 \\ -1 & 4 - \lambda \end{vmatrix} = 0 \iff \lambda^2 - 2\lambda - 15 = 0 \iff \lambda = -3 \text{ or } \lambda = 5.$$

Eigenvectors for $\lambda = -3$: Here we have

$$A + 3I = \begin{bmatrix} 1 & -7 \\ -1 & 7 \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A + 3I)$ as follows: $\mathbf{v}_1 = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$.

Hence, we obtain the first solution to the linear system: $\mathbf{x}_1(t) = e^{-3t} \begin{bmatrix} 7 \\ 1 \end{bmatrix}$.

Eigenvectors for $\lambda = 5$: Here we have

$$A - 5I = \begin{bmatrix} -7 & -7 \\ -1 & -1 \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A - 5I)$ as follows: $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Hence, we obtain the second solution to the linear system: $\mathbf{x}_2(t) = e^{5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Putting the two solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ together, we obtain the general solution to this linear system:

$$\mathbf{x}(t) = c_1 e^{-3t} \begin{bmatrix} 7 \\ 1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

2. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & -4 \\ 4 & -\lambda \end{vmatrix} = 0 \iff \lambda^2 + 16 = 0 \iff \lambda = 4i \text{ or } \lambda = -4i.$$

Eigenvectors for $\lambda = 4i$: Here we have

$$A - 4iI = \begin{bmatrix} -4i & -4 \\ 4 & -4i \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A - 4iI)$ as follows: $\mathbf{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. Hence, we obtain the solution

$$\begin{aligned} \mathbf{x}(t) &= e^{4it} \begin{bmatrix} 1 \\ -i \end{bmatrix} \\ &= (\cos 4t + i \sin 4t) \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) \\ &= \left(\cos 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin 4t \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) + i \left(\cos 4t \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right). \end{aligned}$$

Therefore, we obtain the two real-valued solutions

$$\mathbf{x}_1(t) = \cos 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin 4t \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \cos 4t \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Putting the two solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ together, we obtain the general solution to this linear system:

$$\mathbf{x}(t) = c_1 \left(\cos 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin 4t \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) + c_2 \left(\cos 4t \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).$$

Equivalently, this can be expressed as

$$\mathbf{x}(t) = c_1 \begin{bmatrix} \cos 4t \\ \sin 4t \end{bmatrix} + c_2 \begin{bmatrix} \sin 4t \\ -\cos 4t \end{bmatrix}.$$

Note that one can also obtain the solution by working with the eigenvalue $\lambda = -4i$. The calculations are similar.

3. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1 - \lambda & -2 \\ 5 & -5 - \lambda \end{vmatrix} \iff \lambda^2 + 4\lambda + 5 = 0 \iff \lambda = -2 \pm i.$$

Eigenvectors for $\lambda = -2 + i$: Here we have

$$A - (-2 + i)I = \begin{bmatrix} 3 - i & -2 \\ 5 & -3 - i \end{bmatrix}.$$

The two rows of this matrix must be proportional, since we must be able to find a nonzero vector in its nullspace corresponding to $\lambda = -2 + i$, so concentrating on the first row, we can determine an eigenvector corresponding to $\lambda = -2 + i$ as follows: $\mathbf{v} = \begin{bmatrix} 2 \\ 3 - i \end{bmatrix}$. Note that other choices of $\mathbf{v}$ are possible here, such as $\begin{bmatrix} 3 + i \\ 5 \end{bmatrix}$, obtained by concentrating on the second row instead of the first row. Proceeding with our

choice of $\mathbf{v}$, we obtain the solution

$$\begin{aligned}\mathbf{x}(t) &= e^{(-2+i)t} \begin{bmatrix} 2 \\ 3-i \end{bmatrix} \\ &= e^{-2t} (\cos t + i \sin t) \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) \\ &= e^{-2t} \left[\left(\cos t \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) + i \left(\cos t \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin t \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) \right].\end{aligned}$$

Therefore, we obtain the two real-valued solutions

$$\mathbf{x}_1(t) = e^{-2t} \left(\cos t \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) \quad \text{and} \quad \mathbf{x}_2(t) = e^{-2t} \left(\cos t \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin t \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right).$$

Putting the two solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ together, we obtain the general solution to this linear system:

$$\mathbf{x}(t) = c_1 e^{-2t} \left(\cos t \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) + c_2 e^{-2t} \left(\cos t \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sin t \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right).$$

Equivalently, this can be expressed as

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{bmatrix} 2 \cos t \\ 3 \cos t + \sin t \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 2 \sin t \\ -\cos t + 3 \sin t \end{bmatrix}.$$

Note that one can also obtain the solution by working with the eigenvalue $\lambda = -2 - i$. The calculations are similar.

4. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} -1-\lambda & -2 \\ 2 & -1-\lambda \end{vmatrix} = 0 \iff \lambda^2 + 2\lambda + 5 = 0 \iff \lambda = -1 \pm 2i.$$

Eigenvectors for $\lambda = -1 + 2i$: Here we have

$$A - (-1 + 2i)I = \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}.$$

One choice of eigenvector, found by examining the nullspace of this matrix, is $\mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$. We obtain the solution

$$\begin{aligned}\mathbf{x}(t) &= e^{(-1+2i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} \\ &= e^{-t} (\cos 2t + i \sin 2t) \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= e^{-t} \left[\left(\cos 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin 2t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + i \left(\cos 2t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \right].\end{aligned}$$

Therefore, we obtain the two real-valued solutions

$$\mathbf{x}_1(t) = e^{-t} \left(\cos 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin 2t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \quad \text{and} \quad \mathbf{x}_2(t) = e^{-t} \left(\cos 2t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).$$

Putting the two solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ together, we obtain the general solution to this linear system:

$$\mathbf{x}(t) = c_1 e^{-t} \left(\cos 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin 2t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + c_2 e^{-t} \left(\cos 2t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).$$

Equivalently, this can be expressed as

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}.$$

Note that one can also obtain the solution by working with the eigenvalue $\lambda = -1 - 2i$. The calculations are similar.

5. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & 0 & 0 \\ 0 & 5 - \lambda & -7 \\ 0 & 2 & -4 - \lambda \end{vmatrix} = 0 \iff (2 - \lambda)(\lambda^2 - \lambda - 6) = 0 \iff (2 - \lambda)(\lambda - 3)(\lambda + 2) = 0.$$

Thus, $\lambda = 2, 3$, or -2 .

Eigenvectors for $\lambda = -2$: Here we have

$$A + 2I = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & -7 \\ 0 & 2 & -2 \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A + 2I)$ as follows: $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Hence, we obtain

the first solution to the linear system: $\mathbf{x}_1(t) = e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Eigenvectors for $\lambda = 2$: Here we have

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & -7 \\ 0 & 2 & -6 \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A - 2I)$ as follows: $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Hence, we obtain

the second solution to the linear system: $\mathbf{x}_2(t) = e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Eigenvectors for $\lambda = 3$: Here we have

$$A - 3I = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & -7 \\ 0 & 2 & -7 \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A - 3I)$ as follows: $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$. Hence, we obtain

the third solution to the linear system: $\mathbf{x}_3(t) = e^{3t} \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}$.

Putting the three solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, and $\mathbf{x}_3(t)$ together, we obtain the general solution to this linear system:

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 0 \\ 7 \\ 2 \end{bmatrix}.$$

6. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} -1 - \lambda & 0 & 0 \\ 1 & 5 - \lambda & -1 \\ 1 & 6 & -2 - \lambda \end{vmatrix} = 0 \iff (-1 - \lambda)(\lambda^2 - 3\lambda - 4) = 0 \iff (-1 - \lambda)(\lambda - 4)(\lambda + 1) = 0.$$

Thus, $\lambda = -1, -1$, or 4 .

Eigenvectors for $\lambda = -1$: Here we have

$$A + I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 6 & -1 \\ 1 & 6 & -1 \end{bmatrix},$$

and we must find two linearly independent vectors in the nullspace of this matrix. Setting $z = t$ and $y = s$, the last two rows of the matrix give $x + 6y - z = 0$, or $x = t - 6s$. Thus, vectors in $\text{nullspace}(A + I)$ have

the form $\begin{bmatrix} t - 6s \\ s \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix}$. So we have two linearly independent eigenvectors: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

and $\mathbf{v}_2 = \begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix}$. Therefore, we obtain two solutions to the linear system: $\mathbf{x}_1(t) = e^{-t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and

$$\mathbf{x}_2(t) = e^{-t} \begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix}.$$

Eigenvectors for $\lambda = 4$: Here we have

$$A - 4I = \begin{bmatrix} -5 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 6 & -6 \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A - 4I)$ as follows: $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Hence, we obtain

the third solution to the linear system: $\mathbf{x}_3(t) = e^{4t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Putting the three solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, and $\mathbf{x}_3(t)$ together, we obtain the general solution to this linear system:

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

7. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & 5 - \lambda \end{vmatrix} = 0 \iff (5 - \lambda)(\lambda^2 + 1) = 0 \iff \lambda = 5, i, \text{ or } -i.$$

Eigenvectors for $\lambda = 5$: Here we have

$$A - 5I = \begin{bmatrix} -5 & 1 & 0 \\ -1 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A - 5I)$ as follows: $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Hence, we obtain

the first solution to the linear system: $\mathbf{x}_1(t) = e^{5t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Eigenvectors for $\lambda = i$: Here we have

$$A - iI = \begin{bmatrix} -i & 1 & 0 \\ -1 & -i & 0 \\ 0 & 0 & 5 - i \end{bmatrix},$$

and we obtain an eigenvector corresponding to $\lambda = i$ by computing $\text{nullspace}(A - iI)$. We obtain $\mathbf{v} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$.

We therefore obtain the corresponding solution

$$\begin{aligned} \mathbf{x}(t) &= e^{it} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \\ &= (\cos t + i \sin t) \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) \\ &= \left(\cos t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) + i \left(\cos t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right). \end{aligned}$$

Therefore, we obtain two real-valued solutions corresponding to $\lambda = i$:

$$\mathbf{x}_2(t) = \cos t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_3(t) = \cos t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Putting the solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, and $\mathbf{x}_3(t)$ together, we obtain the general solution to the linear system:

$$\mathbf{x}(t) = c_1 e^{5t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \left(\cos t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) + c_3 \left(\cos t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right).$$

Equivalently, we can express this solution as

$$\mathbf{x}(t) = c_1 e^{5t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \cos t \\ -\sin t \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \sin t \\ \cos t \\ 0 \end{bmatrix}.$$

8. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & 0 & 3 \\ 0 & -4 - \lambda & 0 \\ -3 & 0 & 2 - \lambda \end{vmatrix} = 0 \iff (-4 - \lambda)(\lambda^2 - 4\lambda + 13) = 0.$$

Therefore, $\lambda = -4, 2 + 3i$, or $2 - 3i$.

Eigenvectors for $\lambda = -4$: Here we have

$$A + 4I = \begin{bmatrix} 6 & 0 & 3 \\ 0 & 0 & 0 \\ -3 & 0 & 6 \end{bmatrix},$$

and we obtain an eigenvector corresponding to $\text{nullspace}(A + 4I)$ as follows: $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Hence, we obtain

the first solution to the linear system: $\mathbf{x}_1(t) = e^{-4t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Eigenvectors for $\lambda = 2 + 3i$: Here we have

$$A - (2 + 3i)I = \begin{bmatrix} -3i & 0 & 3 \\ 0 & -6 - 3i & 0 \\ -3 & 0 & -3i \end{bmatrix}.$$

One choice of eigenvector, found by examining the nullspace of this matrix, is $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}$. We obtain the solution

$$\begin{aligned} \mathbf{x}(t) &= e^{(2+3i)t} \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} \\ &= e^{2t} \left[(\cos 3t + i \sin 3t) \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \right] \\ &= e^{2t} \left[\left(\cos 3t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \sin 3t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) + i \left(\cos 3t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \sin 3t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \right]. \end{aligned}$$

Therefore, we obtain the two real-valued solutions

$$\mathbf{x}_2(t) = e^{2t} \left(\cos 3t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \sin 3t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \quad \text{and} \quad \mathbf{x}_3(t) = e^{2t} \left(\cos 3t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \sin 3t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right).$$

Putting the three solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, and $\mathbf{x}_3(t)$ together, we obtain the general solution to this linear system:

$$\mathbf{x}(t) = c_1 e^{-4t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \left(\cos 3t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \sin 3t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) + c_3 e^{2t} \left(\cos 3t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \sin 3t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right).$$

Equivalently, this can be expressed as

$$\mathbf{x}(t) = c_1 e^{-4t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} \cos 3t \\ 0 \\ -\sin 3t \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} \sin 3t \\ 0 \\ \cos 3t \end{bmatrix}.$$

9. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 3 - \lambda & 2 & 6 \\ -2 & 1 - \lambda & -2 \\ -1 & -2 & -4 - \lambda \end{vmatrix} = 0 \iff \lambda = 1, 2, \text{ or } -3.$$

Eigenvectors for $\lambda = 1$: Here we have

$$A - I = \begin{bmatrix} 2 & 2 & 6 \\ -2 & 0 & -2 \\ -1 & -2 & -5 \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A - I)$ as follows: $\mathbf{v}_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$. Hence, we obtain

the first solution to the linear system: $\mathbf{x}_1(t) = e^t \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$.

Eigenvectors for $\lambda = 2$: Here we have

$$A - 2I = \begin{bmatrix} 1 & 2 & 6 \\ -2 & -1 & -2 \\ -1 & -2 & -6 \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A - 2I)$ as follows: $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -10 \\ 3 \end{bmatrix}$. Hence, we

obtain the second solution to the linear system: $\mathbf{x}_2(t) = e^{2t} \begin{bmatrix} 2 \\ -10 \\ 3 \end{bmatrix}$.

Eigenvectors for $\lambda = -3$: Here we have

$$A + 3I = \begin{bmatrix} 6 & 2 & 6 \\ -2 & 4 & -2 \\ -1 & -2 & -1 \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A + 3I)$ as follows: $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Hence, we

obtain the third solution to the linear system: $\mathbf{x}_3(t) = e^{-3t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Putting the three solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, and $\mathbf{x}_3(t)$ together, we obtain the general solution to this linear system:

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2 \\ -10 \\ 3 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

10. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & -3 & 1 \\ -2 & -1-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 \iff (2-\lambda)(\lambda^2 + \lambda - 6) = 0 \iff \lambda = 2 \text{ or } \lambda = -3.$$

Eigenvectors for $\lambda = 2$: Here we have

$$A - 2I = \begin{bmatrix} -2 & -3 & 1 \\ -2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore $-2x - 3y + z = 0$. Setting $x = t$ and $y = s$, we have $z = 2t + 3s$, and so vectors in $\text{nullspace}(A - 2I)$

have the form $\begin{bmatrix} t \\ s \\ 2t + 3s \end{bmatrix}$. Therefore, we have two linearly independent eigenvectors corresponding to $\lambda = 2$:

$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$. Hence, we obtain two linearly independent solutions to the linear system:

$$\mathbf{x}_1(t) = e^{2t} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \text{ and } \mathbf{x}_2(t) = e^{2t} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

Eigenvectors for $\lambda = -3$: Here we have

$$A + 3I = \begin{bmatrix} 3 & -3 & 1 \\ -2 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A + 3I)$ as follows: $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Hence, we obtain

the third linearly independent solution to the linear system: $\mathbf{x}_3(t) = e^{-3t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Putting the three solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, and $\mathbf{x}_3(t)$ together, we obtain the general solution to this linear system:

$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

11. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 3 - \lambda & 0 & -1 \\ 0 & -3 - \lambda & -1 \\ 0 & 2 & -1 - \lambda \end{vmatrix} = 0 \iff (3 - \lambda)(\lambda^2 + 4\lambda + 5) = 0 \iff \lambda = 3, -2 + i, \text{ or } -2 - i.$$

Eigenvectors for $\lambda = 3$: Here we have

$$A - 3I = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -6 & -1 \\ 0 & 2 & -4 \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A - 3I)$ as follows: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Hence, we obtain

the first solution to the linear system: $\mathbf{x}_1(t) = e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Eigenvectors for $\lambda = -2 \pm i$: Here we have

$$A - (-2 + i)I = \begin{bmatrix} 5 - i & 0 & -1 \\ 0 & -1 - i & -1 \\ 0 & 2 & 1 - i \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A - (-2 + i)I)$ as follows: $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 - i \end{bmatrix}$.

Therefore, an eigenvector corresponding to $\text{nullspace}(A - (-2 - i)I)$ can be taken as $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 + i \end{bmatrix}$.

Hence, a complex-valued solution corresponding to $\lambda = -2 \pm i$ can be expressed as

$$\begin{aligned} \mathbf{x}(t) &= e^{-2t} \left\{ (\cos t + i \sin t) \left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right) \right\} \\ &= e^{-2t} \left\{ \left(\cos t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right) + i \left(\cos t \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + \sin t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right) \right\}. \end{aligned}$$

From this, we extract two real-valued solutions:

$$\mathbf{x}_2(t) = e^{-2t} \left(\cos t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \sin t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \quad \text{and} \quad \mathbf{x}_3(t) = e^{-2t} \left(\cos t \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + \sin t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right).$$

These can be compressed to

$$\mathbf{x}_2(t) = e^{-2t} \begin{bmatrix} 0 \\ \cos t \\ -\cos t + \sin t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_3(t) = e^{-2t} \begin{bmatrix} 0 \\ \sin t \\ -\cos t - \sin t \end{bmatrix}.$$

Therefore, the general solution is

$$\mathbf{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 0 \\ \cos t \\ -\cos t + \sin t \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} 0 \\ \sin t \\ -\cos t - \sin t \end{bmatrix}.$$

[NOTE: The answer is the back of the text, although correct, should be changed to conform to this one.]

12. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1 - \lambda & 1 & -1 \\ 1 & 1 - \lambda & 1 \\ -1 & 1 & 1 - \lambda \end{vmatrix} = 0 \iff (\lambda - 2)^2(\lambda + 1) = 0 \iff \lambda = -1 \text{ or } \lambda = 2.$$

Eigenvectors for $\lambda = -1$: Here we have

$$A + I = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix},$$

and we have a corresponding eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Hence, we obtain the first solution to the linear

system: $\mathbf{x}_1(t) = e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$

Eigenvectors for $\lambda = 2$: Here we have

$$A - 2I = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}.$$

The corresponding equation here is $x - y + z = 0$. Setting $z = t$ and $y = s$, we have $x = s - t$. Therefore, vectors in $\text{nullspace}(A - 2I)$ take the form $\begin{bmatrix} s - t \\ s \\ t \end{bmatrix}$. Hence, we obtain two linearly independent eigenvectors

of A corresponding to $\lambda = 2$: $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Therefore, we obtain two more solutions to the linear system: $\mathbf{x}_2(t) = e^{2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}_3(t) = e^{2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Putting the three solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, and $\mathbf{x}_3(t)$ together, we obtain the general solution to this linear system:

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

13. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & -1 & 3 \\ 2 & -1 - \lambda & 3 \\ 2 & -1 & 3 - \lambda \end{vmatrix} = 0 \iff \lambda = 0 \text{ or } \lambda = 4.$$

Eigenvectors for $\lambda = 0$: Here we have

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 2 & -1 & 3 \\ 2 & -1 & 3 \end{bmatrix}.$$

The corresponding equation here is $2x - y + 3z = 0$. Setting $x = t$ and $z = s$, we obtain $y = 2t + 3s$.

Therefore, a typical vector in $\text{nullspace}(A)$ takes the form $\begin{bmatrix} t \\ 2t + 3s \\ s \end{bmatrix}$, and thus we obtain two linearly

independent eigenvectors corresponding to $\lambda = 0$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$. Hence, we obtain two linearly independent solutions to the linear system:

$$\mathbf{x}_1(t) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}.$$

Eigenvectors for $\lambda = 4$: Here we have

$$A - 4I = \begin{bmatrix} -2 & -1 & 3 \\ 2 & -5 & 3 \\ 2 & -1 & -1 \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A - 4I)$ as follows: $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Therefore, we

obtain the third linearly independent solution to the linear system: $\mathbf{x}_3(t) = e^{4t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Putting the three solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, and $\mathbf{x}_3(t)$ together, we obtain the general solution to this linear system:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

14. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 1-\lambda & 2 & 3 & 4 \\ 4 & 3-\lambda & 2 & 1 \\ 4 & 5 & 6-\lambda & 7 \\ 7 & 6 & 5 & 4-\lambda \end{vmatrix} = 0 \stackrel{1}{\iff} \lambda = 0 \text{ or } \lambda = -2 \text{ or } \lambda = 16.$$

1. It is appropriate to use technology to determine the eigenvalues here.

Eigenvectors for $\lambda = 0$: Here we have

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 \end{bmatrix}.$$

A row-echelon form for this matrix is $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, with corresponding equations $x + 2y + 3z + 4w = 0$

and $y + 2z + 3w = 0$. Setting $w = s$ and $z = t$, we have $y = -2t - 3s$ and $x = t + 2s$. Therefore, a

typical vector in $\text{nullspace}(A)$ takes the form $\begin{bmatrix} t+2s \\ -2t-3s \\ t \\ s \end{bmatrix}$, and thus we obtain two linearly independent

eigenvectors corresponding to $\lambda = 0$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$. Hence, we obtain two linearly

independent solutions to the linear system:

$$\mathbf{x}_1(t) = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

Eigenvectors for $\lambda = -2$: Here we have

$$A + 2I = \begin{bmatrix} 3 & 2 & 3 & 4 \\ 4 & 5 & 2 & 1 \\ 4 & 5 & 8 & 7 \\ 7 & 6 & 5 & 6 \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A + 2I)$ as follows: $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$. Hence, we

obtain a third linearly independent solution to the linear system: $\mathbf{x}_3(t) = e^{-2t} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.

Eigenvectors for $\lambda = 16$: Here we have

$$A - 16I = \begin{bmatrix} -15 & 2 & 3 & 4 \\ 4 & -12 & 2 & 1 \\ 4 & 5 & -9 & 7 \\ 7 & 6 & 5 & -12 \end{bmatrix},$$

and so we have an eigenvector corresponding to $\text{nullspace}(A + 16I)$ as follows: $\mathbf{v}_4 = \begin{bmatrix} 17 \\ 13 \\ 35 \\ 31 \end{bmatrix}$. Hence, we

obtain a fourth linearly independent solution to the linear system: $\mathbf{x}_4(t) = e^{16t} \begin{bmatrix} 17 \\ 13 \\ 35 \\ 31 \end{bmatrix}$.

Putting the four solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\mathbf{x}_3(t)$, and $\mathbf{x}_4(t)$ together, we obtain the general solution to this linear system:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_4 e^{16t} \begin{bmatrix} 17 \\ 13 \\ 35 \\ 31 \end{bmatrix}.$$

15. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ -1 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & -1 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} = 0 \iff (\lambda^2 + 1)^2 = 0 \iff \lambda = \pm i.$$

Eigenvectors for $\lambda = i$: Here we have

$$A - iI = \begin{bmatrix} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & -i & -1 \\ 0 & 0 & 1 & -i \end{bmatrix},$$

and the system reduces to the equations $-x - iy = 0$ and $z - iw = 0$. Setting $y = t$ and $w = s$, we have

$x = -it$ and $z = is$. Therefore, a typical vector in $\text{nullspace}(A - iI)$ takes the form $\begin{bmatrix} -it \\ t \\ is \\ s \end{bmatrix}$ and gives two

complex-valued eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}.$$

These eigenvectors correspond to two complex-valued solutions

$$\mathbf{u}_1(t) = e^{it} \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2(t) = e^{it} \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix}.$$

We next seek two real-valued solutions corresponding to each of these complex-valued solutions:

$$\begin{aligned} \mathbf{x}(t) &= e^{it} \begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix} \\ &= (\cos t + i \sin t) \left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \\ &= \left(\cos t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \sin t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + i \left(\cos t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right). \end{aligned}$$

Thus, we obtain the real-valued solutions

$$\mathbf{x}_1(t) = \cos t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \sin t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \cos t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

or

$$\mathbf{x}_1(t) = \begin{bmatrix} \sin t \\ \cos t \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} -\cos t \\ \sin t \\ 0 \\ 0 \end{bmatrix}.$$

Finally, we use $\mathbf{u}_2(t)$ to generate two more real-valued solutions:

$$\begin{aligned}\mathbf{x}(t) &= e^{it} \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix} \\ &= (\cos t + i \sin t) \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) \\ &= \left(\cos t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) + i \left(\cos t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right).\end{aligned}$$

Thus, we obtain the real-valued solutions

$$\mathbf{x}_3(t) = \cos t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_4(t) = \cos t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

or

$$\mathbf{x}_3(t) = \begin{bmatrix} 0 \\ 0 \\ -\sin t \\ \cos t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_4(t) = \begin{bmatrix} 0 \\ 0 \\ \cos t \\ \sin t \end{bmatrix}.$$

Putting together the four real-valued solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\mathbf{x}_3(t)$ and $\mathbf{x}_4(t)$ obtained above, we find the general solution

$$\mathbf{x}(t) = c_1 \begin{bmatrix} \sin t \\ \cos t \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -\cos t \\ \sin t \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ -\sin t \\ \cos t \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ \cos t \\ \sin t \end{bmatrix}.$$

16. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} -1 - \lambda & 4 \\ 2 & -3 - \lambda \end{vmatrix} = 0 \iff \lambda^2 + 4\lambda - 5 = 0 \iff \lambda = 1 \text{ or } \lambda = -5.$$

Eigenvectors for $\lambda = 1$: Here we have

$$A - I = \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix},$$

and so we obtain the first solution to the linear system: $\mathbf{x}_1(t) = e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Eigenvectors for $\lambda = -5$: Here we have

$$A + 5I = \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix},$$

and so we obtain the second solution to the linear system: $\mathbf{x}_2(t) = e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Putting the two solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ together, we obtain the general solution to this linear system:

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Since

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

we can quickly solve for the constants c_1 and c_2 to obtain $c_1 = 1$ and $c_2 = -1$. Thus, the particular solution to this initial-value problem is

$$\mathbf{x}(t) = e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

17. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} -1 - \lambda & -6 \\ 3 & 5 - \lambda \end{vmatrix} = 0 \iff \lambda^2 - 4\lambda + 13 = 0 \iff \lambda = 2 \pm 3i.$$

Eigenvectors for $\lambda = 2 + 3i$: Here we have

$$A - (2 + 3i)I = \begin{bmatrix} -3 - 3i & -6 \\ 3 & 3 - 3i \end{bmatrix}.$$

The two rows of this matrix must be proportional, since we must be able to find a nonzero vector in its nullspace corresponding to $\lambda = 2 + 3i$, so concentrating on the first row, we can determine an eigenvector corresponding to $\lambda = 2 + 3i$ as follows: $\mathbf{v} = \begin{bmatrix} -2 \\ 1 + i \end{bmatrix}$. Note that other choices of $\mathbf{v}$ are possible here. Using our choice, we obtain

$$\begin{aligned} \mathbf{x}(t) &= e^{(2+3i)t} \mathbf{v} \\ &= e^{2t} (\cos 3t + i \sin 3t) \left(\begin{bmatrix} -2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= e^{2t} \left[\left(\cos 3t \begin{bmatrix} -2 \\ 1 \end{bmatrix} - \sin 3t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + i \left(\cos 3t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin 3t \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right) \right]. \end{aligned}$$

Thus, we find two real-valued solutions to the linear system:

$$\mathbf{x}_1(t) = e^{2t} \left(\cos 3t \begin{bmatrix} -2 \\ 1 \end{bmatrix} - \sin 3t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \quad \text{and} \quad \mathbf{x}_2(t) = e^{2t} \left(\cos 3t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin 3t \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right).$$

Therefore, the general solution to the linear system is

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{2t} \left(\cos 3t \begin{bmatrix} -2 \\ 1 \end{bmatrix} - \sin 3t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + c_2 e^{2t} \left(\cos 3t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin 3t \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right) \\ &= c_1 e^{2t} \begin{bmatrix} -2 \cos 3t \\ \cos 3t - \sin 3t \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} -2 \sin 3t \\ \cos 3t + \sin 3t \end{bmatrix}. \end{aligned}$$

Since

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

we can quickly solve for the constants c_1 and c_2 to obtain $c_1 = -1$ and $c_2 = 3$. Thus, the particular solution to this initial-value problem is

$$\mathbf{x}(t) = -e^{2t} \begin{bmatrix} -2 \cos 3t \\ \cos 3t - \sin 3t \end{bmatrix} + 3e^{2t} \begin{bmatrix} -2 \sin 3t \\ \cos 3t + \sin 3t \end{bmatrix}.$$

18. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & -1 & 3 \\ 3 & 1 - \lambda & 0 \\ 2 & -1 & 3 - \lambda \end{vmatrix} = 0 \iff \lambda = 0 \text{ or } \lambda = 2 \text{ or } \lambda = 4.$$

Eigenvectors for $\lambda = 0$: Here we have

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & -1 & 3 \end{bmatrix},$$

and this matrix row-reduces to $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -5 & 9 \\ 0 & 0 & 0 \end{bmatrix}$. Setting $z = 5t$, we have $y = 9t$ and $x = -3t$. Thus,

we have corresponding eigenvector $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 9 \\ 5 \end{bmatrix}$. Thus, we obtain the first solution to the linear system:

$$\mathbf{x}_1(t) = \begin{bmatrix} -3 \\ 9 \\ 5 \end{bmatrix}.$$

Eigenvectors for $\lambda = 2$: Here we have

$$A - 2I = \begin{bmatrix} 0 & -1 & 3 \\ 3 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix},$$

and this matrix row-reduces to $\begin{bmatrix} 3 & -1 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. Setting $z = t$, we have $y = 3t$ and $x = t$. Therefore, we

have corresponding eigenvector $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$. Thus, we obtain the second solution to the linear system:

$$\mathbf{x}_2(t) = e^{2t} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

Eigenvectors for $\lambda = 4$: Here we have

$$A - 4I = \begin{bmatrix} -2 & -1 & 3 \\ 3 & -3 & 0 \\ 2 & -1 & -1 \end{bmatrix},$$

and this matrix row-reduces to $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. Setting $z = t$, we have $x = y = t$. Therefore we have corresponding eigenvector $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Thus, we obtain the third solution to the linear system: $\mathbf{x}_3(t) =$

$$e^{4t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Putting together the three solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, and $\mathbf{x}_3(t)$ above, we obtain the general solution to the linear system:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -3 \\ 9 \\ 5 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Since

$$\begin{bmatrix} -4 \\ 4 \\ 4 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} -3 \\ 9 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

we obtain a linear system of equations with augmented matrix

$$\left[\begin{array}{ccc|c} -3 & 1 & 1 & -4 \\ 9 & 3 & 1 & 4 \\ 5 & 1 & 1 & 4 \end{array} \right].$$

We can solve this system to obtain $c_1 = 1$, $c_2 = -2$, and $c_3 = 1$. Therefore, the particular solution to this initial-value problem is

$$\mathbf{x}(t) = \begin{bmatrix} -3 \\ 9 \\ 5 \end{bmatrix} - 2e^{2t} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + e^{4t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

19. Note that

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} -\lambda & 4 \\ -4 & -\lambda \end{vmatrix} = 0 \iff \lambda^2 + 16 = 0 \iff \lambda = \pm 4i.$$

Eigenvectors for $\lambda = 4i$: Here we have

$$A - 4iI = \begin{bmatrix} -4i & 4 \\ -4 & -4i \end{bmatrix},$$

and focusing on the first row of this matrix (since the second row must be row-equivalent to the first row), we obtain $-4ix + 4y = 0$. We may choose the eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$. Thus, we obtain the solution

$$\begin{aligned} \mathbf{x}(t) &= e^{4it} \mathbf{v} \\ &= (\cos 4t + i \sin 4t) \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= \left(\cos 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin 4t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + i \left(\cos 4t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right). \end{aligned}$$

This generates two real-valued solutions:

$$\mathbf{x}_1(t) = \cos 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin 4t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \cos 4t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

or

$$\mathbf{x}_1(t) = \begin{bmatrix} \cos 4t \\ -\sin 4t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} \sin 4t \\ \cos 4t \end{bmatrix}.$$

Therefore, putting these solutions together, we obtain the general solution to the linear system:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} \cos 4t \\ -\sin 4t \end{bmatrix} + c_2 \begin{bmatrix} \sin 4t \\ \cos 4t \end{bmatrix}.$$

Since

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

we can quickly solve for the constants c_1 and c_2 : $c_1 = c_2 = 1$. Therefore, the particular solution to this initial-value problem is

$$\mathbf{x}(t) = \begin{bmatrix} \cos 4t \\ -\sin 4t \end{bmatrix} + \begin{bmatrix} \sin 4t \\ \cos 4t \end{bmatrix} = \begin{bmatrix} \cos 4t + \sin 4t \\ -\sin 4t + \cos 4t \end{bmatrix}.$$

Since $\mathbf{x}_1^2 + \mathbf{x}_2^2 = 2$ for all $t \in \mathbb{R}$, the graph of the solution is a circle of radius $\sqrt{2}$, centered at $(0, 0)$.

20.

(a) If we let $x_1 = x$ and $x_2 = \frac{dx}{dt}$, then the given differential equation can be replaced by the equivalent first-order system

$$x_1' = x_2 \quad \text{and} \quad x_2' = -cx_1 - bx_2.$$

That is, $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}$.

(b) We have $p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -c & -b - \lambda \end{bmatrix} = \lambda^2 + b\lambda + c$, which coincides with the auxiliary polynomial of $x'' + bx' + cx = 0$.

21. Let us assume that $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{0}$. We must show that $c_1 = c_2 = 0$. Substituting the expressions given in the problem for $\mathbf{x}_1$ and $\mathbf{x}_2$, we obtain

$$\begin{aligned} e^{at} [c_1(\mathbf{r} \cos bt - \mathbf{s} \sin bt) + c_2(\mathbf{r} \sin bt + \mathbf{s} \cos bt)] &= \mathbf{0}, \\ (c_1\mathbf{r} + c_2\mathbf{s}) \cos bt + (c_2\mathbf{r} - c_1\mathbf{s}) \sin bt &= \mathbf{0}, \\ c_1\mathbf{r} + c_2\mathbf{s} = \mathbf{0} \quad \text{and} \quad c_2\mathbf{r} - c_1\mathbf{s} &= \mathbf{0}. \end{aligned}$$

But $\mathbf{r}$ and $\mathbf{s}$ are linearly independent by assumption. Therefore, $c_1 = c_2 = 0$.

22. Suppose that λ_1 and λ_2 are the eigenvalues of A , and suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are the corresponding (linearly independent) eigenvectors of A . Then by Theorem 7.4.3, the general solution to the linear system is

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2.$$

Thus,

$$\lim_{t \rightarrow \infty} \mathbf{x}(t) = \lim_{t \rightarrow \infty} (c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2) = \mathbf{0},$$

since $\operatorname{Re}(\lambda_1) < 0$ and $\operatorname{Re}(\lambda_2) < 0$.

23. Note that $\lim_{t \rightarrow \infty} e^{\lambda t} = 0$ if and only if $\operatorname{Re}(\lambda) < 0$. All solutions to the system are of the form $\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2$, where λ_1 and λ_2 are the eigenvalues of A , and $\mathbf{v}_1$ and $\mathbf{v}_2$ are corresponding linearly independent eigenvectors. Thus we are given that for all values of c_1 and c_2 $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{0}$. In particular,

$$\lim_{t \rightarrow \infty} e^{\lambda_1 t} \mathbf{v}_1 = \mathbf{0} \quad \text{and} \quad \lim_{t \rightarrow \infty} e^{\lambda_2 t} \mathbf{v}_2 = \mathbf{0}.$$

Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are constant vectors, it must therefore be the case that $\lim_{t \rightarrow \infty} e^{\lambda_1 t} = 0$ and $\lim_{t \rightarrow \infty} e^{\lambda_2 t} = 0$. Consequently, $\operatorname{Re}(\lambda_1) < 0$ and $\operatorname{Re}(\lambda_2) < 0$.

24. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} -\lambda & b \\ -b & -\lambda \end{vmatrix} = 0$ if and only if $\lambda^2 + b^2 = 0$. Therefore, $\lambda = \pm bi$.

When $\lambda = bi$, then $A - biI = \begin{bmatrix} -bi & b \\ -b & -bi \end{bmatrix}$, and so the linear system $(A - biI)\mathbf{v} = \mathbf{0}$ has solutions of the form $\mathbf{v} = r(1, i)$, where $r \in \mathbb{C}$. A complex-valued solution of the system is therefore

$$e^{ibt} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = (\cos bt + i \sin bt) \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

This gives rise to two real-valued solutions,

$$\mathbf{x}_1(t) = \begin{bmatrix} \cos bt \\ -\sin bt \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} \sin bt \\ \cos bt \end{bmatrix}.$$

Therefore, the general solution is given by

$$\mathbf{x}(t) = c_1 \begin{bmatrix} \cos bt \\ -\sin bt \end{bmatrix} + c_2 \begin{bmatrix} \sin bt \\ \cos bt \end{bmatrix}.$$

The solutions of this system are periodic with period $2\pi/|b|$.

25. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} a - \lambda & b \\ -b & a - \lambda \end{vmatrix} = 0$ if and only if $\lambda^2 - 2a\lambda + a^2 + b^2 = 0$. Therefore,

$\lambda = a \pm bi$. When $\lambda = a - bi$, then $A - \lambda I$ becomes $\begin{bmatrix} bi & b \\ -b & bi \end{bmatrix}$, and the linear system $(A - \lambda I)\mathbf{v} = \mathbf{0}$ has solutions of the form $\mathbf{v} = r(i, 1)$, where $r \in \mathbb{C}$. A complex-valued solution of the system is therefore

$$e^{(a-bi)t} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = e^{at} (\cos bt - i \sin bt) \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

This gives rise to two real-valued solutions

$$\mathbf{x}_1(t) = e^{at} \begin{bmatrix} \sin bt \\ \cos bt \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{at} \begin{bmatrix} \cos bt \\ -\sin bt \end{bmatrix}.$$

Therefore, the general solution is given by

$$\mathbf{x}(t) = e^{at} \left\{ c_1 \begin{bmatrix} \sin bt \\ \cos bt \end{bmatrix} + c_2 \begin{bmatrix} \cos bt \\ -\sin bt \end{bmatrix} \right\}.$$

Since $a < 0$, as $t \rightarrow \infty$ we have $\mathbf{x}(t) \rightarrow \mathbf{0}$. More specifically, we have

$$x_1(t) = e^{at}(c_1 \sin bt + c_2 \cos bt) \quad \text{and} \quad x_2(t) = e^{at}(c_1 \cos bt - c_2 \sin bt).$$

Hence, $x_1(t)^2 + x_2(t)^2 = e^{2at}(c_1^2 + c_2^2)$. Therefore, the solution curves lie on circles of ever decreasing radii as t increases. Thus, the solution curves in the x_1x_2 -plane spiral towards the origin as t increases.

26. If A has eigenvalue $\lambda = 0$ with corresponding eigenvector $\mathbf{x}_0$, then the system $\mathbf{x}' = A\mathbf{x}$ has a solution $\mathbf{x}(t) = \mathbf{x}_0$ for all t .

27. The given linear system can be expressed in matrix form as $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}$. We have $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -b & -a - \lambda \end{vmatrix} = 0$ if and only if $\lambda^2 + a\lambda + b = 0$ if and only if $\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$. Thus, both eigenvalues have negative real part. From Problem 22, this implies that $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{0}$.

Solutions to Section 7.5

True-False Review:

1. FALSE. Every $n \times n$ linear system of differential equations has n linearly independent solutions (Theorem 7.3.2), regardless of whether A is defective or nondefective.

2. TRUE. This fact is part of the information contained in Theorem 7.5.4.

3. FALSE. The number of linearly independent solutions to $\mathbf{x}' = A\mathbf{x}$ corresponding to λ is equal to the algebraic multiplicity of the eigenvalue λ as a root of the characteristic equation for A , not the dimension of the eigenspace E_λ .

4. FALSE. We have

$$A\mathbf{x} = A(e^{\lambda t}\mathbf{v}_1) = e^{\lambda t}(A\mathbf{v}_1) = e^{\lambda t}(\mathbf{v}_0 + \lambda\mathbf{v}_1) = e^{\lambda t}\mathbf{v}_0 + \lambda e^{\lambda t}\mathbf{v}_1 = e^{\lambda t}\mathbf{v}_0 + \mathbf{x}'(t).$$

Therefore, $A\mathbf{x} \neq \mathbf{x}'$.

Problems:

1. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} -\lambda & -2 \\ 2 & 4 - \lambda \end{vmatrix} = 0$ if and only if $\lambda^2 - 4\lambda + 4 = 0$. Therefore $\lambda = 2$ (with multiplicity two). When $\lambda = 2$, the corresponding eigenvectors take the form $\mathbf{v} = r(-1, 1)$, where $r \in \mathbb{R}$. We will determine a second linearly independent solution of the form

$$\mathbf{x}_1(t) = e^{2t}(\mathbf{v}_1 + t\mathbf{v}_0)$$

where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A - 2I)^2\mathbf{v}_1 = \mathbf{0}, \quad (A - 2I)\mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A - 2I)\mathbf{v}_1.$$

In this case, we have $A - 2I = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}$ and $(A - 2I)^2 = \mathbf{0}_2$. Therefore, we may choose $\mathbf{v}_1$ to be any vector such that $(A - 2I)\mathbf{v}_1 \neq \mathbf{0}$. For simplicity, we take $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus, $\mathbf{v}_0 = (A - 2I)\mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

From the expressions for $\mathbf{v}_0$ and $\mathbf{v}_1$, we can write down two linearly independent solutions to the vector differential equation:

$$\mathbf{x}_0(t) = e^{2t} \begin{bmatrix} -2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_1(t) = e^{2t} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\} = e^{2t} \begin{bmatrix} 1 - 2t \\ 2t \end{bmatrix}.$$

Consequently, the general solution to the vector differential equation is

$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} -2 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 - 2t \\ 2t \end{bmatrix}.$$

2. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} -3 - \lambda & -2 \\ 2 & 1 - \lambda \end{vmatrix} = 0$ if and only if $\lambda^2 + 2\lambda + 1 = 0$. Therefore $\lambda = -1$ (with multiplicity two). When $\lambda = -1$, the corresponding eigenvectors take the form $\mathbf{v} = r(-1, 1)$, where $r \in \mathbb{R}$. We will determine a second linearly independent solution of the form

$$\mathbf{x}_1(t) = e^{-t}(\mathbf{v}_1 + t\mathbf{v}_0)$$

where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A + I)^2 \mathbf{v}_1 = \mathbf{0}, \quad (A + I)\mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A + I)\mathbf{v}_1.$$

In this case, we have $A + I = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}$ and $(A + I)^2 = 0_2$. Therefore, we may choose $\mathbf{v}_1$ to be any vector such that $(A + I)\mathbf{v}_1 \neq \mathbf{0}$. For simplicity, we take $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus, $\mathbf{v}_0 = (A + I)\mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. From the expressions for $\mathbf{v}_0$ and $\mathbf{v}_1$, we can write down two linearly independent solutions to the vector differential equation:

$$\mathbf{x}_0(t) = e^{-t} \begin{bmatrix} -2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_1(t) = e^{-t} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\} = e^{-t} \begin{bmatrix} 1 - 2t \\ 2t \end{bmatrix}.$$

Consequently, the general solution to the vector differential equation is

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} -2 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 - 2t \\ 2t \end{bmatrix}.$$

3. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 1 & -1 - \lambda \end{vmatrix} = 0$ if and only if $(\lambda + 1)(1 - \lambda^2) = 0$. Therefore, the eigenvalues of A are $\lambda = 1$ and $\lambda = -1$ (with multiplicity two).

Eigenvectors for $\lambda = 1$: The corresponding eigenvectors are $\mathbf{v} = r(1, 1, 1)$, where $r \in \mathbb{R}$. Hence, one solution

of the system is $\mathbf{x}(t) = e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Eigenvectors for $\lambda = -1$: The corresponding eigenvectors are $\mathbf{v} = r(1, -1, 1)$, where $r \in \mathbb{R}$. Thus, we obtain one solution corresponding to $\lambda = -1$ of $\mathbf{x}_0(t) = e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. We will determine a second linearly

independent solution corresponding to $\lambda = -1$ of the form

$$\mathbf{x}_1(t) = e^{-t}(\mathbf{v}_1 + t\mathbf{v}_0)$$

where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A + I)^2 \mathbf{v}_1 = \mathbf{0}, \quad (A + I) \mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A + I) \mathbf{v}_1.$$

In this case, we have $A + I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $(A + I)^2 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$. Many choices for $\mathbf{v}_1$ are possible that meet the requirements above. Let us take, as one possibility, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Then $\mathbf{v}_0 = (A + I) \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Hence, we have the solution

$$\mathbf{x}_1(t) = e^{-t} \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right).$$

We have therefore obtained two linearly independent solutions to the system of differential equations corresponding to $\lambda = -1$:

$$\mathbf{x}_0(t) = e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_1(t) = e^{-t} \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right).$$

Putting this together with the solution obtained above corresponding to $\lambda = 1$, we obtain the general solution to this system of differential equations:

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{-t} \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right),$$

or equivalently,

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + e^{-t} \left\{ c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1+t \\ -t \\ -1+t \end{bmatrix} \right\}.$$

4. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} 2-\lambda & 2 & -1 \\ 2 & 1-\lambda & -1 \\ 2 & 3 & -1-\lambda \end{vmatrix} = 0$ if and only if $\lambda^3 - 2\lambda^2 = 0$. Therefore, the eigenvalues of A are $\lambda = 2$ and $\lambda = 0$ (with multiplicity two).

Eigenvectors for $\lambda = 2$: The matrix $A - 2I = \begin{bmatrix} 0 & 2 & -1 \\ 2 & -1 & -1 \\ 2 & 3 & -3 \end{bmatrix}$ row-reduces to $\begin{bmatrix} 2 & -1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, which implies that there is only one free variable (and hence, only one linearly independent eigenvector). Such eigenvector must take the form $\mathbf{v} = r(3, 2, 4)$, where $r \in \mathbb{R}$. Hence, one solution of the system is $\mathbf{x}(t) = e^{2t} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$.

Eigenvectors for $\lambda = 0$: The matrix A can be row-reduced to $\begin{bmatrix} 2 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This implies that there is only one free variable, and hence only one linearly independent eigenvector. Such eigenvector must take the form

$\mathbf{v} = r(1, 0, 2)$, where $r \in \mathbb{R}$. Thus, we obtain one solution corresponding to $\lambda = 0$ of $\mathbf{x}_0(t) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. We will determine a second linearly independent solution corresponding to $\lambda = 0$ of the form

$$\mathbf{x}_1(t) = \mathbf{v}_1 + t\mathbf{v}_0$$

where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$A^2\mathbf{v}_1 = \mathbf{0}, \quad A\mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = A\mathbf{v}_1.$$

In this case, we have $A = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$ and $A^2 = \begin{bmatrix} 6 & 3 & -3 \\ 4 & 2 & -2 \\ 8 & 4 & -4 \end{bmatrix}$. Many choices for $\mathbf{v}_1$ are possible that meet the requirements above. Let us take, as one possibility, $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Then $\mathbf{v}_0 = A\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. Hence,

we have the solution $\mathbf{x}_1(t) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. Therefore, we have two linearly independent solutions to the system of differential equations corresponding to $\lambda = 0$:

$$\mathbf{x}_0(t) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_1(t) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Putting this together with the solution obtained above corresponding to $\lambda = 2$, we obtain the general solution to this system of differential equations:

$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_3 \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right).$$

5. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} -2 - \lambda & 0 & 0 \\ 1 & -3 - \lambda & -1 \\ -1 & 1 & -1 - \lambda \end{vmatrix} = 0$ if and only if $(-2 - \lambda)(\lambda^2 + 4\lambda + 4) = 0$ if and only if $(\lambda + 2)^3 = 0$. This means that the only eigenvalue of A is $\lambda = -2$ (with multiplicity 3).

Eigenvectors for $\lambda = -2$: The matrix $A + 2I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. There are two free variables, and hence we obtain two linearly independent eigenvectors. In fact, the eigenvectors must take the form $(r + s, r, s) = r(1, 1, 0) + s(1, 0, 1)$, where $r, s \in \mathbb{R}$. Therefore, we can construct two linearly independent solutions to the system of differential equations as follows:

$$\mathbf{x}_0^{(1)}(t) = e^{-2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_0^{(2)}(t) = e^{-2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Consequently, we seek a third linearly independent solution $\mathbf{x}_1(t)$ to the vector differential equation of the form

$$\mathbf{x}_1(t) = e^{-2t}(\mathbf{v}_1 + t\mathbf{v}_0)$$

where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A + 2I)^2 \mathbf{v}_1 = \mathbf{0}, \quad (A + 2I) \mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A + 2I) \mathbf{v}_1.$$

In this case, we have $A + 2I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ and $(A + 2I)^2 = 0_3$. Therefore, we can choose $\mathbf{v}_1$ to be any

vector such that $(A + 2I) \mathbf{v}_1 \neq \mathbf{0}$. For instance, we can choose $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then $\mathbf{v}_0 = (A + 2I) \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

Hence, we have the solution $\mathbf{x}_1(t) = e^{-2t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$. Therefore, the general solution to this vector differential equation is

$$\mathbf{x}(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-2t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right),$$

or equivalently,

$$\mathbf{x}(t) = e^{-2t} \left\{ c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ t \\ -t \end{bmatrix} \right\}.$$

6. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} 15 - \lambda & -32 & 12 \\ 8 & -17 - \lambda & 6 \\ 0 & 0 & -1 - \lambda \end{vmatrix} = 0$ if and only if $(-1 - \lambda)(\lambda^2 + 2\lambda + 1) = 0$ if and only if $(\lambda + 1)^3 = 0$. This means that the only eigenvalue of A is $\lambda = -1$ (with multiplicity 3).

Eigenvectors for $\lambda = -1$: The matrix $A + I = \begin{bmatrix} 16 & -32 & 12 \\ 8 & -16 & 6 \\ 0 & 0 & 0 \end{bmatrix}$, which row reduces to $\begin{bmatrix} 4 & -8 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

There are two free variables, and hence we obtain two linearly independent eigenvectors. In fact, the eigenvectors take the form $(2r - 3s, r, 4s) = r(2, 1, 0) + s(-3, 0, 4)$, where $r, s \in \mathbb{R}$. Therefore, we can construct two linearly independent solutions to the system of differential equations as follows:

$$\mathbf{x}_0^{(1)}(t) = e^{-t} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_0^{(2)}(t) = e^{-t} \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}.$$

Consequently, we seek a third linearly independent solution $\mathbf{x}_1(t)$ to the vector differential equation of the form

$$\mathbf{x}_1(t) = e^{-t}(\mathbf{v}_1 + t\mathbf{v}_0)$$

where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A + I)^2 \mathbf{v}_1 = \mathbf{0}, \quad (A + I) \mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A + I) \mathbf{v}_1.$$

In this case, we have $A + I = \begin{bmatrix} 16 & -32 & 12 \\ 8 & -16 & 6 \\ 0 & 0 & 0 \end{bmatrix}$ and $(A + I)^2 = 0_3$. Therefore, we can choose $\mathbf{v}_1$ to be any

vector such that $(A + I) \mathbf{v}_1 \neq \mathbf{0}$. For instance, we can choose $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then $\mathbf{v}_0 = (A + I) \mathbf{v}_1 = \begin{bmatrix} 16 \\ 8 \\ 0 \end{bmatrix}$.

Hence, we have the solution $\mathbf{x}_1(t) = e^{-t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 16 \\ 8 \\ 0 \end{bmatrix} \right)$. Therefore, the general solution to this vector differential equation is

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix} + c_3 e^{-t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 16 \\ 8 \\ 0 \end{bmatrix} \right),$$

or equivalently,

$$\mathbf{x}(t) = e^{-t} \left\{ c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 1 + 16t \\ 8t \\ 0 \end{bmatrix} \right\}.$$

7. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} 4 - \lambda & 0 & 0 \\ 1 & 4 - \lambda & 0 \\ 0 & 1 & 4 - \lambda \end{vmatrix} = 0$ if and only if $(4 - \lambda)^3 = 0$. This means that the only eigenvalue of A is $\lambda = 4$ (with multiplicity 3).

Eigenvectors for $\lambda = 4$: The matrix $A - 4I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Therefore, we obtain eigenvectors of the form $\mathbf{v}_0 = r(0, 0, 1)$, where $r \in \mathbb{R}$, and we obtain a solution to $\mathbf{x}' = A\mathbf{x}$ of the form

$$\mathbf{x}_0(t) = e^{4t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

According to Theorem 7.5.4, there exist two further linearly independent solutions to $\mathbf{x}' = A\mathbf{x}$ of the form

$$\mathbf{x}_1(t) = e^{4t}(\mathbf{v}_1 + t\mathbf{v}_0),$$

$$\mathbf{x}_2(t) = e^{4t}(\mathbf{v}_2 + t\mathbf{v}_1 + \frac{1}{2!}t^2\mathbf{v}_0),$$

where

$$(A - 4I)^3\mathbf{v}_2 = \mathbf{0}, \quad (A - 4I)^2\mathbf{v}_2 \neq \mathbf{0},$$

and

$$\mathbf{v}_1 = (A - 4I)\mathbf{v}_2, \quad \mathbf{v}_0 = (A - 4I)^2\mathbf{v}_2.$$

A short calculation shows that

$$A - 4I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad (A - 4I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad (A - 4I)^3 = 0_3.$$

Therefore, we may take

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_1 = (A - 4I)\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_0 = (A - 4I)^2\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, we obtain

$$\mathbf{x}_1(t) = e^{4t} \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = e^{4t} \begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix}$$

and

$$\mathbf{x}_2(t) = e^{4t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2!} t^2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = e^{4t} \begin{bmatrix} 1 \\ t \\ t^2/2 \end{bmatrix}.$$

Therefore, the general solution to the vector differential equation is

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{4t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} 1 \\ t \\ t^2/2 \end{bmatrix} \\ &= e^{4t} \left\{ c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ t \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ t \\ t^2/2 \end{bmatrix} \right\}. \end{aligned}$$

8. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & 2 \\ 2 & -2 & -1-\lambda \end{vmatrix} = 0$ if and only if $(1-\lambda)(\lambda^2 - 2\lambda + 1) = 0$ if and only if $(\lambda - 1)^3 = 0$. This means that the only eigenvalue of A is $\lambda = 1$ (with multiplicity 3).

Eigenvectors for $\lambda = 1$: The matrix $A - I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 2 & -2 & -2 \end{bmatrix}$. Therefore, we obtain eigenvectors of the form $\mathbf{v}_0 = r(0, -1, 1)$, where $r \in \mathbb{R}$, and we obtain a solution to $\mathbf{x}' = A\mathbf{x}$ of the form

$$\mathbf{x}_0(t) = e^t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

According to Theorem 7.5.4, there exist two further linearly independent solutions to $\mathbf{x}' = A\mathbf{x}$ of the form

$$\mathbf{x}_1(t) = e^t(\mathbf{v}_1 + t\mathbf{v}_0),$$

$$\mathbf{x}_2(t) = e^t(\mathbf{v}_2 + t\mathbf{v}_1 + \frac{1}{2!}t^2\mathbf{v}_0),$$

where

$$(A - I)^3\mathbf{v}_2 = \mathbf{0}, \quad (A - I)^2\mathbf{v}_2 \neq \mathbf{0},$$

and

$$\mathbf{v}_1 = (A - I)\mathbf{v}_2, \quad \mathbf{v}_0 = (A - I)^2\mathbf{v}_2.$$

A short calculation shows that

$$A - I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 2 & -2 & -2 \end{bmatrix}, \quad (A - I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ -4 & 0 & 0 \end{bmatrix}, \quad (A - I)^3 = 0_3.$$

Therefore, we may take

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_1 = (A - I)\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_0 = (A - I)^2\mathbf{v}_2 = \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix}.$$

Hence, we obtain

$$\mathbf{x}_1(t) = e^t \left(\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix} \right) = e^t \begin{bmatrix} 0 \\ 4t \\ 2 - 4t \end{bmatrix}$$

and

$$\mathbf{x}_2(t) = e^t \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + \frac{1}{2!}t^2 \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix} \right) = e^t \begin{bmatrix} 1 \\ 2t^2 \\ 2t - 2t^2 \end{bmatrix}.$$

Therefore, the general solution to the vector differential equation is

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 4t \\ 2 - 4t \end{bmatrix} + c_3 e^t \begin{bmatrix} 1 \\ 2t^2 \\ 2t - 2t^2 \end{bmatrix} \\ &= e^t \left\{ c_1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 4t \\ 2 - 4t \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2t^2 \\ 2t - 2t^2 \end{bmatrix} \right\}. \end{aligned}$$

9. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} 3 - \lambda & 1 & 0 \\ -1 & 5 - \lambda & 0 \\ 0 & 0 & 4 - \lambda \end{vmatrix} = 0$ if and only if $(4 - \lambda)(\lambda^2 - 8\lambda + 16) = 0$ if and only if $(\lambda - 4)^3 = 0$. This means that the only eigenvalue of A is $\lambda = 4$ (with multiplicity 3).

Eigenvectors for $\lambda = 4$: The matrix $A - 4I = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. There are two free variables, and hence we obtain two linearly independent eigenvectors. In fact, the eigenvectors must take the form $(r, r, s) = r(1, 1, 0) + s(0, 0, 1)$, where $r, s \in \mathbb{R}$. Therefore, we can construct two linearly independent solutions to the system of differential equations as follows:

$$\mathbf{x}_0^{(1)}(t) = e^{4t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_0^{(2)}(t) = e^{4t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Consequently, we seek a third linearly independent solution $\mathbf{x}_1(t)$ to the vector differential equation of the form

$$\mathbf{x}_1(t) = e^{4t}(\mathbf{v}_1 + t\mathbf{v}_0)$$

where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A - 4I)^2\mathbf{v}_1 = \mathbf{0}, \quad (A - 4I)\mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A - 4I)\mathbf{v}_1.$$

In this case,

$$A - 4I = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad (A - 4I)^2 = 0_3.$$

Therefore, we may choose any $\mathbf{v}_1$ such that $(A - 4I)\mathbf{v}_1 = \mathbf{0}$. For instance, we may take $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then

$\mathbf{v}_0 = (A - 4I)\mathbf{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$. Hence, we have the solution

$$\mathbf{x}_1(t) = e^{4t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \right).$$

Therefore, the general solution to this vector differential equation is

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{4t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \right) \\ &= e^{4t} \left\{ c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1-t \\ -t \\ 0 \end{bmatrix} \right\}. \end{aligned}$$

10. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} -1-\lambda & 1 & 0 \\ -2 & -3-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} = 0$ if and only if $(\lambda + 2)^3 = 0$. This means that the only eigenvalue of A is $\lambda = -2$ (with multiplicity 3).

Eigenvectors for $\lambda = -2$: The matrix $A + 2I = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. This matrix row reduces to $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

We obtain eigenvectors of the form $\mathbf{v}_0 = r(-1, 1, -1)$, where $r \in \mathbb{R}$, and we obtain a solution to $\mathbf{x}' = A\mathbf{x}$ of the form

$$\mathbf{x}_0(t) = e^{-2t} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.$$

According to Theorem 7.5.4, there exist two further linearly independent solutions to $\mathbf{x}' = A\mathbf{x}$ of the form

$$\mathbf{x}_1(t) = e^{-2t}(\mathbf{v}_1 + t\mathbf{v}_0),$$

$$\mathbf{x}_2(t) = e^{-2t}(\mathbf{v}_2 + t\mathbf{v}_1 + \frac{1}{2!}t^2\mathbf{v}_0),$$

where

$$(A + 2I)^3\mathbf{v}_2 = \mathbf{0}, \quad (A + 2I)^2\mathbf{v}_2 \neq \mathbf{0},$$

and

$$\mathbf{v}_1 = (A + 2I)\mathbf{v}_2, \quad \mathbf{v}_0 = (A + 2I)^2\mathbf{v}_2.$$

In this case

$$A + 2I = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad (A + 2I)^2 = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \quad (A + 2I)^3 = 0_3.$$

Therefore, we may take $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then $\mathbf{v}_1 = (A + 2I)\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_0 = (A + 2I)^2\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$.

Therefore, we obtain the additional solutions to the vector differential equation:

$$\mathbf{x}_1(t) = e^{-2t} \left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right) = e^{-2t} \begin{bmatrix} 1-t \\ -2+t \\ 1-t \end{bmatrix}$$

and

$$\mathbf{x}_2(t) = e^{-2t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \frac{1}{2!} t^2 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right) = e^{-2t} \begin{bmatrix} 1+t-t^2/2 \\ -2t+t^2/2 \\ t-t^2/2 \end{bmatrix}.$$

Therefore, the general solution to the vector differential equation is

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{-2t} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1-t \\ -2+t \\ 1-t \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} 1+t-t^2/2 \\ -2t+t^2/2 \\ t-t^2/2 \end{bmatrix} \\ &= e^{-2t} \left\{ c_1 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1-t \\ -2+t \\ 1-t \end{bmatrix} + c_3 \begin{bmatrix} 1+t-t^2/2 \\ -2t+t^2/2 \\ t-t^2/2 \end{bmatrix} \right\}. \end{aligned}$$

11. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} -\lambda & -1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 1 & 0 & 2-\lambda & 1 \\ 0 & 1 & 0 & 2-\lambda \end{vmatrix} = 0$ if and only if $(2-\lambda)^2(\lambda^2+1) = 0$.

This means that the eigenvalues of A are $\lambda = \pm i$ and $\lambda = 2$ (with multiplicity 2).

Eigenvectors for $\lambda = i$: We have $A - iI = \begin{bmatrix} -i & -1 & 0 & 0 \\ 1 & -i & 0 & 0 \\ 1 & 0 & 2-i & 1 \\ 0 & 1 & 0 & 2-i \end{bmatrix}$. A short calculation shows that a

corresponding complex eigenvector for $\lambda = i$ is $\mathbf{v} = (-5(1+2i), 5(-2+i), -2+4i, 5)$. Therefore, we obtain the complex-valued solution

$$\mathbf{u}(t) = (\cos t + i \sin t) \begin{bmatrix} -5(1+2i) \\ 5(-2+i) \\ -2+4i \\ 5 \end{bmatrix}.$$

We can then derive two real-valued linearly independent solutions:

$$\mathbf{x}_1(t) = \begin{bmatrix} 5(2 \sin t - \cos t) \\ -5(2 \cos t + \sin t) \\ -2(\cos t + 2 \sin t) \\ 5 \cos t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} -5(2 \cos t + \sin t) \\ 5(\cos t - 2 \sin t) \\ 2(\cos t - \sin t) \\ 5 \sin t \end{bmatrix}.$$

Eigenvectors for $\lambda = 2$: We have $A - 2I = \begin{bmatrix} -2 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. A short calculation shows that we obtain

only one linearly independent eigenvector: $\mathbf{v} = r(0, 0, 1, 0)$, where $r \in \mathbb{C}$. Therefore, another real-valued

solution to this system of differential equation is given by

$$\mathbf{x}_3(t) = e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

To find a fourth real-valued linearly independent solution, we seek a solution of the form $\mathbf{x}_4(t) = e^{2t}(\mathbf{v}_1 + t\mathbf{v}_0)$, where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A - 2I)^2 \mathbf{v}_1 = \mathbf{0}, \quad (A - 2I)\mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A - 2I)\mathbf{v}_1.$$

In this case, we have $A - 2I = \begin{bmatrix} -2 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ and $(A - 2I)^2 = \begin{bmatrix} 3 & 4 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix}$. We may choose

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \text{ Then } \mathbf{v}_0 = (A - 2I)\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \text{ Therefore, we have a solution}$$

$$\mathbf{x}_4(t) = e^{2t} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) = e^{2t} \begin{bmatrix} 0 \\ 0 \\ t \\ 1 \end{bmatrix}.$$

Therefore, the general solution to this system of differential equations is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 5(2 \sin t - \cos t) \\ -5(2 \cos t + \sin t) \\ -2(\cos t + 2 \sin t) \\ 5 \cos t \end{bmatrix} + c_2 \begin{bmatrix} -5(2 \cos t + \sin t) \\ 5(\cos t - 2 \sin t) \\ 2(\cos t - \sin t) \\ 5 \sin t \end{bmatrix} + e^{2t} \left\{ c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ t \\ 1 \end{bmatrix} \right\}.$$

12. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} -2 - \lambda & 3 & 0 & 0 \\ 3 & -2 - \lambda & 0 & 0 \\ 1 & 0 & 1 - \lambda & -1 \\ 0 & 1 & 0 & 1 - \lambda \end{vmatrix} = 0$ if and only if $(1 - \lambda)^2(\lambda^2 + 4\lambda - 5) = 0$. This means that the eigenvalues of A are $\lambda = -5$ and $\lambda = 1$ (with multiplicity 3).

Eigenvectors for $\lambda = -5$: We have $A + 5I = \begin{bmatrix} 3 & 3 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 1 & 0 & 6 & -1 \\ 0 & 1 & 0 & 6 \end{bmatrix}$. A short calculation yields corresponding eigenvectors of the form $\mathbf{v} = r(36, -36, -5, 6)$, where $r \in \mathbb{R}$. Hence, a corresponding solution to this system of differential equations is

$$\mathbf{x}_1(t) = e^{-5t} \begin{bmatrix} 36 \\ -36 \\ -5 \\ 6 \end{bmatrix}.$$

Eigenvectors for $\lambda = 1$: We have $A - I = \begin{bmatrix} -3 & 3 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. The only corresponding eigenvector here takes the form $\mathbf{v} = r(0, 0, 1, 0)$, where $r \in \mathbb{R}$. Hence, a corresponding solution to this system of differential equations is

$$\mathbf{x}_2(t) = e^t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

According to Theorem 7.5.4, there exist two further linearly independent solutions to $\mathbf{x}' = A\mathbf{x}$ of the form

$$\mathbf{x}_3(t) = e^t(\mathbf{v}_1 + t\mathbf{v}_0),$$

$$\mathbf{x}_4(t) = e^t(\mathbf{v}_2 + t\mathbf{v}_1 + \frac{1}{2!}t^2\mathbf{v}_0),$$

where

$$(A - I)^3\mathbf{v}_2 = \mathbf{0}, \quad (A - I)^2\mathbf{v}_2 \neq \mathbf{0},$$

and

$$\mathbf{v}_1 = (A - I)\mathbf{v}_2, \quad \mathbf{v}_0 = (A - I)^2\mathbf{v}_2.$$

A short calculation shows that

$$A - I = \begin{bmatrix} -3 & 3 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad (A - I)^2 = \begin{bmatrix} 18 & -18 & 0 & 0 \\ -18 & 18 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 3 & -3 & 0 & 0 \end{bmatrix}, \quad (A - I)^3 = \begin{bmatrix} -108 & 108 & 0 & 0 \\ 108 & -108 & 0 & 0 \\ 15 & -15 & 0 & 0 \\ -18 & 18 & 0 & 0 \end{bmatrix}.$$

We can take $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Then $\mathbf{v}_1 = (A - I)\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_0 = (A - I)^2\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$. Thus, we obtain the additional solutions

$$\mathbf{x}_3(t) = e^t \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right) = e^t \begin{bmatrix} 0 \\ 0 \\ 1 - t \\ 1 \end{bmatrix}$$

and

$$\mathbf{x}_4(t) = e^t \left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2!}t^2 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right) = e^t \begin{bmatrix} 1 \\ 1 \\ t - t^2/2 \\ t \end{bmatrix}.$$

Hence, the general solution to this system of differential equations is

$$\mathbf{x}(t) = c_1 e^{-5t} \begin{bmatrix} 36 \\ -36 \\ -5 \\ 6 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^t \begin{bmatrix} 0 \\ 0 \\ 1 - t \\ 1 \end{bmatrix} + c_4 e^t \begin{bmatrix} 1 \\ 1 \\ t - t^2/2 \\ t \end{bmatrix}.$$

13. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} -\lambda & -1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 1 & 0 & -\lambda & -1 \\ 0 & 1 & 1 & -\lambda \end{vmatrix} = 0$ if and only if $(\lambda^2 + 1)^2 = 0$. This means that the eigenvalues of A are $\lambda = \pm i$ (with multiplicity 2).

Eigenvectors for $\lambda = i$: We have $A - iI = \begin{bmatrix} -i & -1 & 0 & 0 \\ 1 & -i & 0 & 0 \\ 1 & 0 & -i & -1 \\ 0 & 1 & 1 & -i \end{bmatrix}$, which yields corresponding eigenvectors of the form $\mathbf{v} = r(0, 0, i, 1)$, where $r \in \mathbb{C}$. Therefore, we obtain a solution to the linear system $\mathbf{x}' = A\mathbf{x}$ as follows:

$$\begin{aligned} \mathbf{u}(t) &= (\cos t + i \sin t) \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix} \\ &= (\cos t + i \sin t) \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 \\ 0 \\ -\sin t \\ \cos t \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ \cos t \\ \sin t \end{bmatrix}. \end{aligned}$$

This gives rise to the two real-valued solutions

$$\mathbf{x}_1(t) = \begin{bmatrix} 0 \\ 0 \\ -\sin t \\ \cos t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} 0 \\ 0 \\ \cos t \\ \sin t \end{bmatrix}.$$

We seek to find further real-valued and linearly independent solutions to $\mathbf{x}' = A\mathbf{x}$ of the form $\mathbf{x}(t) = e^{it}(\mathbf{v}_1 + t\mathbf{v}_0)$, where

$$(A - iI)^2 \mathbf{v}_1 = \mathbf{0}, \quad (A - iI) \mathbf{v}_1 \neq \mathbf{0}, \quad (A - iI) \mathbf{v}_1 = \mathbf{v}_0.$$

One possible choice for $\mathbf{v}_1$ is $\mathbf{v}_1 = (i, 1, 1, -i)$. Then $\mathbf{v}_0 = (A - iI)\mathbf{v}_1 = (0, 0, i, 1)$. Therefore, we obtain the solution

$$\begin{aligned} \mathbf{u}(t) &= (\cos t + i \sin t) \left(\begin{bmatrix} i \\ 1 \\ 1 \\ -i \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix} \right) \\ &= \left(\cos t \begin{bmatrix} 0 \\ 1 \\ 1 \\ t \end{bmatrix} - \sin t \begin{bmatrix} 1 \\ 0 \\ t \\ -1 \end{bmatrix} \right) + i \left(\cos t \begin{bmatrix} 1 \\ 0 \\ t \\ -1 \end{bmatrix} + \sin t \begin{bmatrix} 0 \\ 1 \\ 1 \\ t \end{bmatrix} \right). \end{aligned}$$

This gives rise to two more real-valued solutions:

$$\mathbf{x}_3(t) = \begin{bmatrix} -\sin t \\ \cos t \\ \cos t - t \sin t \\ \sin t + t \cos t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_4(t) = \begin{bmatrix} \cos t \\ \sin t \\ \sin t + t \cos t \\ t \sin t - \cos t \end{bmatrix}.$$

Therefore, the general solution to this linear system of differential equations is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 0 \\ 0 \\ -\sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ \cos t \\ \sin t \end{bmatrix} + c_3 \begin{bmatrix} -\sin t \\ \cos t \\ \cos t - t \sin t \\ \sin t + t \cos t \end{bmatrix} + c_4 \begin{bmatrix} \cos t \\ \sin t \\ \sin t + t \cos t \\ t \sin t - \cos t \end{bmatrix}.$$

14. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} -2 - \lambda & -1 \\ 1 & -4 - \lambda \end{vmatrix} = 0$ if and only if $\lambda^2 + 6\lambda + 9 = 0$ if and only if $(\lambda + 3)^2 = 0$. This means that the only eigenvalue of A is $\lambda = -3$ (with multiplicity 2).

Eigenvectors for $\lambda = -3$: We have $A + 3I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(1, 1)$, where $r \in \mathbb{R}$. Therefore, we obtain one solution to the linear system of differential equations as $\mathbf{x}_1(t) = e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We next seek a second linearly independent solution to this linear system of the form $\mathbf{x}(t) = e^{-3t}(\mathbf{v}_1 + t\mathbf{v}_0)$, where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A + 3I)^2 \mathbf{v}_1 = \mathbf{0}, \quad (A + 3I) \mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A + 3I) \mathbf{v}_1.$$

In this case, we have $A + 3I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and $(A + 3I)^2 = 0_{\mathbb{R}^2}$. Hence, we may choose $\mathbf{v}_1$ to be any vector such that $(A + 3I) \mathbf{v}_1 \neq \mathbf{0}$. Let us choose, say, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $\mathbf{v}_0 = (A + 3I) \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore, our second solution to this system of differential equations is

$$\mathbf{x}_2(t) = e^{-3t} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right).$$

Hence, the general solution to this system is

$$\mathbf{x}(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right).$$

We have been given that $\mathbf{x}(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, so that

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 \end{bmatrix}.$$

Therefore, $c_1 + c_2 = 0$ and $c_1 = -1$. Therefore, $c_2 = 1$. Hence, the solution to this initial-value problem is

$$\mathbf{x}(t) = -e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = e^{-3t} \begin{bmatrix} t \\ t - 1 \end{bmatrix}.$$

15. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} -2-\lambda & -1 & 4 \\ 0 & -1-\lambda & 0 \\ -1 & -3 & 2-\lambda \end{vmatrix} = 0$ if and only if $\lambda^2(-1-\lambda) = 0$. This means that the eigenvalues of A are $\lambda = -1$ and $\lambda = 0$ (with multiplicity 2).

Eigenvectors for $\lambda = -1$: We have $A + I = \begin{bmatrix} -1 & -1 & 4 \\ 0 & 0 & 0 \\ -1 & -3 & 3 \end{bmatrix}$, which row reduces to $\begin{bmatrix} 1 & 1 & -4 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. This gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(9, -1, 2)$, where $r \in \mathbb{R}$. Taking $r = 1$, we obtain the following solution to the linear system: $\mathbf{x}_1(t) = e^{-t} \begin{bmatrix} 9 \\ -1 \\ 2 \end{bmatrix}$.

Eigenvectors for $\lambda = 0$: We have $A = \begin{bmatrix} -2 & -1 & 4 \\ 0 & -1 & 0 \\ -1 & -3 & 2 \end{bmatrix}$, which row reduces to $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. This gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(2, 0, 1)$, where $r \in \mathbb{R}$. Taking $r = 1$, we obtain the following solution to the linear system: $\mathbf{x}_2(t) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. To find a second linearly independent solution corresponding to $\lambda = 0$, we seek a solution of the form $\mathbf{x}_3(t) = \mathbf{v}_1 + t\mathbf{v}_0$, where

$$A^2\mathbf{v}_1 = \mathbf{0}, \quad A\mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = A\mathbf{v}_1.$$

In this case, we have $A^2 = \begin{bmatrix} 0 & -9 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$. We can choose $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then $\mathbf{v}_0 = A\mathbf{v}_1 = \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}$. Hence, we have another solution $\mathbf{x}_3(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix}$. Therefore, the general solution to this linear system is

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} 9 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} \right).$$

We have been given that $\mathbf{x}(0) = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, so that

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 9 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We quickly solve this system to find $c_1 = -1$, $c_2 = 3$, and $c_3 = 1$. Therefore, the solution to this initial-value problem is

$$\mathbf{x}(t) = -e^{-t} \begin{bmatrix} 9 \\ -1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 7 - 2t - 9e^{-t} \\ e^{-t} \\ 3 - t - 2e^{-t} \end{bmatrix}.$$

16. Since $\mathbf{x}' = A\mathbf{x}$, we have

$$\lambda e^{\lambda t} \left[\mathbf{v}_2 + t\mathbf{v}_1 + \frac{t^2}{2!}\mathbf{v}_0 \right] + e^{\lambda t} [\mathbf{v}_1 + t\mathbf{v}_0] = A e^{\lambda t} \left[\mathbf{v}_2 + t\mathbf{v}_1 + \frac{t^2}{2!}\mathbf{v}_0 \right],$$

or

$$(\lambda \mathbf{v}_2 + \mathbf{v}_1) + t(\lambda \mathbf{v}_1 + \mathbf{v}_0) + \frac{t^2}{2!}\lambda \mathbf{v}_0 = A\mathbf{v}_2 + t(A\mathbf{v}_1) + \frac{t^2}{2!}(A\mathbf{v}_0).$$

In order for this equality to hold for all t , the coefficients of t^2 , t , and 1 on both sides must agree. That is,

$$\lambda \mathbf{v}_2 + \mathbf{v}_1 = A\mathbf{v}_2,$$

$$\lambda \mathbf{v}_1 + \mathbf{v}_0 = A\mathbf{v}_1,$$

and

$$\mathbf{v}_0 = A\mathbf{v}_0.$$

Rearranging these equations, we have

$$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1,$$

$$(A - \lambda I)\mathbf{v}_1 = \mathbf{v}_0,$$

and

$$(A - \lambda I)\mathbf{v}_0 = \mathbf{0},$$

as we were asked to show.

17. All solutions to the system are linear combinations of solutions of the form $\mathbf{x}(t) = e^{\lambda t}(\mathbf{v}_1 + t\mathbf{v}_0)$, where λ is an eigenvalue of A and $\mathbf{v}_0$ may be the zero vector. Writing $\lambda = a + ib$, we have $\mathbf{x}(t) = e^{at}(\cos bt + i \sin bt)(\mathbf{v}_1 + t\mathbf{v}_0)$. Now for $k = 0$ or $k = 1$, we have $\lim_{t \rightarrow \infty} t^k e^{at} = 0$ if and only if $a < 0$. Therefore, it follows that

$$\lim_{t \rightarrow \infty} e^{at}(\cos bt + i \sin bt)(\mathbf{v}_1 + t\mathbf{v}_0) = \mathbf{0} \text{ if and only if } a < 0,$$

and so all solutions to the system satisfy

$$\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{0}$$

if and only if all eigenvalues of A have negative real part.

18. From Theorem 7.5.4, we deduce that the system has n linearly independent solutions of the form

$$\mathbf{x}(t) = e^{\lambda t} \left(\mathbf{v}_i + t\mathbf{v}_{i-1} + \cdots + \frac{1}{i!}t^i\mathbf{v}_0 \right),$$

where $\lambda = a + ib$ is an eigenvalue of A . For any positive integer k , we have

$$\lim_{t \rightarrow \infty} t^k e^{at} = 0 \quad \text{if and only if } a < 0.$$

Therefore, each of the solutions $\mathbf{x}(t)$ above tends to zero if and only if $a < 0$. Therefore, all solutions to the linear system $\mathbf{x}' = A\mathbf{x}$ satisfy

$$\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{0}$$

if and only if $a < 0$; that is, if and only if all eigenvalues of A have negative real part.

19.

(a) The number of cycles of generalized eigenvectors of A corresponding to λ is precisely the number of linearly independent eigenvectors of A corresponding to λ . Since our assumption in the Theorem is that the eigenspace corresponding to λ is r -dimensional, we conclude that we have r cycles of generalized eigenvectors of A corresponding to λ .

(b) The length l_i of a cycle of generalized eigenvectors is given as $k_i + 1$ in the formulation of Theorem 7.5.4. By the fact cited on p. 434 of the text, since the multiplicity of λ is m , the total number of linearly independent generalized eigenvectors corresponding to λ is m . However, the number of such generalized eigenvectors found in part (a) is

$$l_1 + l_2 + \cdots + l_r = (k_1 + 1) + (k_2 + 1) + \cdots + (k_r + 1) = (k_1 + k_2 + \cdots + k_r) + r.$$

Therefore,

$$k_1 + k_2 + \cdots + k_r = m - r.$$

(c) A cycle of generalized eigenvectors corresponding to λ of length $l_i = k_i + 1$ must take the form $\{(A - \lambda I)^{k_i} \mathbf{v}, (A - \lambda I)^{k_i-1} \mathbf{v}, \dots, (A - \lambda I) \mathbf{v}, \mathbf{v}\}$, where $\mathbf{v}$ is a vector chosen such that $(A - \lambda I)^{k_i+1} \mathbf{v} = \mathbf{0}$ and $(A - \lambda I)^{k_i} \mathbf{v} \neq \mathbf{0}$. This vector $\mathbf{v}$ is precisely the vector $\mathbf{v}_{k_i}$ referred to in the statement of Theorem 7.5.4. The statements (7.5.16)-(7.5.19) describe the remaining vectors comprising the cycle.

(d) Assume that

$$c_0 \mathbf{x}_0^{(i)}(t) + c_1 \mathbf{x}_1^{(i)}(t) + \cdots + c_{k_i} \mathbf{x}_{k_i}^{(i)}(t) = \mathbf{0}.$$

We must show that $c_0 = c_1 = \cdots = c_{k_i} = 0$. Substituting the expressions for $\mathbf{x}_j^{(i)}(t)$ from (7.5.11)-(7.5.14), we obtain

$$c_0 e^{\lambda t} \mathbf{v}_0^{(i)} + c_1 e^{\lambda t} (\mathbf{v}_1^{(i)} + t \mathbf{v}_0^{(i)}) + \cdots + c_{k_i} e^{\lambda t} \left(\mathbf{v}_{k_i}^{(i)} + t \mathbf{v}_{k_i-1}^{(i)} + \cdots + \frac{1}{k_i!} t^{k_i} \mathbf{v}_0^{(i)} \right) = \mathbf{0}.$$

Since this must hold for all t , we can choose $t = 0$ and reduce this equation to

$$c_0 \mathbf{v}_0^{(i)} + c_1 \mathbf{v}_1^{(i)} + \cdots + c_{k_i} \mathbf{v}_{k_i}^{(i)} = \mathbf{0}.$$

Since $\{\mathbf{v}_0^{(i)}, \mathbf{v}_1^{(i)}, \dots, \mathbf{v}_{k_i}^{(i)}\}$ is linearly independent, we conclude that $c_0 = c_1 = \cdots = c_{k_i} = 0$, as desired.

(e) We have

$$\begin{aligned}
[\mathbf{x}_j^{(i)}(t)]' &= \lambda e^{\lambda t} \left(\mathbf{v}_j^{(i)} + t\mathbf{v}_{j-1}^{(i)} + \cdots + \frac{1}{j!} t^j \mathbf{v}_0^{(i)} \right) + e^{\lambda t} \left(\mathbf{v}_{j-1}^{(i)} + t\mathbf{v}_{j-2}^{(i)} + \cdots + \frac{1}{(j-1)!} t^{j-1} \mathbf{v}_0^{(i)} \right) \\
&= e^{\lambda t} \left[\lambda \mathbf{v}_j^{(i)} + (\lambda t + 1) \mathbf{v}_{j-1}^{(i)} + \left(\lambda \frac{t^2}{2} + t \right) \mathbf{v}_{j-2}^{(i)} + \cdots + \left(\lambda \frac{t^j}{j!} + \frac{t^{j-1}}{(j-1)!} \right) \mathbf{v}_0^{(i)} \right] \\
&= e^{\lambda t} \left[(\mathbf{v}_{j-1}^{(i)} + \lambda \mathbf{v}_j^{(i)}) + t(\mathbf{v}_{j-2}^{(i)} + \lambda \mathbf{v}_{j-1}^{(i)}) + \frac{t^2}{2!} (\mathbf{v}_{j-3}^{(i)} + \lambda \mathbf{v}_{j-2}^{(i)}) + \cdots + \lambda \frac{t^j}{j!} \mathbf{v}_0^{(i)} \right] \\
&= e^{\lambda t} \left[((A - \lambda I) \mathbf{v}_j^{(i)} + \lambda \mathbf{v}_j^{(i)}) + t((A - \lambda I) \mathbf{v}_{j-1}^{(i)} + \lambda \mathbf{v}_{j-1}^{(i)}) + \frac{t^2}{2!} ((A - \lambda I) \mathbf{v}_{j-2}^{(i)} + \lambda \mathbf{v}_{j-2}^{(i)}) + \cdots \right. \\
&\quad \left. \cdots + \frac{t^j}{j!} ((A - \lambda I) \mathbf{v}_0^{(i)} + \lambda \mathbf{v}_0^{(i)}) \right] \\
&= e^{\lambda t} \left[(A \mathbf{v}_j^{(i)}) + t(A \mathbf{v}_{j-1}^{(i)}) + \frac{t^2}{2!} (A \mathbf{v}_{j-2}^{(i)}) + \cdots + \frac{t^j}{j!} (A \mathbf{v}_0^{(i)}) \right] \\
&= A e^{\lambda t} \left[\mathbf{v}_j^{(i)} + t\mathbf{v}_{j-1}^{(i)} + \cdots + \frac{t^j}{j!} \mathbf{v}_0^{(i)} \right] \\
&= A \mathbf{x}_j^{(i)}(t),
\end{aligned}$$

as desired.

(f) Apply part (1) to each eigenvalue of A . For each eigenvalue, we obtain linearly independent solutions totalling the multiplicity of the eigenvalue. The sum of the multiplicities of all of the eigenvalues is n , thereby yielding n solutions to $\mathbf{x}' = A\mathbf{x}$. The solutions $\mathbf{x}_j^{(i)}(t)$ comprising a given cycle of generalized eigenvectors are linearly independent by part (d), and vectors corresponding to different cycles and different eigenvalues are linearly independent from one another. Therefore, we have n linearly independent solutions to $\mathbf{x}' = A\mathbf{x}$.

Solutions to Section 7.6

True-False Review:

1. FALSE. The particular solution, given by Equation (7.6.4), depends on a fundamental matrix $X(t)$ for the nonhomogeneous linear system, and the columns of any fundamental matrix consist of linearly independent solutions to the corresponding homogeneous vector differential equation $\mathbf{x}' = A\mathbf{x}$.

2. FALSE. The vector function $\mathbf{u}(t)$ is not arbitrary; it must satisfy the equation $X(t)\mathbf{u}'(t) = \mathbf{b}(t)$, according to Theorem 7.6.1.

3. TRUE. By definition, we have $X = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n]$, so

$$X' = [\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_n] = [A\mathbf{x}_1, A\mathbf{x}_2, \dots, A\mathbf{x}_n] = AX.$$

4. FALSE. If we assume that $\mathbf{x}_p$ is a solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$, then note that

$$(c \cdot \mathbf{x}_p)' = c \cdot \mathbf{x}'_p = c \cdot (A\mathbf{x}_p + \mathbf{b}),$$

whereas

$$A(c \cdot \mathbf{x}_p) + \mathbf{b} = c \cdot (A\mathbf{x}_p) + \mathbf{b}.$$

The right-hand sides of the above expressions are not equal. Therefore, $c \cdot \mathbf{x}_p$ is not a solution to $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ in general.

5. FALSE. A formula for the particular solution is given in Theorem 7.6.1:

$$\mathbf{x}_p(t) = X(t) \int^t X^{-1}(s) \mathbf{b}(s) ds.$$

Since an arbitrary integration constant can be applied to this expression, we have an arbitrary number of different choices for the particular solution $\mathbf{x}_p(t)$.

6. TRUE. From $X\mathbf{u}' = \mathbf{b}$, we use the invertibility of X to obtain $\mathbf{u}' = X^{-1}\mathbf{b}$. Therefore, $\mathbf{u}$ is obtained by integrating $X^{-1}\mathbf{b}$:

$$\mathbf{u}(t) = \int^t X^{-1}(s) \mathbf{b}(s) ds.$$

Problems:

1. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$ if and only if $\lambda^2 - 4\lambda + 3 = 0$ if and only if $(\lambda - 3)(\lambda - 1) = 0$. This means that the eigenvalues of A are $\lambda = 1$ and $\lambda = 3$.

Eigenvectors for $\lambda = 1$: We have $A - I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(1, 1)$, where $r \in \mathbb{R}$. Hence, one solution to the associated homogeneous system is $\mathbf{x}_1(t) = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Eigenvectors for $\lambda = 3$: We have $A - 3I = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(-1, 1)$, where $r \in \mathbb{R}$. Hence, a second solution to the associated homogeneous system is $\mathbf{x}_2(t) = e^{3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Therefore, the general solution to the associated homogeneous system is

$$\mathbf{x}_c(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and the fundamental matrix is

$$X(t) = \begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix}.$$

We have

$$X^{-1}(t) = \frac{1}{2} \begin{bmatrix} e^{-t} & e^{-t} \\ -e^{-3t} & e^{-3t} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \mathbf{u}(t) &= \int^t X^{-1}(s) \mathbf{b}(s) ds \\ &= \frac{1}{2} \int^t \begin{bmatrix} e^{-s} & e^{-s} \\ -e^{-3s} & e^{-3s} \end{bmatrix} \begin{bmatrix} 0 \\ 4e^s \end{bmatrix} ds = \frac{1}{2} \int^t \begin{bmatrix} 4 \\ 4e^{-2s} \end{bmatrix} ds \\ &= \begin{bmatrix} 2t \\ -e^{-2t} \end{bmatrix}. \end{aligned}$$

Hence, we obtain the particular solution

$$\mathbf{x}_p(t) = X(t)\mathbf{u}(t) = \begin{bmatrix} e^t & -e^{3t} \\ e^t & e^{3t} \end{bmatrix} \begin{bmatrix} 2t \\ -e^{-2t} \end{bmatrix} = \begin{bmatrix} e^t(2t+1) \\ e^t(2t-1) \end{bmatrix}.$$

Thus, the general solution to the nonhomogeneous system of differential equations is

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} e^t(2t+1) \\ e^t(2t-1) \end{bmatrix}.$$

2. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} 4 - \lambda & -3 \\ 2 & -1 - \lambda \end{vmatrix} = 0$ if and only if $\lambda^2 - 3\lambda + 2 = 0$ if and only if $(\lambda - 2)(\lambda - 1) = 0$. This means that the eigenvalues of A are $\lambda = 1$ and $\lambda = 2$.

Eigenvectors for $\lambda = 1$: We have $A - I = \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(1, 1)$, where $r \in \mathbb{R}$. Hence, one solution to the associated homogeneous system is $\mathbf{x}_1(t) = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Eigenvectors for $\lambda = 2$: We have $A - 2I = \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(3, 2)$, where $r \in \mathbb{R}$. Hence, a second solution to the associated homogeneous system is $\mathbf{x}_2(t) = e^{2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Therefore, the general solution to the associated homogeneous system is

$$\mathbf{x}_c(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

and the fundamental matrix is

$$X(t) = \begin{bmatrix} e^t & 3e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}.$$

We have

$$X^{-1}(t) = \begin{bmatrix} -2e^{-t} & 3e^{-t} \\ e^{-2t} & -e^{-2t} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \mathbf{u}(t) &= \int^t X^{-1}(s)\mathbf{b}(s)ds \\ &= \int^t \begin{bmatrix} -2e^{-t} & 3e^{-t} \\ e^{-2t} & -e^{-2t} \end{bmatrix} \begin{bmatrix} e^{2s} \\ e^s \end{bmatrix} ds = \int^t \begin{bmatrix} 3 - 2e^s \\ 1 - e^{-s} \end{bmatrix} ds \\ &= \begin{bmatrix} 3t - 2e^t \\ t + e^{-t} \end{bmatrix}. \end{aligned}$$

Hence, we obtain the particular solution

$$\mathbf{x}_p(t) = X(t)\mathbf{u}(t) = \begin{bmatrix} e^t & 3e^{2t} \\ e^t & 2e^{2t} \end{bmatrix} \begin{bmatrix} 3t - 2e^t \\ t + e^{-t} \end{bmatrix} = \begin{bmatrix} 3te^t - 2e^{2t} + 3te^{2t} + 3e^t \\ 3te^t - 2e^{2t} + 2te^{2t} + 2e^t \end{bmatrix}.$$

Thus, the general solution to the nonhomogeneous system of differential equations is

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 3te^t - 2e^{2t} + 3te^{2t} + 3e^t \\ 3te^t - 2e^{2t} + 2te^{2t} + 2e^t \end{bmatrix}.$$

3. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} -1 - \lambda & 1 \\ 3 & 1 - \lambda \end{vmatrix} = 0$ if and only if $\lambda^2 - 4 = 0$. This means that the eigenvalues of A are $\lambda = -2$ and $\lambda = 2$.

Eigenvectors for $\lambda = -2$: We have $A + 2I = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(-1, 1)$, where $r \in \mathbb{R}$. Hence, one solution to the associated homogeneous system is $\mathbf{x}_1(t) = e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Eigenvectors for $\lambda = 2$: We have $A - 2I = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(1, 3)$, where $r \in \mathbb{R}$. Hence, a second solution to the associated homogeneous system is $\mathbf{x}_2(t) = e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Therefore, the general solution to the associated homogeneous system is

$$\mathbf{x}_c(t) = c_1 e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

and the fundamental matrix is

$$X(t) = \begin{bmatrix} -e^{-2t} & e^{2t} \\ e^{-2t} & 3e^{2t} \end{bmatrix}.$$

We have

$$X^{-1}(t) = -\frac{1}{4} \begin{bmatrix} 3e^{2t} & -e^{2t} \\ -e^{-2t} & -e^{-2t} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \mathbf{u}(t) &= \int^t X^{-1}(s) \mathbf{b}(s) ds \\ &= -\frac{1}{4} \int^t \begin{bmatrix} 3e^{2s} & -e^{2s} \\ -e^{-2s} & -e^{-2s} \end{bmatrix} \begin{bmatrix} 20e^{3s} \\ 12e^s \end{bmatrix} ds \\ &= -\frac{1}{4} \int^t \begin{bmatrix} 60e^{5s} - 12e^{3s} \\ -20e^s - 12e^{-s} \end{bmatrix} ds = \int^t \begin{bmatrix} -15e^{5s} + 3e^{3s} \\ 5e^s + 3e^{-s} \end{bmatrix} ds \\ &= \begin{bmatrix} -3e^{5t} + e^{3t} \\ 5e^t - 3e^{-t} \end{bmatrix}. \end{aligned}$$

Hence, we obtain the particular solution

$$\mathbf{x}_p(t) = X(t) \mathbf{u}(t) = \begin{bmatrix} -e^{-2t} & e^{2t} \\ e^{-2t} & 3e^{2t} \end{bmatrix} \begin{bmatrix} -3e^{5t} + e^{3t} \\ 5e^t - 3e^{-t} \end{bmatrix} = \begin{bmatrix} 8e^{3t} - 4e^t \\ 12e^{3t} - 8e^t \end{bmatrix}.$$

Thus, the general solution to the nonhomogeneous system of differential equations is

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t) = c_1 e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 8e^{3t} - 4e^t \\ 12e^{3t} - 8e^t \end{bmatrix}.$$

4. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} -1 - \lambda & 2 \\ -2 & 4 - \lambda \end{vmatrix} = 0$ if and only if $\lambda^2 - 3\lambda = 0$. This means that the eigenvalues of A are $\lambda = 0$ and $\lambda = 3$.

Eigenvectors for $\lambda = 0$: We have $A = \begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(2, 1)$, where $r \in \mathbb{R}$. Hence, one solution to the associated homogeneous system is $\mathbf{x}_1(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Eigenvectors for $\lambda = 3$: We have $A - 3I = \begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(1, 2)$, where $r \in \mathbb{R}$. Hence, a second solution to the associated homogeneous system is $\mathbf{x}_2(t) = e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Therefore, the general solution to the associated homogeneous system is

$$\mathbf{x}_c(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and the fundamental matrix is

$$X(t) = \begin{bmatrix} 2 & e^{3t} \\ 1 & 2e^{3t} \end{bmatrix}.$$

We have

$$X^{-1}(t) = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -e^{-3t} & 2e^{-3t} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \mathbf{u}(t) &= \int_0^t X^{-1}(s) \mathbf{b}(s) ds \\ &= \frac{1}{3} \int_0^t \begin{bmatrix} 2 & -1 \\ -e^{-3s} & 2e^{-3s} \end{bmatrix} \begin{bmatrix} 54se^{3s} \\ 9e^{3s} \end{bmatrix} ds = \int_0^t \begin{bmatrix} 36se^{3s} - 3e^{3s} \\ -18s + 6 \end{bmatrix} ds \\ &= \begin{bmatrix} e^{3t}(12t - 5) \\ 3t(2 - 3t) \end{bmatrix}. \end{aligned}$$

Hence, we obtain the particular solution

$$\mathbf{x}_p(t) = X(t) \mathbf{u}(t) = \begin{bmatrix} 2 & e^{3t} \\ 1 & 2e^{3t} \end{bmatrix} \begin{bmatrix} e^{3t}(12t - 5) \\ 3t(2 - 3t) \end{bmatrix} = -e^{3t} \begin{bmatrix} 9t^2 - 30t + 10 \\ 18t^2 - 24t + 5 \end{bmatrix}.$$

Thus, the general solution to the nonhomogeneous system of differential equations is

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - e^{3t} \begin{bmatrix} 9t^2 - 30t + 10 \\ 18t^2 - 24t + 5 \end{bmatrix}.$$

5. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} 2 - \lambda & 4 \\ -2 & -2 - \lambda \end{vmatrix} = 0$ if and only if $\lambda^2 + 4 = 0$. This means that the eigenvalues of A are $\lambda = \pm 2i$.

Eigenvectors for $\lambda = -2i$: We have $A + 2iI = \begin{bmatrix} 2 + 2i & 4 \\ -2 & -2 + 2i \end{bmatrix}$, which gives rise to a complex-valued eigenvector of the form $\mathbf{v} = r(-1 + i, 1)$, where $r \in \mathbb{C}$. Therefore, we get the following complex-valued solution to the system:

$$\mathbf{x}(t) = e^{-2it} \begin{bmatrix} -1 + i \\ 1 \end{bmatrix} = (\cos 2t - i \sin 2t) \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).$$

By computing the real and imaginary parts of this expression, we obtain two real-valued solutions to the associated homogeneous system of differential equations:

$$\mathbf{x}_1(t) = \cos 2t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \sin 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\cos 2t + \sin 2t \\ \cos 2t \end{bmatrix}$$

and

$$\mathbf{x}_2(t) = \cos 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin 2t \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos 2t + \sin 2t \\ -\sin 2t \end{bmatrix}.$$

Therefore, the general solution to the associated homogeneous system is

$$\mathbf{x}_c(t) = c_1 \begin{bmatrix} -\cos 2t + \sin 2t \\ \cos 2t \end{bmatrix} + c_2 \begin{bmatrix} \cos 2t + \sin 2t \\ -\sin 2t \end{bmatrix}$$

and the fundamental matrix is

$$X(t) = \begin{bmatrix} -\cos 2t + \sin 2t & \cos 2t + \sin 2t \\ \cos 2t & -\sin 2t \end{bmatrix}.$$

We have

$$X^{-1}(t) = \begin{bmatrix} \sin 2t & \cos 2t + \sin 2t \\ \cos 2t & \cos 2t - \sin 2t \end{bmatrix}.$$

Therefore

$$\begin{aligned} \mathbf{u}(t) &= \int_0^t X^{-1}(s)\mathbf{b}(s)ds \\ &= \int_0^t \begin{bmatrix} \sin 2s & \cos 2s + \sin 2s \\ \cos 2s & \cos 2s - \sin 2s \end{bmatrix} \begin{bmatrix} 8 \sin 2s \\ 8 \cos 2s \end{bmatrix} ds = \int_0^t \begin{bmatrix} 8 + 8 \sin 2s \cos 2s \\ 8 \cos^2 2s \end{bmatrix} ds \\ &= \begin{bmatrix} 8t + 2 \sin^2 2t \\ 4t + \sin 4t \end{bmatrix}. \end{aligned}$$

Hence, we obtain the particular solution

$$\begin{aligned} \mathbf{x}_p(t) &= X(t)\mathbf{u}(t) = \begin{bmatrix} -\cos 2t + \sin 2t & \cos 2t + \sin 2t \\ \cos 2t & -\sin 2t \end{bmatrix} \begin{bmatrix} 8t + 2 \sin^2 2t \\ 4t + \sin 4t \end{bmatrix} \\ &= \begin{bmatrix} 12t \sin 2t - 4t \cos 2t + 2 \sin 2t \\ 8t \cos 2t - 4t \sin 2t \end{bmatrix}. \end{aligned}$$

Thus, the general solution to the nonhomogeneous system of differential equations is

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t) = c_1 \begin{bmatrix} -\cos 2t + \sin 2t \\ \cos 2t \end{bmatrix} + c_2 \begin{bmatrix} \cos 2t + \sin 2t \\ -\sin 2t \end{bmatrix} + \begin{bmatrix} 12t \sin 2t - 4t \cos 2t + 2 \sin 2t \\ 8t \cos 2t - 4t \sin 2t \end{bmatrix}.$$

Note that we can also express

$$\mathbf{x}_p(t) = \begin{bmatrix} 12t \sin 2t - 4t \cos 2t + 2 \sin 2t \\ 8t \cos 2t - 4t \sin 2t \end{bmatrix} = \begin{bmatrix} (12t + 1) \sin 2t + (1 - 4t) \cos 2t \\ (8t - 1) \cos 2t - 4t \sin 2t \end{bmatrix} + \begin{bmatrix} \sin 2t - \cos 2t \\ \cos 2t \end{bmatrix},$$

and since the latter term is a solution to the associated homogeneous system, we can choose $\mathbf{x}_p(t) = \begin{bmatrix} (12t + 1) \sin 2t + (1 - 4t) \cos 2t \\ (8t - 1) \cos 2t - 4t \sin 2t \end{bmatrix}$ as an alternative particular solution instead of the one used above.

6. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} 3 - \lambda & 2 \\ -2 & -1 - \lambda \end{vmatrix} = 0$ if and only if $\lambda^2 - 2\lambda + 1 = 0$ if and only if $(\lambda - 1)^2 = 0$. Therefore, A has one eigenvalue, $\lambda = 1$ (with multiplicity 2).

Eigenvectors for $\lambda = 1$: We have $A - I = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(-1, 1)$, where $r \in \mathbb{R}$. Therefore, one solution of the associated homogeneous system is $\mathbf{x}_1(t) = e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. We will determine a second linearly independent solution of the form

$$\mathbf{x}_2(t) = e^t(\mathbf{v}_1 + t\mathbf{v}_0)$$

where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A - I)^2\mathbf{v}_1 = \mathbf{0}, \quad (A - I)\mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A - I)\mathbf{v}_1.$$

In this case, we have $A - I = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$ and $(A - I)^2 = 0_2$. Therefore, we may choose $\mathbf{v}_1$ to be any vector such that $(A - I)\mathbf{v}_1 \neq \mathbf{0}$. For simplicity, we take $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus, $\mathbf{v}_0 = (A - I)\mathbf{v}_1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$. From the expressions for $\mathbf{v}_0$ and $\mathbf{v}_1$, we can write down a second linearly independent solution to the vector differential equation:

$$\mathbf{x}_2(t) = e^t \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -2 \end{bmatrix} \right\} = e^t \begin{bmatrix} 1 + 2t \\ -2t \end{bmatrix}.$$

Consequently, the general solution to the associated homogeneous system is

$$\mathbf{x}_c(t) = c_1 e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 + 2t \\ -2t \end{bmatrix}$$

and a fundamental matrix is

$$X(t) = \begin{bmatrix} -e^t & e^t(1 + 2t) \\ e^t & -2te^t \end{bmatrix}.$$

We have

$$X^{-1}(t) = \begin{bmatrix} 2te^{-t} & e^{-t}(1 + 2t) \\ e^{-t} & e^{-t} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \mathbf{u}(t) &= \int^t X^{-1}(s)\mathbf{b}(s)ds \\ &= \int^t \begin{bmatrix} 2se^{-s} & e^{-s}(1 + 2s) \\ e^{-s} & e^{-s} \end{bmatrix} \begin{bmatrix} -3e^s \\ 6se^s \end{bmatrix} ds \\ &= \int^t \begin{bmatrix} 12s^2 \\ -3 + 6s \end{bmatrix} ds = \begin{bmatrix} 4t^3 \\ -3t + 3t^2 \end{bmatrix}. \end{aligned}$$

Hence, we obtain the particular solution

$$\mathbf{x}_p(t) = X(t)\mathbf{u}(t) = \begin{bmatrix} -e^t & e^t(1 + 2t) \\ e^t & -2te^t \end{bmatrix} \begin{bmatrix} 4t^3 \\ -3t + 3t^2 \end{bmatrix} = te^t \begin{bmatrix} 2t^2 - 3t - 3 \\ 6t - 2t^2 \end{bmatrix}.$$

Therefore, the general solution to the nonhomogeneous system of differential equations is

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t) = c_1 e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 + 2t \\ -2t \end{bmatrix} + te^t \begin{bmatrix} 2t^2 - 3t - 3 \\ 6t - 2t^2 \end{bmatrix}.$$

7. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} 1 - \lambda & 0 & 0 \\ 2 & -3 - \lambda & 2 \\ 1 & -2 & 2 - \lambda \end{vmatrix} = 0$ if and only if $(1 - \lambda)(\lambda^2 + \lambda - 2) = 0$. Therefore, the eigenvalues of A are $\lambda = -2$ and $\lambda = 1$ (with multiplicity 2).

Eigenvectors for $\lambda = -2$: We have $A + 2I = \begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 2 \\ 1 & -2 & 4 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(0, 2, 1)$, where $r \in \mathbb{R}$. Therefore, one solution of the associated homogeneous system is $\mathbf{x}_1(t) = e^{-2t} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

Eigenvectors for $\lambda = 1$: We have $A - I = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -4 & 2 \\ 1 & -2 & 1 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(2, 1, 0) + s(-1, 0, 1)$, where $r, s \in \mathbb{R}$. Therefore, we obtain two more linearly independent solutions to the corresponding homogeneous system: $\mathbf{x}_2(t) = e^t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}_3(t) = e^t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. Consequently, the general solution to the associated homogeneous system is

$$\mathbf{x}_c(t) = c_1 e^{-2t} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3 e^t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

and a fundamental matrix is

$$X(t) = \begin{bmatrix} 0 & 2e^t & -e^t \\ 2e^{-2t} & e^t & 0 \\ e^{-2t} & 0 & e^t \end{bmatrix}.$$

We have

$$X^{-1}(t) = -\frac{1}{3} \begin{bmatrix} e^{2t} & -2e^{2t} & e^{2t} \\ -2e^{-t} & e^{-t} & -2e^{-t} \\ -e^{-t} & 2e^{-t} & -4e^{-t} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \mathbf{u}(t) &= \int_0^t X^{-1}(s) \mathbf{b}(s) ds \\ &= -\frac{1}{3} \int_0^t \begin{bmatrix} e^{2s} & -2e^{2s} & e^{2s} \\ -2e^{-s} & e^{-s} & -2e^{-s} \\ -e^{-s} & 2e^{-s} & -4e^{-s} \end{bmatrix} \begin{bmatrix} -e^s \\ 6e^{-s} \\ e^s \end{bmatrix} ds \\ &= \int_0^t \begin{bmatrix} 4e^s \\ -2e^{-2s} \\ 1 - 4e^{-2s} \end{bmatrix} ds \\ &= \begin{bmatrix} 4e^t \\ e^{-2t} \\ t + 2e^{-2t} \end{bmatrix}. \end{aligned}$$

Hence, we obtain the particular solution

$$\mathbf{x}_p(t) = X(t)\mathbf{u}(t) = \begin{bmatrix} 0 & 2e^t & -e^t \\ 2e^{-2t} & e^t & 0 \\ e^{-2t} & 0 & e^t \end{bmatrix} \begin{bmatrix} 4e^t \\ e^{-2t} \\ t + 2e^{-2t} \end{bmatrix} = \begin{bmatrix} -te^t \\ 9e^{-t} \\ te^t + 6e^{-t} \end{bmatrix}.$$

Therefore, the general solution to the nonhomogeneous system of differential equations is

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t) = c_1 e^{-2t} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_3 e^t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -te^t \\ 9e^{-t} \\ te^t + 6e^{-t} \end{bmatrix}.$$

8. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} -1 - \lambda & -2 & 2 \\ 2 & 4 - \lambda & -1 \\ 0 & 0 & 3 - \lambda \end{vmatrix} = 0$ if and only if $(3 - \lambda)(\lambda^2 - 3\lambda) = 0$.

Therefore, the eigenvalues of A are $\lambda = 0$ and $\lambda = 3$ (with multiplicity 2).

Eigenvectors for $\lambda = 0$: Vectors of the form $\mathbf{v} = r(-2, 1, 0)$, where $r \in \mathbb{R}$, form a basis for the nullspace of

A , and therefore one solution of the associated homogeneous system is $\mathbf{x}_1(t) = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

Eigenvectors for $\lambda = 3$: We have $A - 3I = \begin{bmatrix} -4 & -2 & 2 \\ 2 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(1, 0, 2) + s(0, 1, 1)$, where $r, s \in \mathbb{R}$. Therefore, we obtain two more linearly independent solutions to the corresponding homogeneous system: $\mathbf{x}_2(t) = e^{3t} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\mathbf{x}_3(t) = e^{3t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Consequently, the general solution to the associated homogeneous system is

$$\mathbf{x}_c(t) = c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and a fundamental matrix is

$$X(t) = \begin{bmatrix} -2 & e^{3t} & 0 \\ 1 & 0 & e^{3t} \\ 0 & 2e^{3t} & e^{3t} \end{bmatrix}.$$

We have

$$X^{-1}(t) = -\frac{1}{3}e^{-3t} \begin{bmatrix} 2e^{3t} & e^{3t} & -e^{3t} \\ 1 & 2 & -2 \\ -2 & -4 & 1 \end{bmatrix}.$$

Therefore,

$$\begin{aligned}
 \mathbf{u}(t) &= \int^t X^{-1}(s)\mathbf{b}(s)ds \\
 &= -\frac{1}{3} \int^t e^{-3s} \begin{bmatrix} 2e^{3s} & e^{3s} & -e^{3s} \\ 1 & 2 & -2 \\ -2 & -4 & 1 \end{bmatrix} \begin{bmatrix} -e^{3s} \\ 4e^{3s} \\ 3e^{3s} \end{bmatrix} ds \\
 &= -\frac{1}{3} \int^t \begin{bmatrix} -e^{3s} \\ 1 \\ -11 \end{bmatrix} ds \\
 &= -\frac{1}{3} \begin{bmatrix} -\frac{1}{3}e^{3t} \\ t \\ -11t \end{bmatrix}.
 \end{aligned}$$

Hence, we obtain the particular solution

$$\mathbf{x}_p(t) = X(t)\mathbf{u}(t) = -\frac{1}{3} \begin{bmatrix} -2 & e^{3t} & 0 \\ 1 & 0 & e^{3t} \\ 0 & 2e^{3t} & e^{3t} \end{bmatrix} \begin{bmatrix} -\frac{1}{3}e^{3t} \\ t \\ -11t \end{bmatrix} = -\frac{1}{3}e^{3t} \begin{bmatrix} \frac{2}{3} + t \\ -\frac{1}{3} - 11t \\ -9t \end{bmatrix} = e^{3t} \begin{bmatrix} -(3t+2)/9 \\ (33t+1)/9 \\ 3t \end{bmatrix}.$$

Thus, the general solution to the non-homogeneous system is given by

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t) = c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + e^{3t} \begin{bmatrix} -(3t+2)/9 \\ (33t+1)/9 \\ 3t \end{bmatrix}.$$

9. The solution to $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{b}(t)$ can be written

$$\mathbf{x}(t) = X(t)\mathbf{c} + X(t) \int^t X^{-1}(s)\mathbf{b}(s)ds,$$

where $\mathbf{c} = (c_1, c_2, \dots, c_n)$ are arbitrary constants and $X(t)$ is a fundamental matrix for the system. We can rewrite the equation above as

$$\mathbf{x}(t) = X(t)\mathbf{c} + X(t) \int_{t_0}^t X^{-1}(s)\mathbf{b}(s)ds.$$

When $t = t_0$, this reduces to $\mathbf{x}(t_0) = X(t_0)\mathbf{c}$, which means we can write $\mathbf{c} = X^{-1}(t_0)\mathbf{x}(t_0)$. Substituting $\mathbf{x}(t_0) = \mathbf{x}_0$, we have $\mathbf{c} = X^{-1}(t_0)\mathbf{x}_0$. Therefore,

$$\mathbf{x}(t) = X(t)X^{-1}(t_0)\mathbf{x}_0 + X(t) \int_{t_0}^t X^{-1}(s)\mathbf{b}(s)ds.$$

10. Let $A = \begin{bmatrix} 2 & -3 \\ -4 & -2 \end{bmatrix}$ and $\mathbf{b}(t) = \begin{bmatrix} 34 \sin t \\ 17 \cos t \end{bmatrix}$. We may then write the nonhomogeneous system as $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Now,

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2 - \lambda & -3 \\ -4 & -2 - \lambda \end{vmatrix} = 0 \iff \lambda^2 - 16 = 0,$$

from which it follows that the eigenvalues of A are $\lambda = \pm 4$.

Eigenvectors for $\lambda = 4$: We have $A - 4I = \begin{bmatrix} -2 & -3 \\ -4 & -6 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(3, -2)$, where $r \in \mathbb{R}$. Hence, one solution of the associated homogeneous system is $\mathbf{x}_1(t) = e^{4t} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Eigenvectors for $\lambda = -4$: We have $A + 4I = \begin{bmatrix} 6 & -3 \\ -4 & 2 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = s(1, 2)$, where $s \in \mathbb{R}$. Hence, a second linearly independent solution of the associated homogeneous system is $\mathbf{x}_2(t) = e^{-4t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Putting the above results together, we obtain the general solution to the associated homogeneous system:

$$\mathbf{x}_c(t) = c_1 e^{4t} \begin{bmatrix} 3 \\ -2 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

To find a particular solution, we follow the hint and try a solution of the form $\mathbf{x}_p(t) = \begin{bmatrix} A_1 \cos t + B_1 \sin t \\ A_2 \cos t + B_2 \sin t \end{bmatrix}$.

Substituting this expression into both sides of $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$, we find

$$\begin{aligned} \begin{bmatrix} -A_1 \sin t + B_1 \cos t \\ -A_2 \sin t + B_2 \cos t \end{bmatrix} &= \begin{bmatrix} 2 & -3 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} A_1 \cos t + B_1 \sin t \\ A_2 \cos t + B_2 \sin t \end{bmatrix} + \begin{bmatrix} 34 \sin t \\ 17 \cos t \end{bmatrix} \\ &= \begin{bmatrix} (2B_1 - 3B_2) \sin t + (2A_1 - 3A_2) \cos t + 34 \sin t \\ (-4B_1 - 2B_2) \sin t + (-4A_1 - 2A_2) \cos t + 17 \cos t \end{bmatrix}. \end{aligned}$$

Equating like terms simplifying the system, we can quickly determine the solution: $A_1 = 1$, $A_2 = 2$, $B_1 = -4$, and $B_2 = 9$. Consequently, we have $\mathbf{x}_p(t) = \begin{bmatrix} \cos t - 4 \sin t \\ 2 \cos t + 9 \sin t \end{bmatrix}$. The general solution to the system is therefore given by

$$\mathbf{x}(t) = c_1 e^{4t} \begin{bmatrix} 3 \\ -2 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} \cos t - 4 \sin t \\ 2 \cos t + 9 \sin t \end{bmatrix}.$$

Solutions to Section 7.7

True-False Review:

1. FALSE. In order to solve a coupled spring-mass system with two masses and two springs as a first-order linear system, a 4×4 linear system is required. More precisely, if the masses are m_1 and m_2 , and the spring constants are k_1 and k_2 , then the first-order linear system for the spring-mass system is $\mathbf{x}' = A\mathbf{x}$ where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{m_1}(k_1 + k_2) & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0 \end{bmatrix}.$$

2. TRUE. Hooke's Law states that the force due to a spring on a mass is proportional to the displacement of the mass from equilibrium, and oriented in the direction opposite to the displacement: $F = -kx$. The

units of force are Newtons, and the units of displacement are meters. Therefore, in solving for the spring constant k , we have $k = -\frac{F}{x}$, with units of Newtons on top of the fraction and units of meters on the bottom of the fraction.

3. FALSE. The *amount* of chemical in a tank of solution is measured in grams in the metric system, but the concentration is measured in grams/L, since concentration is computed as the amount of substance per unit of volume.

4. FALSE. The correct relation is that $r_{in} + r_{12} - r_{21} = 0$, where r_{12} and r_{21} are the rates of flow from tank 1 to tank 2, and from tank 2 to tank 1, respectively. Example 7.7.2 shows quite clearly that r_{12} and r_{21} need not be the same.

5. FALSE. Although this is a closed system, chemicals still pass between the two tanks. So, for example, if tank 1 contained 100 grams of chemical and tank 2 contained no chemical, then as time passes and fluid flows between the two tanks, some of the chemical will move to tank 2. The amount of chemical in each tank changes over time.

Problems:

1. From Example 7.7.1, we have $\det(A - \lambda I) = 0$ if and only if $\lambda^4 + 5\lambda^2 + 4 = (\lambda^2 + 4)(\lambda^2 + 1) = 0$. Thus, we have $\lambda = \pm i, \pm 2i$.

Eigenvectors for $\lambda = i$: Here we have

$$A - iI = \begin{bmatrix} -i & 1 & 0 & 0 \\ -3 & -i & 1 & 0 \\ 0 & 0 & -i & 1 \\ 2 & 0 & -2 & -i \end{bmatrix},$$

and the reduced row-echelon form of the linear system $(A - iI)\mathbf{v}_1 = \mathbf{0}$ is $\left[\begin{array}{cccc|c} 1 & 0 & 0 & i/2 & 0 \\ 0 & 1 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$. Consequently, $\mathbf{v}_1 = r(-i, 1, -2i, 2)$, where $r \in \mathbb{C}$. It follows that a complex-valued solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{u}_1(t) = e^{it} \begin{bmatrix} -i \\ 1 \\ -2i \\ 2 \end{bmatrix} = (\cos t + i \sin t) \begin{bmatrix} -i \\ 1 \\ -2i \\ 2 \end{bmatrix}.$$

Taking the real and imaginary parts of this complex-valued solution yields two real-valued solutions,

$$\mathbf{x}_1(t) = \begin{bmatrix} \cos t \\ -\sin t \\ 2 \sin t \\ 2 \cos t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} \sin t \\ \cos t \\ -2 \cos t \\ 2 \sin t \end{bmatrix}.$$

Eigenvectors for $\lambda = 2i$: Here we have

$$A - 2iI = \begin{bmatrix} -2i & 1 & 0 & 0 \\ -3 & -2i & 1 & 0 \\ 0 & 0 & -2i & 1 \\ 2 & 0 & -2 & -2i \end{bmatrix},$$

and the reduced row-echelon form of the linear system $(A - 2iI)\mathbf{v}_2 = \mathbf{0}$ is $\left[\begin{array}{cccc|c} -2i & 1 & 0 & 0 & 0 \\ -3 & -2i & 1 & 0 & 0 \\ 0 & 0 & -2i & 1 & 0 \\ 2 & 0 & -2 & -2i & 0 \end{array} \right]$. Consequently, $\mathbf{v}_2 = s(i, -2, -i, 2)$, where $s \in \mathbb{C}$. It follows that a complex-valued solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{u}_2(t) = e^{2it} \begin{bmatrix} i \\ -2 \\ -i \\ 2 \end{bmatrix} = (\cos t + i \sin t) \begin{bmatrix} i \\ -2 \\ -i \\ 2 \end{bmatrix}.$$

Taking the real and imaginary parts of this complex-valued solution yields two more real-valued solutions,

$$\mathbf{x}_3(t) = \begin{bmatrix} -\sin 2t \\ -2 \cos 2t \\ \sin t \\ 2 \cos 2t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_4(t) = \begin{bmatrix} \cos 2t \\ \sin 2t \\ -\cos 2t \\ 2 \sin 2t \end{bmatrix}.$$

2. The motion of the system is governed by

$$\begin{cases} \frac{1}{2}x'' = -3x + \frac{1}{2}(y - x), \\ \frac{1}{12}y'' = -\frac{1}{2}(y - x). \end{cases}$$

Letting $x_1 = x$, $x_2 = x'_1$, $x_3 = y$, and $x_4 = x'_3$, we obtain the equivalent system

$$\begin{cases} x'_1 = x_2, \\ x'_2 = -7x_1 + x_3, \\ x'_3 = x_4, \\ x'_4 = 6x_1 - 6x_3. \end{cases}$$

Writing this problem as $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -7 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 6 & 0 & -6 & 0 \end{bmatrix}.$$

We find that

$$\det(A - \lambda I) = 0 \iff \lambda^4 + 13\lambda^2 + 36 = 0 \iff (\lambda^2 + 9)(\lambda^2 + 4) = 0 \iff \lambda = \pm 2i, \pm 3i.$$

Eigenvectors for $\lambda = 2i$: Here we have

$$A - 2iI = \begin{bmatrix} -2i & 1 & 0 & 0 \\ -7 & -2i & 1 & 0 \\ 0 & 0 & -2i & 1 \\ 6 & 0 & -6 & -2i \end{bmatrix},$$

and the reduced row-echelon form of the linear system $(A - 2iI)\mathbf{v}_1 = \mathbf{0}$ is $\left[\begin{array}{cccc|c} -2i & 1 & 0 & 0 & 0 \\ -7 & -2i & 1 & 0 & 0 \\ 0 & 0 & -2i & 1 & 0 \\ 6 & 0 & -6 & -2i & 0 \end{array} \right]$. Consequently, $\mathbf{v}_1 = r(-i, 2, -3i, 6)$, where $r \in \mathbb{C}$. It follows that a complex-valued solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{u}_1(t) = e^{2it} \begin{bmatrix} -i \\ 2 \\ -3i \\ 6 \end{bmatrix} = (\cos 2t + i \sin 2t) \begin{bmatrix} -i \\ 2 \\ -3i \\ 6 \end{bmatrix}.$$

Taking real and imaginary parts of this complex-valued solution yields two real-valued solutions,

$$\mathbf{x}_1(t) = \begin{bmatrix} \sin 2t \\ 2 \cos 2t \\ 3 \sin 2t \\ 6 \cos 2t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} -\cos 2t \\ 2 \sin 2t \\ -3 \cos 2t \\ 6 \sin 2t \end{bmatrix}.$$

Eigenvectors for $\lambda = 3i$: Here we have

$$A - 3iI = \begin{bmatrix} -3i & 1 & 0 & 0 \\ -7 & -3i & 1 & 0 \\ 0 & 0 & -3i & 1 \\ 6 & 0 & -6 & -3i \end{bmatrix},$$

and the reduced row-echelon form of the linear system $(A - 3iI)\mathbf{v}_2 = \mathbf{0}$ is $\left[\begin{array}{cccc|c} -3i & 1 & 0 & 0 & 0 \\ -7 & -3i & 1 & 0 & 0 \\ 0 & 0 & -3i & 1 & 0 \\ 6 & 0 & -6 & -3i & 0 \end{array} \right]$. Consequently, $\mathbf{v}_2 = s(i, -3, -2i, 6)$, where $s \in \mathbb{C}$. It follows that a complex-valued solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{u}_2(t) = e^{3it} \begin{bmatrix} i \\ -3 \\ -2i \\ 6 \end{bmatrix} = (\cos 3t + i \sin 3t) \begin{bmatrix} i \\ -3 \\ -2i \\ 6 \end{bmatrix}.$$

Taking real and imaginary parts of this complex-valued solution yields two real-valued solutions,

$$\mathbf{x}_3(t) = \begin{bmatrix} -\sin 3t \\ -3 \cos 3t \\ 2 \sin 3t \\ 6 \cos 3t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_4(t) = \begin{bmatrix} \cos 3t \\ -3 \sin 3t \\ -2 \cos 3t \\ 6 \sin 3t \end{bmatrix}.$$

Putting together the four solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\mathbf{x}_3(t)$, and $\mathbf{x}_4(t)$ obtained above, we find the general solution to the linear system:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} \sin 2t \\ 2 \cos 2t \\ 3 \sin 2t \\ 6 \cos 2t \end{bmatrix} + c_2 \begin{bmatrix} -\cos 2t \\ 2 \sin 2t \\ -3 \cos 2t \\ 6 \sin 2t \end{bmatrix} + c_3 \begin{bmatrix} -\sin 3t \\ -3 \cos 3t \\ 2 \sin 3t \\ 6 \cos 3t \end{bmatrix} + c_4 \begin{bmatrix} \cos 3t \\ -3 \sin 3t \\ -2 \cos 3t \\ 6 \sin 3t \end{bmatrix}.$$

Since $x = x_1$ and $y = x_3$, it follows that the motion of the spring-mass system is given by

$$x(t) = c_1 \sin 2t - c_2 \cos 2t - c_3 \sin 3t + c_4 \cos 3t$$

and

$$y(t) = 3c_1 \sin 2t - 3c_2 \cos 2t + 2c_3 \sin 3t - 2c_4 \cos 3t.$$

Given that

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{x}(0) = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 2 & 0 & -3 & 0 \\ 0 & -3 & 0 & -2 \\ 6 & 0 & 6 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix},$$

we find that

$$c_1 = \frac{3}{10}, \quad c_2 = 0, \quad c_3 = -\frac{2}{15}, \quad c_4 = 0.$$

Substituting these results into the expressions for $x(t)$ and $y(t)$ above, we conclude that the motion of the spring-mass system is

$$x(t) = \frac{3}{10} \sin 2t + \frac{2}{15} \sin 3t \quad \text{and} \quad y(t) = \frac{9}{10} \sin 2t - \frac{4}{15} \sin 3t.$$

3. The motion of the system is governed by the

$$\begin{cases} x'' &= -3x + 4(y - x), \\ \frac{4}{3}y'' &= -4(y - x). \end{cases}$$

Letting $x_1 = x$, $x_2 = x'_1$, $x_3 = y$, and $x_4 = y'$ yields the equivalent system

$$\begin{cases} x'_1 &= x_2, \\ x'_2 &= -7x_1 + 4x_3, \\ x'_3 &= x_4, \\ x'_4 &= 3x_1 - 3x_3. \end{cases}$$

Writing this problem as $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -7 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & -3 & 0 \end{bmatrix}$, we find that

$$\det(A - \lambda I) = 0 \text{ if and only if } \lambda^4 + 10\lambda^2 + 9 = 0 \text{ if and only if } (\lambda^2 + 1)(\lambda^2 + 9) = 0.$$

Therefore, we obtain the eigenvalues $\lambda = \pm i, \pm 3i$.

Eigenvectors for $\lambda = i$: Here we have

$$A - iI = \begin{bmatrix} -i & 1 & 0 & 0 \\ -7 & -i & 4 & 0 \\ 0 & 0 & -i & 1 \\ 3 & 0 & -3 & -i \end{bmatrix},$$

and the reduced row-echelon form of the linear system $(A - iI)\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{cccc|c} 1 & -i & 0 & 0 & 0 \\ 0 & 1 & -\frac{2i}{3} & 0 & 0 \\ 0 & 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(-2i, 2, -3i, 3)$, where $r \in \mathbb{C}$. The corresponding complex-valued solution to the system $\mathbf{x}' = A\mathbf{x}$ is

$$\begin{aligned} \mathbf{u}(t) &= e^{it}\mathbf{v} \\ &= (\cos t + i \sin t) \begin{bmatrix} -2i \\ 2 \\ -3i \\ 3 \end{bmatrix} \\ &= (\cos t + i \sin t) \left(\begin{bmatrix} 0 \\ 2 \\ 0 \\ 3 \end{bmatrix} + i \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \end{bmatrix} \right) \\ &= \left(\cos t \begin{bmatrix} 0 \\ 2 \\ 0 \\ 3 \end{bmatrix} + \sin t \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right) + i \left(\cos t \begin{bmatrix} -2 \\ 0 \\ -3 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} 0 \\ 2 \\ 0 \\ 3 \end{bmatrix} \right) \\ &= \begin{bmatrix} 2 \sin t \\ 2 \cos t \\ 3 \sin t \\ 3 \cos t \end{bmatrix} + i \begin{bmatrix} -2 \cos t \\ 2 \sin t \\ -3 \cos t \\ 3 \sin t \end{bmatrix}, \end{aligned}$$

which gives rise to the two real-valued solutions

$$\mathbf{x}_1(t) = \begin{bmatrix} 2 \sin t \\ 2 \cos t \\ 3 \sin t \\ 3 \cos t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} -2 \cos t \\ 2 \sin t \\ -3 \cos t \\ 3 \sin t \end{bmatrix}.$$

Eigenvectors for $\lambda = 3i$: Here we have

$$A - 3iI = \begin{bmatrix} -3i & 1 & 0 & 0 \\ -7 & -3i & 4 & 0 \\ 0 & 0 & -3i & 1 \\ 3 & 0 & -3 & -3i \end{bmatrix},$$

and the reduced row-echelon form of the linear system $(A - 3iI)\mathbf{x} = \mathbf{0}$ is $\left[\begin{array}{cccc|c} 1 & \frac{i}{3} & 0 & 0 & 0 \\ 0 & 1 & -3i & 3 & 0 \\ 0 & 0 & 1 & \frac{i}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(2i, -6, -i, 3)$, where $r \in \mathbb{C}$. The corresponding

complex-valued solution to the system $\mathbf{x}' = A\mathbf{x}$ is

$$\begin{aligned}
 \mathbf{u}(t) &= e^{3it} \mathbf{v} \\
 &= (\cos 3t + i \sin 3t) \begin{bmatrix} 2i \\ -6 \\ -i \\ 3 \end{bmatrix} \\
 &= (\cos 3t + i \sin 3t) \left(\begin{bmatrix} 0 \\ -6 \\ 0 \\ 3 \end{bmatrix} + i \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right) \\
 &= \left(\cos 3t \begin{bmatrix} 0 \\ -6 \\ 0 \\ 3 \end{bmatrix} - \sin 3t \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right) + i \left(\cos 3t \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \sin 3t \begin{bmatrix} 0 \\ -6 \\ 0 \\ 3 \end{bmatrix} \right) \\
 &= \begin{bmatrix} -2 \sin 3t \\ -6 \cos 3t \\ \sin 3t \\ 3 \cos 3t \end{bmatrix} + i \begin{bmatrix} 2 \cos 3t \\ -6 \sin 3t \\ -\cos 3t \\ 3 \sin 3t \end{bmatrix},
 \end{aligned}$$

which gives rise to the two real-valued solutions

$$\mathbf{x}_3(t) = \begin{bmatrix} -2 \sin 3t \\ -6 \cos 3t \\ \sin 3t \\ 3 \cos 3t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_4(t) = \begin{bmatrix} 2 \cos 3t \\ -6 \sin 3t \\ -\cos 3t \\ 3 \sin 3t \end{bmatrix}.$$

Consequently, the general solution to the linear system is

$$\begin{aligned}
 \mathbf{x}(t) &= c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) + c_3 \mathbf{x}_3(t) + c_4 \mathbf{x}_4(t) \\
 &= c_1 \begin{bmatrix} 2 \sin t \\ 2 \cos t \\ 3 \sin t \\ 3 \cos t \end{bmatrix} + c_2 \begin{bmatrix} -2 \cos t \\ 2 \sin t \\ -3 \cos t \\ 3 \sin t \end{bmatrix} + c_3 \begin{bmatrix} -2 \sin 3t \\ -6 \cos 3t \\ \sin 3t \\ 3 \cos 3t \end{bmatrix} + c_4 \begin{bmatrix} 2 \cos 3t \\ -6 \sin 3t \\ -\cos 3t \\ 3 \sin 3t \end{bmatrix}.
 \end{aligned}$$

Since $x = x_1$ and $y = x_3$, it follows that the motion of the spring-mass system is given by

$$x(t) = 2(c_1 \sin t - c_2 \cos t - c_3 \sin 3t + c_4 \cos 3t) \quad \text{and} \quad y(t) = 3c_1 \sin t - 3c_2 \cos t + c_3 \sin 3t - c_4 \cos 3t.$$

4. The system corresponding to the given information is

$$\begin{cases} 2m_2 \frac{d^2 x}{dt^2} = -2k_2 x + k_2(y - x), \\ m_2 \frac{d^2 y}{dt^2} = -k_2(y - x), \end{cases} \quad \text{or} \quad \begin{cases} m_2 \frac{d^2 x}{dt^2} = \frac{1}{2}(-3k_2 x + k_2 y), \\ m_2 \frac{d^2 y}{dt^2} = -k_2(y - x). \end{cases}$$

Letting $x_1 = x$, $x_2 = x'_1$, $x_3 = y$, and $x_4 = x'_3$ yields the equivalent system:

$$\begin{cases} x'_1 = x_2, \\ x'_2 = \frac{k_2}{2m_2}(-3x_1 + x_3), \\ x'_3 = x_4, \\ x'_4 = \frac{k_2}{m_2}(x_1 - x_3). \end{cases} \quad \text{Writing this problem as } \mathbf{x}' = A\mathbf{x} \text{ where } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{ and } A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{3k_2}{2m_2} & 0 & \frac{k_2}{2m_2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & 0 \end{bmatrix}$$

with $\omega^2 = \frac{k_2}{2m_2}$, we find that $\det(A - \lambda I) = 0 \iff \lambda^4 + 5\lambda^2\omega^2 + 4\omega^4 = 0 \iff \lambda \in \{-2i\omega, -i\omega, i\omega, 2i\omega\}$.

When $\lambda = i\omega$, we find a corresponding eigenvector $\mathbf{v}_1 = (-i/\omega, 1, -2i/\omega, 2)$. A complex-valued solution is given by

$$\mathbf{u}_1 = e^{i\omega t} \begin{bmatrix} -i/\omega \\ 1 \\ -2i/\omega \\ 2 \end{bmatrix} = (\cos \omega t + i \sin \omega t) \begin{bmatrix} -i/\omega \\ 1 \\ -2i/\omega \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega} \sin \omega t \\ \cos \omega t \\ 2 \\ \frac{2}{\omega} \sin \omega t \end{bmatrix} + i \begin{bmatrix} -\frac{1}{\omega} \cos \omega t \\ \sin \omega t \\ -2 \\ \frac{2}{\omega} \cos \omega t \end{bmatrix},$$

so that two linearly independent solutions are given by

$$\mathbf{x}_1(t) = \begin{bmatrix} \frac{1}{\omega} \sin \omega t \\ \cos \omega t \\ 2 \\ \frac{2}{\omega} \sin \omega t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} -\frac{1}{\omega} \cos \omega t \\ \sin \omega t \\ -2 \\ \frac{2}{\omega} \cos \omega t \end{bmatrix}.$$

When $\lambda = 2i\omega$, we find a corresponding eigenvector $\mathbf{v}_2 = (i/2\omega, -1, -i/2\omega, 1)$. A complex-valued solution is given by

$$\mathbf{u}_2 = e^{2i\omega t} \begin{bmatrix} i/2\omega \\ -1 \\ -i/2\omega \\ 1 \end{bmatrix} = (\cos 2\omega t + i \sin 2\omega t) \begin{bmatrix} i/2\omega \\ -1 \\ -i/2\omega \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2\omega} \sin 2\omega t \\ -\cos 2\omega t \\ \frac{1}{2\omega} \sin 2\omega t \\ \cos 2\omega t \end{bmatrix} + i \begin{bmatrix} \frac{1}{2\omega} \cos 2\omega t \\ -\sin 2\omega t \\ -\frac{1}{2\omega} \cos 2\omega t \\ \sin 2\omega t \end{bmatrix},$$

so that two more linearly independent solutions are given by

$$\mathbf{x}_3(t) = \begin{bmatrix} -\frac{1}{2\omega} \sin 2\omega t \\ -\cos 2\omega t \\ \frac{1}{2\omega} \sin 2\omega t \\ \cos 2\omega t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_4(t) = \begin{bmatrix} \frac{1}{2\omega} \cos 2\omega t \\ -\sin 2\omega t \\ -\frac{1}{2\omega} \cos 2\omega t \\ \sin 2\omega t \end{bmatrix}.$$

The general solution is given by $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 + c_4\mathbf{x}_4$.

5. We have

$$\begin{aligned}
 \det(A - \lambda I) = 0 &\iff \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ -\frac{k_1 + k_2}{m_1} & -\lambda & \frac{k_2}{m_1} & 0 \\ 0 & 0 & -\lambda & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2}{m_2} & -\lambda \end{vmatrix} = 0 \\
 &\iff -\lambda[-\lambda(\lambda^2 + \frac{k_2}{m_2})] - [-\frac{k_1 + k_2}{m_1}(\lambda^2 + \frac{k_2}{m_2}) + \frac{k_2}{m_2} \frac{k_2}{m_1}] = 0 \\
 &\iff \lambda^4 + \frac{k_2}{m_2}\lambda^2 + \frac{k_1 + k_2}{m_1}\lambda^2 - (-\frac{(k_1 + k_2)k_2 + k_2^2}{m_1 m_2}) = 0 \\
 &\iff \lambda^4 + [\frac{k_2}{m_2} + \frac{k_1 + k_2}{m_1}]\lambda^2 + \frac{k_1 k_2}{m_1 m_2} = 0 \\
 &\iff m_1 m_2 \lambda^4 + [m_1 k_1 + m_2(k_1 + k_2)]\lambda^2 + k_1 k_2 = 0. \tag{5.1}
 \end{aligned}$$

The discriminant of Equation (5.1) is given by

$$[m_1 k_1 + m_2(k_1 + k_2)]^2 - 4m_1 m_2 k_1 k_2 = (m_1 k_1 - m_2 k_2)^2 + m_2^2 k_1^2 + 2m_1 m_2 k_1^2 + 2m_2 k_1 k_2,$$

which is strictly positive. Thus, Equation (5.1) has two distinct real roots for λ^2 . Since all coefficients are positive, both roots must be negative, so letting $\omega_1 = \sqrt{-r_1}$ and $\omega_2 = \sqrt{-r_2}$, we obtain $(\lambda^2 + \omega_1^2)(\lambda^2 + \omega_2^2) = 0$ which implies that $\lambda = \pm i\omega_1$, $\lambda = \pm i\omega_2$.

6. The motion of the system is governed by:
$$\begin{cases} m_1 \frac{d^2 x}{dt^2} = k_1 x + k_2(x - y), \\ m_2 \frac{d^2 y}{dt^2} = k_2(y - x) + k_3 y. \end{cases}$$
 To transform this system to

a first-order system, let $x_1 = x$, $x_2 = x'_1$, $x_3 = y$, and $x_4 = x'_3 \implies \begin{cases} m_1 x'_2 = k_1 x_1 + k_2(x_1 - x_3), \\ m_2 x'_4 = k_2(x_3 - x_1) + k_3 x_3. \end{cases} \implies$

$$\begin{cases} x'_1 = x_2, \\ x'_2 = \frac{k_1 + k_2}{m_1} x_1 - \frac{k_2}{m_1} x_3, \\ x'_3 = x_4, \\ x'_4 = -\frac{k_2}{m_2} x_1 + \frac{k_2 + k_3}{m_2} x_3. \end{cases}$$

7. The assumption that V_1 and V_2 are constant implies that the net rate of change is zero. Thus,

$$\begin{cases} r_{\text{in}} + r_{12} - r_{21} = 0, \\ r_{21} - r_{12} - r_{\text{out}} = 0, \end{cases} \quad \text{or} \quad \begin{cases} r_{\text{in}} + r_{12} = r_{21}, \\ r_{12} + r_{\text{out}} = r_{21}. \end{cases}$$

8. Starting with the system derived in the example,
$$\begin{cases} A'_1 = -\frac{1}{5}A_1 + \frac{1}{20}A_2 + 12, \\ A'_2 = \frac{1}{5}A_1 - \frac{1}{5}A_2, \end{cases}$$
 we can rewrite the

system as

$$\begin{cases} (D + \frac{1}{5})A_1 - \frac{1}{20}A_2 = 12, \\ -\frac{1}{5}A_1 + (D + \frac{1}{5})A_2 = 0. \end{cases} \quad (8.1)$$

Therefore, $(D + \frac{1}{5})^2 A_1 - \frac{1}{20}(D + \frac{1}{5})A_2 = \frac{12}{5}$. From the last two equations we have $(D + \frac{1}{5})^2 A_1 - \frac{1}{100}A_1 = \frac{12}{5} \Rightarrow (D^2 + \frac{2}{5}D + \frac{3}{100})A_1 = \frac{12}{5} \Rightarrow (D + \frac{3}{10})(D + \frac{1}{10})A_1 = \frac{12}{5}$. The complementary function is $A_c(t) = c_1 e^{-t/10} + c_2 e^{-3t/10}$ and an appropriate trial solution is $A_p(t) = A_0$ where A_0 is a constant. Substituting A_p into the preceding differential equation and simplifying, we find that $A_0 = 80$. Consequently,

$$A_1(t) = c_1 e^{-t/10} + c_2 e^{-3t/10} + 80,$$

and the first equation of the system (8.1) yields $A_2 = 20[(D + \frac{1}{5})A_1 - 12]$. Substituting the expression for $A_1(t)$, we find that

$$A_2(t) = 20(-\frac{c_1}{10}e^{-t/10} - \frac{3c_2}{10}e^{-3t/10} + \frac{c_1}{5}e^{-t/10} + \frac{c_2}{5}e^{-3t/10} + 4).$$

The general solution to the system is given by $\begin{cases} A_1(t) = c_1 e^{-t/10} + c_2 e^{-3t/10} + 80, \\ A_2(t) = 2c_1 e^{-t/10} - 2c_2 e^{-3t/10} + 80. \end{cases}$ Now, $A_1(0) = 40$ and $A_2(0) = 20 \Rightarrow c_1 + c_2 + 80 = 40$ and $2c_1 - 2c_2 + 80 = 20 \Rightarrow c_1 = -35$ and $c_2 = -5$. Thus $\begin{cases} A_1(t) = -35e^{-t/10} - 5e^{-3t/10} + 80, \\ A_2(t) = -70e^{-t/10} + 10e^{-3t/10} + 80. \end{cases}$

9. From the given information in the problem we see that the volume of solution in both tanks remains constant at 60 liters. Thus $V_1 = 60 = V_2$. Further, during a short time interval Δt , $\Delta A_1 \approx \left(12 + 2\frac{A_2}{V_2} - 8\frac{A_1}{V_1}\right)\Delta t$, $\Delta A_2 \approx \left(8\frac{A_2}{V_2} - 6\frac{A_2}{V_2}\right)\Delta t$. Substituting in for V_1 and V_2 , dividing by Δt , and taking the limit as $\Delta t \rightarrow 0$, we have

$$\frac{dA_1}{dt} = -\frac{2}{15}A_1 + \frac{1}{30}A_2 + 12 \quad \text{and} \quad \frac{dA_2}{dt} = \frac{2}{15}A_1 - \frac{2}{15}A_2.$$

That is $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$, where

$$\mathbf{x} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad A = \begin{bmatrix} -\frac{2}{15} & \frac{1}{30} \\ \frac{2}{15} & -\frac{2}{15} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 12 \\ 0 \end{bmatrix}.$$

Now,

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} -\frac{2}{15} - \lambda & \frac{1}{30} \\ \frac{2}{15} & -\frac{2}{15} - \lambda \end{vmatrix} = 0 \iff \left(\lambda + \frac{2}{15}\right)^2 - \left(\frac{1}{15}\right)^2 = 0 \iff \lambda \in \left\{-\frac{1}{5}, -\frac{1}{15}\right\}.$$

When $\lambda = -\frac{1}{5}$: $A + \frac{1}{5}I = \begin{bmatrix} \frac{1}{15} & \frac{1}{30} \\ \frac{2}{15} & \frac{1}{15} \end{bmatrix}$, so that the corresponding eigenvectors are of the form $\mathbf{v}_1 = r(-1, 2)$, where $r \in \mathbb{R}$.

When $\lambda = -\frac{1}{15}$: $A + \frac{1}{15}I = \begin{bmatrix} -\frac{1}{15} & \frac{1}{30} \\ \frac{2}{15} & -\frac{1}{15} \end{bmatrix}$, so that the corresponding eigenvectors are of the form $\mathbf{v}_2 = s(1, 2)$, where $s \in \mathbb{R}$. Consequently two linearly independent solutions to the associated homogeneous system are $\mathbf{x}_1(t) = e^{-t/5} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\mathbf{x}_2(t) = e^{-t/15} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. According to the variation-of-parameters technique, a particular solution to the non-homogeneous equation $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ is $\mathbf{x}_p = X\mathbf{u}$, where $\mathbf{u}$ is determined by solving $X\mathbf{u}' = \mathbf{b}$ with $X = \begin{bmatrix} -e^{-t/5} & e^{-t/15} \\ 2e^{-5/t} & 2e^{-t/15} \end{bmatrix}$. The reduced row echelon form of the augmented matrix corresponding to $X\mathbf{u}' = \mathbf{b}$ is $\left[\begin{array}{cc|c} 1 & 0 & -6e^{t/5} \\ 0 & 1 & 6e^{t/15} \end{array} \right]$, so that $\mathbf{u}' = \begin{bmatrix} -6e^{t/5} \\ 6e^{t/15} \end{bmatrix}$, which implies that $\mathbf{u} = \begin{bmatrix} -30e^{t/5} \\ 90e^{t/15} \end{bmatrix}$. Hence, $\mathbf{x}_p(t) = \begin{bmatrix} -e^{-t/5} & e^{-t/15} \\ 2e^{-t/5} & 2e^{-t/15} \end{bmatrix} \begin{bmatrix} -30e^{t/5} \\ 90e^{t/15} \end{bmatrix} = \begin{bmatrix} 120 \\ 120 \end{bmatrix}$. Thus the general solution to the non-homogeneous system is $\mathbf{x}(t) = c_1 e^{-t/5} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{-t/15} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 120 \\ 120 \end{bmatrix}$. Imposing the initial condition $\mathbf{x}(0) = \begin{bmatrix} 60 \\ 200 \end{bmatrix}$ yields $c_1 = 50$ and $c_2 = -10$, so that

$$\mathbf{x}(t) = 50e^{-t/5} \begin{bmatrix} -1 \\ 2 \end{bmatrix} - 10e^{-t/15} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 120 \\ 120 \end{bmatrix}.$$

Thus,

$$A_1(t) = 120 - 50e^{-t/5} - 10e^{-t/15} \quad \text{and} \quad A_2(t) = 120 + 100e^{-t/5} - 20e^{-t/15}.$$

10. Substituting $r_{\text{in}} = r_{\text{out}} = 0$, $r_{12} = r_{21} = 2$ liters/minute, $V_1 = 6$ liters, and $V_2 = 12$ liters into the

$$\text{system } \begin{cases} \frac{dA_1}{dt} = -\frac{r_{21}}{V_1}A_1 + \frac{r_{12}}{V_2}A_2 + c_{\text{in}}r_{\text{in}}, \\ \frac{dA_2}{dt} = \frac{r_{21}}{V_1}A_1 - \frac{r_{21}}{V_2}A_2, \end{cases} \quad \text{yields } \begin{cases} \frac{dA_1}{dt} = -\frac{1}{3}A_1 + \frac{1}{6}A_2, \\ \frac{dA_2}{dt} = \frac{1}{3}A_1 - \frac{1}{6}A_2. \end{cases} \quad \text{We may write this in the}$$

form $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ and $A = \begin{bmatrix} -\frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & -\frac{1}{6} \end{bmatrix}$. Then

$$\det(A - \lambda I) = 0 \iff \begin{vmatrix} -\frac{1}{3} - \lambda & \frac{1}{6} \\ \frac{1}{3} & -\frac{1}{6} - \lambda \end{vmatrix} = 0 \iff \lambda^2 + \frac{1}{2}\lambda = 0 \iff \lambda \in \{0, -1/2\}.$$

When $\lambda = -1/2$: The corresponding eigenvectors take the form $\mathbf{v} = r(-1, 1)$, where $r \in \mathbb{R}$. Therefore,

$$\mathbf{x}_1(t) = e^{-t/2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

When $\lambda = 0$: The corresponding eigenvectors take the form $\mathbf{v} = r(1, 2)$, where $r \in \mathbb{R}$. Therefore, $\mathbf{x}_2(t) =$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The general solution to the system is

$$\mathbf{x}(t) = c_1 e^{-t/2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

$$\text{Now } \mathbf{x}(0) = \begin{bmatrix} 5 \\ 25 \end{bmatrix} \Rightarrow \begin{cases} c_1 - c_2 = -5, \\ c_1 + 2c_2 = 25 \end{cases} \Rightarrow c_1 = 5 \text{ and } c_2 = 10. \text{ Therefore,}$$

$$x(t) = 5e^{-t/2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 10 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{cases} A_1(t) = 10 - 5e^{-t/2}, \\ A_2(t) = 20 + 5e^{-t/2}. \end{cases}$$

11.

(a) In the closed system, $r_{\text{in}} = r_{\text{out}} = 0$ liters/minute and $r_{12} = r_{21}$ so the system

$$\begin{cases} \frac{dA_1}{dt} = -\frac{r_{21}}{V_1}A_1 + \frac{r_{12}}{V_2}A_2 + c_{in}r_{in}, \\ \frac{dA_2}{dt} = \frac{r_{21}}{V_1}A_1 - \frac{r_{21}}{V_2}A_2, \end{cases} \quad \text{becomes} \quad \begin{cases} \frac{dA_1}{dt} = -\frac{r_{21}}{V_1}A_1 + \frac{r_{21}}{V_2}A_2, \\ \frac{dA_2}{dt} = \frac{r_{21}}{V_1}A_1 - \frac{r_{21}}{V_2}A_2. \end{cases}$$

(b) Replace V_2 by βV_1 and write the system in part (a) as $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ and $A =$

$$\begin{bmatrix} -\frac{r_{21}}{V_1} & \frac{r_{21}}{\beta V_1} \\ \frac{r_{21}}{V_1} & -\frac{r_{21}}{\beta V_1} \end{bmatrix}. \text{ Then}$$

$$\det(A - \lambda I) = 0 \iff \det \begin{bmatrix} -\frac{r_{21}}{V_1} - \lambda & \frac{r_{21}}{\beta V_1} \\ \frac{r_{21}}{V_1} & -\frac{r_{21}}{\beta V_1} - \lambda \end{bmatrix} = 0 \iff \lambda^2 + \frac{(1+\beta)r_{21}}{\beta V_1}\lambda = 0$$

$$\iff \lambda \in \{0, -\frac{(1+\beta)}{\beta V_1}r_{21}\}.$$

(c) When $\lambda_1 = 0$, then the corresponding eigenvectors take the form $\mathbf{v} = r(1, \beta)$, where $r \in \mathbb{R}$. When $\lambda_2 = -\frac{(1+\beta)}{\beta V_1}r_{21}$, then the corresponding eigenvectors take the form $\mathbf{v} = s(-1, 1)$, where $s \in \mathbb{R}$. The

general solution for the system is $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ \beta \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Now

$$\mathbf{x}(0) = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \Rightarrow \begin{cases} A_1(t) = \frac{\alpha_1 + \alpha_2}{1 + \beta} - \frac{\alpha_2 - \beta\alpha_1}{1 + \beta}e^{\lambda_2 t}, \\ A_2(t) = \beta \frac{\alpha_1 + \alpha_2}{1 + \beta} + \frac{\alpha_2 - \beta\alpha_1}{1 + \beta}e^{\lambda_2 t}. \end{cases}$$

(d) Since $\lambda_2 < 0$, $\lim_{t \rightarrow \infty} e^{\lambda_2 t} = 0$. Therefore, $\lim_{t \rightarrow \infty} \frac{A_1}{V_1} = \frac{\alpha_1 + \alpha_2}{(1 + \beta)v_1}$ and $\lim_{t \rightarrow \infty} \frac{A_2}{V_2} = \lim_{t \rightarrow \infty} \frac{A_2}{\beta V_1} = \frac{\alpha_1 + \alpha_2}{(1 + \beta)v_1}$. Yes, the result is reasonable, because as the time increases without bound the concentrations in the tanks represented by the two fractions, $\frac{A_1}{V_1}$ and $\frac{A_2}{V_2}$, will become equal.

Solutions to Section 7.8

True-False Review:

1. **TRUE.** This is precisely the content of Equation (7.8.2).
2. **TRUE.** If we denote the fundamental matrix for the linear system $\mathbf{x}' = A\mathbf{x}$ by

$$X(t) = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n],$$

then from Equations (7.8.4) and (7.8.7), we see that

$$e^{At} = X(t)X^{-1}(0) = X_0(t).$$

3. **TRUE.** For each generalized eigenvector $\mathbf{v}$, $e^{At}\mathbf{v}$ is a solution to $\mathbf{x}' = A\mathbf{x}$. If A is an $n \times n$ matrix, then A has n linearly independent generalized eigenvectors, and therefore, the general solution to $\mathbf{x}' = A\mathbf{x}$ can be formulated as linear combinations of n linearly independent solutions of the form $e^{At}\mathbf{v}$. As usual, application of the initial condition yields the unique solution to the given initial-value problem.

4. **TRUE.** Since the transition matrix for $\mathbf{x}' = A\mathbf{x}$ is a fundamental matrix for the system, its columns consist of linearly independent vector functions, and hence, this matrix is invertible.

5. **TRUE.** This follows immediately from Equations (7.8.4) and (7.8.6).

6. **TRUE.** If

$$c_1 e^{At}\mathbf{v}_1 + c_2 e^{At}\mathbf{v}_2 + \cdots + c_n e^{At}\mathbf{v}_n = \mathbf{0},$$

then

$$e^{At} [c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n] = \mathbf{0}.$$

Since e^{At} is invertible for all matrices A , we conclude that

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n = \mathbf{0}.$$

Since $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is linearly independent, $c_1 = c_2 = \cdots = c_n = 0$, and hence, $\{e^{At}\mathbf{v}_1, e^{At}\mathbf{v}_2, \dots, e^{At}\mathbf{v}_n\}$ is linearly independent.

Problems:

1. If $X(t)$ is a fundamental matrix for $\mathbf{x}' = A\mathbf{x}$, then $X(t)B$, where B is an invertible matrix, is also a fundamental matrix. Clearly, $X^{-1}(0)$ is invertible, and its inverse is $X(0)$. Thus, $X_0 = X(t)^{-1}X(0)$ is a fundamental matrix. Since $X_0(0) = X(0)^{-1}X(0) = I_n$, X_0 is the transition matrix, based at $t = 0$, for $\mathbf{x}' = A\mathbf{x}$.

$$2. \det(A - \lambda I) = 0 \iff \begin{vmatrix} 1 - \lambda & 2 \\ 0 & -1 - \lambda \end{vmatrix} = 0 \iff \lambda^2 - 1 = 0 \iff \lambda \in \{-1, 1\}.$$

When $\lambda = 1$: In this case, $A - I = \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix} \Rightarrow \mathbf{v}_1 = r(1, 0)$, where $r \in \mathbb{R} \Rightarrow \mathbf{x}_1(t) = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

When $\lambda = -1$: In this case, $A + I = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v}_2 = s(1, -1)$, where $s \in \mathbb{R} \Rightarrow \mathbf{x}_2(t) = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

A fundamental matrix for $\mathbf{x}' = A\mathbf{x}$ is $X(t) = \begin{bmatrix} e^t & e^{-t} \\ 0 & -e^{-t} \end{bmatrix} \Rightarrow X^{-1}(t) = \begin{bmatrix} e^{-t} & e^{-t} \\ 0 & -e^{-t} \end{bmatrix} \Rightarrow X^{-1}(0) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$.

Thus, $e^{At} = X(t)X^{-1}(0) = \begin{bmatrix} e^t & e^{-t} \\ 0 & -e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} e^t & e^t - e^{-t} \\ 0 & e^{-t} \end{bmatrix}$.

3. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0 \iff (\lambda - 2)^2 = 0$, which implies that the only eigenvalue of A is $\lambda = 2$ (with multiplicity 2).

Eigenvectors for $\lambda = 2$: We have $A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(1, 0)$, where $r \in \mathbb{R}$. Thus, one solution of the linear system $\mathbf{x}' = A\mathbf{x}$ is given by $\mathbf{x}_1(t) = e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$.

We will determine a second linearly independent solution of the form

$$\mathbf{x}_2(t) = e^{2t}(\mathbf{v}_1 + t\mathbf{v}_0)$$

where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A - 2I)^2\mathbf{v}_1 = \mathbf{0}, \quad (A - 2I)\mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A - 2I)\mathbf{v}_1.$$

In this case, we have $A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $(A - 2I)^2 = 0_2$. Therefore, we may choose $\mathbf{v}_1$ to be any vector such that $(A - 2I)\mathbf{v}_1 \neq \mathbf{0}$. For simplicity, we take $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus, $\mathbf{v}_0 = (A - 2I)\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. From the expressions for $\mathbf{v}_0$ and $\mathbf{v}_1$, we can write down a second linearly independent solution to the vector differential equation:

$$\mathbf{x}_2(t) = e^{2t} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = e^{2t} \begin{bmatrix} t \\ 1 \end{bmatrix} = \begin{bmatrix} te^{2t} \\ e^{2t} \end{bmatrix}.$$

A fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$ is therefore given by $X(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$. Therefore, $X^{-1}(t) = \begin{bmatrix} e^{-2t} & -te^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$, and hence, $X^{-1}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus,

$$e^{At} = X(t)X^{-1}(0) = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = e^{2t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

4. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0 \iff (3-\lambda)(\lambda^2 - 4\lambda + 4) = 0 \iff (3-\lambda)(\lambda - 2)^2 = 0$.

Therefore, the eigenvalues of A are $\lambda = 3$ and $\lambda = 2$ (with multiplicity 2).

Eigenvectors for $\lambda = 3$: We have $A - 3I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(1, 0, 0)$, where $r \in \mathbb{R}$. Thus, one solution of the linear system $\mathbf{x}' = A\mathbf{x}$ is given by $\mathbf{x}_1(t) = e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{3t} \\ 0 \\ 0 \end{bmatrix}$.

Eigenvectors for $\lambda = 2$: We have $A - 2I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = s(0, 1, 1)$, where $s \in \mathbb{R}$. Thus, a second solution of the linear system $\mathbf{x}' = A\mathbf{x}$ is given by $\mathbf{x}_2(t) = e^{2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{2t} \\ e^{2t} \end{bmatrix}$.

We will determine a second linearly independent solution corresponding to $\lambda = 2$ of the form

$$\mathbf{x}_3(t) = e^{2t}(\mathbf{v}_1 + t\mathbf{v}_0)$$

where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A - 2I)^2\mathbf{v}_1 = \mathbf{0}, \quad (A - 2I)\mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A - 2I)\mathbf{v}_1.$$

In this case, we have $A - 2I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ and $(A - 2I)^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Let us choose $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Then $\mathbf{v}_0 = (A - 2I)\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. From the expressions for $\mathbf{v}_0$ and $\mathbf{v}_1$, we can write down a second linearly independent solution to the vector differential equation corresponding to $\lambda = 2$:

$$\mathbf{x}_3(t) = e^{2t} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} = e^{2t} \begin{bmatrix} 0 \\ 1+t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ (1+t)e^{2t} \\ te^{2t} \end{bmatrix}.$$

A fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$ is therefore given by $X(t) = \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & (1+t)e^{2t} \\ 0 & e^{2t} & te^{2t} \end{bmatrix}$. Then

$X^{-1}(t) = \begin{bmatrix} e^{-3t} & 0 & 0 \\ 0 & -te^{-2t} & (1+t)e^{2t} \\ 0 & e^{-2t} & -e^{-2t} \end{bmatrix}$, which implies that $X^{-1}(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$. Thus,

$$e^{At} = X(t)X^{-1}(0) = \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{2t} & (1+t)e^{2t} \\ 0 & e^{2t} & te^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & te^{2t} + e^{2t} & -te^{2t} \\ 0 & te^{2t} & e^{2t} - te^{2t} \end{bmatrix}.$$

5. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{vmatrix} = 0 \iff \lambda^2 - 2\lambda + 1 = 0 \iff (\lambda - 1)^2 = 0$. Therefore, the only eigenvalue of A is $\lambda = 1$ (with multiplicity 2).

Eigenvectors for $\lambda = 1$: We have $A - I = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(1, 2)$, where $r \in \mathbb{R}$. Thus, one solution of the linear system $\mathbf{x}' = A\mathbf{x}$ is given by $\mathbf{x}_1(t) = e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} e^t \\ 2e^t \end{bmatrix}$. We will determine a second linearly independent solution corresponding to $\lambda = 1$ of the form $\mathbf{x}_2(t) = e^t(\mathbf{v}_1 + t\mathbf{v}_0)$, where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A - I)^2\mathbf{v}_1 = \mathbf{0}, \quad (A - I)\mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A - I)\mathbf{v}_1.$$

In this case, we have $A - I = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$ and $(A - I)^2 = 0_2$. Therefore, we may choose $\mathbf{v}_1$ to be any vector such that $(A - I)\mathbf{v}_1 \neq \mathbf{0}$. For simplicity, we choose $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $\mathbf{v}_0 = (A - I)\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. It follows that

$$\mathbf{x}_2(t) = e^t \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\} = e^t \begin{bmatrix} 1 + 2t \\ 4t \end{bmatrix} = \begin{bmatrix} (1 + 2t)e^t \\ 4te^t \end{bmatrix}.$$

A fundamental matrix for the system is therefore given by $X(t) = \begin{bmatrix} e^t & (1 + 2t)e^t \\ 2e^t & 4te^t \end{bmatrix}$. Therefore, $X^{-1}(t) = \begin{bmatrix} -2te^{-t} & \frac{1}{2}(1 + 2t)e^{-t} \\ e^{-t} & -\frac{1}{2}e^{-t} \end{bmatrix}$. Thus, $X^{-1}(0) = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$. Thus,

$$e^{At} = X(t)X^{-1}(0) = \begin{bmatrix} e^t & (1 + 2t)e^t \\ 2e^t & 4te^t \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} (1 + 2t)e^t & -te^t \\ e^t - 2te^t & 4te^t \end{bmatrix} = e^t \begin{bmatrix} 1 + 2t & -t \\ 4t & 1 - 2t \end{bmatrix}.$$

6. $\det(A - \lambda I) = 0 \iff \begin{vmatrix} -3 - \lambda & -2 \\ 2 & 1 - \lambda \end{vmatrix} = 0 \iff \lambda^2 + 2\lambda + 1 = 0 \iff (\lambda + 1)^2 = 0$. Therefore, the only eigenvalue of A is $\lambda = -1$ (with multiplicity 2).

Eigenvectors for $\lambda = -1$: We have $A + I = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(-1, 1)$, where $r \in \mathbb{R}$. Thus, one solution of the linear system $\mathbf{x}' = A\mathbf{x}$ is given by $\mathbf{x}_1(t) = e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$. We will determine a second linearly independent solution corresponding to $\lambda = -1$ of the form $\mathbf{x}_2(t) = e^{-t}(\mathbf{v}_1 + t\mathbf{v}_0)$, where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A + I)^2\mathbf{v}_1 = \mathbf{0}, \quad (A + I)\mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A + I)\mathbf{v}_1.$$

In this case, we have $A + I = \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}$ and $(A + I)^2 = 0_2$. Therefore, we may choose $\mathbf{v}_1$ to be any vector such that $(A + I)\mathbf{v}_1 \neq \mathbf{0}$. For simplicity, we choose $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $\mathbf{v}_0 = (A + I)\mathbf{v}_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. It follows that

$$\mathbf{x}_2(t) = e^{-t} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\} = e^{-t} \begin{bmatrix} 1 - 2t \\ 2t \end{bmatrix} = \begin{bmatrix} (1 - 2t)e^{-t} \\ 2te^{-t} \end{bmatrix}.$$

A fundamental matrix for the system is therefore given by $X(t) = \begin{bmatrix} -e^{-t} & (1 - 2t)e^{-t} \\ e^{-t} & 2te^{-t} \end{bmatrix}$. Therefore, $X^{-1}(t) = \begin{bmatrix} -2te^t & (1 - 2t)e^t \\ e^t & e^t \end{bmatrix}$, which implies that $X^{-1}(0) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Thus,

$$e^{At} = X(t)X^{-1}(0) = \begin{bmatrix} -e^{-t} & (1 - 2t)e^{-t} \\ e^{-t} & 2te^{-t} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = e^{-t} \begin{bmatrix} 1 - 2t & -2t \\ 2t & 1 + 2t \end{bmatrix}.$$

$$7. \det(A - \lambda I) = 0 \iff \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & -8 \\ 0 & 2 & -7-\lambda \end{vmatrix} = 0 \iff (2-\lambda)(\lambda^2 + 6\lambda + 9) = 0 \iff (2-\lambda)(\lambda+3)^2 = 0.$$

Therefore, the eigenvalues of A are $\lambda = 2$ and $\lambda = -3$ (with multiplicity 2).

Eigenvectors for $\lambda = 2$: We have $A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -8 \\ 0 & 2 & -9 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(1, 0, 0)$, where $r \in \mathbb{R}$. Thus, one solution of the linear system $\mathbf{x}' = A\mathbf{x}$ is given by $\mathbf{x}_1(t) = e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{2t} \\ 0 \\ 0 \end{bmatrix}$.

Eigenvectors for $\lambda = -3$: We have $A + 3I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & -8 \\ 0 & 2 & -4 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(0, 2, 1)$, where $r \in \mathbb{R}$. Thus, a second solution of the linear system $\mathbf{x}' = A\mathbf{x}$ is given by $\mathbf{x}_2(t) = e^{-3t} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2e^{-3t} \\ e^{-3t} \end{bmatrix}$.

We will determine a second linearly independent solution corresponding to $\lambda = -3$ of the form $\mathbf{x}_3(t) = e^{-3t}(\mathbf{v}_1 + t\mathbf{v}_0)$, where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A + 3I)^2 \mathbf{v}_1 = \mathbf{0}, \quad (A + 3I) \mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A + 3I) \mathbf{v}_1.$$

In this case, we have $A + 3I = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & -8 \\ 0 & 2 & -4 \end{bmatrix}$ and $(A + 3I)^2 = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Let us choose $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Then $\mathbf{v}_0 = (A + 3I) \mathbf{v}_1 = \begin{bmatrix} 0 \\ -8 \\ -4 \end{bmatrix}$. From the expressions for $\mathbf{v}_0$ and $\mathbf{v}_1$, we can write down a second linearly independent solution to the vector differential equation corresponding to $\lambda = -3$:

$$\mathbf{x}_3(t) = e^{-3t} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ -8 \\ -4 \end{bmatrix} \right\} = e^{-3t} \begin{bmatrix} 0 \\ -8t \\ 1 - 4t \end{bmatrix} = \begin{bmatrix} 0 \\ -8te^{-3t} \\ (1 - 4t)e^{-3t} \end{bmatrix}.$$

A fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$ is therefore given by $X(t) = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & 2e^{-3t} & -8te^{-3t} \\ 0 & e^{-3t} & (1 - 4t)e^{-3t} \end{bmatrix}$.

Then $X^{-1}(t) = \begin{bmatrix} e^{-2t} & 0 & 0 \\ 0 & \frac{1}{2}e^{3t}(1 - 4t) & 4te^{3t} \\ 0 & -\frac{1}{2}e^{3t} & e^{3t} \end{bmatrix}$, which implies that $X^{-1}(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$. Thus,

$$e^{At} = X(t)X^{-1}(0) = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & 2e^{-3t} & -8te^{-3t} \\ 0 & e^{-3t} & (1 - 4t)e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-3t}(1 + 4t) & -8te^{-3t} \\ 0 & 2te^{-3t} & (1 - 4t)e^{-3t} \end{bmatrix}.$$

8. We are given the characteristic equation, from which we see that the eigenvalues of A are $\lambda = 4$ and $\lambda = -2$ (with multiplicity 2).

Eigenvectors for $\lambda = 4$: We have $A - 4I = \begin{bmatrix} -12 & 6 & -3 \\ -12 & 6 & -3 \\ -12 & 12 & -6 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(0, 1, 2)$, where $r \in \mathbb{R}$. Thus, one solution of the linear system $\mathbf{x}' = A\mathbf{x}$ is given by $\mathbf{x}_1(t) = e^{4t} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{4t} \\ 2e^{4t} \end{bmatrix}$.

Eigenvectors for $\lambda = -2$: We have $A + 2I = \begin{bmatrix} -6 & 6 & -3 \\ -12 & 12 & -3 \\ -12 & 12 & 0 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(1, 1, 0)$, where $r \in \mathbb{R}$. Therefore, a second linearly independent solution of the linear system $\mathbf{x}' = A\mathbf{x}$ is given by $\mathbf{x}_2(t) = e^{-2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-2t} \\ e^{-2t} \\ 0 \end{bmatrix}$.

We will determine a second linearly independent solution corresponding to $\lambda = -2$ of the form $\mathbf{x}_3(t) = e^{-2t}(\mathbf{v}_1 + t\mathbf{v}_0)$, where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A + 2I)^2\mathbf{v}_1 = \mathbf{0}, \quad (A + 2I)\mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A + 2I)\mathbf{v}_1.$$

In this case, we have $A + 2I = \begin{bmatrix} -6 & 6 & -3 \\ -12 & 12 & -3 \\ -12 & 12 & 0 \end{bmatrix}$ and $(A + 2I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ -36 & 36 & 0 \\ -72 & 72 & 0 \end{bmatrix}$. Let us choose $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Then $\mathbf{v}_0 = (A + 2I)\mathbf{v}_1 = \begin{bmatrix} -3 \\ -3 \\ 0 \end{bmatrix}$. From the expressions for $\mathbf{v}_0$ and $\mathbf{v}_1$, we can write down a second linearly independent solution to the vector differential equation corresponding to $\lambda = -2$:

$$\mathbf{x}_3(t) = e^{-2t} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -3 \\ -3 \\ 0 \end{bmatrix} \right\} = \begin{bmatrix} -3te^{-2t} \\ -3te^{-2t} \\ e^{-2t} \end{bmatrix}.$$

Therefore, the general solution to $\mathbf{x}' = A\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 0 \\ e^{4t} \\ 2e^{4t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-2t} \\ e^{-2t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -3te^{-2t} \\ -3te^{-2t} \\ e^{-2t} \end{bmatrix}.$$

9. We are given the characteristic equation, from which we see that the eigenvalues of A are $\lambda = -1$ and $\lambda = 3$ (with multiplicity 2).

Eigenvectors for $\lambda = -1$: We have $A + I = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 4 & -2 \\ 1 & 1 & 3 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(-7, 4, 1)$, where $r \in \mathbb{R}$. Thus, one solution of the linear system $\mathbf{x}' = A\mathbf{x}$ is given by $\mathbf{x}_1(t) = e^{-t} \begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -7e^{-t} \\ 4e^{-t} \\ e^{-t} \end{bmatrix}$.

Eigenvectors for $\lambda = 3$: We have $A - 3I = \begin{bmatrix} -3 & 1 & 3 \\ 2 & 0 & -2 \\ 1 & 1 & -1 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = s(1, 0, 1)$, where $s \in \mathbb{R}$. Thus, a second linearly independent solution of the linear system $\mathbf{x}' = A\mathbf{x}$ is given by $\mathbf{x}_2(t) = e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{3t} \\ 0 \\ e^{3t} \end{bmatrix}$.

We will determine a second linearly independent solution corresponding to $\lambda = 3$ of the form $\mathbf{x}_3(t) = e^{3t}(\mathbf{v}_1 + t\mathbf{v}_0)$, where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A - 3I)^2 \mathbf{v}_1 = \mathbf{0}, \quad (A - 3I)\mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A - 3I)\mathbf{v}_1.$$

In this case, we have $A - 3I = \begin{bmatrix} -3 & 1 & 3 \\ 2 & 0 & -2 \\ 1 & 1 & -1 \end{bmatrix}$ and $(A - 3I)^2 = \begin{bmatrix} 14 & 0 & -14 \\ -8 & 0 & 8 \\ -2 & 0 & 2 \end{bmatrix}$. Let us choose

$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Then $\mathbf{v}_0 = (A - 3I)\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. From the expressions for $\mathbf{v}_0$ and $\mathbf{v}_1$, we can write down a second linearly independent solution to the vector differential equation corresponding to $\lambda = 3$:

$$\mathbf{x}_3(t) = e^{3t} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} = \begin{bmatrix} te^{3t} \\ e^{3t} \\ te^{3t} \end{bmatrix}.$$

Therefore, the general solution to $\mathbf{x}' = A\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -7e^{-t} \\ 4e^{-t} \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{3t} \\ 0 \\ e^{3t} \end{bmatrix} + c_3 \begin{bmatrix} te^{3t} \\ e^{3t} \\ te^{3t} \end{bmatrix}.$$

10. We are given the characteristic equation, from which we see that the eigenvalues of A are $\lambda = 1$ and $\lambda = 2$ (with multiplicity 3).

Eigenvectors for $\lambda = 1$: We have $A - I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 5 & -7 & 3 \\ 0 & 0 & 2 & -1 \\ 0 & -4 & 9 & -4 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(1, 0, 0, 0)$, where $r \in \mathbb{R}$. Thus, one solution of the linear system $\mathbf{x}' = A\mathbf{x}$ is given by

$$\mathbf{x}_1(t) = e^t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} e^t \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Eigenvectors for $\lambda = 2$: We have $A - 2I = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 4 & -7 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & -4 & 9 & -5 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = s(0, 1, 1, 1)$, where $s \in \mathbb{R}$. Thus, a second linearly independent solution of $\mathbf{x}' = A\mathbf{x}$

$$\text{is given by } \mathbf{x}_2(t) = e^{2t} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}.$$

To obtain two further solutions to $\mathbf{x}' = A\mathbf{x}$ corresponding to $\lambda = 2$, note that

$$(A - 2I)^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -8 & 4 \\ 0 & 4 & -8 & 4 \\ 0 & 4 & -8 & 4 \end{bmatrix} \quad \text{and} \quad (A - 2I)^3 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, if we take $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, then

$$\mathbf{v}_1 = (A - 2I)\mathbf{v}_2 = \begin{bmatrix} 0 \\ 4 \\ 0 \\ -4 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_0 = (A - 2I)^2\mathbf{v}_2 = \begin{bmatrix} 0 \\ 4 \\ 4 \\ 4 \end{bmatrix}.$$

Therefore, using Theorem 7.5.4, we obtain the solutions

$$\mathbf{x}_3(t) = e^{2t} \left\{ \begin{bmatrix} 0 \\ 4 \\ 0 \\ -4 \end{bmatrix} + t \begin{bmatrix} 0 \\ 4 \\ 4 \\ 4 \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ (4 + 4t)e^{2t} \\ 4te^{2t} \\ (-4 + 4t)e^{2t} \end{bmatrix}$$

and

$$\mathbf{x}_4(t) = e^{2t} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 4 \\ 0 \\ -4 \end{bmatrix} + \frac{t^2}{2!} \begin{bmatrix} 0 \\ 4 \\ 4 \\ 4 \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ (1 + 4t + 2t^2)e^{2t} \\ 2t^2e^{2t} \\ (-4t + 2t^2)e^{2t} \end{bmatrix}.$$

Hence, the general solution to $\mathbf{x}' = A\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1 \begin{bmatrix} e^t \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ (4 + 4t)e^{2t} \\ 4te^{2t} \\ (-4 + 4t)e^{2t} \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ (1 + 4t + 2t^2)e^{2t} \\ 2t^2e^{2t} \\ (-4t + 2t^2)e^{2t} \end{bmatrix}.$$

11. From the characteristic polynomial, we find that the eigenvalues of A are $\lambda = i$ (with multiplicity 2) and $\lambda = -i$ (with multiplicity 2).

Eigenvectors for $\lambda = i$: In this case, $A - iI = \begin{bmatrix} -i & -1 & 0 & 0 \\ 1 & -i & 0 & 0 \\ 1 & 0 & -i & -1 \\ 0 & 1 & 1 & -i \end{bmatrix}$ and $(A - iI)^2 = \begin{bmatrix} -2 & 2i & 0 & 0 \\ -2i & -2 & 0 & 0 \\ -2i & -2 & -2 & 2i \\ 2 & -2i & -2i & -2 \end{bmatrix}.$

Now $(A - iI)^2\mathbf{v} = 0$ has two linearly independent solutions. Solving the system we obtain

$$\mathbf{v} = r(i, 1, 0, 0) + s(0, 0, i, 1),$$

where $r, s \in \mathbb{C}$. Let $\mathbf{v}_1 = (i, 1, 0, 0)$, so that

$$\begin{aligned}\mathbf{u}_1(t) &= e^{At}\mathbf{v}_1 = e^{it}[\mathbf{v}_1 + t(A - iI)\mathbf{v}_1] \\ &= e^{it} \left\{ \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -i & -1 & 0 & 0 \\ 1 & -i & 0 & 0 \\ 1 & 0 & -i & -1 \\ 0 & 1 & 1 & -i \end{bmatrix} \begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \\ &= e^{it} \begin{bmatrix} i \\ 1 \\ it \\ t \end{bmatrix} = (\cos t + i \sin t) \begin{bmatrix} i \\ 1 \\ it \\ t \end{bmatrix} = \begin{bmatrix} -\sin t \\ \cos t \\ -t \sin t \\ t \cos t \end{bmatrix} + i \begin{bmatrix} \cos t \\ \sin t \\ t \cos t \\ t \sin t \end{bmatrix}.\end{aligned}$$

Hence, two linearly independent solutions are given by

$$\mathbf{x}_1(t) = \begin{bmatrix} -\sin t \\ \cos t \\ -t \sin t \\ t \cos t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} \cos t \\ \sin t \\ t \cos t \\ t \sin t \end{bmatrix}.$$

Next, let $\mathbf{v}_2 = (0, 0, i, 1)$, so that

$$\begin{aligned}\mathbf{u}_2(t) &= e^{At}\mathbf{v}_2 = e^{it}[\mathbf{v}_2 + t(A - iI)\mathbf{v}_2] \\ &= e^{it} \left\{ \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix} + t \begin{bmatrix} -i & -1 & 0 & 0 \\ 1 & -i & 0 & 0 \\ 1 & 0 & -i & -1 \\ 0 & 1 & 1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix} \right\} \\ &= e^{it} \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix} = (\cos t + i \sin t) \begin{bmatrix} 0 \\ 0 \\ i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\sin t \\ \cos t \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ \cos t \\ \sin t \end{bmatrix}.\end{aligned}$$

Hence, two further linearly independent real-valued solutions are given by

$$\mathbf{x}_3(t) = \begin{bmatrix} 0 \\ 0 \\ -\sin t \\ \cos t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_4(t) = \begin{bmatrix} 0 \\ 0 \\ \cos t \\ \sin t \end{bmatrix}.$$

Solutions to Section 7.9

True-False Review:

- 1. FALSE.** The trajectories need not approach the equilibrium point as $t \rightarrow \infty$. For instance, Figures 7.9.4 and 7.9.8 show equilibrium points for which not all solution trajectories approach the origin as $t \rightarrow \infty$.
- 2. TRUE.** This is well-illustrated in Figure 7.9.7. All trajectories in this case are closed curves, and the equilibrium point is a stable center.
- 3. TRUE.** This is case (b), described below Theorem 7.9.3.

4. TRUE. The *type* of eigenvalues (see Table 7.9.1) for A and $2A$ will be the same, upon examination of the roots of their respective characteristic equations.

5. TRUE. This is essentially the definition of a stable point.

6. FALSE. The only trajectories that approach $(0, 0)$ as $t \rightarrow \infty$ are those pointing along the eigenvector $\mathbf{v}$ corresponding to the *negative* eigenvalue λ .

Problems:

1. Equilibrium solutions arise when $x(x - y + 1) = 0$ and $y(y + 2x) = 0$. Hence, either $y = 0$, in which case $x = 0$ or -1 , or $y = -2x$, in which case $x = -\frac{1}{3}$. Therefore, the equilibrium solutions are $(0, 0)$, $(-1, 0)$ and $(-\frac{1}{3}, \frac{2}{3})$.

2. Equilibrium solutions arise when $x(2x + y) = 0$ and $y(x - 2y + 4) = 0$. Hence, either $x = 0$, in which case $y = 0$ or 2 , or $y = -2x$, in which case $x = -\frac{4}{5}$. Therefore the equilibrium solutions are $(0, 0)$, $(0, 2)$, and $(-\frac{4}{5}, \frac{8}{5})$.

3. Equilibrium solutions arise when $x(x^2 + y^2 - 1) = 0$ and $y(xy - 1) = 0$. Two cases arise:

(a) $y = 0 \implies x(x^2 - 1) = 0 \implies x = 0, \pm 1$. Hence the equilibrium solutions are $(0, 0)$, $(1, 0)$ and $(-1, 0)$.

(b) $y \neq 0 \implies y = \frac{1}{x} \implies x(x^2 + \frac{1}{x^2} - 1) = 0 \implies \frac{1}{x}(x^4 - x^2 + 1) = 0$. Since this equation has no real solution, there are no corresponding equilibrium solutions to the system of differential equations.

4. A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -2$ with corresponding linearly independent eigenvectors $\mathbf{v}_1 = (3, 1)$ and $\mathbf{v}_2 = (-1, 1)$, respectively. Hence $(0, 0)$ is a saddle point.

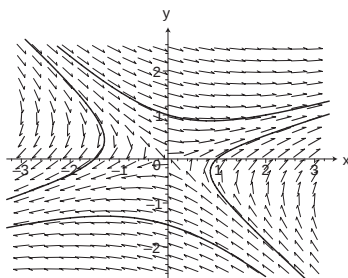


Figure 91: Figure for Problem 4

5. A has eigenvalues $\lambda = \pm 2i$. Hence $(0, 0)$ is a stable center.

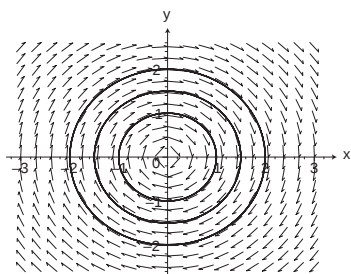


Figure 92: Figure for Problem 5

6. A has a single eigenvalue $\lambda_1 = 1$ with a corresponding eigenvector $\mathbf{v} = (0, 1)$. Hence $(0, 0)$ is a degenerate unstable node.

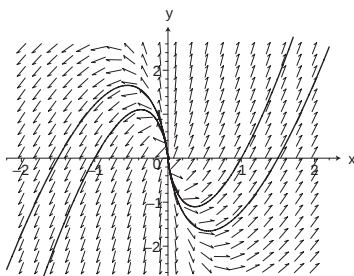


Figure 93: Figure for Problem 6

7. A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ with corresponding linearly independent eigenvectors $\mathbf{v}_1 = (-3, 1)$ and $\mathbf{v}_2 = (1, -1)$. Hence $(0, 0)$ is a saddle point.

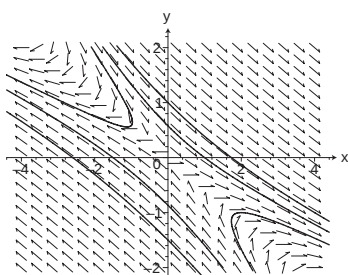


Figure 94: Figure for Problem 7

8. A has eigenvalues $\lambda = -2 \pm 3i$. Hence $(0, 0)$ is a stable spiral.

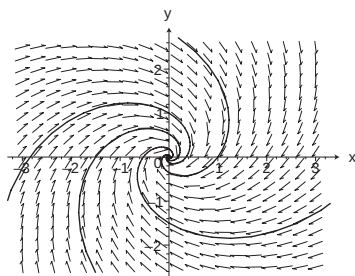


Figure 95: Figure for Problem 8

9. A has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = -1$ with corresponding linearly independent eigenvectors $\mathbf{v}_1 = (-1, 1)$ and $\mathbf{v}_2 = (1, 1)$, respectively. Hence $(0, 0)$ is a stable node.

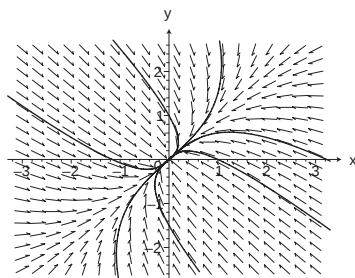


Figure 96: Figure for Problem 9

10. A has eigenvalues $\lambda_1 = 9$ and $\lambda_2 = 1$ with corresponding linearly independent eigenvectors $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (-1, 1)$, respectively. Hence $(0, 0)$ is an unstable node.

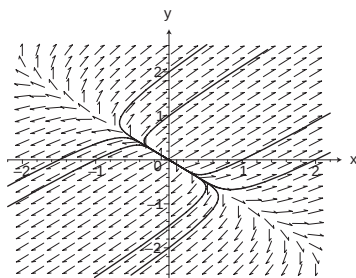


Figure 97: Figure for Problem 10

11. A has eigenvalues $\lambda = \pm i$. Hence, $(0, 0)$ is a stable center.

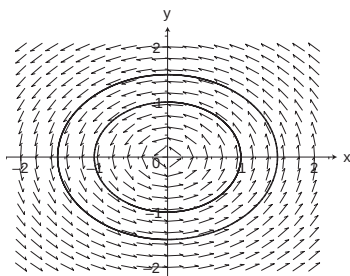


Figure 98: Figure for Problem 11

12. A has a single eigenvalue $\lambda = 1$, with a corresponding eigenvector $\mathbf{v}_1 = (1, 1)$. Hence $(0, 0)$ is a degenerate unstable node.

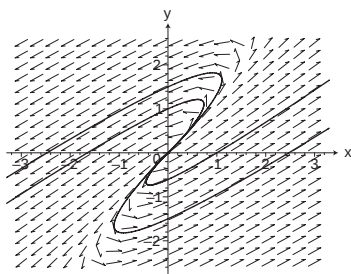


Figure 99: Figure for Problem 12

13. A has eigenvalues $\lambda = 2 \pm i$. Hence $(0, 0)$ is an unstable spiral.

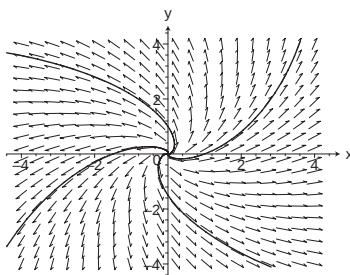


Figure 100: Figure for Problem 13

14. A has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -3$ with corresponding linearly independent eigenvectors $\mathbf{v}_1 = (5, 4)$ and $\mathbf{v}_2 = (1, 1)$, respectively. Hence $(0, 0)$ is a stable node.

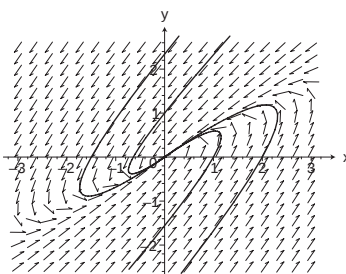


Figure 101: Figure for Problem 14

15. A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 5$ with corresponding linearly independent eigenvectors $\mathbf{v}_1 = (1, -1)$ and $\mathbf{v}_2 = (1, 3)$, respectively. Hence $(0, 0)$ is an unstable node.

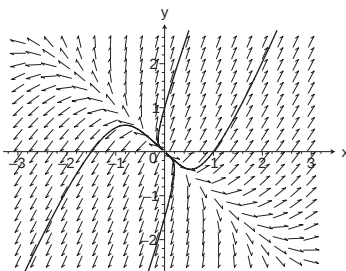


Figure 102: Figure for Problem 15

16. A has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -5$ with corresponding linearly independent eigenvectors $\mathbf{v}_1 = (2, 1)$ and $\mathbf{v}_2 = (1, -2)$, respectively. Hence $(0, 0)$ is a saddle point.

17. A has a single eigenvalue $\lambda = -2$ with a corresponding eigenvector $\mathbf{v}_1 = (1, -3)$. Hence $(0, 0)$ is a stable degenerate node.

18. A has eigenvalues $\lambda = \frac{1}{2}(3 \pm i\sqrt{3})$. Hence $(0, 0)$ is an unstable spiral.

19. A has a single eigenvalue $\lambda = 3$ with corresponding linearly independent eigenvectors $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_2 = (0, 1)$, respectively. Hence $(0, 0)$ is a unstable proper node.

20. A has eigenvalues $\lambda = -1 \pm 2i$. Hence $(0, 0)$ is a saddle point. An eigenvector corresponding to the eigenvalue $\lambda = -1 + 2i$ is $\mathbf{v} = (1, i)$. Hence a complex-valued solution to the system of differential equations is

$$\mathbf{w}(t) = e^{-t}(\cos 2t + i \sin 2t) \begin{bmatrix} 1 \\ i \end{bmatrix}.$$

Therefore two real-valued linearly independent solutions to the system of differential equations are

$$\mathbf{x}_1(t) = e^{-t} \begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-t} \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix},$$

and the general solution to the system is

$$\mathbf{x}(t) = e^{-t} \begin{bmatrix} c_1 \cos 2t + c_2 \sin 2t \\ -c_1 \sin 2t + c_2 \cos 2t \end{bmatrix}.$$

Setting $x(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$ and $y(t) = e^{-t}(-c_1 \sin 2t + c_2 \cos 2t)$ we have $x(t)^2 + y(t)^2 = e^{-2t}(c_1^2 + c_2^2)$. The trajectory in the phase plane is a spiral.

21. Setting $u = y$ and $v = \frac{dy}{dt}$ yields the system

$$\frac{du}{dt} = v, \quad \frac{dv}{dt} = -9u - 6v.$$

This can be written as $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 1 \\ -9 & -6 \end{bmatrix}$. The matrix A has eigenvalue $\lambda = -3$ with corresponding eigenvector $\mathbf{v} = (1, -3)$. Hence, the equilibrium point $(0, 0)$ is a stable degenerate node.

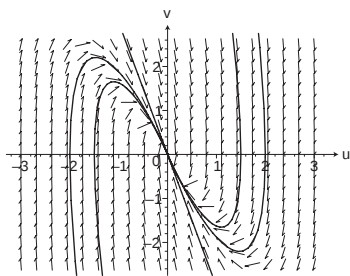


Figure 103: Figure for Problem 21

22. Setting $u = y$ and $v = \frac{dy}{dt}$ yields the system

$$\frac{du}{dt} = v, \quad \frac{dv}{dt} = -16u.$$

This can be written as $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 1 \\ -16 & 0 \end{bmatrix}$. The matrix A has eigenvalues $\lambda = \pm 4i$. Hence, the equilibrium point $(0, 0)$ is a stable center.

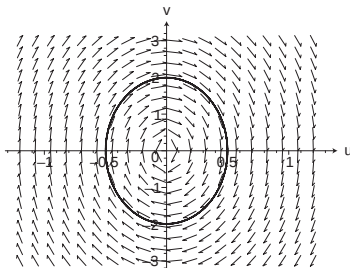


Figure 104: Figure for Problem 22

23. Setting $u = y$ and $v = \frac{dy}{dt}$ yields the system

$$\frac{du}{dt} = v, \quad \frac{dv}{dt} = -5u - 4v.$$

This can be written as $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix}$. The matrix A has eigenvalues $\lambda = -2 \pm i$. Hence, the equilibrium point $(0, 0)$ is a stable spiral point.

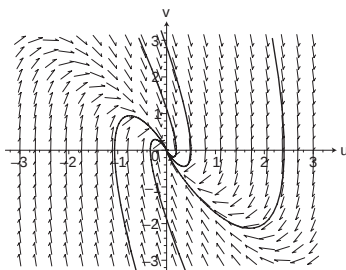


Figure 105: Figure for Problem 23

24. Setting $u = y$ and $v = \frac{dy}{dt}$ yields the system

$$\frac{du}{dt} = v, \quad \frac{dv}{dt} = 25u.$$

This can be written as $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 1 \\ 25 & 0 \end{bmatrix}$. The matrix A has eigenvalues $\lambda_1 = 5$, $\lambda_2 = -5$ with corresponding linearly independent eigenvectors $\mathbf{v}_1 = (1, 5)$ and $\mathbf{v}_2 = (1, -5)$, respectively. Hence, the equilibrium point $(0, 0)$ is a saddle point.

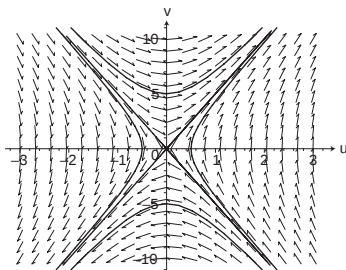


Figure 106: Figure for Problem 24

25. Setting $u = y$ and $v = \frac{dy}{dt}$ yields the system

$$\frac{du}{dt} = v, \quad \frac{dv}{dt} = -ku - 2cv.$$

This can be written as $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 1 \\ -k & -2c \end{bmatrix}$. The matrix A has eigenvalues

$$\lambda = -c \pm \sqrt{c^2 - k}.$$

Consequently there are three cases:

(a) If $c^2 - k > 0$ then there are two real distinct and negative eigenvalues. Consequently, $(0, 0)$ is a stable node. Since $y = u$, the system passes through equilibrium whenever the corresponding trajectory crosses the v -axis. From the phase portrait we see that this happens at most once.

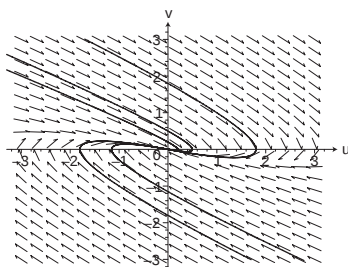


Figure 107: Figure for Problem 25(a)

(b) If $c^2 - k = 0$ then A has the real repeated negative eigenvalue $\lambda = -c$ with a single linearly independent eigenvector. Therefore, $(0, 0)$ is a stable degenerate node. The behavior of the physical system is similar to that in (a).

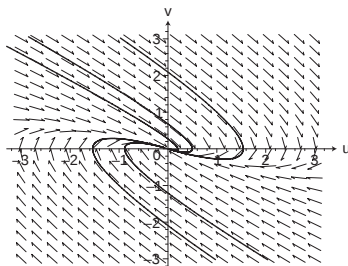


Figure 108: Figure for Problem 25(b)

(c) If $c^2 - k < 0$ then A has complex conjugate eigenvalues with negative real part. Consequently, $(0, 0)$ is a stable spiral. In this case the physical system oscillates about $y = 0$ with decreasing amplitude and velocity. The spring-mass system eventually comes to rest at its equilibrium position.

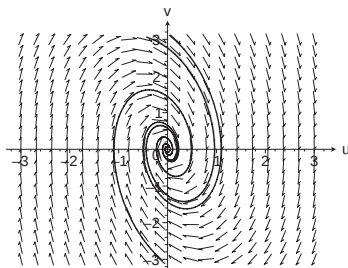


Figure 109: Figure for Problem 25(c)

26. Suppose, to the contrary, that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are proportional. Say $\mathbf{x}_2(t) = k\mathbf{x}_1(t)$, where $k \in \mathbb{R}$. Then from $\mathbf{x}_2(0) = k\mathbf{x}_1(0)$, we obtain $\mathbf{s} = k\mathbf{r}$. In particular, this implies that $\mathbf{r} \neq \mathbf{0}$ (since $\mathbf{v} = \mathbf{r} + i\mathbf{s}$ is an eigenvector, and must therefore be nonzero). Substituting $\mathbf{s} = k\mathbf{r}$ into $\mathbf{x}_2 = k\mathbf{x}_1$, we obtain

$$e^{at}(\sin btr + k \cos btr) = ke^{at}(\cos btr - k \sin btr),$$

or

$$(\sin bt + k \cos bt)\mathbf{r} = (k \cos bt - k^2 \sin bt)\mathbf{r}.$$

Since $\mathbf{r} \neq \mathbf{0}$, we must have

$$\sin bt + k \cos bt = k \cos bt - k^2 \sin bt,$$

or

$$(k^2 + 1) \sin bt = 0.$$

Since $k \in \mathbb{R}$, we conclude that $\sin bt = 0$, which is certainly not true for all $t \in \mathbb{R}$. Therefore, we conclude that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are *not* proportional.

27. Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\det(A - \lambda I) = \det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + (ad - bc).$$

According to the quadratic equation, the roots of this characteristic equation are

$$\lambda = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}.$$

(a) With the assumption that A has a repeated eigenvalue, we conclude that $\sqrt{(a + d)^2 - 4(ad - bc)} = 0$. Therefore, $\lambda = (a + d)/2$ is the repeated eigenvalue.

(b) Since A has two linearly independent eigenvectors corresponding to λ , we know that every nonzero vector in $\mathbb{R}^2$ is an eigenvector of A . Therefore, with $\mathbf{e}_1$ and $\mathbf{e}_2$ denoting the standard basis vectors on $\mathbb{R}^2$, we have

$$(A - \lambda I)\mathbf{e}_1 = \mathbf{0} \quad \text{and} \quad (A - \lambda I)\mathbf{e}_2 = \mathbf{0}.$$

This implies that the $A - \lambda I$ must be the zero matrix. That is, $A = \lambda I$, which means that A is a scalar matrix.

(c) By part (b), if A is a 2×2 matrix of real constants with a repeated eigenvalue λ that is *not* a scalar multiple of the identity matrix, then A cannot have two linearly independent eigenvectors corresponding to λ . That is, A is defective.

Solutions to Section 7.10

True-False Review:

1. TRUE. This is just the definition of the Jacobian matrix given in the text.

2. FALSE. Although the conclusion is valid in most cases, Table 7.10.1 shows that in the case of pure imaginary eigenvalues for the linear system, the behavior of the linear approximation can diverge from that of the given nonlinear system.

3. TRUE. From Equations (7.10.2) and (7.10.3), we find that the Jacobian of the system is

$$J(x, y) = \begin{bmatrix} a - by & -bx \\ cy & cx - d \end{bmatrix},$$

so that $J(0, 0) = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}$. Since $a, d > 0$, we have one positive and one negative eigenvalue, and therefore, the equilibrium point $(0, 0)$ is a saddle point.

4. TRUE. The linear system corresponding to the Van der Pol Equation is given in (7.10.5), and the only equilibrium point of this system occurs at $(0, 0)$.

5. FALSE. The equilibrium point is only an unstable spiral if $\mu < 2$. In this equation, $\mu = 3$, in which case the equilibrium point is an unstable node, as discussed in the text.

Problems:

1. The equilibrium points are obtained by solving

$$y(3x - 2) = 0, \quad 2x + 9y^2 = 0.$$

The only solution to the system is $x = y = 0$, and hence the only equilibrium point is $(0, 0)$. The Jacobian for the given system is

$$J(x, y) = \begin{bmatrix} 3y & 3x - 2 \\ 2 & 18y \end{bmatrix} \implies J(0, 0) = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}.$$

The eigenvalues of $J(0, 0)$ are $\lambda = \pm 2i$, so that the equilibrium point is either a center or a spiral point.

2. The equilibrium points are obtained by solving

$$y(3x - 2) = 0, \quad 2x - 9y^2 = 0.$$

The solutions to this pair of equations are $x = 0, y = 0$, and $x = \frac{2}{3}, y = \pm \frac{2}{3\sqrt{3}}$. Hence, the only equilibrium points are $(0, 0), (2/3, 2/3\sqrt{3}), (2/3, -2/3\sqrt{3})$. The Jacobian for the given system is

$$J(x, y) = \begin{bmatrix} 3y & 3x - 2 \\ 2 & -18y \end{bmatrix}.$$

Consequently, $J(0, 0) = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$ with eigenvalues $\lambda = \pm 2i \implies (0, 0)$ is either a center or a spiral point.

$J(\frac{2}{3}, \frac{2}{3\sqrt{3}}) = \begin{bmatrix} \frac{2}{\sqrt{3}} & 0 \\ 2 & -\frac{12}{\sqrt{3}} \end{bmatrix}$ with eigenvalues $\lambda = \frac{2}{\sqrt{3}}, -\frac{12}{\sqrt{3}} \implies (\frac{2}{3}, \frac{2}{3\sqrt{3}})$ is a saddle point.

$J(\frac{2}{3}, -\frac{2}{3\sqrt{3}}) = \begin{bmatrix} -\frac{2}{\sqrt{3}} & 0 \\ 2 & \frac{12}{\sqrt{3}} \end{bmatrix}$ with eigenvalues $\lambda = -\frac{2}{\sqrt{3}}, \frac{12}{\sqrt{3}} \implies (\frac{2}{3}, -\frac{2}{3\sqrt{3}})$ is a saddle point.

3. The equilibrium points are obtained by solving

$$x - y^2 = 0, \quad y(9x - 4) = 0.$$

The solutions to this system are $x = y = 0$, and $x = \frac{4}{9}$, $y = \pm \frac{2}{3}$. The Jacobian for the given system is

$$J(x, y) = \begin{bmatrix} 1 & -2y \\ 9y & 9x - 4 \end{bmatrix}.$$

Consequently, $J(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$ with eigenvalues $\lambda = 1, -4 \implies (0, 0)$ is a saddle point.

$J(\frac{4}{9}, \frac{2}{3}) = \begin{bmatrix} 1 & -\frac{4}{3} \\ 6 & 0 \end{bmatrix}$ with eigenvalues $\lambda = \frac{1}{2}(1 \pm i\sqrt{31}) \implies (\frac{4}{9}, \frac{2}{3})$ is an unstable spiral point.

$J(\frac{4}{9}, -\frac{2}{3}) = \begin{bmatrix} 1 & \frac{4}{3} \\ -6 & 0 \end{bmatrix}$ with eigenvalues $\lambda = \frac{1}{2}(1 \pm i\sqrt{31}) \implies (\frac{4}{9}, -\frac{2}{3})$ is an unstable spiral point.

4. The equilibrium points are obtained by solving

$$x + 3y^2 = 0, \quad 2y(x - 2) = 0.$$

The only solution to the system is $x = y = 0$, and hence the only equilibrium point is $(0, 0)$. The Jacobian for the given system is

$$J(x, y) = \begin{bmatrix} 1 & 6y \\ y & x - 2 \end{bmatrix} \implies J(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}.$$

The eigenvalues of $J(0, 0)$ are $\lambda = 1, -2$ so that the equilibrium point is a saddle point.

5. The equilibrium points are obtained by solving

$$2x + 5y^2 = 0, \quad y(3 - 4x) = 0.$$

The only solution to this system is $x = y = 0$, and hence the only equilibrium point is $(0, 0)$. The Jacobian for the given system is

$$J(x, y) = \begin{bmatrix} 2 & 10y \\ -4y & 3 \end{bmatrix} \implies J(0, 0) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

The eigenvalues of $J(0, 0)$ are $\lambda = 2, 3$ so that the equilibrium point is an unstable node.

6. The equilibrium points are obtained by solving

$$2y + \sin x = 0, \quad x(\cos y - 2) = 0.$$

The only solution to this system is $x = y = 0$, and hence the only equilibrium point is $(0, 0)$. The Jacobian for the given system is

$$J(x, y) = \begin{bmatrix} \cos x & 2 \\ \cos y - 2 & -x \sin y \end{bmatrix} \implies J(0, 0) = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}.$$

The eigenvalues of $J(0, 0)$ are $\lambda = \frac{1}{2}(1 \pm i\sqrt{7})$ so that the equilibrium point is an unstable spiral.

7. The equilibrium points are obtained by solving

$$x - 2y + 5xy = 0, \quad 2x + y = 0.$$

The solutions to this system are $x = 0$, $y = 0$, and $x = \frac{1}{2}$, $y = -1$. The Jacobian for the given system is

$$J(x, y) = \begin{bmatrix} 1 + 5y & -2 + 5x \\ 2 & 1 \end{bmatrix}.$$

Consequently, $J(0, 0) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ with eigenvalues $\lambda = 1 \pm 2i \implies (0, 0)$ is an unstable spiral point.

$J(\frac{1}{2}, -1) = \begin{bmatrix} -4 & \frac{1}{2} \\ 2 & 1 \end{bmatrix}$ with eigenvalues $\lambda = \frac{1}{2}(1 \pm \sqrt{29}) \implies (\frac{1}{2}, -1)$ is a saddle point.

8. The equilibrium points are obtained by solving

$$x(1 - y) = 0, \quad y(x + 1) = 0.$$

The solutions to this system are $x = 0$, $y = 0$, and $x = -1$, $y = 1$. The Jacobian for the given system is

$$J(x, y) = \begin{bmatrix} 1 - y & -x \\ y & x + 1 \end{bmatrix}.$$

Consequently, $J(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ with eigenvalue $\lambda = 1$ of multiplicity 2. Since we have a repeated eigenvalue, the type and stability of the equilibrium point of the nonlinear system is indeterminate.

$J(-1, 1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with eigenvalues $\lambda = \pm 1 \implies (-1, 1)$ is a saddle point.

9. The equilibrium points are obtained by solving $4x - y - y \sin x = 0$ and $x + 2y = 0$. The only solution to this system is $x = y = 0$. Hence, there is only one equilibrium point, namely $(0, 0)$. The Jacobian for the given system is

$$J(x, y) = \begin{bmatrix} 4 - y \cos x & -1 - \sin x \\ 1 & 2 \end{bmatrix} \implies J(0, 0) = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}.$$

The eigenvalues of $J(0, 0)$ are $\lambda = 3$ of multiplicity 2. Since we have a repeated eigenvalue, the type and stability of the equilibrium point of the nonlinear system is indeterminate.

10. From the phase portrait we would conclude that the equilibrium point $(0, 0)$ is a center.

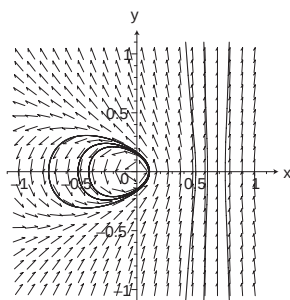


Figure 110: Figure for Problem 10

11. The phase portrait verifies the result obtained in Problem 6, namely that $(0,0)$ is an unstable spiral point.

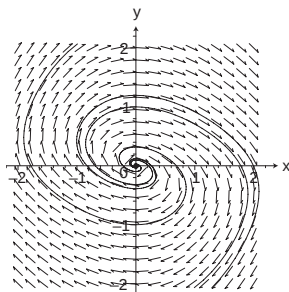


Figure 111: Figure for Problem 11

12. By inspection of the phase portrait it appears that there may be a solution with $y = -x$. From the given system we see that the slope of the trajectories at each point is

$$\frac{dy}{dx} = \frac{y(x+1)}{x(1-y)}$$

which does indeed have the particular solution $y = -x$. To actually find the solution we must solve the system of differential equations when $y = -x$. In this case the differential equation in the system is reduced to

$$\frac{dx}{dt} = x(1+x).$$

Separating the variables in this differential equation gives

$$\frac{1}{x(x+1)}dx = dt$$

which can be written as

$$\left(\frac{1}{x} - \frac{1}{x+1}\right)dx = dt.$$

Integrating both sides of this differential equation yields

$$\ln\left(\frac{x}{x+1}\right) = t + c$$

so that

$$\frac{x}{x+1} = c_1 e^t.$$

Solving for x gives

$$x(t) = \frac{c_1 e^t}{1 - c_1 e^t},$$

and thus,

$$y(t) = -\frac{c_1 e^t}{1 - c_1 e^t}.$$

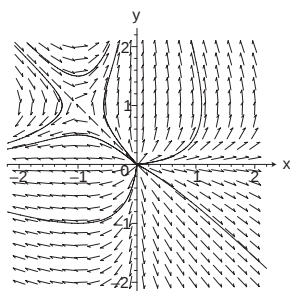


Figure 112: Figure for Problem 12

13. From the phase portrait, we see the center and two saddle points identified in Problem 2.

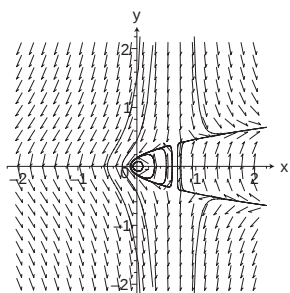


Figure 113: Figure for Problem 13

14. The two spiral points found in Problem 3 are clearly visible in the phase portrait.

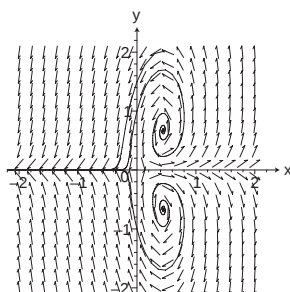


Figure 114: Figure for Problem 14

15. From the phase portrait, we see the saddle point behavior of the trajectories around $(0,0)$.

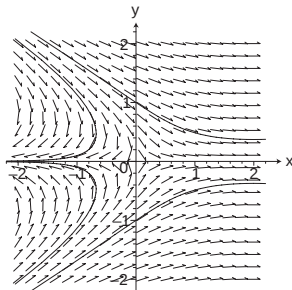


Figure 115: Figure for Problem 15

16. The unstable node found in Problem 8 is clearly visible in the phase portrait.

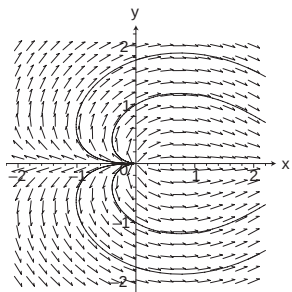


Figure 116: Figure for Problem 16

17. The saddle point and unstable spiral point found in Problem 7 are clearly visible in the phase portrait.

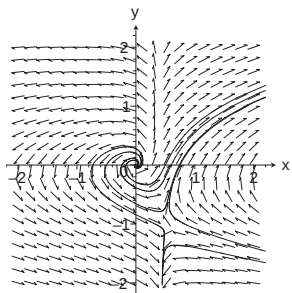


Figure 117: Figure for Problem 17

18.

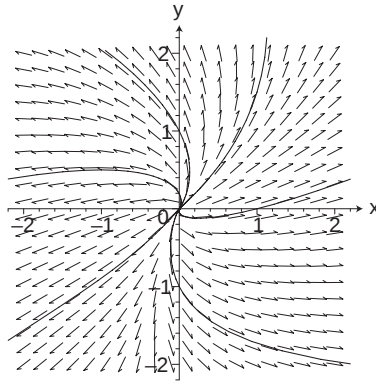
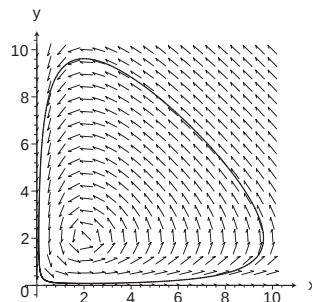


Figure 118: Figure for Problem 18

19. In the case of the initial condition $x(0) = 1$, $y(0) = 0.1$, the initial predator population is only 10% of the initial population. Consequently the prey population grows rapidly initially. The increased number of prey enable the predator population to grow with a corresponding decrease in the prey population. The lack of prey causes the predator population to decrease until we return to the initial situation and the cycle then repeats.

Figure 119: Figure for Problem 19 with initial conditions $x(0) = 1$, $y(0) = 0.1$

In the case of the initial condition $x(0) = 1$, $y(0) = 1$, the general behavior is similar to the preceding case. However, the initial predator and prey populations are equal, and therefore there is not such a wide variation in the predator or prey populations.

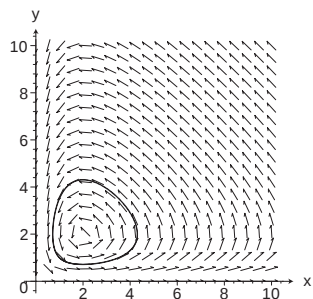


Figure 120: Figure for Problem 19 with initial conditions $x(0) = 1$, $y(0) = 1$

20. From the accompanying figure we see that

$$\lim_{t \rightarrow \infty} x(t) = 1, \quad \text{and} \quad \lim_{t \rightarrow \infty} y(t) = 2.$$

Consequently, the predator and prey populations approach these limiting values as $t \rightarrow \infty$. Unlike the cases considered in the preceding problem, the behavior is not cyclic.

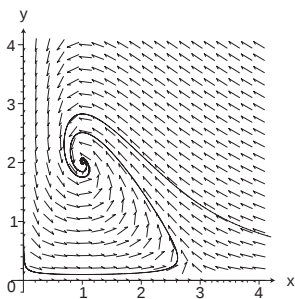


Figure 121: Figure for Problem 20

21.

(a) If we let $u = y$ and $v = \frac{dv}{dt}$, then the given differential equation is replaced by the system

$$\frac{du}{dt} = v, \quad \frac{dv}{dt} = -u - 0.1(u - 4)(u + 1)v.$$

The only equilibrium point of this system is $(0, 0)$, and

$$J(0, 0) = \begin{bmatrix} 0 & 1 \\ -1 & 0.4 \end{bmatrix}$$

which has eigenvalues $\lambda = \frac{1}{5}(1 \pm 2i\sqrt{6})$. Consequently the equilibrium point $(0, 0)$ is an unstable spiral point.

(b) The sketch of the phase plane on the square $-2 \leq u \leq 2$, $-2 \leq v \leq 2$ suggests that the differential equation does not have a limit cycle.

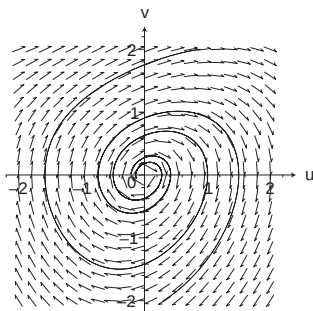


Figure 122: Figure for Problem 21(b)

(c) The sketch of the phase plane on the square $-8 \leq u \leq 8$, $-8 \leq v \leq 8$ suggests that our conclusion in (b) was incorrect, and that in fact the differential equation does have a limit cycle.

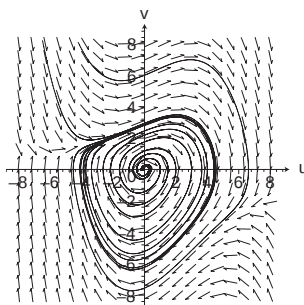


Figure 123: Figure for Problem 21(c)

Solutions to Section 7.11

Problems:

1. We have

$$A\mathbf{x}_1 = \begin{bmatrix} 2t-1 & 0 \\ e^{t-t^2} & 1 \end{bmatrix} \begin{bmatrix} e^{t^2-t} \\ -1 \end{bmatrix} = \begin{bmatrix} (2t-1)e^{t^2-t} \\ 0 \end{bmatrix} = \mathbf{x}'_1(t)$$

and

$$A\mathbf{x}_2 = \begin{bmatrix} 2t-1 & 0 \\ e^{t-t^2} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2e^t \end{bmatrix} = \begin{bmatrix} 0 \\ 2e^t \end{bmatrix} = \mathbf{x}'_2(t).$$

Therefore, $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are indeed solutions to $\mathbf{x}' = A\mathbf{x}$. Furthermore, we can use Theorem 7.2.4 to show that $\{\mathbf{x}_1(t), \mathbf{x}_2(t)\}$ is linearly independent. To do so, we compute the Wronskian of the given vector functions:

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det \begin{bmatrix} e^{t^2-t} & 0 \\ -1 & 2e^t \end{bmatrix} = 2e^{t^2} \neq 0.$$

Therefore, $\{\mathbf{x}_1(t), \mathbf{x}_2(t)\}$ is linearly independent by Theorem 7.2.4. Therefore, by Theorem 7.3.2, since the dimension of the solution space of $\mathbf{x}' = A\mathbf{x}$ is 2 in this case, $\{\mathbf{x}_1(t), \mathbf{x}_2(t)\}$ is a basis for the solution space. Hence, the general solution to $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} e^{t^2-t} \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2e^t \end{bmatrix}.$$

2.

(a) We have

$$\begin{aligned} A\mathbf{x} + \mathbf{b} &= \begin{bmatrix} t \cot(t^2) & 0 & t \cos(t^2)/2 \\ 0 & 1/t & -1 \\ \csc(t^2) & 1 & -1 \end{bmatrix} \begin{bmatrix} \sin(t^2) \\ t \cos t \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 - t \sin t \\ 1 - t \cos t \end{bmatrix} \\ &= \begin{bmatrix} 2t \cos(t^2) \\ \cos t - t \sin t \\ 0 \end{bmatrix} \\ &= \mathbf{x}'(t). \end{aligned}$$

(b) Suppose $\mathbf{x}_0 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Then $A\mathbf{x}_0 + \mathbf{b} = \begin{bmatrix} at \cos(t^2) + ct \cos(t^2)/2 \\ b/t - c + 2 - t \sin t \\ a \csc(t^2) + b - c + 1 - t \cos t \end{bmatrix} \neq \mathbf{0} = \mathbf{x}'_0$, so this calculation shows that it is not possible for a constant vector $\mathbf{x}_0$ to solve the system.

3. We begin by finding the eigenvalues of A . We have

$$\det(A - \lambda I) = \det \begin{bmatrix} -6 - \lambda & 1 \\ 6 & -5 - \lambda \end{bmatrix} = (-6 - \lambda)(-5 - \lambda) - 6 = \lambda^2 + 11\lambda + 24 = (\lambda + 3)(\lambda + 8).$$

Therefore, the eigenvalues of A are $\lambda = -3$ and $\lambda = -8$.

Eigenvectors for $\lambda = -3$: We have $A + 3I = \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(1, 3)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_1(t) = e^{-3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ to the system $\mathbf{x}' = A\mathbf{x}$.

Eigenvectors for $\lambda = -8$: We have $A + 8I = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(-1, 2)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_2(t) = e^{-8t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ to the system $\mathbf{x}' = A\mathbf{x}$.

Putting the results above together, we obtain the general solution to the linear system:

$$\mathbf{x}(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-8t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

4. We begin by finding the eigenvalues of A . We have

$$\det(A - \lambda I) = \det \begin{bmatrix} 9 - \lambda & -2 \\ 5 & -2 - \lambda \end{bmatrix} = (9 - \lambda)(-2 - \lambda) + 10 = \lambda^2 - 7\lambda - 8 = (\lambda - 8)(\lambda + 1).$$

Therefore, the eigenvalues of A are $\lambda = 8$ and $\lambda = -1$.

Eigenvectors for $\lambda = 8$: We have $A - 8I = \begin{bmatrix} 1 & -2 \\ 5 & -10 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(2, 1)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_1(t) = e^{8t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ to the system $\mathbf{x}' = A\mathbf{x}$.

Eigenvectors for $\lambda = -1$: We have $A + I = \begin{bmatrix} 10 & -2 \\ 5 & -1 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(1, 5)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_2(t) = e^{-t} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ to the system $\mathbf{x}' = A\mathbf{x}$.

Putting the results above together, we obtain the general solution to the linear system:

$$\mathbf{x}(t) = c_1 e^{8t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

5. We begin by finding the eigenvalues of A . We have

$$\det(A - \lambda I) = \det \begin{bmatrix} 10 - \lambda & -4 \\ 4 & 2 - \lambda \end{bmatrix} = (10 - \lambda)(2 - \lambda) + 16 = \lambda^2 - 12\lambda + 36 = (\lambda - 6)^2.$$

Therefore, the only eigenvalue of A is $\lambda = 6$ (with multiplicity 2).

Eigenvectors for $\lambda = 6$: We have $A - 6I = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(1, 1)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_1(t) = e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to the system $\mathbf{x}' = A\mathbf{x}$.

We will determine a second linearly independent solution of the form

$$\mathbf{x}_2(t) = e^{6t}(\mathbf{v}_1 + t\mathbf{v}_0)$$

where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A - 6I)^2 \mathbf{v}_1 = \mathbf{0}, \quad (A - 6I) \mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A - 6I) \mathbf{v}_1.$$

In this case, $(A - 6I)^2 = 0_2$. Therefore, we may choose $\mathbf{v}_1$ to be any vector such that $(A - 6I) \mathbf{v}_1 \neq \mathbf{0}$. For simplicity, we take $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus, $\mathbf{v}_0 = (A - 6I) \mathbf{v}_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. From the expressions for $\mathbf{v}_0$ and $\mathbf{v}_1$, we can write down a second (linearly independent) solution to the vector differential equation:

$$\mathbf{x}_2(t) = e^{6t} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 4 \end{bmatrix} \right\} = e^{6t} \begin{bmatrix} 1 + 4t \\ 4t \end{bmatrix}.$$

Consequently, the general solution to the vector differential equation is

$$\mathbf{x}(t) = c_1 e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} 1 + 4t \\ 4t \end{bmatrix}.$$

6. We begin by finding the eigenvalues of A . We have

$$\det(A - \lambda I) = \det \begin{bmatrix} -8 - \lambda & 5 \\ -5 & 2 - \lambda \end{bmatrix} = (-8 - \lambda)(2 - \lambda) + 25 = \lambda^2 + 6\lambda + 9 = (\lambda + 3)^2.$$

Therefore, the only eigenvalue of A is $\lambda = -3$ (with multiplicity 2).

Eigenvectors for $\lambda = -3$: We have $A + 3I = \begin{bmatrix} -5 & 5 \\ -5 & 5 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(1, 1)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_1(t) = e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to the system $\mathbf{x}' = A\mathbf{x}$.

We will determine a second linearly independent solution of the form

$$\mathbf{x}_2(t) = e^{-3t}(\mathbf{v}_1 + t\mathbf{v}_0)$$

where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A + 3I)^2\mathbf{v}_1 = \mathbf{0}, \quad (A + 3I)\mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A + 3I)\mathbf{v}_1.$$

In this case, $(A + 3I)^2 = 0_2$. Therefore, we may choose $\mathbf{v}_1$ to be any vector such that $(A + 3I)\mathbf{v}_1 \neq \mathbf{0}$. For simplicity, we take $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus, $\mathbf{v}_0 = (A + 3I)\mathbf{v}_1 = \begin{bmatrix} -5 \\ -5 \end{bmatrix}$. From the expressions for $\mathbf{v}_0$ and $\mathbf{v}_1$, we can write down a second (linearly independent) solution to the vector differential equation:

$$\mathbf{x}_2(t) = e^{-3t} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ -5 \end{bmatrix} \right\} = e^{-3t} \begin{bmatrix} 1 - 5t \\ -5t \end{bmatrix}.$$

Consequently, the general solution to the vector differential equation is

$$\mathbf{x}(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 - 5t \\ -5t \end{bmatrix}.$$

7. We begin by finding the eigenvalues of A . We have

$$\det(A - \lambda I) = \det \begin{bmatrix} 3 - \lambda & 0 & 4 \\ 0 & 2 - \lambda & 0 \\ -4 & 0 & -5 - \lambda \end{bmatrix} = (2 - \lambda)[(3 - \lambda)(-5 - \lambda) + 16] = (2 - \lambda)[\lambda^2 + 2\lambda + 1] = (2 - \lambda)(\lambda + 1)^2.$$

Therefore, the eigenvalues of A are $\lambda = 2$ and $\lambda = -1$ (with multiplicity 2).

Eigenvectors for $\lambda = 2$: We have $A - 2I = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 0 \\ -4 & 0 & -7 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(0, 1, 0)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_1(t) = e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ to the system $\mathbf{x}' = A\mathbf{x}$.

Eigenvectors for $\lambda = -1$: We have $A + I = \begin{bmatrix} 4 & 0 & 4 \\ 0 & 3 & 0 \\ -4 & 0 & -4 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(-1, 0, 1)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_2(t) = e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ to the system $\mathbf{x}' = A\mathbf{x}$.

We will determine a second linearly independent solution corresponding to $\lambda = -1$ in the form

$$\mathbf{x}_3(t) = e^{-t}(\mathbf{v}_1 + t\mathbf{v}_0)$$

where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A + I)^2 \mathbf{v}_1 = \mathbf{0}, \quad (A + I) \mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A + I) \mathbf{v}_1.$$

In this case, $(A + I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Let us take $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Thus, $\mathbf{v}_0 = (A + I) \mathbf{v}_1 = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}$. From the expressions for $\mathbf{v}_0$ and $\mathbf{v}_1$, we can write down a second (linearly independent) solution to the vector differential equation corresponding to $\lambda = -1$:

$$\mathbf{x}_3(t) = e^{-t} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} \right\} = e^{-t} \begin{bmatrix} 1 + 4t \\ 0 \\ -4t \end{bmatrix}.$$

Consequently, the general solution to the vector differential equation is

$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 1 + 4t \\ 0 \\ -4t \end{bmatrix}.$$

8. We begin by finding the eigenvalues of A . We have

$$\det(A - \lambda I) = \det \begin{bmatrix} -3 - \lambda & -1 & 0 \\ 4 & -7 - \lambda & 0 \\ 6 & 6 & 4 - \lambda \end{bmatrix} = (4 - \lambda) [(-3 - \lambda)(-7 - \lambda) + 4] = (4 - \lambda) [\lambda^2 + 10\lambda + 25] = (4 - \lambda)(\lambda + 5)^2.$$

Therefore, the eigenvalues of A are $\lambda = 4$ and $\lambda = -5$ (with multiplicity 2).

Eigenvectors for $\lambda = 4$: We have $A - 4I = \begin{bmatrix} -7 & -1 & 0 \\ 4 & -11 & 0 \\ 6 & 6 & 0 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(0, 0, 1)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_1(t) = e^{4t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ to the system $\mathbf{x}' = A\mathbf{x}$.

Eigenvectors for $\lambda = -5$: We have $A + 5I = \begin{bmatrix} 2 & -1 & 0 \\ 4 & -2 & 0 \\ 6 & 6 & 9 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(1, 2, -2)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_2(t) = e^{-5t} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ to the system $\mathbf{x}' = A\mathbf{x}$.

We will determine a second linearly independent solution corresponding to $\lambda = -5$ in the form

$$\mathbf{x}_3(t) = e^{-5t}(\mathbf{v}_1 + t\mathbf{v}_0)$$

where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A + 5I)^2 \mathbf{v}_1 = \mathbf{0}, \quad (A + 5I) \mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A + 5I) \mathbf{v}_1.$$

In this case, $(A + 5I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 90 & 36 & 81 \end{bmatrix}$. Let us take $\mathbf{v}_1 = \begin{bmatrix} -9 \\ 0 \\ 10 \end{bmatrix}$. Thus, $\mathbf{v}_0 = (A + 5I)\mathbf{v}_1 = \begin{bmatrix} -18 \\ -36 \\ 36 \end{bmatrix}$.

From the expressions for $\mathbf{v}_0$ and $\mathbf{v}_1$, we can write down a second (linearly independent) solution to the vector differential equation corresponding to $\lambda = -5$:

$$\mathbf{x}_3(t) = e^{-5t} \left\{ \begin{bmatrix} -9 \\ 0 \\ 10 \end{bmatrix} + t \begin{bmatrix} -18 \\ -36 \\ 36 \end{bmatrix} \right\} = e^{-5t} \begin{bmatrix} -9 - 18t \\ -36t \\ 10 + 36t \end{bmatrix}.$$

Consequently, the general solution to the vector differential equation is

$$\mathbf{x}(t) = c_1 e^{4t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + c_3 e^{-5t} \begin{bmatrix} -9 - 18t \\ -36t \\ 10 + 36t \end{bmatrix}.$$

9. We begin by finding the eigenvalues of A . We have

$$\det(A - \lambda I) = \det \begin{bmatrix} 3 - \lambda & 13 \\ -1 & -3 - \lambda \end{bmatrix} = (3 - \lambda)(-3 - \lambda) + 13 = \lambda^2 + 4.$$

Therefore, the eigenvalues of A are $\lambda = \pm 2i$.

Eigenvectors for $\lambda = 2i$: We have $A - 2iI = \begin{bmatrix} 3 - 2i & 13 \\ -1 & -3 - 2i \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(3 + 2i, -1)$, where $r \in \mathbb{C}$. Therefore, we obtain solutions of the form

$$\begin{aligned} \mathbf{x}(t) &= e^{2it} \begin{bmatrix} 3 + 2i \\ -1 \end{bmatrix} \\ &= (\cos 2t + i \sin 2t) \left(\begin{bmatrix} 3 \\ -1 \end{bmatrix} + i \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) \\ &= \left(\cos 2t \begin{bmatrix} 3 \\ -1 \end{bmatrix} - \sin 2t \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) + i \left(\cos 2t \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \sin 2t \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 3 \cos 2t - 2 \sin 2t \\ -\cos 2t \end{bmatrix} + i \begin{bmatrix} 2 \cos 2t + 3 \sin 2t \\ -\sin 2t \end{bmatrix}. \end{aligned}$$

Taking the real and imaginary parts of $\mathbf{x}(t)$, we obtain the two real-valued solutions

$$\mathbf{x}_1(t) = \begin{bmatrix} 3 \cos 2t - 2 \sin 2t \\ -\cos 2t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} 2 \cos 2t + 3 \sin 2t \\ -\sin 2t \end{bmatrix}.$$

Therefore, the general solution to the linear system $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 3 \cos 2t - 2 \sin 2t \\ -\cos 2t \end{bmatrix} + c_2 \begin{bmatrix} 2 \cos 2t + 3 \sin 2t \\ -\sin 2t \end{bmatrix}.$$

10. We begin by finding the eigenvalues of A . We have

$$\det(A - \lambda I) = \det \begin{bmatrix} -3 - \lambda & -10 \\ 5 & 11 - \lambda \end{bmatrix} = (-3 - \lambda)(11 - \lambda) + 50 = \lambda^2 - 8\lambda + 17.$$

Therefore, the eigenvalues of A are $\lambda 4 \pm i$.

Eigenvectors for $\lambda = 4 + i$: We have $A - (4 + i)I = \begin{bmatrix} -7-i & -10 \\ 5 & 7-i \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(7 - i, -5)$, where $r \in \mathbb{C}$. Therefore, we obtain solutions of the form

$$\begin{aligned} \mathbf{x}(t) &= e^{(4+i)t} \begin{bmatrix} 7-i \\ -5 \end{bmatrix} \\ &= e^{4t}(\cos t + i \sin t) \left(\begin{bmatrix} 7 \\ -5 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) \\ &= e^{4t} \left(\cos t \begin{bmatrix} 7 \\ -5 \end{bmatrix} - \sin t \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) + i e^{4t} \left(\cos t \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \sin t \begin{bmatrix} 7 \\ -5 \end{bmatrix} \right) \\ &= e^{4t} \begin{bmatrix} 7 \cos t + \sin t \\ -5 \cos t \end{bmatrix} + i e^{4t} \begin{bmatrix} -\cos t + 7 \sin t \\ -5 \sin t \end{bmatrix}. \end{aligned}$$

Taking the real and imaginary parts of $\mathbf{x}(t)$, we obtain the two real-valued solutions

$$\mathbf{x}_1(t) = e^{4t} \begin{bmatrix} 7 \cos t + \sin t \\ -5 \cos t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{4t} \begin{bmatrix} -\cos t + 7 \sin t \\ -5 \sin t \end{bmatrix}.$$

Therefore, the general solution to the linear system $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{4t} \begin{bmatrix} 7 \cos t + \sin t \\ -5 \cos t \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} -\cos t + 7 \sin t \\ -5 \sin t \end{bmatrix}.$$

11. We begin by finding the eigenvalues of A . We have

$$\det(A - \lambda I) = \det \begin{bmatrix} -1 - \lambda & -5 & 1 \\ 4 & -9 - \lambda & -1 \\ 0 & 0 & 3 - \lambda \end{bmatrix} = (3 - \lambda)(\lambda^2 + 10\lambda + 29).$$

Therefore, the eigenvalues of A are $\lambda = 3$ and $\lambda = -5 \pm 2i$.

Eigenvectors for $\lambda = 3$: We have $A - 3I = \begin{bmatrix} -4 & -5 & 1 \\ 4 & -12 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(1, 0, 4)$, where $r \in \mathbb{C}$. Therefore, we obtain the solution $\mathbf{x}_1(t) = e^{3t} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ to the system $\mathbf{x}' = A\mathbf{x}$.

Eigenvectors for $\lambda = -5 + 2i$: We have $A - (-5 + 2i)I = \begin{bmatrix} 4 - 2i & -5 & 1 \\ 4 & -4 - 2i & -1 \\ 0 & 0 & 8 - 2i \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(4 + 2i, 4, 0)$, where $r \in \mathbb{C}$. Therefore, we obtain solutions of

the form

$$\begin{aligned}
 \mathbf{x}(t) &= e^{(-5+2i)t} \begin{bmatrix} 4+2i \\ 4 \\ 0 \end{bmatrix} \\
 &= e^{-5t}(\cos 2t + i \sin 2t) \left(\begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} + i \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right) \\
 &= e^{-5t} \left(\cos 2t \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} - \sin 2t \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right) + i e^{-5t} \left(\cos 2t \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \sin 2t \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} \right) \\
 &= e^{-5t} \begin{bmatrix} 4 \cos 2t - 2 \sin 2t \\ 4 \cos 2t \\ 0 \end{bmatrix} + i e^{-5t} \begin{bmatrix} 2 \cos 2t + 4 \sin 2t \\ 4 \sin 2t \\ 0 \end{bmatrix}.
 \end{aligned}$$

Taking the real and imaginary parts of $\mathbf{x}(t)$, we obtain the two real-valued solutions

$$\mathbf{x}_2(t) = e^{-5t} \begin{bmatrix} 4 \cos 2t - 2 \sin 2t \\ 4 \cos 2t \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_3(t) = e^{-5t} \begin{bmatrix} 2 \cos 2t + 4 \sin 2t \\ 4 \sin 2t \\ 0 \end{bmatrix}.$$

Therefore, the general solution to the linear system $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 4 \cos 2t - 2 \sin 2t \\ 4 \cos 2t \\ 0 \end{bmatrix} + c_3 e^{-5t} \begin{bmatrix} 2 \cos 2t + 4 \sin 2t \\ 4 \sin 2t \\ 0 \end{bmatrix}.$$

12. We begin by finding the eigenvalues of A . We have

$$\det(A - \lambda I) = \det \begin{bmatrix} -4 - \lambda & 0 & 0 \\ 2 & 5 - \lambda & -9 \\ 0 & 5 & -1 - \lambda \end{bmatrix} = (-4 - \lambda)(\lambda^2 - 4\lambda + 40).$$

Therefore, the eigenvalues of A are $\lambda = -4$ and $\lambda = 2 \pm 6i$.

Eigenvectors for $\lambda = -4$: We have $A + 4I = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 9 & -9 \\ 0 & 5 & 3 \end{bmatrix}$, which gives rise to eigenvectors of the form

$\mathbf{v} = r(36, -3, 5)$, where $r \in \mathbb{C}$. Therefore, we obtain the solution $\mathbf{x}_1(t) = e^{-4t} \begin{bmatrix} 36 \\ -3 \\ 5 \end{bmatrix}$ to the system $\mathbf{x}' = A\mathbf{x}$.

Eigenvectors for $\lambda = 2 + 6i$: We have $A - (2 + 6i)I = \begin{bmatrix} -6 - 6i & 0 & 0 \\ 2 & 3 - 6i & -9 \\ 0 & 5 & -3 - 6i \end{bmatrix}$, which gives rise to

eigenvectors of the form $\mathbf{v} = r(0, 3, 1 - 2i)$, where $r \in \mathbb{C}$. Therefore, we obtain solutions of the form

$$\begin{aligned}\mathbf{x}(t) &= e^{(2+6i)t} \begin{bmatrix} 0 \\ 3 \\ 1 - 2i \end{bmatrix} \\ &= e^{2t}(\cos 6t + i \sin 6t) \left(\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \right) \\ &= e^{2t} \left(\cos 6t \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} - \sin 6t \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \right) + ie^{2t} \left(\cos 6t \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} + \sin 6t \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right) \\ &= e^{2t} \begin{bmatrix} 0 \\ 3 \cos 6t \\ \cos 6t + 2 \sin 6t \end{bmatrix} + ie^{2t} \begin{bmatrix} 0 \\ 3 \sin 6t \\ -2 \cos 6t + \sin 6t \end{bmatrix}.\end{aligned}$$

Taking the real and imaginary parts of $\mathbf{x}(t)$, we obtain the two real-valued solutions

$$\mathbf{x}_2(t) = e^{2t} \begin{bmatrix} 0 \\ 3 \cos 6t \\ \cos 6t + 2 \sin 6t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_3(t) = e^{2t} \begin{bmatrix} 0 \\ 3 \sin 6t \\ -2 \cos 6t + \sin 6t \end{bmatrix}.$$

Therefore, the general solution to the linear system $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{-4t} \begin{bmatrix} 36 \\ -3 \\ 5 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 3 \cos 6t \\ \cos 6t + 2 \sin 6t \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 0 \\ 3 \sin 6t \\ -2 \cos 6t + \sin 6t \end{bmatrix}.$$

13. In the hint, we are given the eigenvalues of A : $\lambda = 2$, $\lambda = -2$, and $\lambda = -5$.

Eigenvectors for $\lambda = 2$: We have $A - 2I = \begin{bmatrix} 0 & -2 & 1 \\ 1 & -6 & 1 \\ 2 & 2 & -5 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(4, 1, 2)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_1(t) = e^{2t} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$.

Eigenvectors for $\lambda = -2$: We have $A + 2I = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -2 & 1 \\ 2 & 2 & -1 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(0, 1, 2)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_2(t) = e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

Eigenvectors for $\lambda = -5$: We have $A + 5I = \begin{bmatrix} 7 & -2 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(1, 2, -3)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_3(t) = e^{-5t} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

Putting the results above together, we obtain the general solution to the linear system:

$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c_3 e^{-5t} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}.$$

14. In the hint, we are given the eigenvalues of A : $\lambda = 6$, $\lambda = -3$, and $\lambda = -1$.

Eigenvectors for $\lambda = 6$: We have $A - 6I = \begin{bmatrix} -4 & -4 & 3 \\ -9 & -9 & -9 \\ 4 & 4 & -3 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(-1, 1, 0)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_1(t) = e^{6t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Eigenvectors for $\lambda = -3$: We have $A + 3I = \begin{bmatrix} 5 & -4 & 3 \\ -9 & 0 & -9 \\ 4 & 4 & 6 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(-2, -1, 2)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_2(t) = e^{-3t} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$.

Eigenvectors for $\lambda = -1$: We have $A + I = \begin{bmatrix} 3 & -4 & 3 \\ -9 & -2 & -9 \\ 4 & 4 & 4 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(-1, 0, 1)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_3(t) = e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Putting the results above together, we obtain the general solution to the linear system:

$$\mathbf{x}(t) = c_1 e^{6t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

15. In the hint, we are given the eigenvalues of A : $\lambda = 4$ and $\lambda = -3$. Since $\det(A) = -48$, we use the fact that the $\det(A)$ is the product of the eigenvalues of A (see Section 5.7, Problem 29(c)) to deduce that $\lambda = 4$ occurs with multiplicity 2.

Eigenvectors for $\lambda = 4$: We have $A - 4I = \begin{bmatrix} -21 & 0 & -42 \\ -7 & 0 & -14 \\ 7 & 0 & 14 \end{bmatrix}$, which gives rise to eigenvectors of the form

$\mathbf{v} = r(0, 1, 0) + s(-2, 0, 1)$, where $r, s \in \mathbb{R}$. Therefore, we obtain the solutions $\mathbf{x}_1(t) = e^{4t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and

$\mathbf{x}_2(t) = e^{4t} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ to the system $\mathbf{x}' = A\mathbf{x}$.

Eigenvectors for $\lambda = -3$: We have $A + 3I = \begin{bmatrix} -14 & 0 & -42 \\ -7 & 7 & -14 \\ 7 & 0 & 21 \end{bmatrix}$, which gives rise to corresponding eigenvec-

tors of the form $\mathbf{v} = r(-3, -1, 1)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_3(t) = e^{3t} \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$ to the system $\mathbf{x}' = A\mathbf{x}$.

Putting the results above together, we obtain the general solution to the linear system:

$$\mathbf{x}(t) = c_1 e^{4t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}.$$

16. In the hint, we are given the eigenvalues of A : $\lambda = 8$, $\lambda = -7$, and $\lambda = 0$.

Eigenvectors for $\lambda = 8$: We have $A - 8I = \begin{bmatrix} -24 & 30 & -18 \\ -8 & 0 & 16 \\ 8 & -15 & 1 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r$ (

17. In the hint, we are given the eigenvalues of A : $\lambda = 0$ and $\lambda = -3$. According to Section 5.7, Problem 29(c), the sum of the eigenvalues of A is $\text{tr}(A) = -3$. Therefore, we conclude that $\lambda = 0$ occurs with multiplicity 2.

Eigenvectors for $\lambda = 0$: The matrix $A - 0I = A$ gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(-1, 0, 1)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_1(t) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ to $\mathbf{x}' = A\mathbf{x}$.

We will determine a second linearly independent solution corresponding to $\lambda = 0$ in the form

$$\mathbf{x}_3(t) = \mathbf{v}_1 + t\mathbf{v}_0$$

where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$A^2\mathbf{v}_1 = \mathbf{0}, \quad A\mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = A\mathbf{v}_1.$$

In this case, $A^2 = \begin{bmatrix} 18 & 18 & 18 \\ 9 & 9 & 9 \\ -18 & -18 & -18 \end{bmatrix}$. Let us choose $\mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. Then $\mathbf{v}_0 = A\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. From the expressions for $\mathbf{v}_0$ and $\mathbf{v}_1$, we can write down a second (linearly independent) solution to the vector differential equation corresponding to $\lambda = 0$:

$$\mathbf{x}_2(t) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -t \\ -1 \\ 1+t \end{bmatrix}.$$

Eigenvectors for $\lambda = -3$: We have $A + 3I = \begin{bmatrix} -4 & -6 & -7 \\ -3 & 0 & -3 \\ 7 & 6 & 10 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(-2, -1, 2)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_3(t) = e^{-3t} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$ to the system $\mathbf{x}' = A\mathbf{x}$.

Putting the results above together, we obtain the general solution to the linear system:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -t \\ -1 \\ 1+t \end{bmatrix} + c_3 e^{-3t} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}.$$

18. In the hint, we are given the eigenvalues of A : $\lambda = 5$ (with multiplicity 3).

Eigenvectors for $\lambda = 5$: The matrix $A - 5I = \begin{bmatrix} -2 & -1 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, which gives rise to corresponding eigen-

vectors of the form $\mathbf{v} = r(-1, 0, 1)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_0(t) = e^{5t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ to the system $\mathbf{x}' = A\mathbf{x}$.

According to Theorem 7.5.4, there exist two further linearly independent solutions to $\mathbf{x}' = A\mathbf{x}$ of the form

$$\mathbf{x}_1(t) = e^{5t}(\mathbf{v}_1 + t\mathbf{v}_0),$$

$$\mathbf{x}_2(t) = e^{5t}(\mathbf{v}_2 + t\mathbf{v}_1 + \frac{1}{2!}t^2\mathbf{v}_0),$$

where

$$(A - 5I)^3\mathbf{v}_2 = \mathbf{0}, \quad (A - 5I)^2\mathbf{v}_2 \neq \mathbf{0},$$

and

$$\mathbf{v}_1 = (A - 5I)\mathbf{v}_2, \quad \mathbf{v}_0 = (A - 5I)^2\mathbf{v}_2.$$

A short calculation shows that

$$A - 5I = \begin{bmatrix} -2 & -1 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad (A - 5I)^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}, \quad (A - 5I)^3 = 0_3.$$

Therefore, we may take

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_1 = (A - 5I)\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_0 = (A - 5I)^2\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Hence, we obtain

$$\mathbf{x}_1(t) = e^{5t} \left(\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) = e^{5t} \begin{bmatrix} -2+t \\ 1 \\ 1-t \end{bmatrix}$$

and

$$\mathbf{x}_2(t) = e^{5t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2!}t^2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) = e^{5t} \begin{bmatrix} 1-2t+t^2/2 \\ t \\ t-t^2/2 \end{bmatrix}.$$

Therefore, the general solution to the vector differential equation is

$$\begin{aligned}\mathbf{x}(t) &= c_1 e^{5t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} -2+t \\ 1 \\ 1-t \end{bmatrix} + c_3 e^{5t} \begin{bmatrix} 1-2t+t^2/2 \\ t \\ t-t^2/2 \end{bmatrix} \\ &= e^{5t} \left\{ c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2+t \\ 1 \\ 1-t \end{bmatrix} + c_3 \begin{bmatrix} 1-2t+t^2/2 \\ t \\ t-t^2/2 \end{bmatrix} \right\}.\end{aligned}$$

19. In the hint, we are given the eigenvalues of A : $\lambda = -1$ and $\lambda = 1 \pm 2i$.

Eigenvectors for $\lambda = -1$: The matrix $A + I = \begin{bmatrix} 0 & -4 & -2 \\ -4 & -4 & -6 \\ 4 & 8 & 8 \end{bmatrix}$, which gives rise to corresponding eigen-

vectors of the form $\mathbf{v} = r(-2, -1, 2)$, where $r \in \mathbb{C}$. Therefore, we obtain the solution $\mathbf{x}_1(t) = e^{-t} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$ to the system $\mathbf{x}' = A\mathbf{x}$.

Eigenvectors for $\lambda = 1 + 2i$: The matrix $A - (1 + 2i)I = \begin{bmatrix} -2-2i & -4 & -2 \\ -4 & -6-2i & -6 \\ 4 & 8 & 6-2i \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(-1, i, 1-i)$, where $r \in \mathbb{C}$. Therefore, we obtain solutions of the form

$$\begin{aligned}\mathbf{x}(t) &= e^{(1+2i)t} \begin{bmatrix} -1 \\ i \\ 1-i \end{bmatrix} \\ &= e^t (\cos 2t + i \sin 2t) \left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right) \\ &= e^t \left(\cos 2t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \sin 2t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right) + ie^t \left(\cos 2t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \sin 2t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right) \\ &= e^t \begin{bmatrix} -\cos 2t \\ -\sin 2t \\ \cos 2t + \sin 2t \end{bmatrix} + ie^t \begin{bmatrix} -\sin 2t \\ \cos 2t \\ -\cos 2t + \sin 2t \end{bmatrix}.\end{aligned}$$

Taking the real and imaginary parts of $\mathbf{x}(t)$, we obtain the two real-valued solutions

$$\mathbf{x}_2(t) = e^t \begin{bmatrix} -\cos 2t \\ -\sin 2t \\ \cos 2t + \sin 2t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_3(t) = e^t \begin{bmatrix} -\sin 2t \\ \cos 2t \\ -\cos 2t + \sin 2t \end{bmatrix}.$$

Putting together the above results, we obtain the general solution to the system $\mathbf{x}' = A\mathbf{x}$:

$$\mathbf{x}(t) = c_1 e^{-t} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} + c_2 e^t \begin{bmatrix} -\cos 2t \\ -\sin 2t \\ \cos 2t + \sin 2t \end{bmatrix} + c_3 e^t \begin{bmatrix} -\sin 2t \\ \cos 2t \\ -\cos 2t + \sin 2t \end{bmatrix}.$$

20. In the hint, we are given that the only eigenvalue of A is $\lambda = 5$.

Eigenvectors for $\lambda = 5$: The matrix $A - 5I = \begin{bmatrix} 2 & -2 & 2 \\ 0 & -1 & -1 \\ -1 & 1 & -1 \end{bmatrix}$, which gives rise to corresponding eigen-

vectors $\mathbf{v} = r(-2, -1, 1)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_0(t) = e^{5t} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ to the system

$\mathbf{x}' = A\mathbf{x}$.

According to Theorem 7.5.4, there exist two further linearly independent solutions to $\mathbf{x}' = A\mathbf{x}$ of the form

$$\mathbf{x}_1(t) = e^{5t}(\mathbf{v}_1 + t\mathbf{v}_0),$$

$$\mathbf{x}_2(t) = e^{5t}(\mathbf{v}_2 + t\mathbf{v}_1 + \frac{1}{2!}t^2\mathbf{v}_0),$$

where

$$(A - 5I)^3\mathbf{v}_2 = \mathbf{0}, \quad (A - 5I)^2\mathbf{v}_2 \neq \mathbf{0},$$

and

$$\mathbf{v}_1 = (A - 5I)\mathbf{v}_2, \quad \mathbf{v}_0 = (A - 5I)^2\mathbf{v}_2.$$

A short calculation shows that

$$A - 5I = \begin{bmatrix} 2 & -2 & 2 \\ 0 & -1 & -1 \\ -1 & 1 & -1 \end{bmatrix}, \quad (A - 5I)^2 = \begin{bmatrix} 2 & 0 & 4 \\ 1 & 0 & 2 \\ -1 & 0 & -2 \end{bmatrix}, \quad (A - 5I)^3 = 0_3.$$

Therefore, we may take

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_1 = (A - 5I)\mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_0 = (A - 5I)^2\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

Hence, we obtain

$$\mathbf{x}_1(t) = e^{5t} \left(\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right) = e^{5t} \begin{bmatrix} -2 + 2t \\ t \\ 1 - t \end{bmatrix}$$

and

$$\mathbf{x}_2(t) = e^{5t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2!}t^2 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right) = e^{5t} \begin{bmatrix} 1 - 2t + t^2 \\ t^2/2 \\ t - t^2/2 \end{bmatrix}.$$

Therefore, the general solution to the vector differential equation is

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{5t} \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} -2 + 2t \\ t \\ 1 - t \end{bmatrix} + c_3 e^{5t} \begin{bmatrix} 1 - 2t + t^2 \\ t^2/2 \\ t - t^2/2 \end{bmatrix} \\ &= e^{5t} \left\{ c_1 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 + 2t \\ t \\ 1 - t \end{bmatrix} + c_3 \begin{bmatrix} 1 - 2t + t^2 \\ t^2/2 \\ t - t^2/2 \end{bmatrix} \right\}. \end{aligned}$$

21. We begin by finding the eigenvalues of A . We have

$$\det(A - \lambda I) = \det \begin{bmatrix} -3 - \lambda & -1 & -2 \\ 1 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{bmatrix} = -1 - 2\lambda - \lambda(\lambda^2 + 3\lambda + 1) = -\lambda^3 - 3\lambda^2 - 3\lambda - 1 = -(\lambda + 1)^3,$$

where we have used cofactor expansion along the third row to compute the determinant. Therefore, we have a single eigenvalue, $\lambda = -1$, with multiplicity 3.

Eigenvectors for $\lambda = -1$: We have $A + I = \begin{bmatrix} -2 & -1 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(-1, 0, 1)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_0(t) = e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ to the system $\mathbf{x}' = A\mathbf{x}$.

According to Theorem 7.5.4, there exist two further linearly independent solutions to $\mathbf{x}' = A\mathbf{x}$ of the form

$$\mathbf{x}_1(t) = e^{-t}(\mathbf{v}_1 + t\mathbf{v}_0),$$

$$\mathbf{x}_2(t) = e^{-t}(\mathbf{v}_2 + t\mathbf{v}_1 + \frac{1}{2!}t^2\mathbf{v}_0),$$

where

$$(A + I)^3\mathbf{v}_2 = \mathbf{0}, \quad (A + I)^2\mathbf{v}_2 \neq \mathbf{0},$$

and

$$\mathbf{v}_1 = (A + I)\mathbf{v}_2, \quad \mathbf{v}_0 = (A + I)^2\mathbf{v}_2.$$

A short calculation shows that

$$A + I = \begin{bmatrix} -2 & -1 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad (A + I)^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix}, \quad (A + I)^3 = 0_3.$$

Therefore, we may take

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_1 = (A + I)\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_0 = (A + I)^2\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Hence, we obtain

$$\mathbf{x}_1(t) = e^{-t} \left(\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) = e^{-t} \begin{bmatrix} -2 + t \\ 1 \\ 1 - t \end{bmatrix}$$

and

$$\mathbf{x}_2(t) = e^{-t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2!}t^2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) = e^{-t} \begin{bmatrix} 1 - 2t + t^2/2 \\ t \\ t - t^2/2 \end{bmatrix}.$$

Therefore, the general solution to the vector differential equation is

$$\begin{aligned}\mathbf{x}(t) &= c_1 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -2+t \\ 1 \\ 1-t \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 1-2t+t^2/2 \\ t \\ t-t^2/2 \end{bmatrix} \\ &= e^{-t} \left\{ c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2+t \\ 1 \\ 1-t \end{bmatrix} + c_3 \begin{bmatrix} 1-2t+t^2/2 \\ t \\ t-t^2/2 \end{bmatrix} \right\}.\end{aligned}$$

22. We begin by finding the eigenvalues of A . We have

$$\det(A - \lambda I) = \det \begin{bmatrix} -2-\lambda & 0 & -1 \\ 0 & -1-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} = -1-\lambda-\lambda(-1-\lambda)(-2-\lambda) = (-1-\lambda)(\lambda^2+2\lambda+1) = -(\lambda+1)^3,$$

where we have used cofactor expansion along the third row to compute the determinant. Therefore, we have a single eigenvalue, $\lambda = -1$ with multiplicity 3.

Eigenvectors for $\lambda = -1$: We have $A + I = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(-1, 0, 1) + s(0, 1, 0)$, where $r, s \in \mathbb{R}$. Therefore, we obtain the solutions $\mathbf{x}_0(t) = e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{y}_0(t) = e^{-t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

We will determine a second linearly independent solution corresponding to $\lambda = -1$ in the form $\mathbf{x}_1(t) = \mathbf{v}_1 + t\mathbf{v}_0$ where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A + I)^2 \mathbf{v}_1 = \mathbf{0}, \quad (A + I) \mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A + I) \mathbf{v}_1.$$

In this case, $(A+I)^2 = 0_3$. Let us choose $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then $\mathbf{v}_0 = (A+I)\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. From the expressions for $\mathbf{v}_0$ and $\mathbf{v}_1$, we can write down a second (linearly independent) solution to the vector differential equation corresponding to $\lambda = -1$:

$$\mathbf{x}_1(t) = e^{-t} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right) = e^{-t} \begin{bmatrix} 1-t \\ 0 \\ t \end{bmatrix}.$$

Putting together the three linearly independent solutions obtained above, we find the general solution to this linear system:

$$\begin{aligned}\mathbf{x}(t) &= c_1 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 1-t \\ 0 \\ t \end{bmatrix} \\ &= e^{-t} \left\{ c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1-t \\ 0 \\ t \end{bmatrix} \right\}.\end{aligned}$$

23. We begin by finding the eigenvalues of A . We have

$$\det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & 13 & 0 & 0 \\ -1 & -2 - \lambda & 0 & 0 \\ 0 & 0 & 2 - \lambda & 4 \\ 0 & 0 & 0 & 2 - \lambda \end{bmatrix} = (2 - \lambda)^2(\lambda^2 + 9),$$

so the eigenvalues of A are $\lambda = 2$ (with multiplicity 2) and $\lambda = \pm 3i$.

Eigenvectors for $\lambda = 2$: We have $A - 2I = \begin{bmatrix} 0 & 13 & 0 & 0 \\ -1 & -4 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(0, 0, 1, 0)$, where $r \in \mathbb{C}$. Therefore, we obtain the solution $\mathbf{x}_1(t) = e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

We will determine a second linearly independent solution corresponding to $\lambda = 2$ in the form $\mathbf{x}_2(t) = \mathbf{v}_1 + t\mathbf{v}_0$, where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A - 2I)^2 \mathbf{v}_1 = \mathbf{0}, \quad (A - 2I) \mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A - 2I) \mathbf{v}_1.$$

In this case, $(A - 2I)^2 = \begin{bmatrix} -13 & -52 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so we can choose $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Then $\mathbf{v}_0 = (A - 2I) \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix}$. From the expressions for $\mathbf{v}_0$ and $\mathbf{v}_1$, we can write down a second (linearly independent) solution to the vector differential equation corresponding to $\lambda = 2$:

$$\mathbf{x}_2(t) = e^{2t} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix} \right) = e^{2t} \begin{bmatrix} 0 \\ 0 \\ 4t \\ 1 \end{bmatrix}.$$

Eigenvectors for $\lambda = 3i$: We have $A - 3iI = \begin{bmatrix} 2 - 3i & 13 & 0 & 0 \\ -1 & -2 - 3i & 0 & 0 \\ 0 & 0 & 2 - 3i & 4 \\ 0 & 0 & 0 & 2 - 3i \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(2 + 3i, -1, 0, 0)$, where $r \in \mathbb{C}$. Hence, we obtain solutions of the

form

$$\begin{aligned}
 \mathbf{x}(t) &= e^{3it} \begin{bmatrix} 2+3i \\ -1 \\ 0 \\ 0 \end{bmatrix} \\
 &= (\cos 3t + i \sin 3t) \left(\begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \\
 &= \left(\cos 3t \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \sin 3t \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) + i \left(\cos 3t \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \sin 3t \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right) \\
 &= \begin{bmatrix} 2 \cos 3t - 3 \sin 3t \\ -\cos 3t \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 3 \cos 3t + 2 \sin 3t \\ -\sin 3t \\ 0 \\ 0 \end{bmatrix}.
 \end{aligned}$$

Taking the real and imaginary parts of $\mathbf{x}(t)$ yields the two real-valued solutions

$$\mathbf{x}_3(t) = \begin{bmatrix} 2 \cos 3t - 3 \sin 3t \\ -\cos 3t \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_4(t) = \begin{bmatrix} 3 \cos 3t + 2 \sin 3t \\ -\sin 3t \\ 0 \\ 0 \end{bmatrix}.$$

Putting together the four solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\mathbf{x}_3(t)$, and $\mathbf{x}_4(t)$ to the linear system $\mathbf{x}' = A\mathbf{x}$ obtained above, we obtain the general solution to the system:

$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 0 \\ 4t \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \cos 3t - 3 \sin 3t \\ -\cos 3t \\ 0 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 3 \cos 3t + 2 \sin 3t \\ -\sin 3t \\ 0 \\ 0 \end{bmatrix}.$$

24. We begin by finding the eigenvalues of A . We have

$$\det(A - \lambda I) = \det \begin{bmatrix} 7-\lambda & 0 & 0 & -1 \\ 0 & 6-\lambda & 0 & 0 \\ 0 & 0 & -1-\lambda & 0 \\ 2 & 0 & 0 & 5-\lambda \end{bmatrix} = (6-\lambda)(-1-\lambda)(\lambda^2 - 12\lambda + 37),$$

so the eigenvalues of A are $\lambda = 6$, $\lambda = -1$, and $\lambda = 6 \pm i$.

Eigenvectors for $\lambda = 6$: We have $A - 6I = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -7 & 0 \\ 2 & 0 & 0 & -1 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(0, 1, 0, 0)$, where $r \in \mathbb{C}$. Therefore, we obtain the solution $\mathbf{x}_1(t) = e^{6t} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Eigenvectors for $\lambda = -1$: We have $A + I = \begin{bmatrix} 8 & 0 & 0 & -1 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(0, 0, 1, 0)$, where $r \in \mathbb{C}$. Therefore, we obtain the solution $\mathbf{x}_2(t) = e^{-t} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

Eigenvectors for $\lambda = 6 + i$: We have $A - (6 + i)I = \begin{bmatrix} 1 - i & 0 & 0 & -1 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -7 - i & 0 \\ 2 & 0 & 0 & -1 - i \end{bmatrix}$, which gives rise to

corresponding eigenvectors of the form $\mathbf{v} = r(1, 0, 0, 1 - i)$, where $r \in \mathbb{C}$. Therefore, we obtain solutions of the form

$$\begin{aligned} \mathbf{x}(t) &= e^{(6+i)t} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 - i \end{bmatrix} \\ &= e^{6t}(\cos t + i \sin t) \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right) \\ &= e^{6t} \left(\cos t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right) + i e^{6t} \left(\cos t \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} + \sin t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) \\ &= e^{6t} \begin{bmatrix} \cos t \\ 0 \\ 0 \\ \cos t + \sin t \end{bmatrix} + i e^{6t} \begin{bmatrix} \sin t \\ 0 \\ 0 \\ -\cos t + \sin t \end{bmatrix}. \end{aligned}$$

Taking the real and imaginary parts of $\mathbf{x}(t)$, we obtain the two real-valued solutions

$$\mathbf{x}_3(t) = e^{6t} \begin{bmatrix} \cos t \\ 0 \\ 0 \\ \cos t + \sin t \end{bmatrix} \quad \text{and} \quad \mathbf{x}_4(t) = e^{6t} \begin{bmatrix} \sin t \\ 0 \\ 0 \\ -\cos t + \sin t \end{bmatrix}.$$

Putting together the four solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\mathbf{x}_3(t)$, and $\mathbf{x}_4(t)$ to the linear system $\mathbf{x}' = A\mathbf{x}$ obtained above, we obtain the general solution to the system:

$$\mathbf{x}(t) = c_1 e^{6t} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{6t} \begin{bmatrix} \cos t \\ 0 \\ 0 \\ \cos t + \sin t \end{bmatrix} + c_4 e^{6t} \begin{bmatrix} \sin t \\ 0 \\ 0 \\ -\cos t + \sin t \end{bmatrix}.$$

25. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} -6 - \lambda & 1 \\ 6 & -5 - \lambda \end{vmatrix} = 0$ if and only if $\lambda^2 + 11\lambda + 24 = 0$ if and

only if $(\lambda + 3)(\lambda + 8) = 0$. This means that the eigenvalues of A are $\lambda = -3$ and $\lambda = -8$.

Eigenvectors for $\lambda = -3$: We have $A + 3I = \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(1, 3)$, where $r \in \mathbb{R}$. Hence, one solution to the associated homogeneous system is $\mathbf{x}_1(t) = e^{-3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Eigenvectors for $\lambda = -8$: We have $A + 8I = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(-1, 2)$, where $r \in \mathbb{R}$. Hence, a second solution to the associated homogeneous system is $\mathbf{x}_2(t) = e^{-8t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Putting the two solutions obtained above together, we obtain the general solution to the associated homogeneous system:

$$\mathbf{x}_c(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-8t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

The fundamental matrix is

$$X(t) = \begin{bmatrix} e^{-3t} & -e^{-8t} \\ 3e^{-3t} & 2e^{-8t} \end{bmatrix}.$$

We have

$$X^{-1}(t) = \frac{1}{5} \begin{bmatrix} 2e^{3t} & e^{3t} \\ -3e^{8t} & e^{8t} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \mathbf{u}(t) &= \int^t X^{-1}(s) \mathbf{b}(s) ds \\ &= \frac{1}{5} \int^t \begin{bmatrix} 2e^{3s} & e^{3s} \\ -3e^{8s} & e^{8s} \end{bmatrix} \begin{bmatrix} 1 \\ e^{-s} \end{bmatrix} ds \\ &= \frac{1}{5} \int^t \begin{bmatrix} 2e^{3s} + e^{2s} \\ -3e^{8s} + e^{7s} \end{bmatrix} ds \\ &= \frac{1}{5} \begin{bmatrix} \frac{2}{3}e^{3t} + \frac{1}{2}e^{2t} \\ -\frac{3}{8}e^{8t} + \frac{1}{7}e^{7t} \end{bmatrix}. \end{aligned}$$

Hence, we obtain the particular solution

$$\mathbf{x}_p(t) = X(t) \mathbf{u}(t) = \begin{bmatrix} e^{-3t} & -e^{-8t} \\ 3e^{-3t} & 2e^{-8t} \end{bmatrix} \frac{1}{5} \begin{bmatrix} \frac{2}{3}e^{3t} + \frac{1}{2}e^{2t} \\ -\frac{3}{8}e^{8t} + \frac{1}{7}e^{7t} \end{bmatrix} = \begin{bmatrix} \frac{5}{24} + \frac{1}{14}e^{-t} \\ \frac{1}{4} + \frac{5}{14}e^{-t} \end{bmatrix}.$$

Thus, the general solution to the nonhomogeneous system of differential equations is

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-8t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} \frac{5}{24} + \frac{1}{14}e^{-t} \\ \frac{1}{4} + \frac{5}{14}e^{-t} \end{bmatrix}.$$

26. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} 9 - \lambda & -2 \\ 5 & -2 - \lambda \end{vmatrix} = 0$ if and only if $\lambda^2 - 7\lambda - 8 = 0$ if and only if $(\lambda - 8)(\lambda + 1) = 0$. This means that the eigenvalues of A are $\lambda = 8$ and $\lambda = -1$.

Eigenvectors for $\lambda = 8$: We have $A - 8I = \begin{bmatrix} 1 & -2 \\ 5 & -10 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(2, 1)$, where $r \in \mathbb{R}$. Hence, one solution to the associated homogeneous system is $\mathbf{x}_1(t) = e^{8t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Eigenvectors for $\lambda = -1$: We have $A + I = \begin{bmatrix} 10 & -2 \\ 5 & -1 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(1, 5)$, where $r \in \mathbb{R}$. Hence, a second solution to the associated homogeneous system is $\mathbf{x}_2(t) = e^{-t} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

Putting the two solutions obtained above together, we obtain the general solution to the associated homogeneous system:

$$\mathbf{x}_c(t) = c_1 e^{8t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

The fundamental matrix is

$$X(t) = \begin{bmatrix} 2e^{8t} & e^{-t} \\ e^{8t} & 5e^{-t} \end{bmatrix}.$$

We have

$$X^{-1}(t) = \frac{1}{9} \begin{bmatrix} 5e^{-8t} & -e^{-8t} \\ -e^t & 2e^t \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \mathbf{u}(t) &= \int^t X^{-1}(s) \mathbf{b}(s) ds \\ &= \frac{1}{9} \int^t \begin{bmatrix} 5e^{-8s} & -e^{-8s} \\ -e^s & 2e^s \end{bmatrix} \begin{bmatrix} 9s \\ 0 \end{bmatrix} ds \\ &= \int^t \begin{bmatrix} 5se^{-8s} \\ -se^s \end{bmatrix} ds \\ &= \begin{bmatrix} -\frac{5}{8}te^{-8t} + \frac{5}{64}e^{-8t} \\ -te^t + e^t \end{bmatrix}. \end{aligned}$$

Hence, we obtain the particular solution

$$\mathbf{x}_p(t) = X(t)\mathbf{u}(t) = \begin{bmatrix} 2e^{8t} & e^{-t} \\ e^{8t} & 5e^{-t} \end{bmatrix} \begin{bmatrix} -\frac{5}{8}te^{-8t} + \frac{5}{64}e^{-8t} \\ -te^t + e^t \end{bmatrix} = \begin{bmatrix} -\frac{9}{4}t + \frac{37}{32} \\ -\frac{45}{8}t + \frac{165}{32} \end{bmatrix}.$$

Thus, the general solution to the nonhomogeneous system of differential equations is

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t) = c_1 e^{8t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} -\frac{9}{4}t + \frac{37}{32} \\ -\frac{45}{8}t + \frac{165}{32} \end{bmatrix}.$$

27. We have $\det(A - \lambda I) = 0$ if and only if $\begin{vmatrix} 10 - \lambda & -4 \\ 4 & 2 - \lambda \end{vmatrix} = 0$ if and only if $\lambda^2 - 12\lambda + 36 = 0$ if and only if $(\lambda - 6)^2 = 0$. This means that the only eigenvalue of A is $\lambda = 6$, with multiplicity 2.

Eigenvectors for $\lambda = 6$: We have $A - 6I = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(1, 1)$, where $r \in \mathbb{R}$. Hence, one solution to the associated homogeneous system is $\mathbf{x}_1(t) =$

$e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We will determine a second linearly independent solution of the form

$$\mathbf{x}_2(t) = e^{6t}(\mathbf{v}_1 + t\mathbf{v}_0)$$

where $\mathbf{v}_1$ and $\mathbf{v}_0$ are determined from

$$(A - 6I)^2\mathbf{v}_1 = \mathbf{0}, \quad (A - 6I)\mathbf{v}_1 \neq \mathbf{0}, \quad \mathbf{v}_0 = (A - 6I)\mathbf{v}_1.$$

In this case, we have $A - 6I = \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}$ and $(A - 6I)^2 = 0_2$. Therefore, we may choose $\mathbf{v}_1$ to be any vector such that $(A - 6I)\mathbf{v}_1 \neq \mathbf{0}$. For simplicity, we take $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus, $\mathbf{v}_0 = (A - 6I)\mathbf{v}_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. From the expressions for $\mathbf{v}_0$ and $\mathbf{v}_1$, we can write down a second solution to the associated homogeneous system:

$$\mathbf{x}_2(t) = e^{6t} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 4 \end{bmatrix} \right\} = e^{6t} \begin{bmatrix} 1 + 4t \\ 4t \end{bmatrix}.$$

Putting the two solutions obtained above together, we obtain the general solution to the associated homogeneous system:

$$\mathbf{x}_c(t) = c_1 e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} 1 + 4t \\ 4t \end{bmatrix}.$$

The fundamental matrix is

$$X(t) = \begin{bmatrix} e^{6t} & (1 + 4t)e^{6t} \\ e^{6t} & 4te^{6t} \end{bmatrix}.$$

We have

$$X^{-1}(t) = -e^{-12t} \begin{bmatrix} 4te^{6t} & -(1 + 4t)e^{6t} \\ -e^{6t} & e^{6t} \end{bmatrix} = \begin{bmatrix} -4te^{-6t} & (1 + 4t)e^{-6t} \\ e^{-6t} & -e^{-6t} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \mathbf{u}(t) &= \int^t X^{-1}(s)\mathbf{b}(s)ds \\ &= \int^t \begin{bmatrix} -4se^{-6s} & (1 + 4s)e^{-6s} \\ e^{-6s} & -e^{-6s} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{s}e^{6s} \end{bmatrix} ds \\ &= \int^t \begin{bmatrix} (1 + 4s)/s \\ -1/s \end{bmatrix} ds \\ &= \begin{bmatrix} \ln|t| + 4t \\ -\ln|t| \end{bmatrix}. \end{aligned}$$

Hence, we obtain the particular solution

$$\mathbf{x}_p(t) = X(t)\mathbf{u}(t) = \begin{bmatrix} e^{6t} & (1 + 4t)e^{6t} \\ e^{6t} & 4te^{6t} \end{bmatrix} \begin{bmatrix} \ln|t| + 4t \\ -\ln|t| \end{bmatrix} = e^{6t} \begin{bmatrix} 4t(1 - \ln|t|) \\ \ln|t| + 4t(1 - \ln|t|) \end{bmatrix}.$$

Thus, the general solution to the nonhomogeneous system of differential equations is

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t) = c_1 e^{6t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} 1 + 4t \\ 4t \end{bmatrix} + e^{6t} \begin{bmatrix} 4t(1 - \ln|t|) \\ \ln|t| + 4t(1 - \ln|t|) \end{bmatrix}.$$

28. Using the hint from Problem 14, the eigenvalues of A are $\lambda = 6$, $\lambda = -3$, and $\lambda = -1$.

Eigenvectors for $\lambda = 6$: We have $A - 6I = \begin{bmatrix} -4 & -4 & 3 \\ -9 & -9 & -9 \\ 4 & 4 & -3 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(-1, 1, 0)$, where $r \in \mathbb{R}$. Therefore, we obtain a solution to the associated homogeneous system: $\mathbf{x}_1(t) = e^{6t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Eigenvectors for $\lambda = -3$: We have $A + 3I = \begin{bmatrix} 5 & -4 & 3 \\ -9 & 0 & -9 \\ 4 & 4 & 6 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(-2, -1, 2)$, where $r \in \mathbb{R}$. Therefore, we obtain another solution to the associated homogeneous system: $\mathbf{x}_2(t) = e^{-3t} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$.

Eigenvectors for $\lambda = -1$: We have $A + I = \begin{bmatrix} 3 & -4 & 3 \\ -9 & -2 & -9 \\ 4 & 4 & 4 \end{bmatrix}$, which gives rise to corresponding eigenvectors of the form $\mathbf{v} = r(-1, 0, 1)$, where $r \in \mathbb{R}$. Therefore, we obtain another solution to the associated homogeneous system: $\mathbf{x}_3(t) = e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Putting the results above together, we obtain the general solution to the associated homogeneous system:

$$\mathbf{x}_c(t) = c_1 e^{6t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

A fundamental matrix is

$$X(t) = \begin{bmatrix} -e^{6t} & -2e^{-3t} & -e^{-t} \\ e^{6t} & -e^{-3t} & 0 \\ 0 & 2e^{-3t} & e^{-t} \end{bmatrix}.$$

Using the adjoint method, we determine

$$X^{-1}(t) = \frac{1}{e^{2t}} \begin{bmatrix} -e^{-4t} & 0 & -e^{-4t} \\ -e^{5t} & -e^{5t} & -e^{5t} \\ 2e^{3t} & 2e^{3t} & 3e^{3t} \end{bmatrix} = \begin{bmatrix} -e^{-6t} & 0 & -e^{-6t} \\ -e^{3t} & -e^{3t} & -e^{3t} \\ 2e^t & 2e^t & 3e^t \end{bmatrix}.$$

Therefore,

$$\begin{aligned}
 \mathbf{u}(t) &= \int^t X^{-1}(s) \mathbf{b}(s) ds \\
 &= \int^t \begin{bmatrix} -e^{-6s} & 0 & -e^{-6s} \\ -e^{3s} & -e^{3s} & -e^{3s} \\ 2e^s & 2e^s & 3e^s \end{bmatrix} \begin{bmatrix} e^{6s} \\ 1 \\ 0 \end{bmatrix} ds \\
 &= \int^t \begin{bmatrix} -1 \\ -e^{9s} - e^{3s} \\ 2e^{7s} + 2e^s \end{bmatrix} ds \\
 &= \begin{bmatrix} -t \\ -\frac{1}{9}e^{9t} - \frac{1}{3}e^{3t} \\ \frac{2}{7}e^{7t} + 2e^t \end{bmatrix}.
 \end{aligned}$$

Hence, we obtain the particular solution

$$\mathbf{x}_p(t) = X(t)\mathbf{u}(t) = \begin{bmatrix} -e^{6t} & -2e^{-3t} & -e^{-t} \\ e^{6t} & -e^{-3t} & 0 \\ 0 & 2e^{-3t} & e^{-t} \end{bmatrix} \begin{bmatrix} -t \\ -\frac{1}{9}e^{9t} - \frac{1}{3}e^{3t} \\ \frac{2}{7}e^{7t} + 2e^t \end{bmatrix} = \begin{bmatrix} e^{6t}(t - \frac{4}{63}) - \frac{4}{3} \\ e^{6t}(-t + \frac{1}{9}) + \frac{1}{3} \\ \frac{4}{63}e^{6t} + \frac{4}{3} \end{bmatrix}.$$

Thus, the general solution to the nonhomogeneous system of differential equations is

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t) = c_1 e^{6t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} e^{6t}(t - \frac{4}{63}) - \frac{4}{3} \\ e^{6t}(-t + \frac{1}{9}) + \frac{1}{3} \\ \frac{4}{63}e^{6t} + \frac{4}{3} \end{bmatrix}.$$

29. Using the hint from Problem 13, the eigenvalues of A are $\lambda = 2$, $\lambda = -2$, and $\lambda = -5$.

Eigenvectors for $\lambda = 2$: We have $A - 2I = \begin{bmatrix} 0 & -2 & 1 \\ 1 & -6 & 1 \\ 2 & 2 & -5 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(4, 1, 2)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_1(t) = e^{2t} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$.

Eigenvectors for $\lambda = -2$: We have $A + 2I = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -2 & 1 \\ 2 & 2 & -1 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(0, 1, 2)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_2(t) = e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

Eigenvectors for $\lambda = -5$: We have $A + 5I = \begin{bmatrix} 7 & -2 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$, which gives rise to corresponding eigenvectors

of the form $\mathbf{v} = r(1, 2, -3)$, where $r \in \mathbb{R}$. Therefore, we obtain the solution $\mathbf{x}_3(t) = e^{-5t} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

Putting the results above together, we obtain the general solution to the associated homogeneous system:

$$\mathbf{x}_c(t) = c_1 e^{2t} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c_3 e^{-5t} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}.$$

A fundamental matrix for this system is

$$X(t) = \begin{bmatrix} 4e^{2t} & 0 & e^{-5t} \\ e^{2t} & e^{-2t} & 2e^{-5t} \\ 2e^{2t} & 2e^{-2t} & -3e^{-5t} \end{bmatrix}.$$

Using the adjoint method, we determine

$$X^{-1}(t) = -\frac{1}{28}e^{5t} \begin{bmatrix} -7e^{-7t} & 2e^{-7t} & -e^{-7t} \\ 7e^{-3t} & -14e^{-3t} & -7e^{-3t} \\ 0 & -8 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}e^{-2t} & -\frac{1}{14}e^{-2t} & \frac{1}{28}e^{-2t} \\ -\frac{1}{4}e^{2t} & \frac{1}{2}e^{2t} & \frac{1}{4}e^{2t} \\ 0 & \frac{2}{7}e^{5t} & -\frac{1}{7}e^{5t} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \mathbf{u}(t) &= \int^t X^{-1}(s)\mathbf{b}(s)s \\ &= \int^t \begin{bmatrix} \frac{1}{4}e^{-2s} & -\frac{1}{14}e^{-2s} & \frac{1}{28}e^{-2s} \\ -\frac{1}{4}e^{2s} & \frac{1}{2}e^{2s} & \frac{1}{4}e^{2s} \\ 0 & \frac{2}{7}e^{5s} & -\frac{1}{7}e^{5s} \end{bmatrix} \begin{bmatrix} s \\ 0 \\ 1 \end{bmatrix} ds \\ &= \int^t \begin{bmatrix} (\frac{1}{4}s + \frac{1}{28})e^{-2s} \\ (-\frac{1}{4}s + \frac{1}{4})e^{2s} \\ -\frac{1}{7}e^{5s} \end{bmatrix} \\ &= \begin{bmatrix} (-\frac{1}{8}t - \frac{9}{112})e^{-2t} \\ (-\frac{1}{8}t + \frac{3}{16})e^{2t} \\ -\frac{1}{35}e^{5t} \end{bmatrix}. \end{aligned}$$

Hence, we obtain the particular solution

$$\mathbf{x}_p(t) = X(t)\mathbf{u}(t) = \begin{bmatrix} 4e^{2t} & 0 & e^{-5t} \\ e^{2t} & e^{-2t} & 2e^{-5t} \\ 2e^{2t} & 2e^{-2t} & -3e^{-5t} \end{bmatrix} \begin{bmatrix} (-\frac{1}{8}t - \frac{9}{112})e^{-2t} \\ (-\frac{1}{8}t + \frac{3}{16})e^{2t} \\ -\frac{1}{35}e^{5t} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}t - \frac{7}{20} \\ -\frac{1}{4}t + \frac{29}{20} \\ -\frac{1}{2}t + \frac{3}{10} \end{bmatrix}.$$

Thus, the general solution to the nonhomogeneous system of differential equations is

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t) = c_1 e^{2t} \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c_3 e^{-5t} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}t - \frac{7}{20} \\ -\frac{1}{4}t + \frac{29}{20} \\ -\frac{1}{2}t + \frac{3}{10} \end{bmatrix}.$$

30. FALSE. We can try a simple example. For instance, if $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, a fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$ is given by $X_A(t) = \begin{bmatrix} e^t & e^{2t} \\ 0 & e^{2t} \end{bmatrix}$. However, for $A^T = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$, a fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$ is given by $X_{A^T}(t) = \begin{bmatrix} -e^t & 0 \\ e^t & e^{2t} \end{bmatrix}$. There is no clear relation between X_A and X_{A^T}

obtained in this example. It is certainly not possible to find a fundamental matrix for A^T in this case which is $X_A(t)^T$.

31. TRUE. We have

$$\mathbf{x}_0'' = (\mathbf{x}_0')' = (A\mathbf{x}_0)' = A\mathbf{x}_0' = A(A\mathbf{x}_0) = A^2\mathbf{x}_0,$$

which shows that $\mathbf{x}_0$ is a solution to the linear system $\mathbf{x}'' = A^2\mathbf{x}$.

32. We must show that the function D respects addition and scalar multiplication:

D respects addition: Suppose that $\mathbf{x}(t)$ and $\mathbf{y}(t)$ belong to $V_n(I)$. Then

$$\begin{aligned} D(\mathbf{x}(t) + \mathbf{y}(t)) &= (\mathbf{x}(t) + \mathbf{y}(t))' \\ &= \mathbf{x}'(t) + \mathbf{y}'(t) \\ &= D(\mathbf{x}(t)) + D(\mathbf{y}(t)), \end{aligned}$$

as needed.

D respects scalar multiplication: Suppose that $\mathbf{x}(t)$ belongs to $V_n(I)$ and k is a scalar. Then

$$D(k\mathbf{x}(t)) = (k\mathbf{x}(t))' = k\mathbf{x}'(t) = kD(\mathbf{x}(t)),$$

as needed.

33.

(a) By setting $x_1 = y$, $x_2 = y'$, and $x_3 = y''$, we obtain the system $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{bmatrix}.$$

(b) Supposing that $y_1 = f_1(t)$, $y_2 = f_2(t)$, and $y_3 = f_3(t)$ are solutions to (7.11.8), we have that

$$\frac{d^3 y_1}{dt^3} + a \frac{d^2 y_1}{dt^2} + b \frac{dy_1}{dt} + cy_1 = 0, \quad \frac{d^3 y_2}{dt^3} + a \frac{d^2 y_2}{dt^2} + b \frac{dy_2}{dt} + cy_2 = 0, \quad \frac{d^3 y_3}{dt^3} + a \frac{d^2 y_3}{dt^2} + b \frac{dy_3}{dt} + cy_3 = 0,$$

which gives

$$f_1'''(t) + af_1''(t) + bf_1'(t) + cf_1(t) = 0, \quad f_2'''(t) + af_2''(t) + bf_2'(t) + cf_2(t) = 0, \quad f_3'''(t) + af_3''(t) + bf_3'(t) + cf_3(t) = 0.$$

Substituting $\mathbf{x}_1(t) = \begin{bmatrix} f_1(t) \\ f_1'(t) \\ f_1''(t) \end{bmatrix}$ into $\mathbf{x}' = A\mathbf{x}$, we obtain

$$\begin{bmatrix} f_1'(t) \\ f_1''(t) \\ f_1'''(t) \end{bmatrix} = \begin{bmatrix} f_1'(t) \\ f_1''(t) \\ -cf_1(t) - bf_1'(t) - af_1''(t) \end{bmatrix},$$

which corresponds precisely to the differential equation for f_1 above. Likewise, substituting $\mathbf{x}_2(t) = \begin{bmatrix} f_2(t) \\ f_2'(t) \\ f_2''(t) \end{bmatrix}$

into $\mathbf{x}' = A\mathbf{x}$, we obtain

$$\begin{bmatrix} f_2'(t) \\ f_2''(t) \\ f_2'''(t) \end{bmatrix} = \begin{bmatrix} f_2'(t) \\ f_2''(t) \\ -cf_2(t) - bf_2'(t) - af_2''(t) \end{bmatrix},$$

which corresponds precisely to the differential equation for f_2 above. Finally, substituting $\mathbf{x}_3(t) = \begin{bmatrix} f_3(t) \\ f'_3(t) \\ f''_3(t) \end{bmatrix}$ into $\mathbf{x}' = A\mathbf{x}$, we obtain

$$\begin{bmatrix} f'_3(t) \\ f''_3(t) \\ f'''_3(t) \end{bmatrix} = \begin{bmatrix} f'_3(t) \\ f''_3(t) \\ -cf_3(t) - bf'_3(t) - af''_3(t) \end{bmatrix},$$

which corresponds precisely to the differential equation for f_3 above.

(c) Using the Wronskian in $V_3(I)$ we have $W[\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3](t) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f'_1 & f'_2 & f'_3 \\ f''_1 & f''_2 & f''_3 \end{vmatrix}$. Similarly, using the Wronskian

$$\text{in } C^1(I), W[y_1, y_2, y_3](t) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f'_1 & f'_2 & f'_3 \\ f''_1 & f''_2 & f''_3 \end{vmatrix}.$$

34. The matrix A has eigenvalues $\lambda_1 = -7$ and $\lambda_2 = 5$ with corresponding nonproportional eigenvectors $\mathbf{v}_1 = (-1, 1)$ and $\mathbf{v}_2 = (1, 2)$, respectively. Since the eigenvalues have different signs, the equilibrium point $(0, 0)$ is a saddle point.

35. The matrix A has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -3$ with corresponding nonproportional eigenvectors $\mathbf{v}_1 = (3, 1)$ and $\mathbf{v}_2 = (2, 1)$, respectively. Since both eigenvalues are real and negative, the equilibrium point is a stable node.

36. The matrix A has eigenvalues $\lambda = 2 \pm 3i$ with corresponding nonproportional eigenvectors $\mathbf{v}_1 = (-3, 1 - i)$ and $\mathbf{v}_2 = (-3, 1 + i)$, respectively. Since the eigenvalues are complex with positive real part, the equilibrium point is an unstable spiral.

37. The matrix A has a single eigenvalue $\lambda = -4$ (with multiplicity 2) with corresponding nonproportional eigenvectors $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_2 = (0, 1)$. In this case, the equilibrium point is called a proper node, with all trajectories approaching the equilibrium point.

38. The matrix A has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 6$ with corresponding nonproportional eigenvectors $\mathbf{v}_1 = (2, 1)$ and $\mathbf{v}_2 = (1, 1)$, respectively. Since both eigenvalues are real and positive, the equilibrium point is an unstable node.

39. The matrix A has eigenvalues $\lambda = -5 \pm i$ with corresponding nonproportional vectors $\mathbf{v}_1 = (2 + i, 1)$ and $\mathbf{v}_2 = (2 - i, 1)$, respectively. Since the eigenvalues are complex with negative real part, the equilibrium point is a stable spiral.

40. The matrix A has a single eigenvalue $\lambda = -3$ (with multiplicity 2) with a single corresponding eigenvector $\mathbf{v} = (1, 1)$. Since $\lambda < 0$, the equilibrium point in this case is a stable degenerate node.

41. The matrix A has a single eigenvalue $\lambda = 6$ (with multiplicity 2) with a single corresponding eigenvector $\mathbf{v} = (2, 1)$. Since $\lambda > 0$, the equilibrium point in this case is an unstable degenerate node.

42. The first-order system is

$$u' = v \quad \text{and} \quad v' = 12u - v.$$

The corresponding matrix formulation of the linear system is $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} u \\ v \end{bmatrix}$ and $A =$

$\begin{bmatrix} 0 & 1 \\ 12 & -1 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = -4$ and $\lambda_2 = 3$, and since they are real and of opposite signs, we conclude that the equilibrium point $(0, 0)$ is a saddle point.

43. The first-order system is

$$u' = v \quad \text{and} \quad v' = -25u.$$

The corresponding matrix formulation of the linear system is $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} u \\ v \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ -25 & 0 \end{bmatrix}$. The eigenvalues of A are $\lambda = \pm 5i$, and since they are pure imaginary, we conclude that the equilibrium point $(0, 0)$ is a stable center.

Solutions to Section 8.1

True-False Review:

- 1. FALSE.** This function has a discontinuity at every integer $n \neq 0, 1$, and therefore, it has infinitely many discontinuities and cannot be piecewise continuous.
- 2. FALSE.** This function has a discontinuity at every multiple of π , and therefore, it has infinitely many discontinuities and cannot be piecewise continuous.
- 3. TRUE.** If f has discontinuities at $a_1, a_2, \dots, a_r$ in I and g has discontinuities at $b_1, b_2, \dots, b_s$ in I , then $f + g$ has at most $r + s$ discontinuities in I . Therefore, we can divide I into finitely many subintervals so that $f + g$ is continuous on each subinterval and f approaches a finite limit as the endpoints of each subinterval are approached from within. Hence, from Definition 8.1.4, f is piecewise continuous on the interval I .
- 4. TRUE.** By definition, we can divide $[a, b]$ into finitely many subintervals so that (1) and (2) in Definition 8.1.4 are satisfied, and likewise, we can divide $[b, c]$ into finitely many subintervals so that (1) and (2) in Definition 8.1.4 are satisfied. Taking all of the subintervals so obtained, we have divided $[a, c]$ into finitely many subintervals so that (1) and (2) in Definition 8.1.4 are satisfied.
- 5. FALSE.** The lower limit of the integral defining the Laplace transform must be 0, not 1 (see Definition 8.1.1).
- 6. TRUE.** The formula $L[e^{at}] = \frac{1}{s-a}$ only applies for $s > a$, for otherwise the improper integral defining $L[e^{at}]$ fails to converge. In this case $a = 1$, so we must restrict s so that $s > 1$.
- 7. FALSE.** We have $L[2 \cos 3t] = 2L[\cos 3t]$, and $L[\cos 3t] = \frac{s}{s^2+9}$ provided that $s > 0$, according to Equation (8.1.4). So this Laplace transform is defined for $s > 0$, not just $s > 3$.
- 8. FALSE.** For instance, if we take $f(t) = e^t$, then $L[f] = \frac{1}{s-1}$, but $L[1/f] = L[e^{-t}] = \frac{1}{s+1}$. Obviously, $L[1/f] \neq 1/L[f]$ in this case.
- 9. FALSE.** For instance, if we take $f(t) = e^t$, then $L[f] = \frac{1}{s-1}$, but $L[f^2] = L[e^{2t}] = \frac{1}{s-2} \neq \left(\frac{1}{s-1}\right)^2$.

Problems:

1. For $s > 2$,

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} e^{2t} dt = \lim_{n \rightarrow \infty} \int_0^n e^{(2-s)t} dt \\ &= \frac{1}{2-s} \lim_{n \rightarrow \infty} \left[e^{(2-s)t} \right]_0^n \\ &= \frac{1}{s-2}. \end{aligned}$$

2. For $s > 0$,

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} (t-1) dt = \lim_{n \rightarrow \infty} \int_0^n e^{-st} (t-1) dt \\ &= \lim_{n \rightarrow \infty} \left[e^{-st} \left(-\frac{t}{s} - \frac{1}{s^2} + \frac{1}{s} \right) \right]_0^n \\ &= \frac{1-s}{s^2}. \end{aligned}$$

3. For $s > 0$,

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} \sin bt dt = \lim_{n \rightarrow \infty} \int_0^n e^{-st} \sin bt dt \\ &= \lim_{n \rightarrow \infty} \left[\frac{e^{-st} (-\sin bt - b \cos bt)}{s^2 + b^2} \right]_0^n \\ &= \frac{b}{s^2 + b^2}. \end{aligned}$$

4. For $s > 1$,

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} t e^t dt = \lim_{n \rightarrow \infty} \int_0^n e^{(1-s)t} t dt \\ &= \lim_{n \rightarrow \infty} \left[\frac{e^{(1-s)t} [(1-s)t - 1]}{(1-s)^2} \right]_0^n \\ &= \frac{1}{(1-s)^2}. \end{aligned}$$

5. For $s > |b|$,

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} \cosh bt dt = \lim_{n \rightarrow \infty} \int_0^n e^{-st} \left(\frac{e^{bt} + e^{-bt}}{2} \right) dt \\ &= \lim_{n \rightarrow \infty} \left[\frac{e^{(b-s)t}}{2(b-s)} - \frac{e^{-(b+s)t}}{2(b+s)} \right]_0^n \\ &= \lim_{n \rightarrow \infty} \left[\frac{e^{(b-s)n} - 1}{2(b-s)} - \frac{e^{-(b+s)n} + 1}{2(b+s)} \right] \\ &= \frac{s}{s^2 - b^2}. \end{aligned}$$

6. For $s > |b|$

$$\begin{aligned}
 F(s) &= \int_0^\infty e^{-st} \sinh bt dt = \lim_{n \rightarrow \infty} \int_0^n e^{-st} \left(\frac{e^{bt} - e^{-bt}}{2} \right) dt \\
 &= \lim_{n \rightarrow \infty} \left[\frac{e^{(b-s)t}}{2(b-s)} + \frac{e^{-(b+s)t}}{2(b+s)} \right]_0^n \\
 &= \lim_{n \rightarrow \infty} \left[\frac{e^{(b-s)n} - 1}{2(b-s)} + \frac{e^{-(b+s)n} - 1}{2(b+s)} \right] \\
 &= \frac{b}{s^2 - b^2}.
 \end{aligned}$$

7. For $s > 0$,

$$\begin{aligned}
 F(s) &= \int_0^\infty e^{-st} 2t dt = 2 \lim_{n \rightarrow \infty} \int_0^n e^{-st} t dt \\
 &= 2 \lim_{n \rightarrow \infty} \left[\frac{e^{-st}(-st - 1)}{s^2} \right]_0^n \\
 &= 2 \lim_{n \rightarrow \infty} \left[\frac{-n}{se^{sn}} - \frac{1}{s^2 e^{sn}} + \frac{1}{s^2} \right] \\
 &= \frac{2}{s^2}.
 \end{aligned}$$

8. For $s > 2$

$$\begin{aligned}
 F(s) &= \int_0^\infty e^{-st} (3e^{2t}) dt = 3 \lim_{n \rightarrow \infty} \int_0^n e^{(2-s)t} dt \\
 &= \frac{3}{2-s} \lim_{n \rightarrow \infty} \left[e^{(2-s)t} \right]_0^n \\
 &= \frac{3}{2-s} \lim_{n \rightarrow \infty} \left[\frac{1}{e^{(s-2)n} - 1} \right] \\
 &= \frac{3}{s-2}.
 \end{aligned}$$

9. For $s > 0$,

$$\begin{aligned}
 F(s) &= \int_0^2 e^{-st} (1) dt + \int_2^\infty e^{-st} (-1) dt \\
 &= \left[\frac{e^{-st}}{-s} \right]_0^2 + \lim_{n \rightarrow \infty} \int_2^n (-e^{-st}) dt \\
 &= \left[-\frac{e^{-2s}}{s} + \frac{1}{s} \right] + \lim_{n \rightarrow \infty} \left[\frac{e^{-st}}{s} \right]_2^n \\
 &= \frac{1}{s} - \frac{1}{se^{2s}} + \lim_{n \rightarrow \infty} \left[\frac{1}{se^{sn}} - \frac{1}{se^{2s}} \right] \\
 &= \frac{1}{s} (1 - 2e^{-2s}).
 \end{aligned}$$

10. For $s > 0$,

$$\begin{aligned}
 F(s) &= \int_0^1 e^{-st} t^2 dt + \int_1^\infty e^{-st} (1) dt \\
 &= \left[-\frac{t^2}{se^{st}} + \frac{2(-st-1)}{s^3 e^{st}} \right]_0^1 + \lim_{n \rightarrow \infty} \left[-\frac{e^{-st}}{s} \right]_0^n \\
 &= \left[-\frac{1}{se^s} - \frac{4}{s^3 e^s} + \frac{2}{s^3} \right] + \lim_{n \rightarrow \infty} \left[-\frac{1}{se^{sn}} + \frac{1}{s} \right] \\
 &= \frac{2e^2 + s^2 e^s - s^2 - 4}{s^3 e^s}.
 \end{aligned}$$

11. For $s > 1$,

$$\begin{aligned}
 F(s) &= \int_0^\infty e^{-st} (e^t \sin t) dt \\
 &= \lim_{n \rightarrow \infty} \int_0^n e^{(1-s)t} \sin t dt \\
 &= \lim_{n \rightarrow \infty} \left[\frac{e^{(1-s)t} (\sin t - \cos t)}{(s-1)^2 + 1} \right]_0^n \\
 &= \lim_{n \rightarrow \infty} \left[\frac{e^{(1-2)n} (\sin n - \cos n)}{(s-1)^2 + 1} + \frac{1}{(s-1)^2 + 1} \right] \\
 &= \frac{1}{(s-1)^2 + 1}.
 \end{aligned}$$

12. For $s > 2$,

$$\begin{aligned}
 F(s) &= \int_0^\infty e^{-st} (e^{2t} \cos 3t) dt \\
 &= \lim_{n \rightarrow \infty} \int_0^n e^{(2-s)t} \cos 3t dt \\
 &= \lim_{n \rightarrow \infty} \left[\frac{e^{(2-s)t} [(2-s) \cos 3t + 3 \sin 3t]}{(s-2)^2 + 9} \right]_0^n \\
 &= \lim_{n \rightarrow \infty} \left[\frac{e^{(2-s)n} [(2-s) \cos 3n + 3 \sin 3n]}{(s-2)^2 + 9} - \frac{2-s}{(s-2)^2 + 9} \right] \\
 &= \frac{s-2}{(s-2)^2 + 9}.
 \end{aligned}$$

13. For $s > 3$,

$$\begin{aligned}
 L[2t - e^{3t}] &= L[2t] - L[e^{3t}] = 2L[t] - L[e^{3t}] \\
 &= 2 \left(\frac{1}{s^2} \right) - \frac{1}{s-3} \\
 &= \frac{2}{s^2} - \frac{1}{s-3}.
 \end{aligned}$$

14. For $s > 0$,

$$\begin{aligned} L[2 \sin 3t + 4t^3] &= 2L[\sin 3t] + 4L[t^3] \\ &= 2 \left(\frac{3}{s^2 + 9} \right) + 4 \left(\frac{3!}{s^4} \right) \\ &= \frac{6}{s^2 + 9} + \frac{24}{s^4} \\ &= \frac{6(s^4 + 4s^2 + 36)}{(s^2 + 9)s^4}. \end{aligned}$$

15. For $s > |b|$,

$$\begin{aligned} L[\cosh bt] &= L \left[\frac{1}{2}e^{bt} + \frac{1}{2}e^{-bt} \right] = \frac{1}{2}L[e^{bt}] + \frac{1}{2}L[e^{-bt}] \\ &= \frac{1}{2} \left(\frac{1}{s-b} \right) + \frac{1}{2} \left(\frac{1}{s+b} \right) \\ &= \frac{s}{s^2 - b^2}. \end{aligned}$$

16. For $s > |b|$,

$$\begin{aligned} L[\sinh bt] &= L \left[\frac{1}{2}e^{bt} - \frac{1}{2}e^{-bt} \right] \\ &= \frac{1}{2}L[e^{bt}] - \frac{1}{2}L[e^{-bt}] \\ &= \frac{1}{2} \left(\frac{1}{s-b} \right) - \frac{1}{2} \left(\frac{1}{s+b} \right) \\ &= \frac{b}{s^2 - b^2}. \end{aligned}$$

17. For $s > 0$,

$$\begin{aligned} L[3t^2 - 5 \cos 2t + \sin 3t] &= 3L[t^2] - 5L[\cos 2t] + L[\sin 3t] \\ &= 3 \left(\frac{2!}{s^3} \right) - 5 \left(\frac{s}{s^2 + 4} \right) + \frac{3}{s^2 + 9} \\ &= \frac{6}{s^3} - \frac{5s}{s^2 + 4} + \frac{3}{s^2 + 9}. \end{aligned}$$

18. For $s > 0$,

$$\begin{aligned} L[7e^{-2t} + 1] &= 7L[e^{-2t}] + L[1] \\ &= 7 \left(\frac{1}{s+2} \right) + \frac{1}{s} \\ &= \frac{2(4s+1)}{s(s+2)}. \end{aligned}$$

19. For $s > 1$,

$$\begin{aligned} L[2e^{-3t} + 4e^t - 5 \sin t] &= 2L[e^{-3t}] + 4L[e^t] - 5L[\sin t] \\ &= \frac{2}{s+3} + \frac{4}{s-1} - \frac{5}{s^2 + 1}. \end{aligned}$$

20. We have,

$$\begin{aligned}
 L[4 \cos t - \pi/4] &= 4L[\cos t - \pi/4] \\
 &= 4L\left[\frac{\sqrt{2}}{2} \cos t + \frac{\sqrt{2}}{2} \sin t\right] \\
 &= 2\sqrt{2}L[\cos t] + 2\sqrt{2}L[\sin t] \\
 &= 2\sqrt{2}\left(\frac{s}{s^2 + 1}\right) + 2\sqrt{2}\left(\frac{1}{s^2 + 1}\right) \\
 &= \frac{2\sqrt{2}(s + 1)}{s^2 + 1}.
 \end{aligned}$$

21. For $s > 0$,

$$\begin{aligned}
 L[4 \cos^2 bt] &= 2L[2 \cos^2 bt] = 2L[\cos 2bt + 1] \\
 &= 2L[\cos 2bt] + 2L[1] \\
 &= 2\left(\frac{s}{4b^2 + s^2}\right) + 2\left(\frac{1}{s}\right) \\
 &= \frac{4(s^2 + 2b^2)}{s(s^2 + 4b^2)}.
 \end{aligned}$$

22. For $s > 0$,

$$\begin{aligned}
 L[2 \sin^2 4t - 3] &= L[2 \sin^2 4t] - 3L[1] \\
 &= L[1 - \cos 8t] - 3\left(\frac{1}{s}\right) \\
 &= \frac{1}{s} - \frac{s}{s^2 + 64} - \frac{3}{s} \\
 &= -\frac{2}{s} - \frac{s}{s^2 + 64} \\
 &= -\frac{128 + 3s^2}{s(s^2 + 64)}.
 \end{aligned}$$

23. The graph of f is piece-wise continuous on $[0, \infty)$.

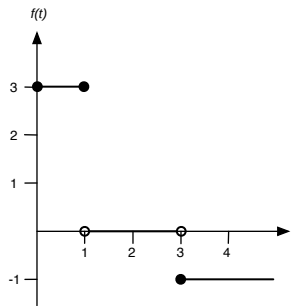


Figure 124: Figure for Problem 23

24. The graph of f is piece-wise continuous on $[0, \infty)$.

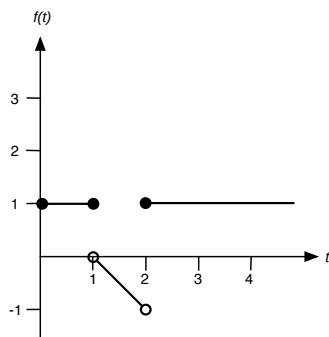


Figure 125: Figure for Problem 24

25. The graph of f is not piece-wise continuous on $[0, \infty)$, because $\lim_{t \rightarrow 1^+} f(t) = \infty$, i.e. the right-hand limit as t approaches 1 from above is not finite.

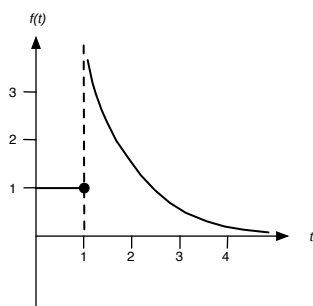


Figure 126: Figure for Problem 25

26. The graph of f is piece-wise continuous on $[0, \infty)$.

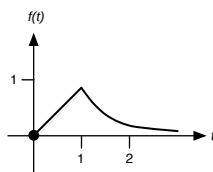


Figure 127: Figure for Problem 26

27. The graph of f is piece-wise continuous on $[0, \infty)$.

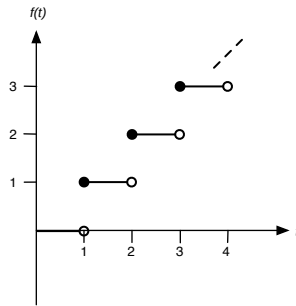


Figure 128: Figure for Problem 27

28. The graph of f is piece-wise continuous on $[0, \infty)$.

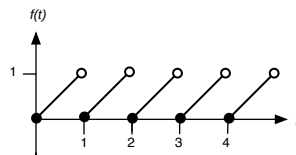


Figure 129: Figure for Problem 28

29. The graph of f is not piece-wise continuous on $[0, \infty)$.

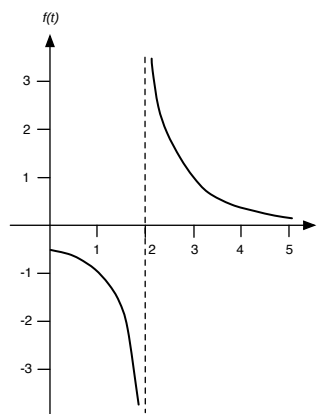


Figure 130: Figure for Problem 29

30. The graph of f is piece-wise continuous on $[0, \infty)$.

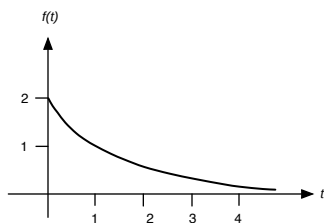


Figure 131: Figure for Problem 30

31. We have,

$$\begin{aligned}
 F(s) &= \int_0^1 e^{-st} t dt + \int_1^\infty e^{-st} (0) dt \\
 &= \left[\frac{e^{-st}}{s^2} (-st - 1) \right]_0^1 \\
 &= \frac{1}{s^2} [1 - e^{-s}(s + 1)].
 \end{aligned}$$

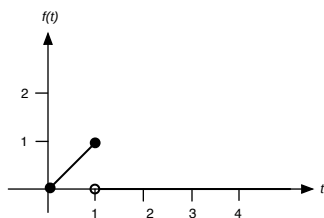


Figure 132: Figure for Problem 31

32. We have,

$$\begin{aligned}
 F(s) &= \int_0^2 e^{-st} (1) dt + \int_2^\infty e^{-st} (-1) dt \\
 &= \left[-\frac{1}{s} e^{-st} \right]_0^2 + \frac{1}{s} \lim_{n \rightarrow \infty} [e^{-sn} - e^{-2s}] \\
 &= -\frac{e^{-2s}}{s} + \frac{1}{s} - \frac{1}{s} e^{-2s} \\
 &= \frac{1 - 2e^{-2s}}{s}.
 \end{aligned}$$

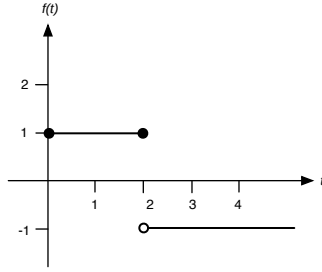


Figure 133: Figure for Problem 32

33. We have,

$$\begin{aligned}
 F(s) &= \int_0^1 e^{-st}(0)dt + \int_1^2 e^{-st}(t)dt + \int_2^\infty e^{-st}(0)dt \\
 &= \left[\frac{1}{s^2} e^{-st} (-st - 1) \right]_1^2 \\
 &= \frac{1}{s^2} e^{-2s} [e^s(s+1) - (2s+1)].
 \end{aligned}$$

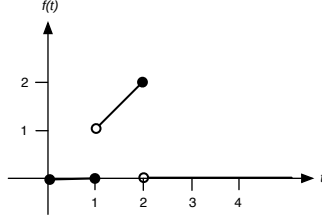


Figure 134: Figure for Problem 33

34. We have,

$$\begin{aligned}
 F(s) &= \int_0^1 e^{-st}t dt + \int_1^3 e^{-st}(1)dt + \int_3^\infty e^{-st}e^{(t-3)}dt \\
 &= \left[\frac{e^{-st}}{s^2} (-st - 1) \right]_0^1 - \left[\frac{e^{-st}}{s} \right]_1^3 + \lim_{n \rightarrow \infty} \left[e^{(1-s)t} \right]_3^n \\
 &= \frac{1}{s^2} [1 - e^{-s}(s+1)] - \frac{1}{s} [e^{-3s} - e^{-s}] + \frac{1}{e^3(1-s)} \lim_{n \rightarrow \infty} [e^{(1-s)t}]_3^n \\
 &= \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} - \frac{e^{-3s}}{s} + \frac{e^{-s}}{s} + \frac{1}{e^3(1-s)} [-e^{3(1-s)}] \\
 &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-3s}}{s(1-s)}.
 \end{aligned}$$

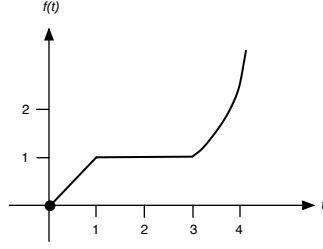


Figure 135: Figure for Problem 34

35. We have,

$$L[e^{ibt}] = \frac{1}{s - bi} = \frac{s + bi}{s^2 + b^2} = \frac{s}{s^2 + b^2} + \frac{b}{s^2 + b^2}i. \quad (0.0.48)$$

But

$$e^{ibt} = \cos bt + i \sin bt;$$

hence,

$$L[e^{ibt}] = L[\cos bt + i \sin bt] = L[\cos bt] + iL[\sin bt]. \quad (0.0.49)$$

From (0.0.48) and (0.0.49) we have, on equating real and imaginary parts,

$$L[\cos bt] = \frac{s}{s^2 + b^2} \quad \text{and} \quad L[\sin bt] = \frac{b}{s^2 + b^2}.$$

36. We have,

$$L[e^{(a+bi)t}] = \frac{1}{s - (a + bi)} = \frac{1}{(s - a) - bi} = \frac{s - a + bi}{(s - a)^2 + b^2} = \frac{s - a}{(s - a)^2 + b^2} + \frac{b}{(s - a)^2 + b^2}i. \quad (0.0.50)$$

Also,

$$L[e^{(a+bi)t}] = L[e^{at} \cos bt + ie^{at} \sin bt] = L[e^{at} \cos bt] + iL[e^{at} \sin bt]. \quad (0.0.51)$$

From (0.0.50) and (0.0.51) we have, on equating real and imaginary parts,

$$L[e^{at} \cos bt] = \frac{s - a}{(s - a)^2 + b^2} \quad \text{and} \quad L[e^{at} \sin bt] = \frac{b}{(s - a)^2 + b^2}.$$

37. If $n = 1$, then

$$\begin{aligned} L[t^n] &= L[t] = \int_0^\infty e^{-st} t dt = \lim_{n \rightarrow \infty} \int_0^n e^{-st} t dt \\ &= \lim_{n \rightarrow \infty} \left\{ \left[-\frac{te^{-st}}{s} \right]_0^n + \frac{1}{s} \int_0^n e^{-st} dt \right\} \\ &= \lim_{n \rightarrow \infty} \left[-\frac{ne^{-sn}}{s} - \frac{e^{-sn}}{s^2} + \frac{1}{s^2} \right] = \frac{1}{s^2} = \frac{n!}{s^{n+1}}, \end{aligned}$$

so the proposition is true for $n = 1$. Now suppose that for some positive integer k , it is true that

$$L[t^k] = \frac{k!}{s^{k+1}}.$$

Then integration by parts yields

$$\begin{aligned}
 L[t^{k+1}] &= \int_0^\infty e^{-st} t^{k+1} dt \\
 &= \lim_{n \rightarrow \infty} \left\{ \left[-\frac{e^{-st} t^{k+1}}{s} \right]_0^n + \frac{k+1}{s} \int_0^n e^{-st} t^k dt \right\} \\
 &= \frac{k+1}{s} \lim_{n \rightarrow \infty} \int_0^n e^{-st} t^k dt = \frac{k+1}{s} L[t^k] \\
 &= \frac{k+1}{s} \cdot \frac{k!}{s^{k+1}} = \frac{(k+1)!}{s^{(k+1)+1}}.
 \end{aligned}$$

Thus the proposition is true for every positive integer $k+1$ given that it is true for the positive integer k . Consequently, by mathematical induction it follows that $L[t^n] = \frac{n!}{s^{n+1}}$ for every positive integer n .

38. (a) Let $t = \frac{x^2}{s}$ where $s > 0$ so $t^{-1/2} = \frac{s^{1/2}}{|x|} = \frac{s^{1/2}}{x}$ as $x \in (0, \infty)$ and $dt = \frac{2x}{s} dx$. Substituting these results into the Laplace transform definition we obtain

$$\begin{aligned}
 L[t^{-1/2}] &= \int_0^\infty e^{-st} t^{-1/2} dt \\
 &= \int_0^\infty e^{-x^2} \frac{s^{1/2}}{x} \frac{2x}{s} dx \\
 &= \frac{2}{s^{1/2}} \int_0^\infty e^{-x^2} dx.
 \end{aligned}$$

(b) Since x is a dummy variable in part (a) we may write

$$\begin{aligned}
 \left(L[t^{-1/2}] \right)^2 &= \left(\frac{2}{s^{1/2}} \int_0^\infty e^{-x^2} dx \right) \left(\frac{2}{s^{1/2}} \int_0^\infty e^{-y^2} dy \right) \\
 &= \frac{4}{s} \int_0^\infty \left[\int_0^\infty e^{-y^2} dy \right] e^{-x^2} dx.
 \end{aligned}$$

The last equality is true because the integral in y yields a constant to carry over into the integral in x . We can write this in the equivalent form

$$\left(L[t^{-1/2}] \right)^2 = \frac{4}{s} \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$$

(c) Converting the results in (b) to polar coordinates yields

$$\begin{aligned}
 (L[t^{-1/2}])^2 &= \frac{4}{s} \int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta \\
 &= \frac{4}{s} \int_0^{\pi/2} \left[-\frac{1}{2} e^{-r^2} \right]_0^\infty d\theta \\
 &= \frac{2}{s} \int_0^{\pi/2} d\theta = \frac{\pi}{s}.
 \end{aligned}$$

Consequently, $L[t^{-1/2}] = \sqrt{\frac{\pi}{s}}$ since $L[t^{-1/2}] \geq 0$.

Solutions to Section 8.2

True-False Review:

1. TRUE. If f is of exponential order, then by Definition 8.2.1, we have $|f(t)| \leq Me^{\alpha t}$ ($t > 0$) for some constants M and α . But then $|g(t)| < f(t) \leq |f(t)| \leq Me^{\alpha t}$ for all $t > 0$, and so g is also of exponential order.

2. TRUE. Suppose that $|f(t)| \leq M_1 e^{\alpha_1 t}$ and $|g(t)| \leq M_2 e^{\alpha_2 t}$ for $t > 0$, for some constants $M_1, M_2, \alpha_1, \alpha_2$. Then

$$|(f+g)(t)| = |f(t) + g(t)| \leq |f(t)| + |g(t)| \leq M_1 e^{\alpha_1 t} + M_2 e^{\alpha_2 t}.$$

If we set $M = \max\{M_1, M_2\}$ and $\alpha = \max\{\alpha_1, \alpha_2\}$, then the last expression on the right above is $\leq 2Me^{\alpha t}$ for all $t > 0$. Therefore, $f+g$ is of exponential order.

3. FALSE. Looking at the Comparison Test for Improper Integrals, we should conclude from the assumptions that if $\int_0^\infty H(t)dt$ converges, then so does $\int_0^\infty G(t)dt$. As a specific counterexample, we could take $G(t) = 0$ and $H(t) = t$. Then although $G(t) \leq H(t)$ for all $t \geq 0$ and $\int_0^\infty G(t)dt = 0$ converges, $\int_0^\infty H(t)dt$ does not converge.

4. TRUE. Assume that $L^{-1}[F](t) = f(t)$ and $L^{-1}[G](t) = g(t)$. Then we have

$$L^{-1}[F+G](t) = (f+g)(t) = f(t) + g(t) = L^{-1}[F](t) + L^{-1}[G](t).$$

Therefore, $L^{-1}[F+G] = L^{-1}[F] + L^{-1}[G]$. Moreover, we have

$$L^{-1}[cF](t) = (cf)(t) = cf(t) = cL^{-1}[F](t),$$

and therefore, $L^{-1}[cF] = cL^{-1}[F]$. Hence, L^{-1} is a linear transformation.

5. FALSE. From the formulas preceding Example 8.2.6, we see that the inverse Laplace transform of $\frac{s}{s^2+9}$ is $f(t) = \cos 3t$.

6. FALSE. From the formulas preceding Example 8.2.6, we see that the inverse Laplace transform of $\frac{1}{s+3}$ is $f(t) = e^{-3t}$.

Problems:

1. $|f(t)| = |\cos 2t| \leq 1e^t$ for all $t > 0$ so $\cos 2t$ is of exponential order.

2. $|f(t)| = |e^{2t}| \leq 1e^{2t}$ for all $t > 0$ so e^{2t} is of exponential order.

3. $|f(t)| = |e^{3t} \sin 4t| \leq 1e^{3t}$ for all $t > 0$ so $e^{3t} \sin 4t$ is of exponential order.

4. $|f(t)| = |te^{-2t}| \leq 1e^t$ for all $t > 0$ so te^{-2t} is of exponential order.

5. $|f(t)| = |t^n e^{at}| \leq (e^t)^n e^{at}$ for all $t > 0$ so $t^n e^{at}$ is of exponential order.

6. Recall that $E(0, \infty)$ represents the set of all functions that are both piece-wise continuous on $[0, \infty)$ and of exponential order. Let $f, g \in E(0, \infty)$. Then f and g are piece-wise continuous on $[0, \infty)$; hence, so is $f+g$. Also there exists constants M_1, M_2, α, β such that $|f(t)| \leq M_1 e^{\alpha t}$ and $|g(t)| \leq M_2 e^{\beta t}$. Thus $|f(t) + g(t)| \leq |f(t)| + |g(t)| \leq M_1 e^{\alpha t} + M_2 e^{\beta t} \leq (M_1 + M_2)e^{\gamma t} \leq Me^{\gamma t}$ where $\gamma = \max\{\alpha, \beta\}$ and $M = M_1 + M_2$ so $f+g$ is of exponential order. Thus $f+g \in E(0, \infty)$. Now let c be any scalar and $f \in E(0, \infty)$. Since f is piece-wise continuous on $[0, \infty)$ then so is cf ; moreover, there exists constants M

and α such that $|cf(t)| = |c||f(t)| \leq |c|Me^{\alpha t} = M_1e^{\alpha t}$ where $M_1 = |c|M$ so cf is of exponential order. Hence $cf \in E(0, \infty)$.

$$7. L^{-1} \left[\frac{2}{s} \right] = 2L^{-1} \left[\frac{1}{s} \right] = (2)(1) = 2.$$

$$8. L^{-1} \left[\frac{3}{s-2} \right] = 3L^{-1} \left[\frac{1}{s-2} \right] = 3e^{2t}.$$

$$9. L^{-1} \left[\frac{5}{s+3} \right] = 5L^{-1} \left[\frac{1}{s-(-3)} \right] = 5e^{-3t}.$$

$$10. L^{-1} \left[\frac{1}{s^2+4} \right] = \frac{1}{2}L^{-1} \left[\frac{2}{s^2+4} \right] = \frac{1}{2} \sin 2t.$$

$$11. L^{-1} \left[\frac{2s}{s^2+9} \right] = 2L^{-1} \left[\frac{s}{s^2+9} \right] = 2 \cos 3t.$$

$$12. L^{-1} \left[\frac{4}{s^3} \right] = 2L^{-1} \left[\frac{2}{s^3} \right] = 2t^2.$$

$$13. L^{-1} \left[\frac{s+6}{s^2+1} \right] = 2L^{-1} \left[\frac{s}{s^2+1} \right] + 6L^{-1} \left[\frac{1}{s^2+1} \right] = \cos t + 6 \sin t.$$

$$14. L^{-1} \left[\frac{2s+1}{s^2+16} \right] = 2L^{-1} \left[\frac{s}{s^2+16} \right] + \frac{1}{4}L^{-1} \left[\frac{4}{s^2+16} \right] = 2 \cos 4t + \frac{1}{4} \sin 4t.$$

$$15. L^{-1} \left[\frac{2}{s} - \frac{3}{s+1} \right] = 2L^{-1} \left[\frac{1}{s} \right] - 3L^{-1} \left[\frac{1}{s+1} \right] = 2 - 3e^{-t}.$$

$$16. L^{-1} \left[\frac{4}{s^2} - \frac{s+2}{s^2+9} \right] = 4L^{-1} \left[\frac{1}{s^2} \right] - L^{-1} \left[\frac{s}{s^2+9} \right] - \frac{2}{3}L^{-1} \left[\frac{3}{s^2+9} \right] = 4t - \cos 3t - \frac{2}{3} \sin 3t.$$

$$17. L^{-1} \left[\frac{1}{s(s+1)} \right] = L^{-1} \left[\frac{1}{s} - \frac{1}{s+1} \right] = L^{-1} \left[\frac{1}{s} \right] - L^{-1} \left[\frac{1}{s+1} \right] = 1 - e^{-t}.$$

18. Using the linearity of the inverse Laplace transform,

$$\begin{aligned} L^{-1} \left[\frac{1}{(s+1)(s^2+4)} \right] &= L^{-1} \left[-\frac{3}{5} \frac{1}{s+1} + \frac{1}{5} \frac{3s+2}{s^2+4} \right] \\ &= \frac{1}{5} \left(-3L^{-1} \left[\frac{1}{s+1} \right] + 3L^{-1} \left[\frac{s}{s^2+4} \right] + L^{-1} \left[\frac{2}{s^2+4} \right] \right) \\ &= \frac{1}{5} (-3e^{-t} + 3 \cos 2t + \sin 2t). \end{aligned}$$

19. Using the linearity of the inverse Laplace transform,

$$\begin{aligned} L^{-1} \left[\frac{2s+3}{(s-2)(s^2+1)} \right] &= L^{-1} \left[\frac{1}{5} \frac{7}{s-2} - \frac{1}{5} \frac{7s+4}{s^2+1} \right] \\ &= \frac{1}{5} \left\{ 7L^{-1} \left[\frac{1}{s-2} \right] - 7L^{-1} \left[\frac{s}{s^2+1} \right] + 4L^{-1} \left[\frac{1}{s^2+1} \right] \right\} \\ &= \frac{1}{5} (7e^{2t} - 7 \cos t - 4 \sin t). \end{aligned}$$

20. Using the linearity of the inverse Laplace transform,

$$\begin{aligned} L^{-1} \left[\frac{s+4}{(s-1)(s+2)(s-3)} \right] &= L^{-1} \left[-\frac{5}{6(s-1)} + \frac{2}{15(s+2)} + \frac{7}{10(s-3)} \right] \\ &= \frac{1}{30} \left\{ -25L^{-1} \left[\frac{1}{s-1} \right] + 4L^{-1} \left[\frac{1}{s+2} \right] + 21L^{-1} \left[\frac{1}{s-3} \right] \right\} \\ &= \frac{1}{30} (-25e^t + 4e^{-2t} + 21e^{3t}). \end{aligned}$$

21. Using the linearity of the inverse Laplace transform,

$$\begin{aligned} L^{-1} \left[\frac{2s+3}{(s^2+4)(s^2+1)} \right] &= L^{-1} \left[-\frac{2s+3}{3(s^2+4)} \right] + L^{-1} \left[\frac{2s+3}{3(s^2+1)} \right] \\ &= -\frac{2}{3} L^{-1} \left[\frac{s}{s^2+4} \right] - \frac{1}{2} L^{-1} \left[\frac{2}{s^2+4} \right] + \frac{2}{3} L^{-1} \left[\frac{s}{s^2+1} \right] + L^{-1} \left[\frac{1}{s^2+1} \right] \\ &= -\frac{2}{3} \cos 2t - \frac{1}{2} \sin 2t + \frac{2}{3} \cos t + \sin t \\ &= \frac{1}{6} (4 \cos t + 6 \sin t - 4 \cos 2t - 3 \sin 2t). \end{aligned}$$

22.

Solutions to Section 8.3

True-False Review:

1. **FALSE.** The period of f is required to be the *smallest* positive real number T such that $f(t+T) = f(t)$ for all $t \geq 0$. There can be only *one* smallest positive real number T , so the period, if it exists, is uniquely determined.

2. **TRUE.** The functions f and g are simply horizontally shifted from one another. Therefore, if f has period T , then $f(t+T) = f(t)$, and so then $g(t+T) = f(t+T+c) = f(t+c) = g(t)$. Thus, g also has period T .

3. **FALSE.** The function $f(t) = \cos(2t)$ has period π , not $\pi/2$.

4. **FALSE.** For all $T > 0$, we have $f(t+T) = \sin((t+T)^2) \neq \sin(t^2)$, so f is not periodic.

5. **FALSE.** Even a continuous function need not be periodic. For instance, the function $f(t) = \sin(t^2)$ in the previous item is continuous, hence piecewise continuous, but not periodic.

6. FALSE. If $f(t) = \cos t$ and $g(t) = \sin t$, then f and g are periodic with period $m = n = 2\pi$. However, $(f + g)(t) = \cos t + \sin t$ is periodic with period 2π , not period 4π .

7. FALSE. If $f(t) = \cos t$ and $g(t) = \sin t$, then f and g are periodic with period $m = n = 2\pi$. However, $(fg)(t) = \cos t \sin t$ is periodic with period 2π , not period $(2\pi)^2$.

8. FALSE. Many examples are possible. The function f given in Example 8.3.4 is one. More simply, we simply note that $L[\sin t] = \frac{1}{s^2+1}$, and $F(s) = \frac{1}{s^2+1}$ is not a periodic function.

Problems:

1. We have,

$$\begin{aligned} L[f] &= \frac{1}{1-e^{-s}} \int_0^1 e^{-st} t dt = \frac{1}{1-e^{-s}} \left[\frac{e^{-st}}{s^2} (-st-1) \right]_0^1 \\ &= \frac{1}{s^2(1-e^{-s})} [1 - e^{-s}(s+1)]. \end{aligned}$$

2. We have,

$$\begin{aligned} L[f] &= \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} t^2 dt = \frac{1}{1-e^{-2s}} \left\{ \left[-\frac{t^2 e^{-st}}{s} \right]_0^2 + \frac{2}{s} \int_0^2 t e^{-st} dt \right\} \\ &= \frac{1}{1-e^{-2s}} \left\{ -\frac{4e^{-2s}}{s} + \frac{2}{s} \left[\frac{e^{-st}}{s^2} (-st-1) \right]_0^2 \right\} \\ &= \frac{2}{s^3(1-e^{-2s})} [1 - e^{-2s}(2s^2 + 2s + 1)]. \end{aligned}$$

3. We have,

$$\begin{aligned} L[f] &= \frac{1}{1-e^{-\pi s}} \int_0^\pi e^{-st} \sin t dt = \frac{1}{1-e^{-\pi s}} \left[\frac{e^{-st}(-s \sin t - \cos t)}{s^2 + 1} \right]_0^\pi \\ &= \frac{1 + e^{-\pi s}}{(1 - e^{-\pi s})(s^2 + 1)}. \end{aligned}$$

4. We have,

$$\begin{aligned} L[f] &= \frac{1}{1-e^{-\pi s}} \int_0^\pi e^{-st} \cos t dt = \frac{1}{1-e^{-\pi s}} \left[\frac{e^{-st}(-s \cos t + \sin t)}{s^2 + 1} \right]_0^\pi \\ &= \frac{s(1 + e^{-\pi s})}{(1 - e^{-\pi s})(s^2 + 1)}. \end{aligned}$$

5. We have,

$$\begin{aligned} L[f] &= \frac{1}{1-e^{-s}} \int_0^1 e^{-st} e^t dt = \frac{1}{1-e^{-s}} \int_0^1 e^{(1-s)t} dt \\ &= \frac{1}{(1-e^{-s})(1-s)} \left[e^{(1-s)t} \right]_0^1 = \frac{e^{1-s} - 1}{(1-s)(1-e^{-s})}. \end{aligned}$$

6. We have,

$$\begin{aligned}
 L[f] &= \frac{1}{1 - e^{-2s}} \left[\int_0^1 e^{-st} dt + \int_1^2 e^{-st} (-1) dt \right] \\
 &= \frac{1}{1 - e^{-2s}} \left\{ \left[-\frac{e^{-st}}{s} \right]_0^1 + \left[\frac{e^{-st}}{s} \right]_1^2 \right\} \\
 &= \frac{1}{1 - e^{-2s}} \left[\frac{1 - e^{-s}}{s} + \frac{e^{-2s} - e^{-s}}{s} \right] \\
 &= \frac{e^{-2s} - 2e^{-s} + 1}{s(1 - e^{-2s})} \\
 &= \frac{(1 - e^{-s})^2}{s(1 + e^{-s})(1 + e^{-s})} \\
 &= \frac{1 - e^{-s}}{s(1 + e^{-s})} = \frac{1}{s} \tanh(s/2).
 \end{aligned}$$

7. We have,

$$\begin{aligned}
 L[f] &= \frac{1}{1 - e^{-\pi s}} \left[\int_0^{\pi/2} e^{-st} t dt + \int_{\pi/2}^{\pi} e^{-st} \sin t dt \right] \\
 &= \frac{1}{1 - e^{-\pi s}} \left\{ \frac{2}{\pi} \left[\frac{e^{-st}(-st - 1)}{s^2} \right]_0^{\pi/2} + \frac{e^{-st}}{s^2 + 1} [-s \sin t - \cos t]_{\pi/2}^{\pi} \right\} \\
 &= \frac{1}{1 - e^{-\pi s}} \left[\frac{2 - e^{-\pi s/2}(\pi s + 2)}{\pi s^2} + \frac{se^{-\pi s/2} + e^{-\pi s}}{s^2 + 1} \right].
 \end{aligned}$$

8. We have,

$$\begin{aligned}
 L[f] &= \frac{1}{1 - e^{-\pi s}} \left[\int_0^{\pi/2} e^{-st} \cos t dt + \int_{\pi/2}^{\pi} e^{-st} (-\cos t) dt \right] \\
 &= \frac{1}{1 - e^{-\pi s}} \left\{ \left[\frac{e^{-st}(-s \cos t + \sin t)}{s^2 + 1} \right]_0^{\pi/2} - \left[\frac{e^{-st}(-s \cos t + \sin t)}{s^2 + 1} \right]_{\pi/2}^{\pi} \right\} \\
 &= \frac{2e^{-s\pi/2} + s(1 - e^{-\pi s})}{(1 - e^{-\pi s})(s^2 + 1)}.
 \end{aligned}$$

9. We have,

$$\begin{aligned}
 L[f] &= \frac{1}{1 - e^{-2as}} \left[\int_0^a e^{-st} (t/a) dt + \int_a^{2a} e^{-st} \left(\frac{2a - t}{a} \right) dt \right] \\
 &= \frac{1}{1 - e^{-2as}} \left[\frac{e^{-st}(-st - 1)}{as^2} \right]_0^a - \frac{2}{s} [e^{-st}] - \left[\frac{1}{a} \right]_a^{2a} \left[\frac{e^{-st}(-st - 1)}{s^2} \right]_a^{2a} \\
 &= \frac{1 - 2e^{-as} + e^{-2as}}{as^2(1 - e^{-2as})} = \frac{e^{as/2} - e^{-as/2}}{as^2(e^{as/2} + e^{-as/2})} = \frac{1}{as^2} \tanh(as/2).
 \end{aligned}$$

10. We have,

$$L[f] = \frac{1}{1 - e^{-2\pi s/a}} \int_0^{2\pi/a} e^{-st} \sin at \, dt = \frac{1}{1 - e^{-2\pi s/a}} \left[\frac{e^{-st}(-s \sin at - a \cos at)}{s^2 + a^2} \right]_0^{2\pi/a} = \frac{a}{s^2 + a^2}.$$

11. We have,

$$L[f] = \frac{1}{1 - e^{-2\pi s/a}} \int_0^{2\pi/a} e^{-st} \cos at \, dt = \frac{1}{1 - e^{-2\pi s/a}} \left[\frac{e^{-st}(-s \cos at + a \sin at)}{s^2 + a^2} \right]_0^{2\pi/a} = \frac{s}{s^2 + a^2}.$$

Solutions to Section 8.4

True-False Review:

1. **FALSE.** The Laplace transform of f may not exist unless we assume additionally that f is of exponential order.

2. **TRUE.** This is illustrated in the examples throughout this section. The procedure is outlined above Figure 8.4.1 in the text and indicates that the initial condition is imposed at the outset of the solution.

3. **TRUE.** If we leave the initial conditions as arbitrary constants, the solution technique presented in this section will result in the general solution to the differential equation.

4. **FALSE.** The expression for $Y(s)$ is affected by the initial conditions, because the initial conditions arise in the formulas for the Laplace transforms of the derivatives of the unknown function y .

Problems:

1. We apply the Laplace transform to both sides of the differential equation:

$$L[y'] + L[y] = 8L[e^{3t}].$$

Using the rule for the transform of the derivative, we obtain

$$sY(s) - y(0) + Y(s) = \frac{8}{s-3}$$

Substituting $y(0) = 2$, this becomes

$$Y(s)(s+1) - 2 = \frac{8}{s-3}.$$

Solving for $Y(s)$, we have

$$Y(s) = \frac{8}{(s-3)(s+1)} + \frac{2}{s+1},$$

which simplifies to

$$Y(s) = \frac{2}{s-3}.$$

Taking the inverse Laplace transform of both sides of this equation yields,

$$L^{-1}[Y(s)] = 2L^{-1}\left[\frac{1}{s-3}\right],$$

so that

$$y(t) = 2e^{3t}.$$

2. We apply the Laplace transform to both sides of the differential equation:

$$L[y'] + 3L[y] = 2L[e^{-t}].$$

Using the rule for the transform of the derivative, we obtain

$$sY(s) - y(0) + 3Y(s) = \frac{2}{s+1}.$$

Substituting $y(0) = 3$, this becomes

$$sY(s) - 3 + 3Y(s) = \frac{2}{s+1}.$$

Solving for $Y(s)$, we have

$$Y(s) = \frac{1}{s+1} + \frac{2}{s+3}.$$

Taking the inverse Laplace transform of both sides of this equation yields,

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{2}{s+3}\right]$$

so that

$$y(t) = e^{-t} + 2e^{-3t}.$$

3. We apply the Laplace transform to both sides of the differential equation:

$$L[y'] + 2L[y] = 4L[t].$$

Using the rule for the transform of the derivative, we obtain

$$sY(s) - y(0) + 2Y(s) = \frac{4}{s^2}.$$

Substituting $y(0) = 1$, this becomes

$$Y(s)(s+2) - 1 = \frac{4}{s^2}.$$

Solving for $Y(s)$, we have

$$Y(s) = -\frac{1}{s} + \frac{2}{s^2} + \frac{2}{s+2}.$$

Taking the inverse Laplace transform of both sides of this equation yields,

$$L^{-1}[Y(s)] = -L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{2}{s^2}\right] + L^{-1}\left[\frac{2}{s+2}\right]$$

so that

$$y(t) = 2t - 1 + 2e^{-2t}.$$

4. We apply the Laplace transform to both sides of the differential equation:

$$L[y'] - L[y] = 6L[\cos t].$$

Using the rule for the transform of the derivative, we obtain

$$sY(s) - y(0) - Y(s) = \frac{6s}{s^2 + 1}.$$

Substituting $y(0) = 2$, and solving for $Y(s)$, we have

$$Y(s) = \frac{2(s^2 + 3s + 1)}{(s - 1)(s^2 + 1)}.$$

Decomposing the right-hand side of this equation into partial fractions yields

$$Y(s) = \frac{5}{s - 1} - \frac{3s}{s^2 + 1} + \frac{3}{s^2 + 1}.$$

Taking the inverse Laplace transform of both sides of this equation gives,

$$L^{-1}[Y(s)] = 5L^{-1}\left[\frac{1}{s - 1}\right] - 3L^{-1}\left[\frac{s}{s^2 + 1}\right] + 3L^{-1}\left[\frac{1}{s^2 + 1}\right]$$

so that

$$y(t) = 5e^t - 3\cos t + 3\sin t.$$

5. We apply the Laplace transform to both sides of the differential equation:

$$L[y'] - L[y] = 5L[\sin 2t].$$

Using the rule for the transform of the derivative, we obtain

$$sY(s) - y(0) - Y(s) = 5\frac{2}{s^2 + 4}.$$

Substituting $y(0) = -1$, and solving for $Y(s)$, we have

$$Y(s) = \frac{10}{(s^2 + 4)(s - 1)} - \frac{1}{s - 1}.$$

Decomposing the right-hand side of this equation into partial fractions yields

$$Y(s) = \frac{1}{s - 1} - \frac{2s}{s^2 + 4} - \frac{2}{s^2 + 4}.$$

Taking the inverse Laplace transform of both sides of this equation gives,

$$L^{-1}[Y(s)] = L^{-1}\left[\frac{1}{s - 1}\right] - 2L^{-1}\left[\frac{s}{s^2 + 4}\right] - L^{-1}\left[\frac{2}{s^2 + 4}\right]$$

so that

$$y(t) = e^t - \sin 2t - 2\cos 2t.$$

6. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivative to obtain

$$sY(s) - y(0) + Y(s) = \frac{5}{(s-1)^2 + 1}.$$

Substituting $y(0) = 1$, and solving for $Y(s)$, we have

$$Y(s) = \frac{5}{[(s-1)^2 + 1](s+1)} + \frac{1}{s+1}.$$

Decomposing the right-hand side of this equation into partial fractions yields

$$Y(s) = -\frac{s}{(s-1)^2 + 1} + \frac{3}{(s-1)^2 + 1} + \frac{2}{s+1}.$$

Taking the inverse Laplace transform of both sides of this equation gives,

$$L^{-1}[Y(s)] = -L^{-1}\left[\frac{s}{(s-1)^2 + 1}\right] + 3L^{-1}\left[\frac{1}{(s-1)^2 + 1}\right] + 2L^{-1}\left[\frac{1}{s+1}\right]$$

so that

$$y(t) = -e^t \sin t + 3e^t \cos t + 2e^{-t}.$$

7. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] - 2Y(s) = 0.$$

Substituting $y(0) = 1$, $y'(0) = 4$ gives

$$[s^2Y(s) - s - 4] + [sY(s) - 1] - 2Y(s) = 0,$$

so that

$$Y(s) = \frac{s+5}{(s-1)(s+2)}.$$

Decomposing the right-hand side of this equation into partial fractions yields

$$Y(s) = \frac{2}{s-1} - \frac{1}{s+2}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = 2e^t - e^{-2t}.$$

8. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2Y(s) - 5s - 1] + 4Y(s) = 0,$$

where we have incorporated the initial conditions $y(0) = 5$, $y'(0) = 1$. Solving for $Y(s)$ yields

$$Y(s) = \frac{5s+1}{s^2+4}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$L^{-1}[Y(s)] = 5L^{-1}\left[\frac{s}{s^2+4}\right] + \frac{1}{2}L^{-1}\left[\frac{2}{s^2+4}\right],$$

so that

$$y(t) = 5 \cos 2t + \frac{1}{2} \sin 2t.$$

9. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2Y(s) - 1] - 3sY(s) + 2Y(s) = \frac{4}{s},$$

where we have incorporated the initial conditions $y(0) = 0$, $y'(0) = 1$. Solving for $Y(s)$ yields

$$Y(s) = \frac{4}{s(s-1)(s-2)} + \frac{1}{(s-1)(s-2)}.$$

Decomposing the right-hand side of this equation into partial fractions gives

$$Y(s) = \frac{2}{s} - \frac{5}{s-1} + \frac{3}{s-2}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$L^{-1}[Y(s)] = 2L^{-1}\left[\frac{1}{s}\right] - 5L^{-1}\left[\frac{1}{s-1}\right] + 3L^{-1}\left[\frac{1}{s-2}\right],$$

so that

$$y(t) = 2 - 5e^t + 3e^{2t}.$$

10. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2Y(s) - sy(0) - y'(0)] - [sY(s) - y(0)] - 12Y(s) = \frac{36}{s}.$$

Imposing the initial conditions $y(0) = 0$, $y'(0) = 12$ and solving for $Y(s)$ yields

$$Y(s) = \frac{12}{s(s-4)},$$

or equivalently,

$$Y(s) = \frac{3}{s-4} - \frac{3}{s}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$L^{-1}[Y(s)] = 3L^{-1}\left[\frac{1}{s-4}\right] - 3L^{-1}\left[\frac{1}{s-1}\right],$$

so that

$$y(t) = 3(e^{4t} - 1).$$

11. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2Y(s) - sy(0) - y'(0)] + [sY(s) - y(0)] - 2Y(s) = \frac{10}{s+1}.$$

Imposing the initial conditions $y(0) = 0$, $y'(0) = 1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{4}{(s+1)(s-1)(s+2)},$$

or equivalently,

$$Y(s) = \frac{2}{s-1} - \frac{5}{s+1} + \frac{3}{s+2}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$L^{-1}[Y(s)] = 2L^{-1}\left[\frac{1}{s-1}\right] - 5L^{-1}\left[\frac{1}{s+1}\right] + 3L^{-1}\left[\frac{1}{s+2}\right],$$

so that

$$y(t) = 2e^t - 5e^{-t} + 3e^{-2t}.$$

12. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s) = \frac{4}{s-3}.$$

Imposing the initial conditions $y(0) = 0$, $y'(0) = 0$ and solving for $Y(s)$ yields

$$Y(s) = \frac{4}{(s-1)(s-2)(s-3)},$$

or equivalently,

$$Y(s) = \frac{2}{s-3} - \frac{4}{s-2} + \frac{2}{s-1}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$L^{-1}[Y(s)] = 2L^{-1}\left[\frac{1}{s-3}\right] - 4L^{-1}\left[\frac{1}{s-2}\right] + 2L^{-1}\left[\frac{1}{s-1}\right],$$

so that

$$y(t) = 2e^{3t} - 4e^{2t} + 2e^t.$$

13. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2Y(s) - y(0) - y'(0)] - 2[sY(s) - y(0)] = \frac{30}{s+3}.$$

Imposing the initial conditions $y(0) = 1$, $y'(0) = 0$ and solving for $Y(s)$ yields

$$Y(s) = -\frac{4}{s} + \frac{3}{s-2} + \frac{2}{s+3}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = -4 + 3e^{2t} + 2e^{-3t}.$$

14. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2Y(s) - sy(0) - y'(0)] - Y(s) = \frac{12}{s-2}.$$

Imposing the initial conditions $y(0) = 1$, $y'(0) = 1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{4}{s-2} - \frac{5}{s-1} + \frac{2}{s+1}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = 4e^{2t} - 5e^t + 2e^{-t}.$$

15. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2Y(s) - sy(0) - y'(0)] + 4Y(s) = \frac{10}{s+1}.$$

Imposing the initial conditions $y(0) = 4$, $y'(0) = 0$ and solving for $Y(s)$ yields

$$Y(s) = \frac{2+2s}{s^2+4} + \frac{2}{s+1}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = 2\cos 2t + \sin 2t + 2e^{-t}.$$

16. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2Y(s) - sy(0) - y'(0)] - [sY(s) - y(0)] - 6Y(s) = \frac{12}{s} - \frac{6}{s-1}.$$

Imposing the initial conditions $y(0) = 5$, $y'(0) = -3$ and solving for $Y(s)$ yields

$$Y(s) = \frac{12}{5(s-3)} + \frac{28}{5(s+2)} - \frac{2}{s} + \frac{1}{s-1}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = \frac{12}{5}e^{3t} + \frac{28}{5}e^{-2t} - 2 + e^t.$$

17. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2Y(s) - sy(0) - y'(0)] - Y(s) = \frac{6s}{s^2+1}.$$

Imposing the initial conditions $y(0) = 0$, $y'(0) = 4$ and solving for $Y(s)$ yields

$$Y(s) = \frac{7}{2(s-1)} - \frac{1}{2(s+1)} - \frac{3s}{s^2+1}$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = \frac{7}{2}e^t - \frac{1}{2}e^{-t} - 3\cos t.$$

18. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2Y(s) - sy(0) - y'(0)] - 9Y(s) = \frac{26}{s^2+4}.$$

Imposing the initial conditions $y(0) = 3$, $y'(0) = 1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{2}{s-3} + \frac{1}{s+3} - \frac{2}{s^2+4}$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = 2e^{3t} + e^{-3t} - \sin 2t.$$

19. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2Y(s) - sy(0) - y'(0)] - Y(s) = \frac{8}{s^2+1} - \frac{6s}{s^2+1}.$$

Imposing the initial conditions $y(0) = 2$, $y'(0) = -1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{1}{s-1} - \frac{2}{s+1} + \frac{3s}{s^2+1} - \frac{4}{s^2+1}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = e^t - 2e^{-t} + 3\cos t - 4\sin t.$$

20. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2Y(s) - sy(0) - y'(0)] - [sY(s) - y(0)] - 2Y(s) = \frac{10s}{s^2+1}.$$

Imposing the initial conditions $y(0) = 0$, $y'(0) = -1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{-1-3s}{s^2+1} + \frac{1}{s-2} + \frac{2}{s+1}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = -\sin t - 3\cos t + e^{2t} + 2e^{-t}.$$

21. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2Y(s) - sy(0) - y'(0)] + 5[sY(s) - y(0)] + 4Y(s) = \frac{40}{s^2 + 4}.$$

Imposing the initial conditions $y(0) = -1$, $y'(0) = 2$ and solving for $Y(s)$ yields

$$Y(s) = \frac{2}{s+1} - \frac{1}{s+4} - \frac{2s}{s^2+4}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = 2e^{-t} - e^{-4t} - 2\cos 2t.$$

22. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2Y(s) - sy(0) - y'(0)] + 5[sY(s) - y(0)] + 4Y(s) = \frac{40}{s^2 + 4}.$$

Imposing the initial conditions $y(0) = 1$, $y'(0) = -2$ and solving for $Y(s)$ yields

$$Y(s) = \frac{10}{3(s+1)} - \frac{1}{3(s+4)} - \frac{2s}{s^2+4}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = \frac{10}{3}e^{-t} - \frac{1}{3}e^{-4t} - 2\cos 2t.$$

23. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2Y(s) - sy(0) - y'(0)] - [sY(s) - y(0)] + 2Y(s) = \frac{3s}{s^2+1} + \frac{1}{s^2+1}.$$

Imposing the initial conditions $y(0) = 1$, $y'(0) = 1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{7}{5(s-2)} - \frac{1}{s-1} + \frac{3s-4}{5(s^2+1)}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = \frac{1}{5}(7e^{2t} - 5e^t + 3\cos t - 4\sin t).$$

24. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2Y(s) - sy(0) - y'(0)] + 4Y(s) = \frac{9}{s^2+1}.$$

Imposing the initial conditions $y(0) = 1$, $y'(0) = -1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{s}{s^2 + 4} - \frac{4}{s^2 + 4} + \frac{3}{s^2 + 1}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = \cos 2t - 2 \sin 2t + 3 \sin t.$$

25. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2 Y(s) - sy(0) - y'(0)] + Y(s) = \frac{6s}{s^2 + 4}.$$

Imposing the initial conditions $y(0) = 0$, $y'(0) = 2$ and solving for $Y(s)$ yields

$$Y(s) = \frac{2}{s^2 + 1} + \frac{2s}{s^2 + 1} - \frac{2s}{s^2 + 4}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = 2(\cos t + \sin t - \cos 2t).$$

26. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2 Y(s) - sy(0) - y'(0)] + 9Y(s) = \frac{14(s + 2)}{s^2 + 16}.$$

Imposing the initial conditions $y(0) = 1$, $y'(0) = 2$ and solving for $Y(s)$ yields

$$Y(s) = \frac{3(s + 2)}{s^2 + 9} - \frac{2(s + 2)}{s^2 + 16}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = 3 \cos 3t + 2 \sin 3t - 2 \cos 4t - \sin 4t.$$

27. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2 Y(s) - sy(0) - y'(0)] - Y(s) = 0.$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{sy(0) + y'(0)}{(s + 1)(s - 1)},$$

which can be written in the equivalent form

$$Y(s) = \frac{y(0) - y'(0)}{2(s - 1)} + \frac{y(0) + y'(0)}{2(s + 1)}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = \frac{y(0) + y'(0)}{2}e^t + \frac{y(0) - y'(0)}{2}e^{-t}.$$

28. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$[s^2Y(s) - sy(0) - y'(0)] - \omega^2Y(s) = \frac{A\omega_0 + Bs}{s^2 + \omega_0^2}.$$

Imposing the initial conditions $y(0) = y_0$, $y'(0) = y_1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{A\omega_0 + Bs}{(s^2 + \omega_0^2)(s^2 + \omega^2)} + \frac{y_0s + y_1}{s^2 + \omega^2},$$

which can be written in the equivalent form

$$\begin{aligned} Y(s) = & \left(\frac{A\omega_0}{\omega_0^2 - \omega^2} + y_1 \right) \cdot \frac{1}{s^2 + \omega^2} + \left(\frac{B}{\omega_0^2 - \omega^2} + y_0 \right) \cdot \frac{s}{s^2 + \omega^2} \\ & - \left(\frac{A}{\omega_0^2 - \omega^2} \right) \cdot \frac{\omega_0}{s^2 + \omega_0^2} - \left(\frac{B}{\omega_0^2 - \omega^2} \right) \cdot \frac{s}{s^2 + \omega_0^2}. \end{aligned}$$

Taking the inverse Laplace transform of both sides of this equation gives

$$\begin{aligned} L^{-1}[Y(s)] = & \left(\frac{A\omega_0}{\omega_0^2 - \omega^2} + y_1 \right) \cdot L^{-1} \left[\frac{1}{s^2 + \omega^2} \right] + \left(\frac{B}{\omega_0^2 - \omega^2} + y_0 \right) \cdot L^{-1} \left[\frac{s}{s^2 + \omega^2} \right] \\ & - \left(\frac{A}{\omega_0^2 - \omega^2} \right) \cdot L^{-1} \left[\frac{\omega_0}{s^2 + \omega_0^2} \right] - \left(\frac{B}{\omega_0^2 - \omega^2} \right) \cdot L^{-1} \left[\frac{s}{s^2 + \omega_0^2} \right]. \end{aligned}$$

Therefore,

$$y(t) = \frac{1}{\omega} \left(\frac{A\omega_0}{\omega_0^2 - \omega^2} + y_1 \right) \sin \omega t + \left(\frac{B}{\omega_0^2 - \omega^2} + y_0 \right) \cos \omega t - \left(\frac{A}{\omega_0^2 - \omega^2} \right) \sin \omega_0 t - \left(\frac{B}{\omega_0^2 - \omega^2} \right) \cos \omega_0 t$$

29.

(a) We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$sI(s) - i(0) + \frac{R}{L}I(s) = \frac{E_0}{L} \cdot \frac{1}{s}.$$

Imposing the initial conditions $i(0) = 0$, and solving for $I(s)$ yields

$$I(s) = \frac{E_0}{Ls(s + R/L)},$$

which can be written in the equivalent form

$$I(s) = \frac{E_0}{Rs} - \frac{E_0}{R(s + R/L)}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$i(t) = \frac{E_0}{R} - \frac{E_0}{R} e^{-Rt/L} = \frac{E_0}{R} \left(1 - e^{-Rt/L}\right).$$

(b) We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$sI(s) - i(0) + \frac{R}{L}I(s) = \frac{E_0}{L} \cdot \frac{\omega}{s^2 + \omega^2}.$$

Imposing the initial conditions $i(0) = 0$, and solving for $I(s)$ yields

$$I(s) = \frac{E_0\omega}{(s^2 + \omega^2)(Ls + R)},$$

which can be written in the equivalent form

$$I(s) = \frac{E_0\omega}{R^2 + L^2\omega^2} \left(\frac{R}{s^2 + \omega^2} - \frac{Ls}{s^2 + \omega^2} + \frac{L}{s + R/L} \right).$$

Taking the inverse Laplace transform of both sides of this equation gives

$$i(t) = \frac{E_0\omega}{R^2 + L^2\omega^2} \left(\frac{R}{\omega} \sin \omega t - L \cos \omega t + L e^{-Rt/L} \right),$$

that is,

$$i(t) = \frac{E_0}{R^2 + L^2\omega^2} \left(R \sin \omega t - L \omega \cos \omega t + L \omega e^{-Rt/L} \right). \quad (0.0.52)$$

We now define the acute angle θ by

$$\tan \theta = \frac{\omega L}{R},$$

in which case

$$\cos \theta = \frac{R}{\sqrt{R^2 + \omega^2 L^2}}, \quad \sin \theta = \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}}.$$

Solving the preceding equations for R and ωL yields

$$R = \sqrt{R^2 + \omega^2 L^2} \cos \theta, \quad \omega L = \sqrt{R^2 + \omega^2 L^2} \sin \theta.$$

We now insert these results into (0.0.52) to obtain

$$i(t) = \frac{E_0}{R^2 + L^2\omega^2} \left(\sqrt{R^2 + \omega^2 L^2} \cos \theta \sin \omega t - \sqrt{R^2 + \omega^2 L^2} \sin \theta \cos \omega t + L \omega e^{-Rt/L} \right),$$

or, equivalently,

$$i(t) = \frac{E_0}{\sqrt{R^2 + L^2\omega^2}} \sin(\omega t - \theta) + \frac{E_0 L \omega}{R^2 + L^2\omega^2} e^{-Rt/L}.$$

30. Taking the Laplace transform of the given system of differential equation yields

$$sX_1 - X_1(0) = a_{11}X_1 + a_{12}X_2 + B_1(s),$$

$$sX_2 - X_2(0) = a_{21}X_1 + a_{22}X_2 + B_2(s),$$

that is

$$\begin{aligned}(s - a_{11})X_1(s) - a_{12}X_2(s) &= \alpha_1 + B_1(s), \\ -a_{21}X_1(s) + (s - a_{22})X_2(s) &= \alpha_2 + B_2(s),\end{aligned}$$

where $\alpha_1 = X_1(0)$ and $\alpha_2 = X_2(0)$.

31. Taking the Laplace transform of the given system yields

$$sX_1 = -4X_1 - 2X_2, \quad sX_2 - 1 = X_1 - X_2,$$

that is,

$$(s + 4)X_1 + 2X_2 = 0, \quad -X_1 + (s + 1)X_2 = 1.$$

Solving this linear algebraic system using Cramer's rule yields

$$\begin{aligned}X_1 &= -\frac{2}{s^2 + 5s + 6} = -\frac{2}{(s + 2)(s + 3)} = 2 \left[\frac{1}{s + 3} - \frac{1}{s + 2} \right], \\ X_2 &= \frac{s + 4}{s^2 + 5s + 6} = \frac{s + 4}{(s + 2)(s + 3)} = \frac{2}{s + 2} - \frac{1}{s + 3}.\end{aligned}$$

Taking the inverse Laplace transform of these equation we obtain

$$x_1(t) = 2(e^{-3t} - e^{-2t}), \quad x_2(t) = 2e^{-2t} - e^{-3t}.$$

32. Taking the Laplace transform of the given system yields

$$(s + 3)X_1 - 4X_2 = 2, \quad X_1 + (s - 2)X_2 = 1.$$

Solving this system we obtain

$$X_1(s) = \frac{4}{s(s + 2)} + \frac{2}{3(s - 1)}, \quad X_2(s) = \frac{1}{3(s + 2)} + \frac{2}{3(s - 1)}.$$

Consequently,

$$x_1(t) = \frac{2}{3}(2e^{-2t} + e^t), \quad x_2(t) = \frac{1}{3}(e^{-2t} + 2e^t).$$

33. Let $P(n)$ be the statement:

$$L[f^{(n)}] = s^n L[f] - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Then we have established in Theorem 8.4.1 that

$$L[f'] = sL[f] - f(0),$$

which means that $P(1)$ is true. Let k be an arbitrary positive integer, and assume that

$$P(k) : L[f^{(k)}] = s^k L[f] - s^{k-1}f(0) - s^{k-2}f'(0) - \dots - sf^{(k-2)}(0) - f^{(k-1)}(0) \quad (0.0.53)$$

is true. Then, since $f^{(k+1)} = (f^{(k)})'$, we have

$$\begin{aligned} L[f^{(k+1)}] &= L\left[(f^{(k)})'\right] \\ &= sL[f^{(k)}] - f^{(k)}(0), \end{aligned}$$

where we have applied $P(1)$ in the second step. Inserting the expression for $L[f^{(k)}]$ given in (0.0.53) into the preceding equation yields

$$\begin{aligned} L[f^{(k+1)}] &= s\left\{s^k L[f] - s^{k-1}f(0) - s^{k-2}f'(0) - \dots - sf^{(k-2)}(0) - f^{(k-1)}(0)\right\} - f^{(k)}(0) \\ &= s^{k+1}L[f] - s^k f(0) - s^{k-1}f'(0) - \dots - sf^{(k-1)}(0) - f^{(k)}(0), \end{aligned}$$

so that $P(k+1)$ is true. We have therefore established that

1. $P(1)$ is true.
2. For each positive integer k , if $P(k)$ is true then $P(k+1)$ is true.

It follows from mathematical induction, that $P(n)$ is true for all positive integers n .

Solutions to Section 8.5

True-False Review:

1. **TRUE.** This follows at once from the first formula in the First Shifting Theorem, with a replaced by $-a$.
2. **FALSE.** For instance, if $f(t) = e^t$, then $f(t-1) = e^{t-1} \neq e^t - 1$.
3. **TRUE.** The formula for $f(t)$ is obtained from the formula for $f(t+2)$ by replacing $t+2$ with t (and hence $t+3$ with $t+1$ and t with $t-2$).
4. **FALSE.** If we take $f(x) = x$ and $g(x) = x-3$, for example, then

$$\int_0^1 f(t)dt = \frac{1}{2} \quad \text{and} \quad \int_0^1 g(t)dt = -\frac{5}{2},$$

and so

$$\int_0^1 f(t)dt \neq \int_0^1 g(t)dt - 3.$$

5. **FALSE.** The correct formula is

$$L[e^{-t} \sin 2t] = \frac{2}{(s+1)^2 + 4}.$$

6. **TRUE.** The Laplace transform of $f(t) = t^3$ is $F(s) = \frac{3!}{s^4} = \frac{6}{s^4}$, and so the First Shifting Theorem with $a = 2$ gives $L[e^{2t}t^3] = \frac{6}{(s-2)^4}$.
7. **TRUE.** The inverse Laplace transform of $F(s) = \frac{s}{s^2+9}$ is $f(t) = \cos 3t$, and so the First Shifting Theorem with $a = -4$ gives the indicated inverse Laplace transform.
8. **FALSE.** The correct formula is

$$L^{-1}\left[\frac{3}{(s+1)^2 + 36}\right] = \frac{1}{2}e^{-t} \sin 6t.$$

Problems:

1. $f(t-a) = f(t-1) = t-1$.

2. $f(t-a) = f(t-3) = 1$.

3. $f(t-a) = f(t+2) = (t+2)^2 - 2(t+2) = t(t+2)$.

4. $f(t-a) = f(t-2) = e^{3(t-2)}$.

5. $f(t-a) = f(t-\pi) = e^{2(t-\pi)} \cos(t-\pi) = -e^{2(t-\pi)} \cos t$.

6. $f(t-a) = f(t+1) = (t+1)e^{2(t+1)}$.

7. $f(t-a) = f(t-\pi/6) = e^{-(t-\pi/6)} \sin[2(t-\pi/6)] = e^{-(t-\pi/6)} [\sin(2t) \cos(\pi/3) - \cos(2t) \sin(\pi/3)] = e^{-(t-\pi/6)} \left(\frac{1}{2} \sin 2t - \frac{\sqrt{3}}{2} \cos 2t \right) = \frac{e^{-(t-\pi/6)}}{2} (\sin 2t - \sqrt{3} \cos 2t)$.

8. $f(t-a) = f(t-1) = \frac{t-1}{(t-1)^2+4}$.

9. $f(t-a) = f(t-2) = \frac{(t-2)+1}{(t-2)^2-2(t-2)+2} = \frac{t-1}{t^2-6t+10}$.

10. $f(t-a) = f(t-\pi/4) = e^{-(t-\pi/4)} [\sin(2(t-\pi/4)) + \cos(2(t-\pi/4))] = e^{-(t-\pi/4)} [\sin(2t-\pi/2) + \cos(2t-\pi/2)] = e^{-(t-\pi/4)} (\sin 2t - \cos 2t)$.

11. $f(t-1) = (t-1)^2$. Replace $t-1$ by t ; $f(t) = t^2$.

12. $f(t-1) = (t-2)^2 \implies f(t-1) = [(t-1)-1]^2$. Replace $t-1$ by t ; $f(t) = (t-1)^2$.

13. $f(t-2) = (t-2)e^{3(t-2)}$. Replace $t-2$ by t : $f(t) = te^{3t}$.

14. $f(t-1) = t \sin[3(t-1)]$. Replace t by $t+1$: $f(t) = (t+1) \sin 3t$.

15. $f(t-3) = te^{-(t-3)}$. Replace t by $t+3$: $f(t) = (t+3)e^{-t}$.

16. $f(t-4) = \frac{t+1}{(t-1)^2+4}$. Replace t by $t+4$: $f(t) = \frac{t+5}{(t+3)^2+4}$.

17. $L[\cos 4t] = \frac{s}{s^2+16} \implies L[e^{3t} \cos 4t] = \frac{s-3}{(s-3)^2+16}$.

18. $L[\sin 5t] = \frac{5}{s^2+25} \implies L[e^{-4t} \sin 5t] = \frac{5}{(s+4)^2+25}$.

19. $L[t] = \frac{1}{s^2} \implies L[te^{2t}] = \frac{1}{(s-2)^2}$.

20. $L[3t] = 3L[t] = \frac{3}{s^2} \implies L[3e^{-t}t] = \frac{3}{(s+1)^2}$.

$$21. L[t^3] = \frac{3!}{s^4} \implies L[e^{-4t}t^3] = \frac{3!}{(s+4)^4}.$$

$$22. L[t] = \frac{1}{s^2} \implies L[e^{-2t}t] = \frac{1}{(s+2)^2}. \text{ Therefore,}$$

$$L[e^t - te^{-2t}] = L[e^t] - L[te^{-2t}] = \frac{1}{s-1} - \frac{1}{(s+2)^2} = \frac{s^2 + 3s + 5}{(s-1)(s+2)^2}.$$

23.

$$\begin{aligned} L[2e^{3t} \sin t + 4e^{3t} \cos 3t] &= 2L[e^{3t} \sin t] + 4L[\cos 3t] \\ &= 2 \left[\frac{1}{(s-3)^2 + 1} \right] + 4 \left[\frac{s+1}{(s+1)^2 + 9} \right] \\ &= \frac{2[2s^3 - 9s^2 + 10s + 30]}{[(s-3)^2 + 1][(s+1)^2 + 9]}. \end{aligned}$$

24.

$$\begin{aligned} L[e^{2t}(1 - \sin^2 t)] &= L[e^{2t}(1 - (\cos 2t - 1)/2)] \\ &= \frac{1}{2}L[e^{2t}(1 - \cos 2t)] = \frac{1}{2}(L[e^{2t}] - L[e^{2t} \cos 2t]) \\ &= \frac{1}{2} \left[\frac{1}{s-2} - \frac{s-2}{(s-2)^2 + 4} \right] = \frac{2}{(s-2)[(s-2)^2 + 4]} \end{aligned}$$

$$25. L[t^2(e^t - 3)] = L[e^t t^2] - 3L[t^2] = \frac{2}{(s-1)^3} - \frac{6}{s^3}.$$

26.

$$\begin{aligned} L[e^{-2t} \sin(t - \pi/4)] &= L \left[e^{-2t} \left(\frac{\sqrt{2}}{2} \sin t - \frac{\sqrt{2}}{2} \cos t \right) \right] \\ &= \frac{\sqrt{2}}{2} \{ L[e^{-2t} \sin t] - L[e^{-2t} \cos t] \} \\ &= \frac{\sqrt{2}}{2} \left[\frac{1}{(s+2)^2 + 1} - \frac{s+2}{(s+2)^2 + 1} \right] = -\frac{\sqrt{2}(s+1)}{2[(s+2)^2 + 1]}. \end{aligned}$$

$$27. L^{-1} \left[\frac{1}{s^2} \right] = t \implies L^{-1} \left[\frac{1}{(s-3)^2} \right] = te^{3t}.$$

$$28. L^{-1} \left[\frac{2}{s^3} \right] = t^2 \implies L^{-1} \left[\frac{4}{(s+2)^3} \right] = 2t^2 e^{-2t}.$$

29. Since

$$L^{-1}[\sqrt{\pi/s}] = t^{-1/2}$$

it follows that

$$L^{-1} \left[\frac{1}{s^{1/2}} \right] = \pi^{-1/2} t^{-1/2}.$$

Therefore,

$$L^{-1} \left[\frac{2}{s^{1/2}} \right] = \frac{2}{\sqrt{\pi}} t^{-1/2}.$$

Consequently,

$$L^{-1} \left[\frac{2}{(s+3)^{1/2}} \right] = \frac{2}{\sqrt{\pi}} e^{-3t} t^{-1/2}.$$

$$\mathbf{30.} \quad L^{-1} \left[\frac{2}{s^2 + 4} \right] = \sin 2t \implies L^{-1} \left[\frac{2}{(s-1)^2 + 4} \right] = e^t \sin 2t.$$

$$\mathbf{31.} \quad L^{-1} \left[\frac{s}{s^2 + 9} \right] = \cos 3t \implies L^{-1} \left[\frac{s+2}{(s+2)^2 + 9} \right] = e^{-2t} \cos 3t.$$

32.

$$\begin{aligned} L^{-1} \left[\frac{s}{(s-3)^2 + 4} \right] &= L^{-1} \left[\frac{s-3}{(s-3)^2 + 4} + \frac{3}{(s-3)^2 + 4} \right] \\ &= L^{-1} \left[\frac{s-3}{(s-3)^2 + 4} \right] + L^{-1} \left[\frac{3}{(s-3)^2 + 4} \right] \\ &= e^{3t} \cos 2t + \frac{3}{2} e^{3t} \sin 2t = \frac{e^{3t}}{2} (2 \cos 2t + 3 \sin 2t). \end{aligned}$$

$$\mathbf{33.} \quad L^{-1} \left[\frac{5}{(s-2)^2 + 16} \right] = \frac{5}{4} L^{-1} \left[\frac{4}{(s-2)^2 + 16} \right] = \frac{5}{4} e^{2t} \sin 4t.$$

$$\mathbf{34.} \quad L^{-1} \left[\frac{6}{(s+1)^2 + 1} \right] = 6 L^{-1} \left[\frac{1}{(s+1)^2 + 1} \right] = 6 e^{-t} \sin t.$$

35.

$$\begin{aligned} L^{-1} \left[\frac{s-2}{s^2 + 2s + 26} \right] &= L^{-1} \left[\frac{s-2}{(s+1)^2 + 25} \right] \\ &= L^{-1} \left[\frac{s+1}{(s+1)^2 + 25} - \frac{3}{(s+1)^2 + 25} \right] \\ &= L^{-1} \left[\frac{s+1}{(s+1)^2 + 25} \right] + L^{-1} \left[\frac{3}{(s+1)^2 + 25} \right] \\ &= e^{-t} \cos 5t - \frac{3}{5} e^{-t} \sin 5t = \frac{1}{5} e^{-t} (5 \cos 5t - 3 \sin 5t). \end{aligned}$$

36.

$$\begin{aligned} L^{-1} \left[\frac{2s}{s^2 - 4s + 13} \right] &= L^{-1} \left[\frac{2s}{(s-2)^2 + 9} \right] \\ &= L^{-1} \left[\frac{2s-4}{(s-2)^2 + 9} + \frac{4}{(s-2)^2 + 9} \right] \\ &= 2 L^{-1} \left[\frac{s-2}{(s-2)^2 + 9} \right] + \frac{4}{3} L^{-1} \left[\frac{3}{(s-2)^2 + 9} \right] \\ &= 2 e^{2t} \cos 3t + \frac{4}{3} e^{2t} \sin 3t = \frac{2}{3} e^{2t} (3 \cos 3t + 2 \sin 3t). \end{aligned}$$

37.

$$\begin{aligned}
L^{-1} \left[\frac{s}{(s+1)^2 + 4} \right] &= L^{-1} \left[\frac{s+1}{(s+1)^2 + 4} - \frac{1}{(s+1)^2 + 4} \right] \\
&= L^{-1} \left[\frac{s+1}{(s+1)^2 + 4} \right] - \frac{1}{2} L^{-1} \left[\frac{2}{(s+1)^2 + 4} \right] \\
&= e^{-t} \cos 2t - \frac{e^{-t}}{2} \sin 2t = \frac{1}{2} e^{-t} (2 \cos 2t - \sin 2t).
\end{aligned}$$

38.

$$\begin{aligned}
L^{-1} \left[\frac{2s+3}{(s+5)^2 + 49} \right] &= L^{-1} \left[\frac{2s+10}{(s+5)^2 + 49} - \frac{7}{(s+5)^2 + 49} \right] \\
&= 2L^{-1} \left[\frac{s+5}{(s+5)^2 + 49} \right] + L^{-1} \left[\frac{7}{(s+5)^2 + 49} \right] \\
&= 2e^{-5t} \cos 7t - e^{-5t} \sin 7t = e^{-5t} (2 \cos 7t - \sin 7t).
\end{aligned}$$

39.

$$\begin{aligned}
L^{-1} \left[\frac{4}{s(s+2)^2} \right] &= L^{-1} \left[\frac{1}{s} - \frac{1}{s+2} - \frac{2}{(s+2)^2} \right] \\
&= L^{-1} \left[\frac{1}{s} \right] - L^{-1} \left[\frac{1}{s+2} \right] - 2L^{-1} \left[\frac{1}{(s+2)^2} \right] \\
&= 1 - e^{-2t} - 2te^{-2t} = 1 - e^{-2t} (1 + 2t).
\end{aligned}$$

40.

$$\begin{aligned}
L^{-1} \left[\frac{2s+1}{(s-1)^2(s+2)} \right] &= L^{-1} \left[\frac{1}{3(s-1)} + \frac{1}{(s-1)^2} - \frac{1}{3(s+2)} \right] \\
&= \frac{1}{3} L^{-1} \left[\frac{1}{s-1} \right] + L^{-1} \left[\frac{1}{(s-1)^2} \right] - \frac{1}{3} L^{-1} \left[\frac{1}{s+2} \right] \\
&= \frac{1}{3} e^t + te^t - \frac{1}{3} e^{-2t} = \frac{1}{3} [e^t (1 + 3t) - e^{-2t}].
\end{aligned}$$

41.

$$\begin{aligned}
L^{-1} \left[\frac{2s+3}{s(s^2-2s+5)} \right] &= L^{-1} \left[\frac{3}{5s} - \frac{3s-16}{5[(s-1)^2+4]} \right] \\
&= L^{-1} \left[\frac{3}{5s} - \frac{3(s-1)}{5[(s-1)^2+4]} + \frac{26}{10[(s-1)^2+4]} \right] \\
&= \frac{3}{5} L^{-1} \left[\frac{1}{s} \right] - \frac{3}{5} L^{-1} \left[\frac{s-1}{(s-1)^2+4} \right] + \frac{13}{10} L^{-1} \left[\frac{2}{(s-1)^2+4} \right] \\
&= \frac{3}{5} - \frac{3}{5} e^t \cos 2t + \frac{13}{10} e^t \sin 2t = \frac{1}{10} [6 + e^t (13 \sin 2t - 6 \cos 2t)].
\end{aligned}$$

42. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$s^2 Y(s) - sy(0) - y'(0) - Y(s) = \frac{8}{s-1}.$$

Imposing the initial conditions $y(0) = 0$, $y'(0) = 0$ and solving for $Y(s)$ yields

$$Y(s) = \frac{8}{(s-1)^2(s+1)},$$

which can be written in the equivalent form

$$Y(s) = \frac{4}{(s-1)^2} - \frac{2}{s-1} + \frac{2}{s+1}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = 4te^t - 2e^t + 2e^{-t} = 2[e^{-t} - e^t(1-2t)].$$

43. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$s^2Y(s) - sy(0) - y'(0) - 4Y(s) = \frac{12}{s-2}.$$

Imposing the initial conditions $y(0) = 2$, $y'(0) = 3$ and solving for $Y(s)$ yields

$$Y(s) = \frac{2s^2 - s + 6}{(s+2)(s-2)^2},$$

which can be written in the equivalent form

$$Y(s) = \frac{3}{(s-2)^2} + \frac{1}{s-2} + \frac{1}{s+2}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = 3te^{2t} + e^{2t} + e^{-2t} = e^{2t}(1+3t) + e^{-2t}.$$

44. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$s^2Y(s) - sy(0) - y'(0) - [sY(s) - y(0)] - 2Y(s) = \frac{6}{s+1}.$$

Imposing the initial conditions $y(0) = 0$, $y'(0) = 1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{s+7}{(s+1)^2(s-2)},$$

which can be written in the equivalent form

$$Y(s) = \frac{1}{s-2} - \frac{2}{(s+1)^2} - \frac{1}{s+1}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = e^{2t} - 2te^{-t} - e^{-t} = e^{2t} - e^{-t}(1+2t).$$

45. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) - 2Y(s) = \frac{3}{s+1}.$$

Imposing the initial conditions $y(0) = 3$, $y'(0) = -1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{3s^2 + 8s + 7}{(s+2)^2(s-1)},$$

which can be written in the equivalent form

$$Y(s) = \frac{2}{s-1} - \frac{1}{(s+2)^2} + \frac{1}{s+2}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = 2e^t - te^{-2t} + e^{-2t} = 2e^t + e^{-2t}(1-t).$$

46. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$s^2Y(s) - sy(0) - y'(0) - 4[sY(s) - y(0)] + 4Y(s) = \frac{6}{s-2}.$$

Imposing the initial conditions $y(0) = 1$, $y'(0) = 0$ and solving for $Y(s)$ yields

$$Y(s) = \frac{s^2 - 6s + 14}{(s-2)^3},$$

which can be written in the equivalent form

$$Y(s) = \frac{6}{(s-2)^3} - \frac{2}{(s-2)^2} + \frac{1}{s-2}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = 3t^2e^{2t} - 2te^{2t} + e^{2t} = e^{2t}(1 - 2t + 3t^2).$$

47. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] + Y(s) = \frac{2}{s+1}.$$

Imposing the initial conditions $y(0) = 2$, $y'(0) = 1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{1}{(s+1)^2} \left(\frac{2}{s+1} + 2s + 5 \right),$$

which can be written in the equivalent form

$$Y(s) = \frac{2}{(s+1)^3} + \frac{3}{(s+1)^2} + \frac{2}{s+1}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = t^2 e^{-t} + 3te^{-t} + 2e^{-t} = e^{-t}(2 + 3t + t^2).$$

48. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$s^2 Y(s) - sy(0) - y'(0) - 4Y(s) = \frac{2}{(s-1)^2}.$$

Imposing the initial conditions $y(0) = 0$, $y'(0) = 0$ and solving for $Y(s)$ yields

$$Y(s) = \frac{2}{(s-1)^2(s^2+4)},$$

which can be written in the equivalent form

$$Y(s) = \frac{1}{2(s-2)} - \frac{2}{3(s-1)^2} - \frac{4}{9(s-1)} - \frac{1}{18(s+2)}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = \frac{1}{2}e^{2t} - \frac{2}{3}te^t - \frac{4}{9}e^t - \frac{1}{18}e^{-2t} = \frac{1}{2}e^{2t} - \frac{1}{18}e^{-2t} - \frac{2e^t}{9}(2+3t).$$

49. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$s^2 Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = \frac{12}{(s-2)^2}.$$

Imposing the initial conditions $y(0) = 0$, $y'(0) = 1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{s^2 - 4s + 16}{(s-2)^2(s+1)(s+2)},$$

which can be written in the equivalent form

$$Y(s) = \frac{1}{(s-2)^2} - \frac{7}{12(s-2)} + \frac{7}{3(s+1)} - \frac{7}{4(s+2)}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = te^{2t} - \frac{7}{12}e^{2t} + \frac{7}{3}e^{-t} - \frac{7}{4}e^{-2t} = \frac{7}{3}e^{-t} - \frac{7}{4}e^{-2t} + \frac{1}{12}e^{2t}(12t-7).$$

50. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{5}{(s+3)^2}.$$

Imposing the initial conditions $y(0) = 2$, $y'(0) = 0$ and solving for $Y(s)$ yields

$$Y(s) = \frac{5}{(s+3)^2(s^2+1)} + \frac{2s}{s^2+1},$$

which can be written in the equivalent form

$$Y(s) = \frac{1}{2(s+3)^2} + \frac{3}{10(s+3)} + \frac{4-3s}{10(s^2+1)} + \frac{2s}{s^2+1}.$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = \frac{1}{2}te^{-3t} + \frac{3}{10}e^{-3t} + \frac{17}{10}\cos t + \frac{2}{5}\sin t = \frac{17}{10}\cos t + \frac{2}{5}\sin t + \frac{1}{10}e^{-3t}(3+5t).$$

51. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$s^2Y(s) - sy(0) - y'(0) - Y(s) = \frac{16}{(s-1)^2+4}.$$

Imposing the initial conditions $y(0) = 2$, $y'(0) = -2$ and solving for $Y(s)$ yields

$$Y(s) = \frac{16}{(s+1)(s-1)(s^2-2s+5)} + \frac{2}{s+1},$$

which can be written in the equivalent form

$$Y(s) = \frac{2}{s-1} + \frac{1}{s+1} - \frac{s-1}{(s-1)^2+4} - \frac{2}{(s-1)^2+4}$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = 2e^t + e^{-t} - e^t(\cos 2t + \sin 2t).$$

52. We apply the Laplace transform to both sides of the differential equation and use the rule for the transform of the derivatives to obtain

$$s^2Y(s) - sy(0) - y'(0) + 2[sY(s) - y(0)] - 3Y(s) = \frac{26(s-2)}{(s-2)^2+1}.$$

Imposing the initial conditions $y(0) = 1$, $y'(0) = 0$ and solving for $Y(s)$ yields

$$Y(s) = \frac{26(s-2)}{(s-1)(s+3)(s^2-4s+5)} + \frac{s+2}{(s-1)(s+3)},$$

which can be written in the equivalent form

$$Y(s) = -\frac{5}{2(s-1)} + \frac{3}{2(s+3)} + \frac{2s-4}{(s-2)^2+1} + \frac{3}{(s-2)^2+1}$$

Taking the inverse Laplace transform of both sides of this equation gives

$$y(t) = -\frac{5}{2}e^t + \frac{3}{2}e^{-3t} + 2e^{2t}\cos t + 3e^{2t}\sin t = \frac{3}{2}e^{-3t} - \frac{5}{2}e^t + e^{2t}(2\cos t + 3\sin t).$$

53. Taking the Laplace transform of each differential equation and using the given initial conditions yields:

$$sX_1(s) - 1 = 2X_1(s) - X_2(s), \quad sX_2(s) = X_1(s) + 2X_2(s),$$

or equivalently,

$$(2 - s)X_1(s) - X_2(s) = -1, \quad X_1(s) + (2 - s)X_2(s) = 0.$$

Solving this algebraic system of equations we obtain:

$$X_1(s) = \frac{s - 2}{s^2 - 4s + 5} = \frac{s - 2}{(s - 2)^2 + 1}, \quad X_2(s) = \frac{1}{(s - 2)^2 + 1}.$$

Consequently,

$$x_1(t) = e^{2t} \cos t, \quad x_2(t) = e^{2t} \sin t.$$

54. Taking the Laplace transform of each differential equation and using the given initial conditions yields:

$$sX_1(s) + 1 = 3X_1(s) + 2X_2(s), \quad sX_2(s) - 1 = -X_1(s) + 4X_2(s),$$

or equivalently,

$$(s + 3)X_1(s) + 2X_2(s) = 2, \quad -2X_1(s) + (s - 1)X_2(s) = 1.$$

Solving this algebraic system of equations using Cramer's rule we obtain:

$$\begin{aligned} X_1(s) &= \frac{\begin{vmatrix} -1 & -2 \\ 1 & s-4 \end{vmatrix}}{\begin{vmatrix} s-3 & -2 \\ 1 & s-4 \end{vmatrix}} = \frac{6-s}{s^2-7s+14} = \frac{6-s}{\left(s-\frac{7}{2}\right)^2 + \frac{7}{4}} \\ &= -\frac{\left(s-\frac{7}{2}\right)}{\left(s-\frac{7}{2}\right)^2 + \frac{7}{4}} + \frac{\left(\frac{5}{2}\right)}{\left(s-\frac{7}{2}\right)^2 + \frac{7}{4}} \\ &= -\frac{\left(s-\frac{7}{2}\right)}{\left(s-\frac{7}{2}\right)^2 + \frac{7}{4}} + \frac{5}{\sqrt{7}} \cdot \frac{\left(\frac{\sqrt{7}}{2}\right)}{\left(s-\frac{7}{2}\right)^2 + \frac{7}{4}}, \\ X_2(s) &= \frac{\begin{vmatrix} s-3 & -1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} s-3 & -2 \\ 1 & s-4 \end{vmatrix}} = \frac{s-2}{s^2-7s+14} = \frac{s-2}{\left(s-\frac{7}{2}\right)^2 + \frac{7}{4}} \\ &= \frac{\left(s-\frac{7}{2}\right)}{\left(s-\frac{7}{2}\right)^2 + \frac{7}{4}} + \frac{\left(\frac{3}{2}\right)}{\left(s-\frac{7}{2}\right)^2 + \frac{7}{4}} \\ &= \frac{\left(s-\frac{7}{2}\right)}{\left(s-\frac{7}{2}\right)^2 + \frac{7}{4}} + \frac{3}{\sqrt{7}} \cdot \frac{\left(\frac{\sqrt{7}}{2}\right)}{\left(s-\frac{7}{2}\right)^2 + \frac{7}{4}} \end{aligned}$$

Taking the inverse Laplace transform of $X_1(s)$ and $X_2(s)$ yields, respectively,

$$\begin{aligned} x_1(t) &= -e^{\frac{7}{2}t} \cos\left(\frac{\sqrt{7}}{2}t\right) + \frac{5}{\sqrt{7}}e^{\frac{7}{2}t} \sin\left(\frac{\sqrt{7}}{2}t\right), \\ x_2(t) &= e^{\frac{7}{2}t} \cos\left(\frac{\sqrt{7}}{2}t\right) + \frac{3}{\sqrt{7}}e^{\frac{7}{2}t} \sin\left(\frac{\sqrt{7}}{2}t\right). \end{aligned}$$

Solutions to Section 8.6

True-False Review:

1. **FALSE.** The given function is not well-defined at $t = a$. The unit step function is 0 for $0 \leq t < a$, not $0 \leq t \leq a$.
2. **FALSE.** As Figure 8.6.2 shows, the value of $u_a(t) - u_b(t)$ at $t = b$ is 0, not 1.
3. **FALSE.** For values of t with $a < t < b$, we have $u_a(t) = 1$ and $u_b(t) = 0$, so the given inequality does not hold for such t .
4. **FALSE.** The given function is precisely the one sketched in Figure 8.6.2, which is $u_a(t) - u_b(t)$, not $u_b(t) - u_a(t)$.

Problems:

$$1. f(t) = 2u_1(t) - 4u_3(t) = \begin{cases} 0, & \text{if } 0 \leq t < 1, \\ 2, & \text{if } 1 \leq t < 3, \\ -2, & \text{if } 3 \leq t. \end{cases}$$

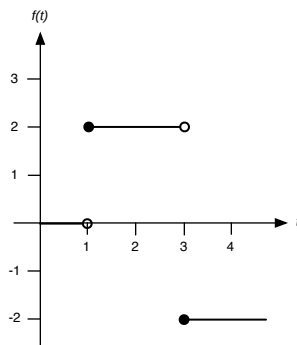


Figure 136: Figure for Problem 1

$$2. f(t) = 1 + (t - 1)u_1(t) = \begin{cases} 1, & \text{if } 0 \leq t < 1, \\ t, & \text{if } 1 \leq t. \end{cases}$$

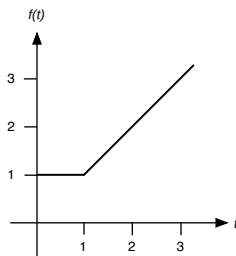


Figure 137: Figure for Problem 2

$$3. f(t) = t(1 - u_1(t)) = \begin{cases} t, & \text{if } 0 \leq t < 1, \\ 0, & \text{if } 1 \leq t. \end{cases}$$

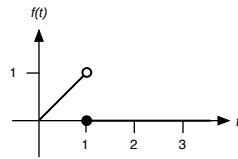


Figure 138: Figure for Problem 3

$$4. f(t) = u_1(t) + u_2(t) + u_3(t) + u_4(t) = \begin{cases} 0, & \text{if } 0 \leq t < 1, \\ 1, & \text{if } 1 \leq t < 2, \\ 2, & \text{if } 2 \leq t < 3, \\ 3, & \text{if } 3 \leq t < 4, \\ 4, & \text{if } 4 \leq t. \end{cases}$$

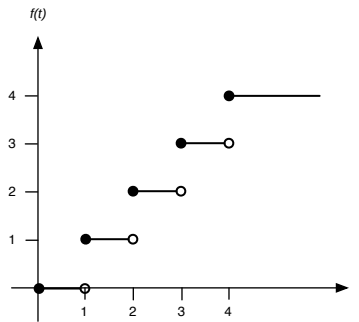


Figure 139: Figure for Problem 4

5. Let $[t]$ represent the greatest integer equal to or less than the real number t . Extending Problem 4,

$$f(t) = \sum_{i=1}^{\infty} u_i(t) = [t], \quad i-1 \leq t < i \text{ where } i \in \{1, 2, 3, \dots\}.$$

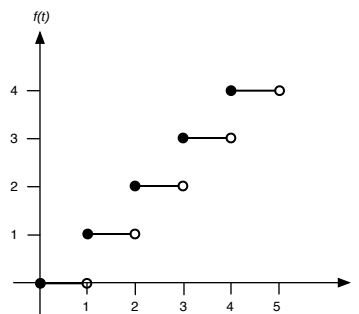


Figure 140: Figure for Problem 5

$$6. f(t) = \sum_{i=1}^{\infty} (-1)^{i+1} u_i(t) = \begin{cases} 0, & i-1 \leq t < i \quad \text{when } i \text{ is odd,} \\ 1, & i-1 \leq t < i \quad \text{when } i \text{ is even.} \end{cases}$$

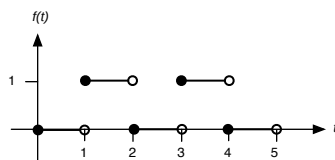


Figure 141: Figure for Problem 6

$$7. f(t) = [3 - 3u_1(t)] + (-1)u_1(t) \implies f(t) = 3 - 4u_1(t).$$

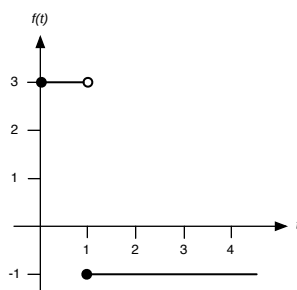


Figure 142: Figure for Problem 7

$$8. f(t) = [t^2 - t^2 u_1(t)] + u_1(t) \implies f(t) = t^2 + (1 - t^2)u_1(t).$$

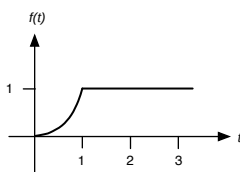


Figure 143: Figure for Problem 8

$$9. f(t) = [2 - 2u_2(t)] + [u_2(t) - u_4(t)] - u_4(t) = 2 - u_2(t) - 2u_4(t).$$

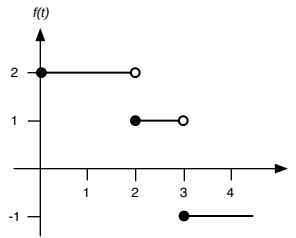


Figure 144: Figure for Problem 9

10. $f(t) = [2 - 2u_1(t)] + 2e^{t-1}u_1(t) = 2[1 + (e^{t-1} - 1)u_1(t)].$

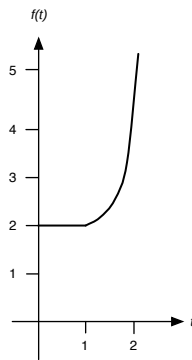


Figure 145: Figure for Problem 10

11. $f(t) = [t - tu_3(t)] + [(6 - t)u_3(t) - (6 - t)u_6(t)] = t + 2(3 - t)u_3(t) - (6 - t)u_6(t).$

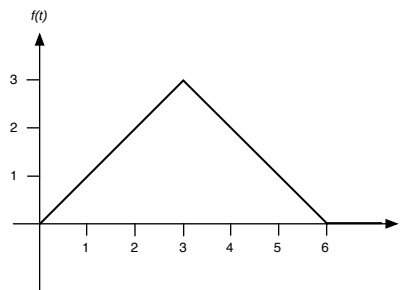


Figure 146: Figure for Problem 11

12. $f(t) = [(3 - t)u_2(t) - (3 - t)u_4(t)] + (-1)u_4(t) = (3 - t)u_2(t) + (t - 4)u_4(t).$

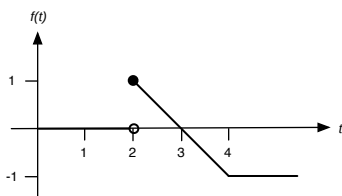


Figure 147: Figure for Problem 12

13.

$$\begin{aligned} f(t) &= [1 - u_{\pi/2}(t)] + \sin t \cdot u_{\pi/2}(t) - \sin t \cdot u_{3\pi/2}(t) + (-1)u_{3\pi/2}(t) \\ &= 1 + (\sin t - 1)u_{\pi/2}(t) - (\sin t + 1)u_{3\pi/2}(t). \end{aligned}$$

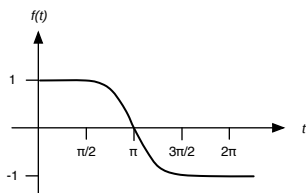


Figure 148: Figure for Problem 13

$$14. f(t) = \sin t[u_0(t) - u_{\pi}(t)] + \sin t[u_{2\pi}(t) - u_{3\pi}(t)] + \cdots \implies f(t) = \sin t \sum_{i=0}^{\infty} [u_{2i\pi}(t) - u_{(2i+1)\pi}(t)].$$

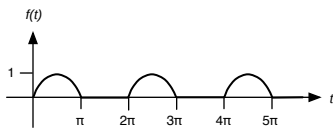


Figure 149: Figure for Problem 14

Solutions to Section 8.7

True-False Review:

1. **FALSE.** According to Equation (8.7.2), the inverse Laplace transform of $e^{as}F(s)$ is $u_{-a}(t)f(t+a)$. Note that this requires $a < 0$.
2. **TRUE.** This is Corollary 8.7.2.
3. **TRUE.** This is an immediate consequence of the Second Shifting Theorem.
4. **FALSE.** According to the Second Shifting Theorem, we have

$$L[u_2(t) \cos 4(t-2)] = \frac{se^{-2s}}{s^2 + 16},$$

not

$$L[u_2(t) \cos 4t] = \frac{se^{-2s}}{s^2 + 16}.$$

5. FALSE. We have

$$L[u_3(t)e^t] = L[u_3(t)e^{(t+3)-3}] = e^{-3s}L[e^{t+3}] = e^{-3s}e^3 \frac{1}{s-1} = \frac{e^3}{e^{3s}(s-1)}.$$

6. TRUE. This is an immediate consequence of the Second Shifting Theorem.

7. FALSE. The correct formula is

$$L^{-1} \left[\frac{1}{se^{2s}} \right] = u_2(t).$$

Problems:

1. Letting $g(t) = t$, we have $L[f(t)] = L[(t-1)u_1(t)] = L[u_1(t)g(t-1)] = e^{-s}L[g(t)] = e^{-s}L[t] = \frac{e^{-s}}{s^2}.$

2. Letting $g(t) = e^{3t}$, we have $L[f(t)] = L[u_2(t)g(t-2)] = e^{-2s}L[g(t)] = e^{-2s}L[e^{3t}] = \frac{e^{-2s}}{s-3}.$

3. Letting $g(t) = \sin t$, we have $L[f(t)] = L[u_{\pi/4}(t)g(t-\pi/4)] = e^{-\pi s/4}L[g(t)] = e^{-\pi s/4}L[\sin t] = \frac{e^{-\pi s/4}}{s^2+1}.$

4. Letting $g(t) = \cos t$, we have

$$\begin{aligned} L[f(t)] &= L[u_{\pi}(t)g(t)] \\ &= L[u_{\pi}(t) \cos t] \\ &= L[-u_{\pi}(t) \cos(t-\pi)] \\ &= -e^{-\pi s}L[\cos t] \\ &= -\frac{se^{-\pi s}}{s^2+1}. \end{aligned}$$

5. Letting $g(t) = t^2$, we have $L[f(t)] = L[u_2(t)g(t-2)] = e^{-2s}L[g(t)] = e^{-2s}L[t^2] = \frac{2e^{-2s}}{s^3}.$

6. We have $f(t) = tu_3(t) = (t-3)u_3(t) + 3u_3(t)$, and therefore,

$$\begin{aligned} L[f(t)] &= L[(t-3)u_3(t)] + 3L[u_3(t)] \\ &= e^{-3s}L[t] + 3L[u_3(t)] \\ &= \frac{e^{-3s}}{s^2} + \frac{3e^{-3s}}{s}. \end{aligned}$$

7. Letting $g(t) = (t+1)^2$, we have

$$\begin{aligned}
 L[f(t)] &= L[u_2(t)g(t-2)] \\
 &= e^{-2s}L[g(t)] \\
 &= e^{-2s}L[(t+1)^2] \\
 &= e^{-2s}L[t^2 + 2t + 1] \\
 &= e^{-2s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) \\
 &= \frac{2 + 2s + s^2}{s^3}e^{-2s}.
 \end{aligned}$$

8. Letting $g(t) = e^t t^3$, we have

$$\begin{aligned}
 L[f(t)] &= L[u_4(t)e^{t-4}(t-4)^3] \\
 &= L[u_4(t)g(t-4)] \\
 &= e^{-4s}L[g(t)] \\
 &= e^{-4s}L[e^t t^3] \\
 &= e^{-4s}\frac{6}{(s-1)^4}.
 \end{aligned}$$

9. Letting $g(t) = e^{-2t} \sin 3t$, we have

$$\begin{aligned}
 L[f(t)] &= L[u_1(t)e^{-2(t-1)} \sin [3(t-1)]] \\
 &= L[u_1(t)g(t-1)] \\
 &= e^{-s}L[g(t)] \\
 &= e^{-s}L[e^{-2t} \sin 3t] \\
 &= e^{-s}\frac{3}{(s+2)^2 + 9}.
 \end{aligned}$$

10. Letting $g(t) = e^{at} \cos bt$, we have

$$\begin{aligned}
 L[f(t)] &= L[u_c(t)e^{a(t-c)} \cos [b(t-c)]] \\
 &= L[u_c(t)g(t-c)] \\
 &= e^{-cs}L[g(t)] \\
 &= e^{-cs}L[e^{at} \cos bt] \\
 &= e^{-cs}\frac{s-a}{(s-a)^2 + b^2}.
 \end{aligned}$$

$$11. L^{-1}[F(s)] = L^{-1}\left[\frac{e^{-2s}}{s^2}\right] = L^{-1}[e^{-2s}L[t]] = u_2(t)(t-2).$$

$$12. L^{-1}[F(s)] = L^{-1}\left[\frac{e^{-s}}{s+1}\right] = L^{-1}[e^{-s}L[e^{-t}]] = u_1(t)e^{-(t-1)}.$$

$$13. L^{-1}[F(s)] = L^{-1} \left[\frac{e^{-3s}}{s+4} \right] = L^{-1}[e^{-3s}L[e^{-4t}]] = u_3(t)e^{-4(t-3)}.$$

$$14. L^{-1}[F(s)] = L^{-1} \left[\frac{se^{-s}}{s^2+4} \right] = L^{-1}[e^{-s}L[\cos 2t]] = u_1(t) \cos [2(t-1)].$$

$$15. L^{-1}[F(s)] = L^{-1} \left[\frac{e^{-3s}}{s^2+1} \right] = L^{-1}[e^{-3s}L[\sin t]] = u_3(t) \sin (t-3).$$

$$16. L^{-1}[F(s)] = L^{-1} \left[\frac{e^{-2s}}{s+2} \right] = L^{-1}[e^{-2s}L[e^{-2t}]] = u_2(t)e^{-2(t-2)}.$$

17.

$$\begin{aligned} L^{-1}[F(s)] &= L^{-1} \left[\frac{e^{-s}}{(s+1)(s-4)} \right] \\ &= L^{-1} \left[\frac{1}{5}e^{-s} \left(\frac{1}{s-4} - \frac{1}{s+1} \right) \right] \\ &= \frac{1}{5}L^{-1}[e^{-s}L[e^{4t}] - e^{-s}L[e^{-t}]] \\ &= \frac{1}{5}u_1(t)[e^{4(t-1)} - e^{-(t-1)}]. \end{aligned}$$

18.

$$\begin{aligned} L^{-1}[F(s)] &= L^{-1} \left[\frac{e^{-2s}}{s^2+2s+2} \right] \\ &= L^{-1} \left[e^{-2s} \frac{1}{(s+1)^2+1} \right] \\ &= L^{-1}[e^{-2s}L[e^{-t} \sin t]] \\ &= u_2(t)e^{-(t-2)} \sin (t-2). \end{aligned}$$

19.

$$\begin{aligned} L^{-1}[F(s)] &= L^{-1} \left[\frac{e^{-s}(s+6)}{s^2+9} \right] \\ &= L^{-1} \left[\frac{se^{-s}}{s^2+9} + \frac{6e^{-s}}{s^2+9} \right] \\ &= L^{-1}[e^{-s}L[\cos 3t] + 2e^{-s}L[\sin 3t]] \\ &= u_1(t)[\cos 3(t-1) + 2 \sin 3(t-1)]. \end{aligned}$$

$$20. L^{-1}[F(s)] = L^{-1} \left[\frac{e^{-5s}}{s^2+16} \right] = \frac{1}{4}L^{-1}[e^{-5s}L[\sin 4t]] = \frac{1}{4} \sin [4(t-5)]u_5(t).$$

$$21. L^{-1}[F(s)] = L^{-1} \left[\frac{e^{-2s}}{(s-3)^3} \right] = \frac{1}{2}L^{-1}[e^{-2s}L[e^{3t}t^2]] = \frac{1}{2}u_2(t)e^{3(t-2)}(t-2)^2.$$

22.

$$\begin{aligned}
L^{-1}[F(s)] &= L^{-1} \left[\frac{e^{-4s}(s+3)}{s^2-6s+13} \right] \\
&= L^{-1} \left[\frac{e^{-4s}(s+3)}{(s-3)^2+4} \right] \\
&= L^{-1} \left[\frac{e^{-4s}([s-3]+6)}{(s-3)^2+4} \right] \\
&= L^{-1} \left[\frac{e^{-4s}(s-3)}{(s-3)^2+4} + 3e^{-4s} \frac{2}{(s-3)^2+4} \right] \\
&= L^{-1} \left[\frac{e^{-4s}(s-3)}{(s-3)^2+4} \right] + 3L^{-1} \left[e^{-4s} \frac{2}{(s-3)^2+4} \right] \\
&= L^{-1}[e^{-4s}L[e^{3t} \cos 2t]] + L^{-1}[3e^{-4s}L[e^{3t} \sin 2t]] \\
&= u_4(t) \left(e^{3(t-4)} \cos [2(t-4)] + 3e^{3(t-4)} \sin [2(t-4)] \right) \\
&= u_4(t)e^{3(t-4)} (\cos [2(t-4)] + 3 \sin [2(t-4)]).
\end{aligned}$$

23.

$$\begin{aligned}
L^{-1}[F(s)] &= L^{-1} \left[\frac{e^{-s}(2s-1)}{s^2+4s+5} \right] \\
&= L^{-1} \left[\frac{e^{-s}(2s-1)}{(s+2)^2+1} \right] \\
&= L^{-1} \left[\frac{2e^{-s}(s+2)}{(s+2)^2+1} - \frac{5e^{-s}}{(s+2)^2+1} \right] \\
&= 2L^{-1} \left[\frac{e^{-s}(s+2)}{(s+2)^2+1} \right] - 5L^{-1} \left[\frac{e^{-s}}{(s+2)^2+1} \right] \\
&= 2L^{-1}[e^{-s}L[e^{-2t} \cos t]] - 5L^{-1}[e^{-s}L[e^{-2t} \sin t]] \\
&= u_1(t) \left(2e^{-2(t-1)} \cos (t-1) - 5e^{-2(t-1)} \sin (t-1) \right) \\
&= u_1(t)e^{-2(t-1)} \left(2 \cos (t-1) - 5 \sin (t-1) \right).
\end{aligned}$$

24.

$$\begin{aligned}
L^{-1}[F(s)] &= L^{-1} \left[\frac{2e^{-2s}}{(s-1)(s^2+1)} \right] \\
&= L^{-1} \left[\frac{e^{-2s}}{s-1} - \frac{e^{-2s}(s+1)}{s^2+1} \right] \\
&= L^{-1} \left[e^{-2s} \frac{1}{s-1} - e^{-2s} \frac{s}{s^2+1} - e^{-2s} \frac{1}{s^2+1} \right] \\
&= L^{-1}[e^{-2s}L[e^t] - e^{-2s}L[\cos t] - e^{-2s}L[\sin t]] \\
&= u_2(t) \left(e^{t-2} - \cos (t-2) - \sin (t-2) \right).
\end{aligned}$$

25.

$$\begin{aligned}
L^{-1}[F(s)] &= L^{-1}\left[\frac{50e^{-3s}}{(s+1)^2(s^2+4)}\right] \\
&= L^{-1}\left[e^{-3s}\left(\frac{10}{(s+1)^2} + \frac{4}{s+1} - \frac{4s+6}{s^2+4}\right)\right] \\
&= L^{-1}[e^{-3s}(10L[e^{-t}] + 4L[e^{-t}] - 4L[\cos 2t] - 3L[\sin 2t])] \\
&= u_3(t) \left\{ 10e^{-(t-3)}(t-3) + 4e^{-(t-3)} - 4\cos[2(t-3)] - 3\sin[2(t-3)] \right\} \\
&= u_3(t) \left\{ 2e^{-(t-3)}(5t-13) - 4\cos[2(t-3)] - 3\sin[2(t-3)] \right\}.
\end{aligned}$$

26. Applying the Laplace transform to both sides of the differential equation $y' + 2y = 2u_1(t)$, we have $sY(s) - y(0) + 2Y(s) = \frac{2e^{-s}}{s}$. Solving for $Y(s)$ and substituting the initial values, we have

$$Y(s) = \frac{2e^{-s}}{s(s+2)} + \frac{1}{s+2} \implies Y(s) = \frac{e^{-s}}{s} - \frac{e^{-s}}{s+2} + \frac{1}{s+2}.$$

Taking the inverse Laplace transform, we conclude that

$$\begin{aligned}
y(t) &= L^{-1}[e^{-s}L[1]] - L^{-1}[e^{-s}L[e^{-2t}]] + L^{-1}[L[e^{-2t}]] \\
&= u_1(t) - e^{-2(t-1)}u_1(t) + e^{-2t} \\
&= u_1(t)(1 - e^{-2(t-1)}) + e^{-2t}.
\end{aligned}$$

27. Applying the Laplace transform to both sides of the differential equation $y' - 2y = u_2(t)e^{t-2}$, we have $sY(s) - y(0) - 2Y(s) = \frac{e^{-2s}}{s-1}$. Solving for $Y(s)$ and substituting the initial values, we have

$$Y(s) = \frac{e^{-2s}}{(s-1)(s-2)} + \frac{2}{s-2} \implies Y(s) = e^{-2s} \left(\frac{1}{s-2} - \frac{1}{s-1} \right) + \frac{2}{s-2}.$$

Taking the inverse Laplace transform, we conclude that

$$\begin{aligned}
y(t) &= L^{-1}[Y(s)] \\
&= L^{-1}[e^{-2s}L[e^{2t}]] - L^{-1}[e^{-2s}L[e^t]] + 2e^{2t} \\
&= 2e^{2t} - u_2(t)[e^{t-2} - e^{2(t-2)}].
\end{aligned}$$

28. Applying the Laplace transform to both sides of the differential equation $y' - y = 4u_{\pi/4}(t)\cos(t - \pi/4)$, we have $sY(s) - y(0) - Y(s) = 4\frac{e^{-\pi s/4}s}{s^2+1}$. Solving for $Y(s)$ and substituting the initial values, we have

$$Y(s) = 4\frac{e^{-\pi s/4}s}{(s^2+1)(s-1)} + \frac{1}{s-1} \implies Y(s) = 4e^{-\pi s/4} \left[\frac{1}{2(s-1)} - \frac{s-1}{2(s^2+1)} \right] + \frac{1}{s-1}.$$

Taking the inverse Laplace transform, we conclude that

$$\begin{aligned}
y(t) &= [2e^{t-\pi/4} - 2\cos(t - \pi/4) + 2\sin(t - \pi/4)]u_{\pi/4}(t) + e^t \\
&= e^t + 2[e^{t-\pi/4} - \cos(t - \pi/4) + \sin(t - \pi/4)]u_{\pi/4}(t).
\end{aligned}$$

29. Applying the Laplace transform to both sides of the differential equation $y' + 2y = u_\pi(t) \sin 2t$, we have

$$sY(s) - y(0) + 2Y(s) = L[u_\pi(t) \sin [2(t - \pi)]].$$

Solving for $Y(s)$ and substituting the initial values, we have

$$Y(s)(s+2) = 3 + \frac{2e^{-\pi s}}{s^2 + 4} \implies Y(s) = \frac{3}{s+2} + \frac{2e^{-\pi s}}{(s^2 + 4)(s+2)} \implies Y(s) = \frac{3}{s+2} + e^{-\pi s} \left[\frac{1}{4(s+2)} - \frac{s-2}{4(s^2 + 4)} \right].$$

Therefore, taking the inverse Laplace transform, we conclude that

$$\begin{aligned} y(t) &= 3e^{-2t} + \left[\frac{e^{-2(t-\pi)}}{4} - \frac{1}{4} \cos(2[t-\pi]) + \frac{1}{4} \sin(2[t-\pi]) \right] u_\pi(t) \\ &= 3e^{-2t} + \frac{1}{4} u_\pi(t) \left[e^{-2(t-\pi)} - \cos 2t + \sin 2t \right]. \end{aligned}$$

30. We have $f(t) = 1 - u_1(t)$, and taking the Laplace transform of both sides of the differential equation $y' + 3y = 1 - u_1(t)$, we have $sY(s) - y(0) + 3Y(s) = \frac{1}{s} - \frac{e^{-s}}{s}$. Solving for $Y(s)$ and substituting the initial values, we have

$$Y(s) = \frac{1}{s(s+3)} - \frac{e^{-s}}{s(s+3)} + \frac{1}{s+3} = \frac{1}{3s} + \frac{2}{3(s+3)} - \frac{e^{-s}}{3s} + \frac{e^{-s}}{3(s+3)}.$$

Taking the inverse Laplace transform, we conclude that

$$y(t) = \frac{1}{3}(1 + 2e^{-3t}) - \frac{1}{3}u_1(t) \left[1 - e^{-3(t-1)} \right].$$

31. We have $f(t) = \sin t + (1 - \sin t)u_{\pi/2}(t) = \sin t + u_{\pi/2}(t) - u_{\pi/2}(t) \cos(t - \pi/2)$. Thus, taking the Laplace transform of each of the differential equation $y' - 3y = \sin t + u_{\pi/2}(t) - u_{\pi/2}(t) \cos(t - \pi/2)$, we obtain

$$sY(s) - y(0) - 3Y(s) = \frac{1}{s^2 + 1} + \frac{e^{-\pi s/2}}{s} - \frac{se^{-\pi s/2}}{s^2 + 1}.$$

Solving for $Y(s)$ and substituting the initial values, we have

$$Y(s) = \frac{1}{(s^2 + 1)(s-3)} + \frac{e^{-\pi s/2}}{s(s-3)} - \frac{se^{-\pi s/2}}{30(s-3)} - \frac{e^{-\pi s/2}}{3s} + \frac{e^{-\pi s/2}(3s-1)}{10(s^2 + 1)}.$$

Taking the inverse Laplace transform, we conclude that

$$\begin{aligned} y(t) &= \frac{21}{10}e^{3t} - \frac{1}{10} \cos t - \frac{3}{10} \sin t + \left[\frac{1}{30}e^{3(t-\pi/2)} - \frac{1}{3} + \frac{3}{10} \cos(t - \pi/2) - \frac{1}{10} \sin(t - \pi/2) \right] u_{\pi/2}(t) \\ &= \frac{1}{10} \left\{ 21e^{3t} - \cos t - 3 \sin t + \frac{1}{3} \left[e^{3(t-\pi/2)} - 10 + 9 \sin t + 3 \cos t \right] u_{\pi/2}(t) \right\}. \end{aligned}$$

32. Taking the Laplace transform of both sides of the differential equation $y' - 3y = 10e^{-(t-a)} \sin(2[t-a])u_a(t)$ and substituting the initial values, we obtain

$$sY(s) - y(0) - 3Y(s) = 10e^{-as}L[e^{-t} \sin 2t] \implies sY(s) - 5 - 3Y(s) = 10e^{-as} \frac{2}{(s+1)^2 + 4}.$$

Solving for $Y(s)$, we have

$$Y(s) = e^{-as} \left(\frac{1}{s-3} - \frac{s+5}{s^2+2s+5} \right) + \frac{5}{s-3}.$$

Taking the inverse Laplace transform, we obtain

$$\begin{aligned} y(t) &= e^{3(t-a)}u_a(t) - L^{-1} \left[e^{-as} \frac{s+1}{(s+1)^2+4} + 4e^{-as} \frac{1}{(s+1)^2+4} \right] + 5e^{3t} \\ &= e^{3(t-a)}u_a(t) - e^{-(t-a)} \cos 2(t-a)u_a(t) - 4e^{-(t-a)} \sin 2(t-a)u_a(t) + 5e^{3t} \\ &= 5e^{3t} + u_a(t) \left\{ e^{3(t-a)} - e^{-(t-a)} [\cos 2(t-a) + 2 \sin 2(t-a)] \right\}. \end{aligned}$$

33. Taking the Laplace transform of both sides of the differential equation $y'' - y = u_1(t)$, we obtain $s^2Y(s) - sy(0) - y'(0) - Y(s) = \frac{e^{-s}}{s}$. Solving for $Y(s)$ and substituting the initial values, we have

$$Y(s) = e^{-s} \frac{1}{s(s^2-1)} + \frac{2s}{s^2-1} = e^{-s} \left[\frac{1}{2(s-1)} + \frac{1}{2(s+1)} - \frac{1}{s} \right] + \frac{1}{s+1} + \frac{1}{s-1}.$$

Taking the inverse Laplace transform, we obtain

$$y(t) = \left(\frac{e^{t-1}}{2} + \frac{e^{-t+1}}{2} - 1 \right) u_1(t) + e^{-t} + e^t = u_1(t) [\cosh(t-1) - 1] + 2 \cosh t.$$

34. Taking the Laplace transform of both sides of the differential equation $y'' - y' - 2y = 1 - 3u_2(t)$, we obtain $s^2Y(s) - sy(0) - y'(0) - [sY(s) - y(0)] - 2Y(s) = \frac{1}{s} - \frac{3e^{-2s}}{s}$. Solving for $Y(s)$ and substituting the initial values, we have

$$\begin{aligned} Y(s) &= \frac{1}{s(s+1)(s-2)} - \frac{3e^{-2s}}{s(s+1)(s-2)} + \frac{s-3}{(s+1)(s-2)} \\ &= -\frac{1}{2s} - \frac{1}{6(s-2)} + \frac{5}{3(s+1)} + \frac{3e^{-2s}}{2s} - \frac{e^{-2s}}{2(s-2)} - \frac{e^{-2s}}{s+1}. \end{aligned}$$

Taking the inverse Laplace transform, we obtain

$$y(t) = -\frac{1}{2} + \frac{5}{3}e^{-t} - \frac{1}{6}e^{2t} + \frac{u_2(t)}{2} [3 - e^{2(t-2)} - 2e^{-(t-2)}].$$

35. Taking the Laplace transform of both sides of the differential equation $y'' - 4y = u_1(t) - u_2(t)$, we obtain $s^2Y(s) - sy(0) - y'(0) - 4Y(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$. Solving for $Y(s)$ and substituting the initial values, we have

$$\begin{aligned} Y(s) &= \frac{4}{s^2-4} + \frac{e^{-s}}{s(s^2-4)} - \frac{e^{-2s}}{s(s^2-4)} \\ &= \frac{1}{s-2} - \frac{1}{s+2} + \frac{e^{-s}}{4} \left[-\frac{1}{s} + \frac{1}{2(s-2)} + \frac{1}{2(s+2)} \right] - \frac{e^{-2s}}{4} \left[-\frac{1}{s} + \frac{1}{2(s-2)} + \frac{1}{2(s+2)} \right]. \end{aligned}$$

Taking the inverse Laplace transform, we obtain

$$\begin{aligned} y(t) &= e^{2t} - e^{-2t} - \frac{1}{4}u_1(t) + \frac{1}{8}e^{2(t-1)}u_1(t) + \frac{1}{8}e^{-2(t-1)}u_1(t) + \frac{1}{4}u_2(t) - \frac{1}{8}e^{2(t-2)}u_2(t) - \frac{1}{8}e^{-2(t-2)}u_2(t) \\ &= 2\sinh 2t + \frac{1}{4}u_1(t)[\cosh 2(t-1) - 1] - \frac{1}{4}u_2(t)[\cosh 2(t-2) - 1]. \end{aligned}$$

36. Taking the Laplace transform of both sides of the differential equation $y'' + y = t - u_1(t)(t-1)$, we obtain $s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2}$. Solving for $Y(s)$ and substituting the initial values, we have

$$\begin{aligned} Y(s) &= \frac{1}{s^2(s^2+1)} + \frac{2s+1}{s^2+1} - e^{-s}\frac{1}{s^2(s^2+1)} \\ &= \frac{1}{s^2} + \frac{2s}{s^2+1} - \frac{e^{-s}}{s^2} + \frac{e^{-s}}{s^2+1}. \end{aligned}$$

Taking the inverse Laplace transform, we obtain

$$\begin{aligned} y(t) &= t + 2\cos t - (t-1)u_1(t) + \sin(t-1)u_1(t) \\ &= t + 2\cos t - u_1(t)[t-1-\sin(t-1)]. \end{aligned}$$

37. Taking the Laplace transform of both sides of the differential equation $y'' + 3y' + 2y = 10u_{\pi/4}(t)\sin(t-\pi/4)$, we obtain $s^2Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = \frac{10e^{-\pi s/4}}{s^2+1}$. Solving for $Y(s)$ and substituting the initial values, we have

$$Y(s) = \frac{s+3}{s^2+3s+2} + \frac{10e^{-\pi s/4}}{(s^2+1)(s^2+3s+2)} = \frac{2}{s+1} - \frac{1}{s+2} + \frac{10e^{-\pi s/4}}{(s^2+1)(s^2+3s+2)}.$$

Taking the inverse Laplace transform, we obtain

$$\begin{aligned} y(t) &= 2e^{-t} - e^{-2t} + 5e^{-(t-\pi/4)}u_{\pi/4}(t) - 2e^{-2(t-\pi/4)}u_{\pi/4}(t) + \sin(t-\pi/4)u_{\pi/4}(t) - 3\cos(t-\pi/4)u_{\pi/4}(t) \\ &= 2e^{-t} - e^{-2t} + u_{\pi/4}(t)\left[5e^{-(t-\pi/4)} - 2e^{-2(t-\pi/4)} - 3\cos(t-\pi/4) + \sin(t-\pi/4)\right]. \end{aligned}$$

38. Taking the Laplace transform of both sides of the differential equation $y'' + y' - 6y = 30u_1(t)e^{-(t-1)}$, we obtain $s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) - 6Y(s) = \frac{30e^{-s}}{s+1}$. Solving for $Y(s)$ and substituting the initial values, we have

$$Y(s) = \frac{3s-1}{(s+3)(s-2)} + \frac{30e^{-s}}{(s+1)(s+3)(s-2)} = \frac{1}{s-2} + \frac{2}{s+3} + e^{-s}\left(\frac{2}{s-2} - \frac{5}{s+1} + \frac{3}{s+3}\right).$$

Taking the inverse Laplace transform, we obtain

$$y(t) = 2e^{-3t} + e^{2t} + u_1(t)\left[2e^{2(t-1)} + 3e^{-3(t-1)} - 5e^{-(t-1)}\right].$$

39. Taking the Laplace transform of both sides of the differential equation $y'' + 4y' + 5y = 5u_3(t)$, we obtain $s^2Y(s) - sy(0) - y'(0) + 4[sY(s) - y(0)] + 5Y(s) = \frac{5e^{-3s}}{s}$. Solving for $Y(s)$ and substituting the initial values, we have

$$Y(s) = \frac{9 + 2s}{s^2 + 4s + 5} + \frac{5e^{-3s}}{s(s^2 + 4s + 5)} = \frac{2s + 4}{(s + 2)^2 + 1} + \frac{5}{(s + 2)^2 + 1} + e^{-3s} \left[\frac{1}{s} - \frac{s + 2}{(s + 2)^2 + 1} - \frac{2}{(s + 2)^2 + 1} \right].$$

Therefore, taking the inverse Laplace transform, we obtain

$$\begin{aligned} y(t) &= 2e^{-2t} \cos t + 5e^{-2t} \sin t + L^{-1}[e^{-3s}L[1]] - L^{-1}[e^{-3s}L[e^{-2t} \cos t]] - 2L^{-1}[e^{-3s}L[e^{-2t} \sin t]] \\ &= e^{-2t}(2 \cos t + 5 \sin t) + u_3(t) \left\{ 1 - e^{-2(t-3)} \left[\cos(t-3) - 2 \sin(t-3) \right] \right\}. \end{aligned}$$

40. Taking the Laplace transform of both sides of the differential equation

$$y'' - 2y' + 5y = 2 \sin t + u_{\pi/2}(t)[1 - \sin(t - \pi/2)],$$

we obtain

$$s^2Y(s) - sy(0) - y'(0) - 2[sY(s) - y(0)] + 5Y(s) = \frac{2}{s^2 + 1} + \frac{e^{-\pi s/2}}{s} - \frac{e^{-\pi s/2}}{s^2 + 1}.$$

Solving for $Y(s)$ and substituting the initial values, we have

$$\begin{aligned} Y(s) &= \frac{2}{(s^2 + 1)(s^2 - 2s + 5)} + e^{-\pi s/2} \left[\frac{1}{s(s^2 - 2s + 5)} - \frac{1}{(s^2 + 1)(s^2 - 2s + 5)} \right] \\ &= \frac{s}{5(s^2 + 1)} + \frac{2}{5(s^2 + 1)} - \frac{s - 1}{5[(s - 1)^2 + 4]} - \frac{1}{5[(s - 1)^2 + 4]} \\ &\quad + e^{-\pi s/2} \left\{ \frac{1}{5s} - \frac{s - 1}{5[(s - 1)^2 + 4]} + \frac{1}{5[(s - 1)^2 + 4]} - \frac{s}{10(s^2 + 1)} - \frac{1}{5(s^2 + 1)} \right\}. \end{aligned}$$

Therefore, taking the inverse Laplace transform, we obtain

$$\begin{aligned} y(t) &= \frac{1}{5} (\cos t + 2 \sin t - e^t \cos 2t - e^t \sin 2t) \\ &\quad + \frac{1}{10} u_{\pi/2}(t) \left\{ 2 - 2e^{t-\pi/2} \cos[2(t - \pi/2)] + e^{t-\pi/2} \sin[2(t - \pi/2)] - \cos(t - \pi/2) - 2 \sin(t - \pi/2) \right\} \\ &= \frac{1}{5} [\cos t + 2 \sin t - e^t (\cos 2t + \sin 2t)] + \frac{1}{10} u_{\pi/2}(t) [2 - \sin t + 2 \cos t + e^{t-\pi/2} (2 \cos 2t - \sin 2t)]. \end{aligned}$$

41. We have

$$f(t) = \begin{cases} 0, & \text{if } 0 \leq t < 1, \\ t - 1, & \text{if } 1 \leq t < 2, \\ 3 - t, & \text{if } 2 \leq t < 3, \\ 0, & \text{if } 3 \leq t. \end{cases}.$$

Thus,

$$\begin{aligned} f(t) &= [(t - 1)u_1(t) - (t - 1)u_2(t)] + [(3 - t)u_2(t) - (3 - t)u_3(t)] \\ &= u_1(t)(t - 1) - 2u_2(t)(t - 2) + u_3(t)(t - 3). \end{aligned}$$

Taking the Laplace transform of both sides of the differential equation

$$y' + y = u_1(t)(t-1) - 2u_2(t)(t-2) + u_3(t)(t-3),$$

we obtain $sY(s) - y(0) + Y(s) = \frac{e^{-s}}{s^2} - \frac{2e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$. Solving for $Y(s)$ and substituting the initial values, we obtain

$$Y(s) = \frac{2}{s+1} + (e^{-s} - 2e^{-2s} + e^{-3s}) \frac{1}{s^2(s+1)} = \frac{2}{s+1} + (e^{-s} - 2e^{-2s} + e^{-3s}) \left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} \right).$$

Taking the inverse Laplace transform, we have

$$\begin{aligned} y(t) &= 2e^{-t} + [(t-1) - 1 + e^{-(t-1)}]u_1(t) - 2[(t-2) - 1 + e^{-(t-2)}]u_2(t) + [(t-3) - 1 + e^{-(t-3)}]u_3(t) \\ &= 2e^{-t} + u_1(t)[e^{-(t-1)} + t - 2] - 2u_2(t)[e^{-(t-2)} + t - 3] + u_3(t)[e^{-(t-3)} + t - 4]. \end{aligned}$$

42. We have $f(t) = \begin{cases} 1-t, & \text{if } 0 \leq t < 1, \\ 0, & \text{if } 1 \leq t. \end{cases}$ Thus, $f(t) = (1-t) - (1-t)u_1(t) = 1-t + u_1(t)(t-1)$.

Taking the Laplace transform of both sides of the differential equation $y' + 2y = 1-t + u_1(t)(t-1)$, we get

$$sY(s) - y(0) + 2Y(s) = \frac{s-1}{s^2} + \frac{e^{-s}}{s^2}.$$

Solving for $Y(s)$ and substituting the initial values, we obtain

$$Y(s) = -\frac{1}{2s^2} + \frac{3}{4s} - \frac{3}{4(s+2)} + e^{-s} \left[\frac{1}{2s^2} - \frac{1}{4s} + \frac{1}{4(s+2)} \right].$$

Therefore, taking the inverse Laplace transform, we find

$$y(t) = \frac{1}{4} \left[3 - 3e^{-2t} - 2t + u_1(t)(e^{-2(t-1)} + 2t - 3) \right].$$

43. We have

$$\begin{aligned} f(t) &= (t - tu_1(t)) + u_1(t)e^{-(t-1)} \\ &= t + (e^{-(t-1)} - t)u_1(t) \\ &= t + e^{-(t-1)}u_1(t) - (t-1)u_1(t) - u_1(t). \end{aligned}$$

Taking the Laplace transform of both sides of the differential equation

$$y' - y = t + e^{-(t-1)}u_1(t) - (t-1)u_1(t) - u_1(t)$$

yields

$$sY(s) - 2 - Y(s) = \frac{1}{s^2} + \frac{e^{-s}}{s+1} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}.$$

Solving for $Y(s)$ and substituting the initial values, we obtain

$$\begin{aligned} Y(s) &= \frac{2}{s-1} + \frac{1}{s^2(s-1)} + e^{-s} \left[\frac{1}{(s+1)(s-1)} - \frac{1}{s^2(s-1)} - \frac{1}{s-1} \right] \\ &= \frac{3}{s-1} - \frac{1}{s^2} - \frac{1}{s} + e^{-s} \left[-\frac{3}{2(s-1)} - \frac{1}{2(s+1)} + \frac{1}{s^2} + \frac{1}{s} \right]. \end{aligned}$$

Taking the inverse Laplace transform, we obtain

$$y(t) = 3e^t - t - 1 - \frac{1}{2}u_1(t) \left[3e^{t-1} + e^{-(t-1)} - 2 - 2t \right].$$

44. We have

$$f(t) = \begin{cases} t, & \text{if } 0 \leq t < 1, \\ 1, & \text{if } 1 \leq t < 2, \\ 3-t, & \text{if } 2 \leq t < 3, \\ 0, & \text{if } t \geq 3. \end{cases}.$$

Thus,

$$\begin{aligned} f(t) &= (t - tu_1(t)) + (u_1(t) - u_2(t)) + (3-t)u_2(t) - (3-t)u_3(t) \\ &= t + (1-t)u_1(t) + (2-t)u_2(t) + (t-3)u_3(t). \end{aligned}$$

Taking the Laplace transform of both sides of the differential equation

$$y' - 2y = t + (1-t)u_1(t) + (2-t)u_2(t) + (t-3)u_3(t),$$

we obtain

$$sY(s) - y(0) - 2Y(s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}.$$

Solving for $Y(s)$ and substituting the initial values, we have

$$\begin{aligned} Y(s) &= \frac{1}{s^2(s-2)} - (e^{-s} + e^{-2s} - e^{-3s}) \frac{1}{s^2(s-2)} \\ &= -\frac{1}{2s^2} - \frac{1}{4s} + \frac{1}{4(s-2)} - (e^{-s} + e^{-2s} - e^{-3s}) \left[-\frac{1}{2s^2} - \frac{1}{4s} + \frac{1}{4(s-2)} \right]. \end{aligned}$$

Taking the inverse Laplace transform, we find

$$\begin{aligned} y(t) &= -\frac{t}{2} - \frac{1}{4} + \frac{e^{2t}}{4} - \left[-\frac{t-1}{2} - \frac{1}{4} + \frac{e^{2(t-1)}}{4} \right] u_1(t) - \left[-\frac{t-2}{2} - \frac{1}{4} + \frac{e^{2(t-2)}}{4} \right] u_2(t) \\ &\quad + \left[-\frac{t-3}{2} - \frac{1}{4} + \frac{e^{2(t-3)}}{4} \right] u_3(t) \\ &= \frac{1}{4} \left[e^{2t} - 1 - 2t - u_1(t)(e^{2(t-1)} - 2t + 1) - u_2(t)(e^{2(t-2)} - 2t + 3) + u_3(t)(e^{2(t-3)} - 2t + 5) \right]. \end{aligned}$$

45. We have $f(t) = \begin{cases} 2, & \text{if } 0 \leq t < 1, \\ -1, & \text{if } 1 \leq t. \end{cases}$ Thus, $f(t) = 2 - 3u_1(t)$.

(a) Taking the Laplace transform of the differential equation $y' - y = 2 - 3u_1(t)$, we get

$$sY(s) - y(0) - Y(s) = \frac{2}{s} - \frac{3e^{-s}}{s}.$$

Solving for $Y(s)$ and substituting the initial values, we have

$$\begin{aligned} Y(s) &= \frac{1}{s-1} + \frac{2}{s(s-1)} - \frac{3e^{-s}}{s(s-1)} \\ &= \frac{3}{s-1} - \frac{2}{s} + 3e^{-s} \left[\frac{1}{s} - \frac{1}{s-1} \right]. \end{aligned}$$

Taking the inverse Laplace transform, we obtain

$$y(t) = 3e^t - 2 + 3(1 - e^{t-1})u_1(t).$$

(b) In order to use the techniques of Chapter 1 we must solve the differential equation on the interval $[0, 1)$ and $[1, \infty)$ separately.

On $[0, 1)$ we must solve $y' - y = 2$, where $y(0) = 1$. An integrating factor is $I(t) = e^{-t}$ so

$$\frac{d}{dt}(e^{-t}y) = 2e^{-t} \implies y(t) = -2 + c_1e^t.$$

Substituting $y(0) = 1$, we have $c_1 = 3$. Thus, $y(t) = 3e^t - 2$ on $[0, 1)$.

Now consider $y' - y = -1$ on $[1, \infty)$. Again, an integrating factor is $I(t) = e^{-t}$. Thus,

$$\frac{d}{dt}(e^{-t}y) = -e^{-t} \implies y(t) = 1 + ce^t$$

on $[1, \infty)$. If the solutions are continuous at $t = 1$, then $\lim_{t \rightarrow 1} y(t) = y(1)$. From $y(t) = 3e^t - 2$ on $[0, 1)$, we have that $y(1) = 3e - 2$ whereas from $y(t) = 1 + ce^t$ on $[1, \infty)$, we have $\lim_{t \rightarrow 1} y(t) = 1 + ce = 3e - 2 \implies c = 3 - 3e^{-1}$ so from $y(t) = 1 + ce^t$, we obtain $y(t) = 1 + 3e^t - 3e^{t-1}$ on $[1, \infty)$. Thus

$$y(t) = \begin{cases} 3e^t - 2, & \text{if } 0 \leq t < 1, \\ 1 + 3e^t - 3e^{t-1}, & \text{if } 1 \leq t. \end{cases}$$

46. We have $E(t) = \begin{cases} 2t, & \text{if } 0 \leq t < 5, \\ 10, & \text{if } 5 \leq t. \end{cases}$ Thus,

$$E(t) = 2t - 2tu_5(t) + 10u_5(t) = 2t - 2u_5(t)(t - 5).$$

Substituting this into the given differential equation, we have

$$\frac{di}{dt} + \frac{R}{L}i = \frac{1}{L} \left[2t - 2u_5(t)(t - 5) \right].$$

Taking the Laplace transform of both sides and substituting the initial values, we have

$$sI(s) - 0 + \frac{R}{L}I(s) = \frac{2}{L}L[t] - \frac{2}{L}L[u_5(t)(t - 5)].$$

Solving for $I(s)$, we have

$$\begin{aligned} I(s) &= \frac{2}{L} \frac{1}{s^2(s + R/L)} - \frac{2}{L} \frac{e^{-5s}}{s^2(s + R/L)} \\ &= \frac{2}{L} \left[\frac{L}{R} \frac{1}{s^2} - \frac{L^2}{R^2} \frac{1}{s} + \frac{L^2}{R^2} \frac{1}{(s + R/L)} \right] - \frac{2}{L} e^{-5s} \left[\frac{L}{R} \frac{1}{s^2} - \frac{L^2}{R^2} \frac{1}{s} + \frac{L^2}{R^2} \frac{1}{(s + R/L)} \right] \\ &= \frac{2}{R} \frac{1}{s^2} - \frac{2L}{R^2} \frac{1}{s} + \frac{2L}{R^2} \frac{1}{(s + R/L)} - \frac{2e^{-5s}}{R} \left[\frac{1}{s^2} - \frac{L}{R} \frac{1}{s} + \frac{L}{R} \frac{1}{(s + R/L)} \right]. \end{aligned}$$

Taking the inverse Laplace transform, we obtain

$$\begin{aligned} i(t) &= \frac{2}{R}t - \frac{2L}{R^2} + \frac{2L}{R^2}e^{-Rt/L} - \frac{2}{R}u_5(t) \left[t - 5 - \frac{L}{R} + \frac{L}{R}e^{-R(t-5)/L} \right] \\ &= \frac{2}{R}t + \frac{2L}{R^2}(e^{-Rt/L} - 1) - u_5(t) \left[\frac{2}{R}(t - 5) + \frac{2L}{R^2}(e^{-R(t-5)/L} - 1) \right] \\ &= g(t) - u_5(t)g(t - 5), \end{aligned}$$

where $g(t) = \frac{2}{R}t + \frac{2L}{R^2}(e^{-Rt/L} - 1)$.

47. We have $E(t) = \begin{cases} 20, & \text{if } 0 \leq t < 10, \\ 20e^{-(t-10)}, & \text{if } 10 \leq t. \end{cases}$ Thus,

$$E(t) = 20(1 - u_{10}(t)) + 20u_{10}(t)e^{-(t-10)}.$$

Substituting this into the given differential equation, we have

$$\frac{dq}{dt} + \frac{1}{RC}q = \frac{20}{R} \left[1 - u_{10}(t) + u_{10}(t)e^{-(t-10)} \right].$$

Taking the Laplace transform of both sides and substituting the initial values, we have

$$sQ(s) + \frac{1}{RC}Q(s) = \frac{20}{R} \left[\frac{1}{s} + e^{-10s} \left(\frac{1}{s+1} - \frac{1}{s} \right) \right].$$

Solving for $Q(s)$, we have

$$\begin{aligned} Q(s) &= \frac{20}{R} \left[\frac{1}{s(s+a)} + e^{-10s} \left(\frac{1}{(s+1)(s+a)} - \frac{1}{s(s+a)} \right) \right] \\ &= \frac{20}{R} \left[\frac{1}{as} - \frac{1}{a(s+a)} + e^{-10s} \left(\frac{1}{(a-1)(s+1)} + \frac{1}{(1-a)(s+a)} - \frac{1}{as} + \frac{1}{a(s+a)} \right) \right], \end{aligned}$$

where $a = \frac{1}{RC}$. Thus, taking the inverse Laplace transform, we have

$$q(t) = \frac{20}{R} \left[\frac{1}{a} - \frac{1}{a}e^{-at} + u_{10}(t) \left(\frac{1}{a-1}(e^{-(t-10)} - e^{-a(t-10)}) - \frac{1}{a}(1 - e^{-at}) \right) \right].$$

Now, since $i(t) = \frac{dq(t)}{dt}$ it follows that

$$i(t) = \frac{20}{R} \left[e^{-at} + u_{10}(t) \left(\frac{1}{a-1}(ae^{-a(t-10)} - e^{-(t-10)}) - e^{-at} \right) \right].$$

Solutions to Section 8.8

True-False Review:

1. TRUE. We have

$$I = \int_{-\infty}^{\infty} F(t)dt,$$

where $F(t)$ is the magnitude of the applied force at time t .

2. TRUE. A unit impulse force acts instantaneously on an object and delivers a unit impulse.

3. FALSE. The correct formula is $L[\delta(t - a)] = e^{-as}$.

4. TRUE. An excellent illustration of this is given in Example 8.8.3. A spring-mass system that receives an instantaneous blow experiences an acceleration, which is the second derivative of the position.

5. FALSE. The initial conditions are unrelated to the instantaneous blow. They are the pre-blow position and velocity of the mass, and these are not related in any way to the instantaneous blow.

Problems:

1. Taking the Laplace transform of both sides of the differential equation $y' - 2y = \delta(t - 2)$, we obtain

$$L[y'] - 2L[y] = L[\delta(t - 2)] \implies sY(s) - y(0) - 2Y(s) = e^{-2s}.$$

Substituting the initial value, we have $sY(s) - 1 - 2Y(s) = e^{-2s}$. Solving for $Y(s)$, we have

$$Y(s)(s - 2) = 1 + e^{-2s} \implies Y(s) = \frac{1}{s - 2} + \frac{e^{-2s}}{s - 2}.$$

Taking the inverse Laplace transform, we conclude that $y(t) = e^{2t} + u_2(t)e^{2(t-2)}$.

2. Taking the Laplace transform of both sides of the differential equation $y' + 4y = 3\delta(t - 1)$ and inserting the initial value, we obtain $sY(s) - 2 + 4Y(s) = 3e^{-s}$. Solving for $Y(s)$, we have $Y(s) = \frac{2}{s + 4} + 3e^{-s}\frac{1}{s + 4}$. Taking the inverse Laplace transform, we conclude that $y(t) = 2e^{-4t} + 3u_1(t)e^{-4(t-1)}$.

3. Taking the Laplace transform of both sides of the differential equation $y' - 5y = 2e^{-t} + \delta(t - 3)$ and inserting the initial value, we obtain $sY(s) - 5Y(s) = \frac{2}{s + 1} + e^{-3s}$. Solving for $Y(s)$, we have

$$\begin{aligned} Y(s) &= \frac{2}{(s + 1)(s - 5)} + e^{-3s}\frac{1}{s - 5} \\ &= \frac{1}{3(s - 5)} - \frac{1}{3(s + 1)} + e^{-3s}\frac{1}{s - 5}. \end{aligned}$$

Taking the inverse Laplace transform, we conclude that $y(t) = \frac{1}{3}(e^{5t} - e^{-t}) + u_3(t)e^{5(t-3)}$.

4. Taking the Laplace transform of both sides of the differential equation $y'' - 3y' + 2y = \delta(t - 1)$ and inserting the initial values, we obtain $s^2Y(s) - s - 0 - 3(sY(s) - 1) + 2Y(s) = e^{-s}$. Solving for $Y(s)$, we have

$$\begin{aligned} Y(s) &= \frac{s - 3}{s^2 - 3s + 2} + \frac{e^{-s}}{s^2 - 3s + 2} \\ &= \frac{2}{s - 1} - \frac{1}{s - 2} + e^{-s} \left[\frac{1}{s - 2} - \frac{1}{s - 1} \right]. \end{aligned}$$

Taking the inverse Laplace transform, we conclude that $y(t) = 2e^t - e^{2t} + u_1(t)[e^{2(t-1)} - e^{t-1}]$.

5. Taking the Laplace transform of both sides of the differential equation $y'' - 4y = \delta(t - 3)$ and inserting the initial values, we obtain $s^2Y(s) - 0s - 1 - 4Y(s) = e^{-3s}$. Solving for $Y(s)$, we have

$$\begin{aligned} Y(s) &= \frac{1}{s^2 - 4} + \frac{e^{-3s}}{s^2 - 4} \\ &= \frac{1}{4(s-2)} - \frac{1}{4(s+2)} + e^{-3s} \left[\frac{1}{4(s-2)} - \frac{1}{4(s+2)} \right]. \end{aligned}$$

Taking the inverse Laplace transform, we conclude that

$$\begin{aligned} y(t) &= \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t} + \frac{1}{4}u_3(t)[e^{2(t-3)} - e^{-2(t-3)}] \\ &= \frac{1}{2} \left[\frac{e^{2t} - e^{-2t}}{2} + u_3(t) \left(\frac{e^{2(t-3)} - e^{-2(t-3)}}{2} \right) \right] \\ &= \frac{1}{2} \left[\sinh 2t + u_3(t) \sinh [2(t-3)] \right]. \end{aligned}$$

6. Taking the Laplace transform of both sides of the differential equation $y'' + 2y' + 5y = \delta(t - \pi/2)$ and inserting the initial values, we have $s^2Y(s) - 0s - 2 + 2(sY(s) - 0) + 5Y(s) = e^{-\pi s/2}$. Solving for $Y(s)$, we have $Y(s) = \frac{2}{s^2 + 2s + 5} + \frac{e^{-\pi s/2}}{s^2 + 2s + 5}$. Taking the inverse Laplace transform, we conclude that

$$\begin{aligned} y(t) &= e^{-t} \sin 2t + u_{\pi/2}(t)e^{-(t-\pi/2)} \sin [2(t - \pi/2)] \\ &= e^{-t} \sin 2t - u_{\pi/2}(t)e^{-(t-\pi/2)} \sin 2t. \end{aligned}$$

7. Taking the Laplace transform of both sides of the differential equation $y'' - 4y' + 13y = \delta(t - \pi/4)$ and inserting the initial values, we have $s^2Y(s) - 3s - 0 - 4(sY(s) - 3) + 13Y(s) = e^{-\pi s/4}$. Solving for $Y(s)$, we obtain

$$\begin{aligned} Y(s) &= \frac{3s - 12}{(s-2)^2 + 9} + \frac{e^{-\pi s/4}}{(s-2)^2 + 9} \\ &= 3 \left[\frac{s-2}{(s-2)^2 + 9} \right] - 2 \left[\frac{3}{(s-2)^2 + 9} \right] + \frac{1}{3}e^{-\pi s/4} \left[\frac{3}{(s-2)^2 + 9} \right]. \end{aligned}$$

Taking the inverse Laplace transform, we conclude that

$$y(t) = 3e^{2t} \cos 3t - 2e^{2t} \sin 3t + \frac{1}{3}u_{\pi/4}(t)e^{2(t-\pi/4)} \sin [3(t - \pi/4)].$$

8. Taking the Laplace transform of both sides of the differential equation $y'' + 4y' + 3y = \delta(t - 2)$ and inserting the initial values, we have $s^2Y(s) - 1s - (-1) + 4(sY(s) - 1) + 3Y(s) = e^{-2s}$. Solving for $Y(s)$, we have

$$Y(s) = \frac{1}{s+1} + e^{-2s} \left[\frac{1}{2(s+1)} - \frac{1}{2(s+3)} \right].$$

Hence, taking the inverse Laplace transform, we have $y(t) = e^{-t} + \frac{1}{2}u_2(t)[e^{-(t-2)} - e^{-3(t-2)}]$.

9. Taking the Laplace transform of both sides of the differential equation $y'' + 6y' + 13y = \delta(t - \pi/4)$ and inserting the initial values, we have $s^2Y(s) - 5s - 5 + 6(sY(s) - 5) + 13Y(s) = e^{-\pi s/4}$. Solving for $Y(s)$, we have

$$\begin{aligned} Y(s) &= \frac{5s + 35}{s^2 + 6s + 13} + \frac{e^{-\pi s/4}}{s^2 + 6s + 13} \\ &= \frac{5(s + 3)}{(s + 3)^2 + 4} + \frac{20}{(s + 3)^2 + 4} + \frac{e^{-\pi s/4}}{(s + 3)^2 + 4}. \end{aligned}$$

Taking the inverse Laplace transform, we conclude that

$$\begin{aligned} y(t) &= 5e^{-3t} \cos 2t + 10e^{-3t} \sin 2t + \frac{1}{2}e^{-3(t-\pi/4)}u_{\pi/4}(t) \sin [2(t - \pi/4)] \\ &= 5e^{-3t}(\cos 2t + 2 \sin 2t) - \frac{1}{2}e^{-3(t-\pi/4)}u_{\pi/4}(t) \cos 2t. \end{aligned}$$

10. Taking the Laplace transform of both sides of the differential equation $y'' + 9y = 15 \sin 2t + \delta(t - \pi/6)$ and inserting the initial values, we have $s^2Y(s) - 0s - 0 + 9Y(s) = \frac{30}{s^2 + 4} + e^{-\pi s/6}$. Solving for $Y(s)$, we have

$$\begin{aligned} Y(s) &= \frac{30}{(s^2 + 4)(s^2 + 9)} + \frac{e^{-\pi s/6}}{s^2 + 9} \\ &= \frac{6}{s^2 + 4} - \frac{6}{s^2 + 9} + \frac{e^{-\pi s/6}}{s^2 + 9}. \end{aligned}$$

Therefore, taking the inverse Laplace transform, we conclude that

$$\begin{aligned} y(t) &= 3 \sin 2t - 2 \sin 3t + \frac{1}{3}u_{\pi/6}(t) \sin [3(t - \pi/6)] \\ &= 3 \sin 2t - 2 \sin 3t - \frac{1}{3}u_{\pi/6}(t) \cos 3t. \end{aligned}$$

11. Taking the Laplace transform of both sides of the differential equation $y'' + 16y = 4 \cos 3t + \delta(t - \pi/3)$ and inserting the initial values, we have $s^2Y(s) - 0s - 0 + 16Y(s) = \frac{4s}{s^2 + 9} + e^{-\pi s/3}$. Solving for $Y(s)$, we find that

$$\begin{aligned} Y(s) &= \frac{4s}{(s^2 + 9)(s^2 + 16)} + \frac{e^{-\pi s/3}}{s^2 + 16} \\ &= \frac{4s}{7(s^2 + 9)} - \frac{4s}{7(s^2 + 16)} + \frac{e^{-\pi s/3}}{s^2 + 16}. \end{aligned}$$

Taking the inverse Laplace transform, we conclude that

$$y(t) = \frac{4}{7}(\cos 3t - \cos 4t) + \frac{1}{4} \sin [4(t - \pi/3)]u_{\pi/3}(t).$$

12. Taking the Laplace transform of both sides of the differential equation $y'' + 2y' + 5y = 4 \sin t + \delta(t - \pi/6)$ and inserting the initial values, we have $s^2Y(s) - 0s - 1 + 2(sY(s) - 0) + 5Y(s) = \frac{4}{s^2 + 1} + e^{-\pi s/6}$. Solving

for $Y(s)$, we have

$$\begin{aligned} Y(s) &= \frac{1}{s^2 + 2s + 5} + \frac{4}{(s^2 + 1)(s^2 + 2s + 5)} + \frac{e^{-\pi s/6}}{s^2 + 2s + 5} \\ &= \frac{3}{10} \frac{2}{(s+1)^2 + 4} + \frac{4}{5} \frac{1}{s^2 + 1} - \frac{2}{5} \frac{s}{s^2 + 1} + \frac{2}{5} \frac{s+1}{(s+1)^2 + 4} + \frac{e^{-\pi s/6}}{(s+1)^2 + 4}. \end{aligned}$$

Therefore, taking the inverse Laplace transform, we conclude that

$$y(t) = \frac{3}{10} e^{-t} \sin 2t + \frac{4}{5} \sin t - \frac{2}{5} \cos t + \frac{2}{5} e^{-t} \cos 2t + \frac{1}{2} u_{\pi/6}(t) e^{-(t-\pi/6)} \sin [2(t - \pi/6)].$$

13. Taking the Laplace transform of both sides of the differential equation $\frac{d^2 y}{dt^2} + 4y = F_0 \cos 3t - 4\delta(t - 5)$ and inserting the initial values, we have $s^2 Y(s) - 0s - 0 + 4Y(s) = \frac{F_0 s}{s^2 + 9} - 4e^{-5s}$. Solving for $Y(s)$, we obtain

$$\begin{aligned} Y(s) &= \frac{F_0 s}{(s^2 + 9)(s^2 + 4)} - \frac{4e^{-5s}}{s^2 + 4} \\ &= \frac{F_0}{5} \left(\frac{s}{s^2 + 4} - \frac{s}{s^2 + 9} \right) - \frac{4e^{-5s}}{s^2 + 4}. \end{aligned}$$

Taking the inverse Laplace transform, we have

$$y(t) = \frac{F_0}{5} (\cos 2t - \cos 3t) - 2u_5(t) \sin [2(t - 5)].$$

14. Taking the Laplace transform of both sides of the differential equation

$$\frac{d^2 y}{dt^2} + 4 \frac{dy}{dt} + 13y = 10 \sin 5t + 2\delta(t - 10),$$

we have

$$s^2 Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 13Y(s) = \frac{50}{s^2 + 25} + 2e^{-10s}.$$

Inserting the initial values and solving for $Y(s)$, we obtain

$$\begin{aligned} Y(s) &= \frac{50}{(s^2 + 25)(s^2 + 4s + 13)} + \frac{2e^{-10s}}{s^2 + 4s + 13} \\ &= -\frac{25}{68} \frac{s}{s^2 + 25} - \frac{15}{68} \frac{5}{s^2 + 25} + \frac{25}{68} \frac{s+2}{(s+2)^2 + 9} + \frac{125}{208} \frac{3}{(s+2)^2 + 9} + \frac{2}{3} \frac{3e^{-10s}}{(s+2)^2 + 9}. \end{aligned}$$

Taking the inverse Laplace transform, we conclude that

$$y(t) = -\frac{25}{68} \cos 5t - \frac{15}{68} \sin 5t + \frac{25}{68} e^{-2t} \cos 3t + \frac{125}{208} e^{-2t} \sin 3t + \frac{2}{3} u_{10}(t) e^{-2(t-10)} \sin [3(t - 10)].$$

15. Taking the Laplace transform of both sides of the differential equation $\frac{d^2 y}{dt^2} + \omega_0 y = F_0 \sin \omega t + A\delta(t - t_0)$,

we obtain $s^2Y(s) - 0s - 0 + \omega_0^2Y(s) = \frac{F_0\omega}{s^2 + \omega^2} + Ae^{-t_0s}$. Solving for $Y(s)$, we have

$$\begin{aligned} Y(s) &= \frac{F_0\omega}{(s^2 + \omega_0^2)(s^2 + \omega^2)} + \frac{Ae^{-t_0s}}{s^2 + \omega_0^2} \\ &= \frac{F_0\omega}{\omega_0^2 - \omega^2} \left(\frac{1}{s^2 + \omega^2} - \frac{1}{s^2 + \omega_0^2} \right) + \frac{Ae^{-t_0s}}{s^2 + \omega_0^2}. \end{aligned}$$

Taking the inverse Laplace transform, we have

$$\begin{aligned} y(t) &= \frac{F_0}{(\omega_0^2 - \omega^2)\omega_0} (\omega_0 \sin \omega t - \omega \sin \omega_0 t) + \frac{A}{\omega_0} u_{t_0}(t) \sin(\omega_0[t - t_0]) \\ &= \frac{F_0}{\omega_0(\omega^2 - \omega_0^2)} (\omega \sin \omega_0 t - \omega_0 \sin \omega t) + \frac{A}{\omega_0} u_{t_0}(t) \sin(\omega_0[t - t_0]). \end{aligned}$$

Solutions to Section 8.9

True-False Review:

- 1. TRUE.** We have $(f * g)(t) = \int_0^t f(t - \tau)g(\tau)d\tau$ and $(g * f)(t) = \int_0^t g(t - \tau)f(\tau)d\tau$. The latter integral becomes the same as the first one via the u -substitution $u = t - \tau$.
- 2. TRUE.** Since both f and g are positive functions, the expression $f(t - \tau)g(\tau)$ is positive for all t, τ . Thus, if we increase the interval of integration $[0, t]$ by increasing t , the value of $(f * g)(t)$ also increases.
- 3. FALSE.** The Convolution Theorem states that $L[f * g] = L[f]L[g]$.
- 4. TRUE.** This is exactly the form of the equation in (8.9.3).
- 5. FALSE.** For instance, if f is identically zero, then $f * g = f * h = 0$ for all functions g and h .
- 6. TRUE.** This is expressed mathematically in Equation (8.9.2).
- 7. TRUE.** This follows from the fact that constants can be factored out of integrals, and $a(f * g)$, $(af) * g$, and $f * (ag)$ can be expressed as integrals. In short, integration is linear.

Problems:

- 1.** $(f * g)(t) = \int_0^t f(t - x)g(x)dx = \int_0^t (t - x)dx = \left[tx - \frac{x^2}{2} \right]_0^t = t^2 - \frac{t^2}{2} = \frac{t^2}{2}.$
- 2.** $(f * g)(t) = \int_0^t f(t - x)g(x)dx = \int_0^t \cos(t - x)xdx = \int_t^0 \cos z(t - z)(-dz) = t \int_0^t \cos z dz - \int_0^t z \cos z dz = [t \sin z - (\cos z + z \sin z)]_0^t = 1 - \cos t.$
- 3.** $(f * g)(t) = \int_0^t f(t - x)g(x)dx = \int_0^t e^{t-x}xdx = e^t \int_0^t e^{-x}xdx = e^t[e^{-x}(-x - 1)]_0^t = e^t - 1 - t.$
- 4.** $(f * g)(t) = \int_0^t f(t - x)g(x)dx = \int_0^t (t - x)^2 e^x dx = t^2 \int_0^t e^x dx - 2t \int_0^t x e^x dx + \int_0^t x^2 e^x dx = [t^2 e^x]_0^t - 2t[e^x(x - 1)]_0^t + [x^2 e^x - 2x e^x + 2e^x]_0^t = t^2 e^t - t^2 - 2te^t(t - 1) - 2t + t^2 e^t - 2te^t + 2e^t - 2 = 2e^t - t^2 - 2t - 2.$
- 5.** $(f * g)(t) = \int_0^t f(t - x)g(x)dx = \int_0^t e^{t-x}e^x \sin x dx = e^t \int_0^t \sin x dx = [-e^t \cos x]_0^t = e^t(1 - \cos t).$

6. Let $w = t - x$. Then $dw = -dx$ and $x = t - w$. Note that $x = 0$ if and only if $w = t$, and that $x = t$ if and only if $w = 0$. Then $(f * g)(t) = \int_0^t f(t-x)g(x)dx = \int_t^0 f(w)g(t-w)(-dw) = \int_0^t g(t-w)f(w)dw = (g * f)(t)$. Therefore $f * g = g * f$.

7. We have

$$\begin{aligned}(f * (g * h))(t) &= \int_0^t f(t-v)(g * h)(v)dv \\ &= \int_0^t f(t-v) \left[\int_0^v g(v-u)h(u)du \right] dv \\ &= \int_0^t \int_0^v f(t-v)g(v-u)h(u)dudv.\end{aligned}$$

The limits of integration are $0 \leq u \leq v$ and $0 \leq v \leq t$ so that the region of integration is that part of the $v-u$ plane that lies just above the v -axis and below the line $u = v$. Reversing the order of integration, the new limits are $u \leq v \leq t$ and $0 \leq u \leq t$. Thus the double integral above can be written

$$(f * (g * h))(t) = \int_0^t \int_u^t f(t-v)g(v-u)h(u)dvdu.$$

We now make a change of variable $w = v - u$ in the first iterated integral. Then $dw = dv$ and the new w -limits are $0 \leq w \leq t - u$. Thus

$$\begin{aligned}(f * (g * h))(t) &= \int_0^t \int_0^{t-u} [f(t-(u+w))g(w)dw]h(u)du \\ &= \int_0^t \int_0^{t-u} [f((t-u)-w)g(w)dw]h(u)du \\ &= \int_0^t (f * g)(t-u)h(u)du \\ &= ((f * g) * h)(t).\end{aligned}$$

Therefore, $(f * g) * h = f * (g * h)$.

8.

$$\begin{aligned}(f * (g + h))(t) &= \int_0^t f(t-x)(g+h)(x)dx \\ &= \int_0^t [f(t-x)g(x) + f(t-x)h(x)]dx \\ &= \int_0^t f(t-x)g(x)dx + \int_0^t f(t-x)h(x)dx \\ &= (f * g)(t) + (f * h)(t).\end{aligned}$$

Thus $f * (g + h) = f * g + f * h$.

$$9. L[f * g] = L[f]L[g] = L[t]L[\sin t] = \frac{1}{s^2} \cdot \frac{1}{s^2 + 1} = \frac{1}{s^2(s^2 + 1)}.$$

$$10. L[f * g] = L[f]L[g] = L[e^{2t}]L[1] = \frac{1}{s-2} \cdot \frac{1}{s} = \frac{1}{s(s-2)}.$$

$$11. L[f * g] = L[f]L[g] = L[\sin t]L[\cos 2t] = \frac{1}{s^2 + 1} \cdot \frac{s}{s^2 + 4} = \frac{s}{(s^2 + 1)(s^2 + 4)}.$$

$$12. L[f * g] = L[f]L[g] = L[e^t]L[te^{2t}] = \frac{1}{s - 1} \cdot \frac{1}{(s - 2)^2} = \frac{1}{(s - 1)(s - 2)^2}.$$

$$13. L[f * g] = L[f]L[g] = L[t^2]L[e^{3t} \sin 2t] = \frac{2}{s^3} \cdot \frac{2}{(s - 3)^2 + 4} = \frac{4}{s^3[(s - 3)^2 + 4]}.$$

14.

(a)

$$\begin{aligned} L^{-1}[F(s)G(s)] &= L^{-1}\left[\frac{1}{s} \frac{1}{s - 2}\right] \\ &= L^{-1}\left[\frac{1}{s}\right] * L^{-1}\left[\frac{1}{s - 2}\right] \\ &= 1 * e^{2t} \\ &= \int_0^t e^{2x} dx = \left[\frac{e^{2x}}{2}\right]_0^t = \frac{1}{2}(e^{2t} - 1). \end{aligned}$$

(b)

$$\begin{aligned} L^{-1}[F(s)G(s)] &= L^{-1}\left[\frac{1}{s} \frac{1}{s - 2}\right] \\ &= L^{-1}\left[\frac{1}{2(s - 2)} - \frac{1}{2s}\right] \\ &= \frac{1}{2}L^{-1}\left[\frac{1}{s - 2}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s}\right] \\ &= \frac{1}{2}(e^{2t} - 1). \end{aligned}$$

15.

(a)

$$\begin{aligned} L^{-1}[F(s)G(s)] &= L^{-1}\left[\frac{1}{s + 1} \frac{1}{s}\right] \\ &= L^{-1}\left[\frac{1}{s + 1}\right] * L^{-1}\left[\frac{1}{s}\right] \\ &= e^{-t} * 1 \\ &= \int_0^t e^{-(t-x)} dx = e^{-t} \int_0^t e^x dx = e^{-t}[e^x]_0^t = 1 - e^{-t}. \end{aligned}$$

(b)

$$\begin{aligned}
L^{-1}[F(s)G(s)] &= L^{-1}\left[\frac{1}{s+1}\frac{1}{s}\right] \\
&= L^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right] \\
&= L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] \\
&= 1 - e^{-t}.
\end{aligned}$$

16.

(a)

$$\begin{aligned}
L^{-1}[F(s)G(s)] &= L^{-1}\left[\frac{s}{s^2+4} \cdot \frac{2}{s}\right] \\
&= L^{-1}\left[\frac{s}{s^2+4}\right] * L^{-1}\left[\frac{2}{s}\right] \\
&= \cos 2t * 2 \\
&= 2 * \cos 2t \\
&= \int_0^t 2 \cos 2x dx = [\sin 2x]_0^t = \sin 2t.
\end{aligned}$$

(b)

$$\begin{aligned}
L^{-1}[F(s)G(s)] &= L^{-1}\left[\frac{s}{s^2+4} \cdot \frac{2}{s}\right] \\
&= L^{-1}\left[\frac{2}{s^2+4}\right] \\
&= \sin 2t.
\end{aligned}$$

17.

(a)

$$\begin{aligned}
L^{-1}[F(s)G(s)] &= L^{-1}\left[\frac{1}{s+2} \cdot \frac{s+2}{s^2+4s+13}\right] \\
&= L^{-1}\left[\frac{1}{s+2}\right] * L^{-1}\left[\frac{s+2}{s^2+4s+13}\right] \\
&= e^{-2t} * e^{-2t} \cos 3t \\
&= \int_0^t e^{-2(t-x)} e^{-2x} \cos 3x dx = e^{-2t} \int_0^t \cos 3x dx = \left[\frac{e^{-2t}}{3} \sin 3x\right]_0^t = \frac{e^{-2t}}{3} \sin 3t.
\end{aligned}$$

(b)

$$\begin{aligned}
L^{-1}[F(s)G(s)] &= L^{-1}\left[\frac{1}{s+2} \cdot \frac{s+2}{s^2+4s+13}\right] \\
&= L^{-1}\left[\frac{1}{s^2+4s+13}\right] \\
&= \frac{1}{3}L^{-1}\left[\frac{3}{(s+2)^2+9}\right] \\
&= \frac{e^{-2t}}{3} \sin 3t.
\end{aligned}$$

18. (a)

$$\begin{aligned}
L^{-1}[F(s)G(s)] &= L^{-1}\left[\frac{1}{s^2+9} \cdot \frac{2}{s^3}\right] \\
&= L^{-1}\left[\frac{1}{s^2+9}\right] * L^{-1}\left[\frac{2}{s^3}\right] \\
&= \frac{1}{3} \sin 3t * t^2 \\
&= t^2 * \frac{1}{3} \sin 3t \\
&= \int_0^t (t-x)^2 \frac{\sin 3x}{3} dx \\
&= \frac{1}{81} [2(\cos 3t - 1) + 9t^2].
\end{aligned}$$

(b)

$$\begin{aligned}
L^{-1}[F(s)G(s)] &= L^{-1}\left[\frac{1}{s^2+9} \cdot \frac{2}{s^3}\right] \\
&= L^{-1}\left[\frac{2}{(s^2+9)s^3}\right] \\
&= 2L^{-1}\left[\frac{1}{9s^3} - \frac{1}{81s} + \frac{s}{81(s^2+9)}\right] \\
&= \frac{1}{9}L^{-1}\left[\frac{2}{s^3}\right] - \frac{2}{81}L^{-1}\left[\frac{1}{s}\right] + \frac{2}{81}L^{-1}\left[\frac{s}{s^2+9}\right] \\
&= \frac{t^2}{9} - \frac{2}{81} + \frac{2}{81} \cos 3t \\
&= \frac{1}{81} [2(\cos 3t - 1) + 9t^2].
\end{aligned}$$

19. (a)

$$\begin{aligned}
 L^{-1}[F(s)G(s)] &= L^{-1}\left[\frac{1}{s^2} \cdot \frac{e^{-\pi s}}{s^2+1}\right] \\
 &= L^{-1}\left[\frac{1}{s^2}\right] * L^{-1}\left[\frac{e^{-\pi s}}{s^2+1}\right] \\
 &= t * u_{\pi}(t) \sin(t - \pi) \\
 &= \int_0^t (t-x)u_{\pi}(x) \sin(x - \pi) dx \\
 &= \int_0^t (x-t)u_{\pi}(x) \sin x dx = u_{\pi}(t)[-x \cos x + \sin x + t \cos x]_{\pi}^t = u_{\pi}(t)(\sin t - \pi + t).
 \end{aligned}$$

(b)

$$\begin{aligned}
 L^{-1}[F(s)G(s)] &= L^{-1}\left[\frac{1}{s^2} \frac{e^{-\pi s}}{s^2+1}\right] \\
 &= L^{-1}\left[e^{-\pi s} \left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)\right] \\
 &= L^{-1}\left[e^{-\pi s} \frac{1}{s^2}\right] - L^{-1}\left[e^{-\pi s} \frac{1}{s^2+1}\right] \\
 &= u_{\pi}(t)(t - \pi) - u_{\pi}(t) \sin(t - \pi) \\
 &= u_{\pi}(t)(t - \pi + \sin t).
 \end{aligned}$$

20. We have

$$\begin{aligned}
 L^{-1}[F(s)G(s)] &= L^{-1}\left[\frac{4}{s^3} \cdot \frac{s-1}{s^2-2s+5}\right] \\
 &= L^{-1}\left[\frac{4}{s^3}\right] * L^{-1}\left[\frac{s-1}{s^2-2s+5}\right] \\
 &= 2t^2 * e^t \cos 2t \\
 &= 2 \int_0^t (t-x)^2 e^x \cos 2x dx.
 \end{aligned}$$

21. We have

$$\begin{aligned}
 L^{-1}[F(s)G(s)] &= L^{-1}\left[\frac{s+1}{s^2+2s+2} \cdot \frac{1}{(s+3)^2}\right] \\
 &= L^{-1}\left[\frac{s+1}{s^2+2s+2}\right] * L^{-1}\left[\frac{1}{(s+3)^2}\right] \\
 &= (e^{-t} \cos t) * (e^{-3t} t) \\
 &= \int_0^t e^{-(t-x)} \cos(t-x) x e^{-3x} dx \\
 &= \int_0^t e^{-(t+2x)} x \cos(t-x) dx.
 \end{aligned}$$

22. We have

$$\begin{aligned}
 L^{-1}[F(s)G(s)] &= L^{-1}\left[\frac{2}{s^2+6s+10} \cdot \frac{2}{s-4}\right] \\
 &= 4L^{-1}\left[\frac{1}{(s+3)^2+1}\right] * L^{-1}\left[\frac{1}{s-4}\right] \\
 &= 4(e^{-3t}\sin t) * (e^{4t}) \\
 &= 4\int_0^t e^{-3(t-x)}\sin(t-x)e^{4x}dx \\
 &= 4\int_0^t e^{-(3t-7x)}\sin(t-x)dx.
 \end{aligned}$$

23. We have

$$\begin{aligned}
 L^{-1}[F(s)G(s)] &= L^{-1}\left[\frac{s+4}{s^2+8s+25} \cdot \frac{se^{-\pi s/2}}{s^2+16}\right] \\
 &= L^{-1}\left[\frac{s+4}{s^2+8s+25}\right] * L^{-1}\left[\frac{se^{-\pi s/2}}{s^2+16}\right] \\
 &= (e^{-4t}\cos 3t) * u_{\pi/2}(t)\cos(4[t-\pi/2]) \\
 &= (e^{-4t}\cos 3t) * (u_{\pi/2}(t)\cos 4t) \\
 &= \int_{\pi/2}^t e^{-4(t-x)}\cos[3(t-x)]\cos 4x dx,
 \end{aligned}$$

where $\pi/2 \leq t$.

24. We have

$$\begin{aligned}
 L^{-1}[F(s)G(s)] &= L^{-1}\left[\frac{1}{s-4}G(s)\right] \\
 &= L^{-1}\left[\frac{1}{s-4}\right] * L^{-1}[G(s)] \\
 &= e^{4t} * g(t) \\
 &= \int_0^t e^{4(t-x)}g(x)dx.
 \end{aligned}$$

25. Applying the Laplace transform to both sides of the differential equation $y'' + y = e^{-t}$ yields

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s+1}.$$

Substituting the initial values and solving for $Y(s)$, we have

$$Y(s) = \frac{1}{s^2+1} + \frac{1}{(s+1)(s^2+1)}.$$

Applying the inverse Laplace transform yields

$$\begin{aligned}
 y(t) &= L^{-1}[Y(s)] \\
 &= L^{-1}\left[\frac{1}{s^2+1}\right] + L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right] \\
 &= \sin t + L^{-1}\left[\frac{1}{s+1}\right] * L^{-1}\left[\frac{1}{s^2+1}\right] \\
 &= \sin t + e^{-t} * \sin t \\
 &= \sin t + \int_0^t \sin(t-\tau)e^{-\tau}d\tau.
 \end{aligned}$$

26. Applying the Laplace transform to both sides of the differential equation $y'' - 2y' + 10y = \cos 2t$ yields

$$s^2Y(s) - sy(0) - y'(0) - 2Y(s) + 2y(0) + 10Y(s) = \frac{s}{s^2+4}.$$

Substituting the initial values and solving for $Y(s)$ we find

$$Y(s) = \frac{1}{s^2-2s+10} + \frac{s}{(s^2+4)(s^2-2s+10)}.$$

Taking the inverse Laplace transform, gives

$$\begin{aligned}
 y(t) &= L^{-1}[Y(s)] \\
 &= \frac{1}{3}L^{-1}\left[\frac{3}{(s-1)^2+9}\right] + \frac{1}{3}L^{-1}\left[\frac{s}{s^2+4}\right] * L^{-1}\left[\frac{3}{(s-1)^2+9}\right] \\
 &= \frac{1}{3}e^t \sin 3t + \frac{1}{3} \cos 2t * e^t \sin 3t \\
 &= \frac{1}{3}e^t \sin 3t + \frac{1}{3} \int_0^t \cos[2(t-x)]e^x \sin 3x dx.
 \end{aligned}$$

27. Applying the Laplace transform to both sides of the differential equation $y'' + 16y = f(t)$, yields

$$s^2Y(s) - sy(0) - y'(0) + 16Y(s) = F(s).$$

Substituting the initial values and solving for $Y(s)$ we find

$$Y(s) = \frac{\alpha s + \beta}{s^2+16} + \frac{F(s)}{s^2+16}.$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
 y(t) &= L^{-1}[Y(s)] \\
 &= \alpha L^{-1}\left[\frac{s}{s^2+16}\right] + \frac{\beta}{4}L^{-1}\left[\frac{4}{s^2+16}\right] + L^{-1}[F(s)] * L^{-1}\left[\frac{1}{s^2+16}\right] \\
 &= \alpha \cos 4t + \frac{\beta}{4} \sin 4t + \frac{1}{4}f(t) * \sin 4t \\
 &= \frac{1}{4}(4\alpha \cos 4t + \beta \sin 4t) + \frac{1}{4} \int_0^t f(t-x) \sin 4x dx.
 \end{aligned}$$

28. Applying the Laplace transform to both sides of the differential equation $y' - ay = f(t)$ yields

$$sY(s) - y(0) - aY(s) = F(s).$$

Substituting the initial value and solving for $Y(s)$ we have

$$Y(s) = \frac{\alpha}{s-a} + \frac{F(s)}{s-a}.$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y(t) &= L^{-1}[Y(s)] \\ &= \alpha L^{-1}\left[\frac{1}{s-a}\right] + L^{-1}[F(s)]L^{-1}\left[\frac{1}{s-a}\right] \\ &= \alpha e^{at} + f(t) * e^{at} \\ &= \alpha e^{at} + \int_0^t f(t-x)e^{ax}dx. \end{aligned}$$

29. Applying the Laplace transform to both sides of the differential equation $y'' - a^2y = f(t)$ and substituting the initial values yields

$$s^2Y(s) - \alpha s - \beta - a^2Y(s) = F(s).$$

Solving for $Y(s)$ we have

$$Y(s) = \frac{\alpha s + \beta}{2a} \left(\frac{1}{s-a} - \frac{1}{s+a} \right) + \frac{F(s)}{s^2 - a^2}.$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y(t) &= L^{-1}[Y(s)] \\ &= \frac{1}{2a} L^{-1}\left[\frac{\alpha s}{s-a} + \frac{\beta}{s-a} - \frac{\alpha s}{s+a} - \frac{\beta}{s+a}\right] + \frac{1}{2a} L^{-1}[F(s)] * L^{-1}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] \\ &= \frac{1}{2a} L^{-1}\left[\frac{\alpha s}{s-a} + \frac{\beta}{s-a} - \frac{\alpha s}{s+a} - \frac{\beta}{s+a}\right] + f(t) * \frac{e^{at} - e^{-at}}{2a} \\ &= \frac{1}{2a} (\alpha a e^{at} + \beta e^{at} - \alpha a e^{-at} - \beta e^{-at}) + \frac{1}{a} f(t) * \sinh at \\ &= \frac{\alpha a + \beta}{2a} e^{at} - \frac{\alpha a + \beta}{2a} e^{-at} + \frac{1}{a} \int_0^t f(t-x) \sinh(ax) dx \\ &= \frac{\alpha a + \beta}{a} \sinh at + \frac{1}{a} \int_0^t f(t-x) \sinh(ax) dx. \end{aligned}$$

30. Applying the Laplace transform to both sides of the differential equation $y'' - (a+b)y' + aby = f(t)$ and substituting the initial values yields

$$s^2Y(s) - \alpha s - \beta - (a+b)(sY(s) - \alpha) + abY(s) = F(s).$$

Solving for $Y(s)$ we have

$$\begin{aligned} Y(s) &= \frac{\alpha s + \beta - (a+b)\alpha}{(s-a)(s-b)} + \frac{F(s)}{(s-a)(s-b)} \\ &= \frac{1}{a-b} \left(\frac{\beta - b\alpha}{s-a} + \frac{a\alpha - \beta}{s-b} \right) + \frac{F(s)}{a-b} \left(\frac{1}{s-a} - \frac{1}{s-b} \right). \end{aligned}$$

Taking the inverse Laplace transform yields

$$\begin{aligned} y(t) &= \frac{1}{a-b}[(\beta - b\alpha)e^{at} + (a\alpha - \beta)e^{bt}] + \frac{1}{a-b}f(t) * (e^{at} - e^{bt}) \\ &= \frac{\beta - b\alpha}{a-b}e^{at} + \frac{a\alpha - \beta}{a-b}e^{bt} + \frac{1}{a-b} \int_0^t f(t-x)(e^{ax} - e^{bx})dx. \end{aligned}$$

31. Applying the Laplace transform to both sides of the differential equation $y'' - 2ay' + (a^2 + b^2)y = f(t)$ and substituting the initial values yields

$$s^2Y(s) - \alpha s - \beta - 2a(sY(s) - \alpha) + (a^2 + b^2)Y(s) = F(s).$$

Solving for $Y(s)$ we have

$$Y(s) = \frac{\alpha s + \beta - 2a\alpha}{(s-a)^2 + b^2} + \frac{F(s)}{(s-a)^2 + b^2}.$$

Taking the inverse Laplace transform gives

$$\begin{aligned} y(t) &= L^{-1}[Y(s)] \\ &= \alpha L^{-1}\left[\frac{s-a}{(s-a)^2 + b^2}\right] + \frac{\beta - a\alpha}{b}L^{-1}\left[\frac{b}{(s-a)^2 + b^2}\right] + \frac{1}{b}L^{-1}[F(s)] * L^{-1}\left[\frac{b}{(s-a)^2 + b^2}\right] \\ &= \alpha e^{at} \cos bt + \frac{\beta - a\alpha}{b}e^{at} \sin bt + \frac{1}{b}f(t) * e^{at} \sin bt \\ &= e^{at} \left(\alpha \cos bt + \frac{\beta - a\alpha}{b} \sin bt \right) + \frac{1}{b} \int_0^t f(t-x)e^{ax} \sin bx \, dx. \end{aligned}$$

32. Taking the Laplace transform of the given integral equation yields

$$L[x(t)] = L[e^{-t}] + 4L\left[\int_0^t (t-\tau)x(\tau)d\tau\right]$$

so that

$$X(s) = \frac{1}{s+1} + 4L[t * x(t)].$$

Applying the convolution theorem gives

$$X(s) = \frac{1}{s+1} + \frac{4}{s^2} \cdot X(s)$$

Solving for $X(s)$ gives

$$X(s) = \frac{s^2}{(s^2-4)(s+1)} = \frac{1}{3(s-2)} + \frac{1}{s+2} - \frac{1}{3(s+1)}.$$

Consequently,

$$x(t) = \frac{1}{3}e^{2t} + e^{-2t} - \frac{1}{3}e^{-t}.$$

33. Taking the Laplace transform of the given integral equation yields

$$L[x(t)] = L\left[2e^{3t} - \int_0^t e^{2(t-\tau)}x(\tau)d\tau\right]$$

so that

$$X(s) = \frac{2}{s-3} - L[e^{2t} * x(t)].$$

Applying the convolution theorem gives

$$X(s) = \frac{2}{s-3} - \frac{1}{s-2} \cdot X(s).$$

Solving for $X(s)$ gives

$$X(s) = \frac{2(s-2)}{(s-1)(s-3)} = \frac{1}{s-3} + \frac{1}{s-1}.$$

Consequently,

$$x(t) = e^{3t} + e^t.$$

34. Taking the Laplace transform of the given integral equation yields

$$L[x(t)] = L\left[4e^t + 3 \int_0^t e^{-(t-\tau)} x(\tau) d\tau\right]$$

so that

$$X(s) = \frac{4}{s-1} + 3L[e^{-t} * x(t)].$$

Applying the convolution theorem gives

$$X(s) = \frac{4}{s-1} + \frac{3}{s+1} \cdot X(s).$$

Solving for $X(s)$ gives

$$X(s) = \frac{4(s+1)}{(s-2)(s-1)} = \frac{12}{s-2} - \frac{8}{s-1}.$$

Consequently,

$$x(t) = 12e^{2t} - 8e^t.$$

35. Taking the Laplace transform of the given integral equation yields

$$L[x(t)] = L\left[1 + 2 \int_0^t \sin(t-\tau)x(\tau) d\tau\right]$$

so that

$$X(s) = \frac{1}{s} + 2L[\sin t * x(t)].$$

Applying the convolution theorem gives

$$X(s) = \frac{1}{s} + \frac{2}{s^2+1} \cdot X(s).$$

Solving for $X(s)$ gives

$$X(s) = \frac{s^2+1}{s(s^2-1)} = \frac{1}{s-1} + \frac{1}{s+1} - \frac{1}{s}.$$

Consequently,

$$x(t) = e^t + e^{-t} - 1 = 2 \cosh t - 1.$$

36. Taking the Laplace transform of the given integral equation yields

$$L[x(t)] = L \left[e^{2t} + 5 \int_0^t \cos(2[t-\tau])x(\tau)d\tau \right]$$

so that

$$X(s) = \frac{1}{s-2} + 5L[\cos 2t * x(t)].$$

Applying the convolution theorem gives

$$X(s) = \frac{1}{s-2} + \frac{5s}{s^2+4} \cdot X(s).$$

Solving for $X(s)$ we find

$$X(s) = \frac{s^2+4}{(s-2)(s-1)(s-4)} = \frac{10}{3(s-4)} - \frac{4}{s-2} + \frac{5}{3(s-1)}.$$

Consequently,

$$x(t) = \frac{1}{3}(5e^t + 10e^{4t} - 12e^{2t}).$$

37. Taking the Laplace transform of the given integral equation yields

$$L[x(t)] = L[2 + 2 \int_0^t \cos(2[t-\tau])x(\tau)d\tau]$$

so that

$$X(s) = \frac{2}{s} + 2L[\cos(2t) * x(t)].$$

Applying the convolution theorem gives

$$X(s) = \frac{2}{s} + \frac{2s}{s^2+4} \cdot X(s).$$

Solving for $X(s)$ we find

$$X(s) = \frac{2(s^2+4)}{s(s^2-2s+4)} = \frac{2}{s} - \frac{4}{s^2-2s+4}.$$

Consequently,

$$x(t) = 2 + \frac{4}{\sqrt{3}}e^t \sin(\sqrt{3}t).$$

38. Let $y(t)$ be any twice differentiable function that satisfies the initial conditions

$$y(0) = 0, \quad y'(0) = 0, \tag{0.0.54}$$

and let $x(t) = y''$. Then, taking the Laplace transform of the preceding equation yields

$$X(s) = L[x(t)] = L[y''] = s^2Y(s) - sy'(0) - y(0),$$

which, upon using (0.0.54) reduces to

$$X(s) = s^2 Y(s),$$

or, equivalently,

$$Y(s) = \frac{1}{s^2} \cdot X(s).$$

Taking the inverse Laplace transform then gives

$$y(t) = L^{-1} \left[\frac{1}{s^2} \cdot X(s) \right],$$

that is, using the convolution theorem,

$$y(t) = \int_0^t (t - \tau)x(\tau)d\tau.$$

Therefore, the initial-value problem $y'' + y = f(t)$, together with initial conditions $y(0) = 0$ and $y'(0) = 0$ can be replaced by

$$x(t) + \int_0^t (t - \tau)x(\tau)d\tau = f(t),$$

that is,

$$x(t) = f(t) - \int_0^t (t - \tau)x(\tau)d\tau.$$

39. THIS PROBLEM IS THE SAME AS #8 AND SHOULD BE REMOVED.

Solutions to Section 8.10

Problems:

1.

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st}(3t - 4)dt \\ &= 3 \int_0^\infty te^{-st}dt - 4 \int_0^\infty e^{-st}dt \\ &= 3 \lim_{n \rightarrow \infty} \int_0^n te^{-st}dt - 4 \lim_{n \rightarrow \infty} \int_0^n e^{-st}dt \\ &= 3 \lim_{n \rightarrow \infty} \left[-\frac{t}{s}e^{-st} \right]_0^n + 3 \lim_{n \rightarrow \infty} \int_0^n \frac{1}{s}e^{-st}dt - 4 \lim_{n \rightarrow \infty} \int_0^n e^{-st}dt \\ &= 3 \lim_{n \rightarrow \infty} \left[-\frac{n}{s}e^{-sn} \right] - \frac{3}{s^2} \lim_{n \rightarrow \infty} [e^{-st}]_0^n + \frac{4}{s} \lim_{n \rightarrow \infty} [e^{-st}]_0^n \\ &= 3 \lim_{n \rightarrow \infty} \left[-\frac{n}{s}e^{-sn} \right] - \frac{3}{s^2} \lim_{n \rightarrow \infty} [e^{-sn} - 1] + \frac{4}{s} \lim_{n \rightarrow \infty} [e^{-sn} - 1] \\ &= \frac{3}{s^2} - \frac{4}{s}. \end{aligned}$$

2.

$$\begin{aligned}
 F(s) &= \int_0^{\infty} e^{-st} \sin 2t \, dt \\
 &= \lim_{n \rightarrow \infty} \int_0^n e^{-st} \sin 2t \, dt \\
 &= \lim_{n \rightarrow \infty} \frac{-1}{s^2 + 4} \left[e^{-st} (s \cdot \sin 2t + 2 \cos 2t) \right]_0^n \\
 &= \frac{2}{s^2 + 4}.
 \end{aligned}$$

3.

$$\begin{aligned}
 F(s) &= \int_0^{\infty} e^{-st} (4t^2) dt \\
 &= 4 \lim_{n \rightarrow \infty} \int_0^n t^2 e^{-st} dt \\
 &= 4 \lim_{n \rightarrow \infty} -e^{-st} \left[\frac{t^2}{s} + \frac{2t}{s^2} + \frac{2}{s^3} \right]_0^n \\
 &= 4 \lim_{n \rightarrow \infty} \left[\frac{2}{s^3} - e^{-sn} \left[\frac{n^2}{s} + \frac{2n}{s^2} + \frac{2}{s^3} \right] \right] \\
 &= \frac{8}{s^3}.
 \end{aligned}$$

4.

$$\begin{aligned}
 F(s) &= \int_0^{\infty} e^{-st} 5e^{-3t} dt \\
 &= 5 \int_0^{\infty} e^{-(s+3)t} dt \\
 &= 5 \lim_{n \rightarrow \infty} \int_0^n e^{-(s+3)t} dt \\
 &= -\frac{5}{s+3} \lim_{n \rightarrow \infty} \left[e^{-(s+3)t} \right]_0^n \\
 &= -\frac{5}{s+3} \lim_{n \rightarrow \infty} \left[e^{-(s+3)n} - 1 \right] \\
 &= \frac{5}{s+3}.
 \end{aligned}$$

5.

$$\begin{aligned}
F(s) &= \int_0^{\infty} e^{-st} (7te^{-t}) dt \\
&= 7 \int_0^{\infty} te^{-(s+1)t} dt \\
&= 7 \lim_{n \rightarrow \infty} \int_0^n te^{-(s+1)t} dt \\
&= 7 \left[\lim_{n \rightarrow \infty} \left(-\frac{t}{s+1} e^{-(s+1)t} \right)_0^n + \frac{1}{s+1} \int_0^n e^{-(s+1)t} dt \right] \\
&= 7 \left[\lim_{n \rightarrow \infty} \left(-\frac{t}{s+1} e^{-(s+1)t} \right)_0^n - \left(\frac{1}{(s+1)^2} e^{-(s+1)t} \right)_0^n \right] \\
&= \frac{7}{(s+1)^2}.
\end{aligned}$$

6.

$$\begin{aligned}
F(s) &= \int_0^{\infty} e^{-st} \sin at \cos bt \, dt \\
&= \lim_{n \rightarrow \infty} \int_0^n e^{-st} \sin at \cos bt \, dt.
\end{aligned}$$

We evaluate this integral via integration by parts with

$$u = e^{-st} \quad \text{and} \quad dv = \sin at \cos bt \, dt = \frac{1}{2} [\sin(a-b)t + \sin(a+b)t] \, dt,$$

which gives

$$du = -se^{-st} dt \quad \text{and} \quad v = -\frac{\cos(a-b)t}{2(a-b)} - \frac{\cos(a+b)t}{2(a+b)}.$$

Hence, we obtain

$$\begin{aligned}
F(s) &= \lim_{n \rightarrow \infty} \left\{ -e^{-st} \left[\frac{\cos(a-b)t}{2(a-b)} + \frac{\cos(a+b)t}{2(a+b)} \right]_0^n - s \int_0^n e^{-st} \left[\frac{\cos(a-b)t}{2(a-b)} + \frac{\cos(a+b)t}{2(a+b)} \right] dt \right\} \\
&= \frac{1}{2(a-b)} + \frac{1}{2(a+b)} - \lim_{n \rightarrow \infty} \frac{s}{2(a-b)} \int_0^n e^{-st} \cos(a-b)t \, dt - \lim_{n \rightarrow \infty} \frac{s}{2(a+b)} \int_0^n e^{-st} \cos(a+b)t \, dt \\
&= \frac{a}{a^2 - b^2} - \lim_{n \rightarrow \infty} \frac{se^{-st}}{2(a-b)(s^2 + (a-b)^2)} \left[-s \cos(a-b)t + (a-b) \sin(a-b)t \right]_0^n \\
&\quad - \lim_{n \rightarrow \infty} \frac{se^{-st}}{2(a+b)(s^2 + (a+b)^2)} \left[-s \cos(a+b)t + (a+b) \sin(a+b)t \right]_0^n \\
&= \frac{a}{a^2 - b^2} - s^2 \left[\frac{1}{2(a-b)(s^2 + (a-b)^2)} + \frac{1}{2(a+b)(s^2 + (a+b)^2)} \right].
\end{aligned}$$

7.

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} \sin^2 at \, dt \\
&= \lim_{n \rightarrow \infty} \int_0^n e^{-st} \sin^2 at \, dt \\
&= \lim_{n \rightarrow \infty} \int_0^n e^{-st} \frac{1 - \cos 2at}{2} \, dt \\
&= \frac{1}{2} \left[\lim_{n \rightarrow \infty} \int_0^n e^{-st} \, dt - \int_0^n e^{-st} \cos 2at \, dt \right] \\
&= \frac{1}{2s} - \frac{1}{2} \lim_{n \rightarrow \infty} \int_0^n e^{-st} \cos 2at \, dt \\
&= \frac{1}{2s} - \frac{1}{2} \lim_{n \rightarrow \infty} \left[\frac{e^{-st}}{s^2 + 4a^2} [-s \cdot \cos(2at) + 2a \sin(2at)] \right]_0^n \\
&= \frac{1}{2s} - \frac{s}{2(s^2 + 4a^2)}.
\end{aligned}$$

8.

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t) \, dt \\
&= \int_0^1 2e^{-st} \, dt + \int_1^\infty te^{-st} \, dt \\
&= \frac{2}{s} [1 - e^{-s}] + \lim_{n \rightarrow \infty} \int_1^n te^{-st} \, dt \\
&= \frac{2}{s} [1 - e^{-s}] + \frac{1}{s} e^{-s} + \frac{1}{s^2} e^{-s} \\
&= \frac{2}{s} + e^{-s} \left[\frac{1}{s^2} - \frac{1}{s} \right].
\end{aligned}$$

9.

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t) \, dt \\
&= \int_0^3 e^{-st} (t+1) \, dt + \int_3^\infty e^{-st} (t^2 - 1) \, dt \\
&= \int_0^3 e^{-st} (t+1) \, dt + \lim_{n \rightarrow \infty} \int_3^n e^{-st} (t^2 - 1) \, dt \\
&= \left[-\frac{t+1}{s} e^{-st} \right]_0^3 + \frac{1}{s} \int_0^3 e^{-st} \, dt + \lim_{n \rightarrow \infty} \left[\left[-\frac{t^2-1}{s} e^{-st} \right]_3^n + \frac{2}{s} \int_3^n te^{-st} \, dt \right] \\
&= \left[-\frac{t+1}{s} e^{-st} \right]_0^3 - \frac{1}{s^2} [e^{-st}]_0^3 + \lim_{n \rightarrow \infty} \left[\left[-\frac{t^2-1}{s} e^{-st} \right]_3^n + \frac{2}{s} \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right]_3^n \right] \\
&= \left[-\frac{4}{s} e^{-3s} + \frac{1}{s} \right] + \left[-\frac{1}{s^2} e^{-3s} + \frac{1}{s^2} \right] + \left[\frac{8}{s} e^{-3s} + \frac{6}{s^2} e^{-3s} + \frac{2}{s^3} e^{-3s} \right] \\
&= \frac{e^{-3s}(4s^2 + 5s + 2)}{s^3} + \frac{s+1}{s^2}.
\end{aligned}$$

10.

$$\begin{aligned}
F(s) &= \int_0^{\infty} e^{-st} f(t) dt \\
&= \int_0^1 2e^{-st} dt + \int_1^2 (1-t)e^{-st} dt \\
&= -\left[\frac{2}{s}e^{-st}\right]_0^1 + \left[\frac{t-1}{s}e^{-st} + \frac{1}{s^2}e^{-st}\right]_1^2 \\
&= \frac{2}{s} - \frac{2s+1}{s^2}e^{-s} + \frac{s+1}{s^2}e^{-2s}.
\end{aligned}$$

11. We have

$$\begin{aligned}
L[f] &= 5L[\cos 2t] - 7L[e^{-t}] - 3L[t^6] \\
&= \frac{5s}{s^2+4} - \frac{7}{s+1} - \frac{3 \cdot 6!}{s^7}.
\end{aligned}$$

12. CHANGE PROBLEM TO: $f(t) = \int_0^t (t-\tau)^4 \delta(\tau+\pi) d\tau$.**SOLUTION:**

$$L[f] = L[t^4 * \delta(t+\pi)] = L[t^4]L[\delta(t+\pi)] = \frac{24e^{\pi s}}{s^5}.$$

13. We have

$$L[f] = \frac{s-3}{(s-3)^2+25} - \frac{2}{(s+1)^2+4}.$$

14. We have

$$\begin{aligned}
L[f] &= 6 \cdot \frac{24}{(s+2)^5} - \frac{2e}{(s-1)^2} + \sqrt{10}L[\sqrt{t}] \\
&= \frac{144}{(s+2)^5} - \frac{2e}{(s-1)^2} + \sqrt{10} \int_0^{\infty} \sqrt{t}e^{-st} dt \\
&= \frac{144}{(s+2)^5} - \frac{2e}{(s-1)^2} + \sqrt{10} \lim_{n \rightarrow \infty} \int_0^n \sqrt{t}e^{-st} dt \\
&= \frac{144}{(s+2)^5} - \frac{2e}{(s-1)^2} + \sqrt{10} \lim_{n \rightarrow \infty} \left[\left[-\frac{\sqrt{t}}{s}e^{-st} \right]_0^n + \frac{1}{2s} \int_0^n \frac{1}{\sqrt{t}}e^{-st} dt \right] \\
&= \frac{144}{(s+2)^5} - \frac{2e}{(s-1)^2} + \frac{\sqrt{10}}{2s} L[t^{-1/2}] \\
&= \frac{144}{(s+2)^5} - \frac{2e}{(s-1)^2} + \frac{\sqrt{10}}{2s} \sqrt{\frac{\pi}{s}} \\
&= \frac{144}{(s+2)^5} - \frac{2e}{(s-1)^2} + \sqrt{\frac{5\pi}{2s^3}}.
\end{aligned}$$

15. We have

$$L[f] = \sqrt{\frac{\pi}{s+5}}.$$

16. According to the Second Shifting Theorem, we have

$$L[f] = 2 \frac{e^{-5s}}{s^2}.$$

17. We have

$$L[f] = 2L[1] + 2L[e^{-t}u_1(t)] - 2L[u_1(t)].$$

For the middle term, we wish to apply the Second Shifting Theorem. In order to do so, we note that $e^{-t} = \frac{e^{-(t-1)}}{e}$. Thus, if we write $f(t-1) = \frac{e^{-(t-1)}}{e}$, then $f(t) = \frac{e^{-t}}{e}$. Hence,

$$L[u_1(t)e^{-t}] = L[u_1(t)f(t-1)] = e^{-s} \cdot \frac{1}{e} \cdot \frac{1}{s+1} = \frac{e^{-s}}{e(s+1)}.$$

Therefore,

$$\begin{aligned} L[f] &= 2L[1] + 2L[e^{-t}u_1(t)] - 2L[u_1(t)] \\ &= \frac{2}{s} + \frac{2e^{-s}}{e(s+1)} - \frac{2e^{-s}}{s} \\ &= \frac{2}{s}(1 - e^{-s}) + \frac{2e^{-s}}{e(s+1)} \\ &= \frac{2}{s}(1 - e^{-s}) + \frac{2e^{-(s+1)}}{s+1}. \end{aligned}$$

18. In this case, we observe that $f(t) = t * \cos 2t$. Therefore, by the convolution theorem, we have

$$L[f] = L[t]L[\cos 2t] = \frac{1}{s^2} \cdot \frac{s}{s^2 + 4} = \frac{1}{s^3 + 4s}.$$

19. In this case, we observe that $f(t) = t^2 * e^t$. Therefore, by the convolution theorem, we have

$$L[f] = L[t^2]L[e^t] = \frac{2}{s^3} \cdot \frac{1}{s-1} = \frac{2}{s^3(s-1)}.$$

20. We have

$$\begin{aligned} L^{-1}[F(s)] &= 3 \cdot L^{-1}\left[\frac{1}{s^2}\right] \\ &= 3t. \end{aligned}$$

21. We have

$$\begin{aligned} L^{-1}[F(s)] &= L^{-1}\left[\frac{4s}{s^2+9}\right] + L^{-1}\left[\frac{5}{s^2+9}\right] \\ &= 4L^{-1}\left[\frac{s}{s^2+9}\right] + 5L^{-1}\left[\frac{1}{s^2+9}\right] \\ &= 4 \cos 3t + \frac{5}{3} \sin 3t. \end{aligned}$$

22. We have

$$\begin{aligned} L^{-1}[F(s)] &= L^{-1} \left[\frac{s-2}{s^2+2s+2} \right] \\ &= L^{-1} \left[\frac{s+1}{(s+1)^2+1} \right] - 3L^{-1} \left[\frac{1}{(s+1)^2+1} \right] \\ &= e^{-t} \cos t - 3e^{-t} \sin t. \end{aligned}$$

23. We recognize $F(s) = \frac{2}{s(s^2+16)}$ as a convolution product, with $g(t) = \frac{1}{2t}$ and $h(t) = \frac{4}{s^2+16}$. By the convolution theorem, we have

$$\begin{aligned} L^{-1}[F(s)] &= L^{-1} \left[\frac{1}{2s} \right] * L^{-1} \left[\frac{4}{s^2+16} \right] \\ &= \frac{1}{2} * \sin 4t \\ &= \int_0^t \frac{1}{2} \sin 4w \, dw \\ &= -\frac{1}{8} (\cos 4w) \Big|_0^t \\ &= \frac{1}{8} [1 - \cos 4t]. \end{aligned}$$

24. *CHANGE PROBLEM TO:* $F(s) = \frac{4s^2-s+8}{s^3-4s}$.

SOLUTION: Using a partial fractions decomposition, we have

$$\begin{aligned} \frac{4s^2-s+8}{s^3-4s} &= \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+2} \\ &= \frac{A(s^2-4) + Bs(s+2) + Cs(s-2)}{s^3-4s} \\ &= \frac{(A+B+C)s^2 + (2B-2C)s - 4A}{s^3-4s}. \end{aligned}$$

Equating numerators on both sides according to powers of s , we have

$$A+B+C=4, \quad 2B-2C=-1, \quad -4A=8.$$

Solving this system of equations, we find that $A = -2$, $B = \frac{11}{4}$, and $C = \frac{13}{4}$. Hence,

$$\begin{aligned} f(t) &= L^{-1}[F(s)] \\ &= L^{-1} \left[\frac{4s^2-s+8}{s^3-4s} \right] \\ &= -2L^{-1} \left[\frac{1}{s} \right] + \frac{11}{4} L^{-1} \left[\frac{1}{s-2} \right] + \frac{13}{4} L^{-1} \left[\frac{1}{s+2} \right] \\ &= -2 + \frac{11}{4} e^{2t} + \frac{13}{4} e^{-2t}. \end{aligned}$$

25. We have

$$\begin{aligned} L^{-1}[F(s)] &= L^{-1}\left[\frac{2s+5}{s(s^2+4s+20)}\right] \\ &= \frac{1}{2}L^{-1}\left[\frac{4}{(s+2)^2+16}\right] + \frac{5}{4}L^{-1}\left[\frac{4}{s[(s+2)^2+16]}\right] \end{aligned}$$

The First Shifting Theorem can be used to evaluate the first inverse Laplace transform, while the second inverse Laplace transform can be evaluated by using the First Shifting Theorem along with the convolution theorem. Continuing, we have

$$\begin{aligned} L^{-1}[F(s)] &= \frac{1}{2}L^{-1}\left[\frac{4}{(s+2)^2+16}\right] + \frac{5}{4}[1 * e^{-2t} \sin 4t] \\ &= \frac{1}{2}e^{-2t} \sin 4t + \frac{5}{4} \int_0^t e^{-2w} \sin 4w \, dw \\ &= \frac{1}{2}e^{-2t} \sin 4t + \frac{5}{4} \left[\frac{e^{-2w}}{20} (-2 \sin 4w - 4 \cos 4w) \right]_0^t \\ &= \frac{1}{2}e^{-2t} \sin 4t + \frac{5}{4} \left[\frac{1}{5} - \frac{e^{-2t}}{10} (\sin 4t + 2 \cos 4t) \right] \\ &= \frac{1}{4} + \frac{3}{8}e^{-2t} \sin 4t - \frac{1}{4}e^{-2t} \cos 4t. \end{aligned}$$

26. We have

$$f(t) = (1-t)u_1(t) + 2,$$

so that

$$L[f] = L[(1-t)u_1(t)] + L[2] = -L[(t-1)u_1(t)] + L[2] = -e^{-s} \cdot \frac{1}{s^2} + \frac{2}{s} = \frac{2s - e^{-s}}{s^2}.$$

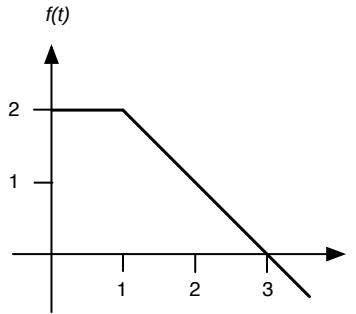


Figure 150: Figure for Problem 26

27. We have

$$f(t) = (2e^{-t} - 1)u_{\ln 2}(t) + 1,$$

so that

$$\begin{aligned}
 L[f] &= L[(2e^{-t} - 1)u_{\ln 2}(t)] + L[1] \\
 &= 2L[e^{-t}u_{\ln 2}(t)] - L[u_{\ln 2}(t)] + L[1] \\
 &= 2e^{-\ln 2 \cdot s} \cdot \frac{1}{2(s+1)} - e^{-\ln 2 \cdot s} \cdot \frac{1}{s} + \frac{1}{s} \\
 &= \frac{1}{2^s(s+1)} - \frac{1}{s2^s} + \frac{1}{s} \\
 &= \frac{1}{s} - \frac{1}{s(s+1)2^s}.
 \end{aligned}$$

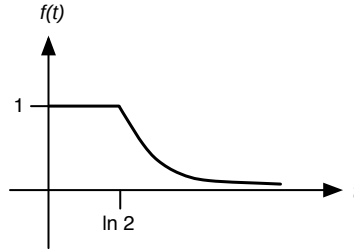


Figure 151: Figure for Problem 27

28. We have

$$f(t) = (1 - u_3(t)) [t + (1 - t)u_1(t) + (2 - t)u_2(t)].$$

We expand this expression, using the fact that $u_a(t)u_b(t) = u_b(t)$ whenever $a \leq b$:

$$\begin{aligned}
 f(t) &= t + (1 - t)u_1(t) + (2 - t)u_2(t) - tu_3(t) - (1 - t)u_3(t) - (2 - t)u_3(t) \\
 &= t - (t - 1)u_1(t) - (t - 2)u_2(t) - tu_3(t) + (t - 1)u_3(t) + (t - 2)u_3(t).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 L[f] &= L[t] - L[(t - 1)u_1(t)] - L[(t - 2)u_2(t)] - L[tu_3(t)] + L[(t - 1)u_3(t)] + L[(t - 2)u_3(t)] \\
 &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} - e^{-3s} \left[\frac{1}{s^2} + \frac{3}{s} \right] + e^{-3s} \left[\frac{1}{s^2} + \frac{2}{s} \right] + e^{-3s} \left[\frac{1}{s^2} + \frac{1}{s} \right] \\
 &= \frac{1 - e^{-s} - e^{-2s} + e^{-3s}}{s^2}.
 \end{aligned}$$

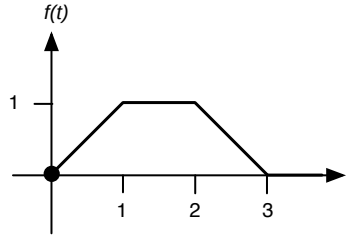


Figure 152: Figure for Problem 28

29. Making the substitution $u = t - a$, we have

$$\begin{aligned}
 L[f_a(t)] &= \int_0^{\infty} e^{-st} f_a(t) dt \\
 &= \int_a^{\infty} e^{-st} f(t-a) dt \\
 &= \int_0^{\infty} e^{-s(u+a)} f(u) du \\
 &= e^{-as} \int_0^{\infty} e^{-su} f(u) du \\
 &= e^{-as} L[f]
 \end{aligned}$$

30. Let $f(x) = x^a$ and let $g(x) = x^b$. Then

$$\int_0^x (x-w)^a w^b dx = (f * g)(x) = L^{-1}[L(f * g)] = L^{-1}\left[\frac{a!}{s^{a+1}} \cdot \frac{b!}{s^{b+1}}\right] = a!b!L^{-1}\left[\frac{1}{s^{a+b+2}}\right] = \frac{a!b!}{(a+b+1)!} x^{a+b+1},$$

as desired.

31.

(a) We have $f'(x) = e^{ax} + axe^{ax}$. Therefore, using the table of Laplace transforms, we immediately find that

$$L[f'(x)] = L[e^{ax} + axe^{ax}] = L[e^{ax}] + aL[xe^{ax}] = \frac{1}{s-a} + aL[f(x)].$$

(b) We have $L[f'] = sL[f] - f(0)$. Equating this with the result for $L[f']$ obtained in part (a), we have

$$sL[f] - f(0) = aL[f] + \frac{1}{s-a},$$

so that

$$(s-a)L[f] = \frac{1}{s-a} + f(0).$$

In this case, $f(0) = 0$, so therefore,

$$L[f] = \frac{1}{(s-a)^2}.$$

(c) Part (b) establishes this result for $n = 1$. Now assume that

$$L[x^{n-1}e^{ax}] = \frac{(n-1)!}{(s-a)^n}.$$

To evaluate $L[x^n e^{ax}]$, we first observe that $(x^n e^{ax})' = nx^{n-1}e^{ax} + ax^n e^{ax}$. Again using $L[f'] = sL[f] - f(0)$ and the fact that $f(0) = 0$ in this case, we have

$$sL[x^n e^{ax}] = L[(x^n e^{ax})'] = nL[x^{n-1}e^{ax}] + aL[x^n e^{ax}].$$

Rearranging and substituting the induction hypothesis, we obtain

$$(s-a)L[x^n e^{ax}] = n \cdot \frac{(n-1)!}{(s-a)^n} = \frac{n!}{(s-a)^n},$$

so that

$$L[x^n e^{ax}] = \frac{n!}{(s-a)^{n+1}},$$

as required.

32. We apply the Laplace transform to both sides of the differential equation $y' + ay = f(t)$ to obtain

$$sY(s) - y(0) + aY(s) = F(s),$$

solving for $Y(s)$ yields,

$$Y(s) = \frac{F(s)}{s+a} + \frac{y_0}{s+a}.$$

Taking the inverse Laplace transform of both sides gives:

$$\begin{aligned} y(t) &= L^{-1} \left[\frac{1}{s+a} \cdot L[f] \right] + L^{-1} \left[\frac{y_0}{s+a} \right] \\ &= L^{-1} \left[\frac{1}{s+a} \right] * f + L^{-1} \left[\frac{y_0}{s+a} \right] \\ &= e^{-at} * f + y_0 e^{-at} \\ &= \int_0^t e^{-a(t-w)} f(w) dw + y_0 e^{-at}, \end{aligned}$$

as required.

33. In Section 8.4, Problem 33, it is established that

$$L[f^{(n)}] = s^n L[f] - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

Therefore, taking the Laplace transform of both sides of

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = f(t),$$

and using the given initial conditions yields

$$(s^n + a_1 s^{n-1} + \dots + a_n) Y(s) = F(s).$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{1}{P(s)} \cdot F(s),$$

where

$$P(s) = s^n + a_1 s^{n-1} + \cdots + a_n.$$

Consequently,

$$y(t) = L^{-1} \left[\frac{1}{P(s)} \cdot F(s) \right] = K(t) * f(t) = \int_0^t K(t-w)f(w)dw,$$

where $K(t) = L^{-1} \left[\frac{1}{P(s)} \right]$.

34. Taking the Laplace transform of both sides of the given differential equation we obtain

$$[s^2 Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] - 4Y(s) = \frac{4}{s+1}.$$

Substituting the initial values $y(0) = 1$ and $y'(0) = 1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{s^2 - s + 2}{(s+1)(s^2 - 3s - 4)} = \frac{s^2 - s + 2}{(s+1)^2(s-4)}.$$

Next, we must determine a partial fractions decomposition of the right-hand side. We have

$$\begin{aligned} \frac{s^2 - s + 2}{(s+1)^2(s-4)} &= \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-4} \\ &= \frac{A(s+1)(s-4) + B(s-4) + C(s+1)^2}{(s+1)^2(s-4)} \\ &= \frac{(A+C)s^2 + (-3A+B+2C)s + (-4A-4B+C)}{(s+1)^2(s-4)}. \end{aligned}$$

Equating the numerators on both sides according to powers of s , we obtain

$$A + C = 1, \quad -3A + B + 2C = -1, \quad -4A - 4B + C = 2.$$

Solving this system, we find

$$A = \frac{11}{25}, \quad B = -\frac{4}{5}, \quad C = \frac{14}{25}.$$

Thus,

$$\frac{s^2 - s + 2}{(s+1)^2(s-4)} = \frac{11}{25} \cdot \frac{1}{(s+1)} - \frac{4}{5} \cdot \frac{1}{(s+1)^2} + \frac{14}{25} \cdot \frac{1}{(s-4)}.$$

We can solve

$$Y(s) = \frac{11}{25} \cdot \frac{1}{(s+1)} - \frac{4}{5} \cdot \frac{1}{(s+1)^2} + \frac{14}{25} \cdot \frac{1}{(s-4)}$$

for $y(t)$ by taking the inverse Laplace transform of both sides:

$$y(t) = \frac{11}{25}e^{-t} - \frac{4}{5}te^{-t} + \frac{14}{25}e^{4t}.$$

35. Taking the Laplace transform of both sides of the given differential equation we obtain

$$[s^2 Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] - 8Y(s) = \frac{5}{s}.$$

Substituting the initial values $y(0) = 1$ and $y'(0) = 0$ and solving for $Y(s)$ yields

$$Y(s) = \frac{s^2 - 2s + 5}{s(s^2 - 2s - 8)} = \frac{s^2 - 2s + 5}{s(s - 4)(s + 2)}.$$

Next, we must determine a partial fractions decomposition of the right-hand side. We have

$$\begin{aligned} \frac{s^2 - 2s + 5}{s(s - 4)(s + 2)} &= \frac{A}{s} + \frac{B}{s - 4} + \frac{C}{s + 2} \\ &= \frac{A(s - 4)(s + 2) + Bs(s + 2) + Cs(s - 4)}{s(s - 4)(s + 2)} \\ &= \frac{(A + B + C)s^2 + (-2A + 2B - 4C)s - 8A}{s(s - 4)(s + 2)}. \end{aligned}$$

Equating numerators on both sides of the equation according to powers of s , we obtain

$$A + B + C = 1, \quad -2A + 2B - 4C = -2, \quad -8A = 5.$$

Solving this system, we find

$$A = -\frac{5}{8}, \quad B = \frac{13}{24}, \quad C = \frac{13}{12}.$$

Thus,

$$\frac{s^2 - 2s + 5}{s(s - 4)(s + 2)} = -\frac{5}{8} \cdot \frac{1}{s} + \frac{13}{24} \cdot \frac{1}{(s - 4)} + \frac{13}{12} \cdot \frac{1}{(s + 2)}.$$

We can solve

$$Y(s) = -\frac{5}{8} \cdot \frac{1}{s} + \frac{13}{24} \cdot \frac{1}{(s - 4)} + \frac{13}{12} \cdot \frac{1}{(s + 2)}$$

for $y(t)$ by taking the inverse Laplace transform of both sides:

$$y(t) = -\frac{5}{8} + \frac{13}{24}e^{4t} + \frac{13}{12}e^{-2t}.$$

36. Taking the Laplace transform of both sides of the given differential equation we obtain

$$[s^2Y(s) - sy(0) - y'(0)] + 9Y(s) = \frac{8s}{s^2 + 9}.$$

Substituting the initial values $y(0) = 1$ and $y'(0) = 0$ and solving for $Y(s)$ yields

$$Y(s) = \frac{s^3 + 17s}{(s^2 + 9)^2}.$$

Next, we must determine a partial fractions decomposition of the right-hand side. We have

$$\begin{aligned} \frac{s^3 + 17s}{(s^2 + 9)^2} &= \frac{As + B}{s^2 + 9} + \frac{Cs + D}{(s^2 + 9)^2} \\ &= \frac{(As + B)(s^2 + 9) + Cs + D}{(s^2 + 9)^2} \\ &= \frac{As^3 + Bs^2 + (9A + C)s + (9B + D)}{(s^2 + 9)^2}. \end{aligned}$$

Equating the numerators on both sides according to powers of s , we obtain

$$A = 1, \quad B = 0, \quad 9A + C = 17, \quad 9B + D = 0.$$

Solving this system, we find

$$A = 1, \quad B = 0, \quad C = 8, \quad D = 0.$$

Thus,

$$\frac{s^3 + 17s}{(s^2 + 9)^2} = \frac{s}{s^2 + 9} + \frac{8s}{(s^2 + 9)^2}.$$

We can solve

$$Y(s) = \frac{s}{s^2 + 9} + \frac{8s}{(s^2 + 9)^2}$$

for $y(t)$ by taking the inverse Laplace transform of both sides:

$$\begin{aligned} y(t) &= L^{-1} \left[\frac{s}{s^2 + 9} \right] + 8 \left\{ L^{-1} \left[\frac{s}{s^2 + 9} \right] * L^{-1} \left[\frac{1}{s^2 + 9} \right] \right\} \\ &= \cos 3t + 8 \left(\cos 3t * \frac{1}{3} \sin 3t \right) \\ &= \cos 3t + \frac{8}{3} \int_0^t \cos 3(t - \tau) \sin 3\tau \, d\tau \\ &= \cos 3t + \frac{8}{3} \int_0^t (\cos 3t \cos 3\tau \sin 3\tau + \sin 3t \sin^2 3\tau) \, d\tau \\ &= \cos 3t + \frac{8}{3} \left[\frac{1}{12} \cos 3t (1 - \cos 6t) + \frac{t \sin 3t}{2} - \frac{1}{12} \sin 3t \sin 6t \right] \\ &= \cos 3t + \frac{4}{3} t \sin 3t + \frac{2}{9} [\cos 3t - (\cos 6t \cos 3t + \sin 6t \sin 3t)] \\ &= \cos 3t + \frac{4}{3} t \sin 3t + \frac{2}{9} (\cos 3t - \cos 3t) \\ &= \cos 3t + \frac{4}{3} t \sin 3t. \end{aligned}$$

37. Note that $f(t) = 1 - u_{\pi/2}(t)$, so that we must solve

$$y'' + y = 1 - u_{\pi/2}(t).$$

Taking the Laplace transform of both sides of the given differential equation we obtain

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{1 - e^{-\pi s/2}}{s}.$$

Substituting the initial values $y(0) = 0$ and $y'(0) = 1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{s + 1 - e^{-\pi s/2}}{s(s^2 + 1)} = \frac{s + 1}{s(s^2 + 1)} - e^{-\pi s/2} \left[\frac{1}{s(s^2 + 1)} \right].$$

Next, we determine a partial fractions decomposition for the first term on the right-hand side:

$$\begin{aligned}\frac{s+1}{s(s^2+1)} &= \frac{A}{s} + \frac{Bs+C}{s^2+1} \\ &= \frac{A(s^2+1) + (Bs+C)s}{s(s^2+1)} \\ &= \frac{(A+B)s^2 + Cs + A}{s(s^2+1)}.\end{aligned}$$

Equating numerators on both sides according to powers of s , we obtain

$$A+B=0, \quad C=1, \quad A=1.$$

Therefore, $A=1$, $B=-1$, and $C=1$. Hence,

$$\frac{s+1}{s(s^2+1)} = \frac{1}{s} + \frac{-s+1}{s^2+1}.$$

Thus,

$$\begin{aligned}L^{-1}\left[\frac{s+1}{s(s^2+1)}\right] &= L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{-s+1}{s^2+1}\right] \\ &= L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s^2+1}\right] - L^{-1}\left[\frac{s}{s^2+1}\right] \\ &= 1 + \sin t - \cos t.\end{aligned}$$

Next, we do a partial fractions decomposition to help us determine the inverse Laplace transform of the second term on the right-hand side above:

$$\begin{aligned}\frac{1}{s(s^2+1)} &= \frac{A}{s} + \frac{Bs+C}{s^2+1} \\ &= \frac{A(s^2+1) + (Bs+C)s}{s(s^2+1)} \\ &= \frac{(A+B)s^2 + Cs + A}{s(s^2+1)}.\end{aligned}$$

Equating numerators on both sides according to powers of s , we obtain

$$A+B=0, \quad C=0, \quad A=1.$$

Therefore, $A=1$, $B=-1$, and $C=0$. Hence,

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}.$$

Hence,

$$\begin{aligned}L^{-1}\left[e^{-\pi s/2}\left[\frac{1}{s(s^2+1)}\right]\right] &= L^{-1}\left[e^{-\pi s/2}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\right] \\ &= L^{-1}\left[\frac{e^{-\pi s/2}}{s}\right] - L^{-1}\left[\frac{se^{-\pi s/2}}{s^2+1}\right] \\ &= u_{\pi/2}(t) - u_{\pi/2}(t) \cos(t - \pi/2) \\ &= u_{\pi/2}(t)[1 - \cos(t - \pi/2)].\end{aligned}$$

Putting this together with the inverse Laplace transform for the first term obtained above, we obtain the final answer:

$$y(t) = 1 + \sin t - \cos t - u_{\pi/2}(t)[1 - \cos(t - \pi/2)].$$

38. Taking the Laplace transform of both sides of the given differential equation we obtain

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{4}{s^2 + 1} + 3e^{-2s}.$$

Substituting the initial values $y(0) = 2$ and $y'(0) = 1$ and solving for $Y(s)$ yields

$$Y(s) = \frac{2s}{s^2 + 4} + \frac{1}{s^2 + 4} + \frac{4}{(s^2 + 1)(s^2 + 4)} + \frac{3e^{-2s}}{s^2 + 4}.$$

Therefore,

$$\begin{aligned} y(t) &= 2L^{-1}\left[\frac{s}{s^2 + 4}\right] + L^{-1}\left[\frac{1}{s^2 + 4}\right] + 4L^{-1}\left[\frac{1}{(s^2 + 1)(s^2 + 4)}\right] + 3L^{-1}\left[\frac{e^{-2s}}{s^2 + 4}\right] \\ &= 2L^{-1}\left[\frac{s}{s^2 + 4}\right] + \frac{1}{2}L^{-1}\left[\frac{2}{s^2 + 4}\right] + 4\left\{L^{-1}\left[\frac{1}{s^2 + 1}\right] * L^{-1}\left[\frac{1}{s^2 + 4}\right]\right\} + \frac{3}{2}L^{-1}\left[\frac{2e^{-2s}}{s^2 + 4}\right] \\ &= 2\cos 2t + \frac{1}{2}\sin 2t + 4[\sin t * \frac{1}{2}\sin 2t] + \frac{3}{2}u_2(t)\sin[2(t - 2)] \\ &= 2\cos 2t + \frac{1}{2}\sin 2t + 2\int_0^t \sin(t - \tau)\sin 2\tau d\tau + \frac{3}{2}u_2(t)\sin[2(t - 2)] \\ &= 2\cos 2t + \frac{1}{2}\sin 2t + \frac{4}{3}\sin t[1 - \cos t] + \frac{3}{2}u_2(t)\sin[2(t - 2)] \\ &= 2\cos 2t + \frac{4}{3}\sin t - \frac{1}{3}\sin t\cos t + \frac{3}{2}u_2(t)\sin[2(t - 2)] \\ &= 2\cos 2t + \frac{4}{3}\sin t - \frac{1}{6}\sin 2t + \frac{3}{2}u_2(t)\sin[2(t - 2)]. \end{aligned}$$

39. Taking the Laplace transform of both sides of the given differential equation we obtain

$$[s^2Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + Y(s) = e^{-4s}.$$

Substituting the initial values $y(0) = 0$ and $y'(0) = 0$ and solving for $Y(s)$ yields

$$Y(s) = \frac{e^{-4s}}{(s + 1)^2}.$$

Now, since

$$L^{-1}\left[\frac{1}{(s + 1)^2}\right] = te^{-t},$$

we have

$$y(t) = u_4(t)(t - 4)e^{-(t-4)}.$$

40. Taking the Laplace transform of both sides of the given differential equation we obtain

$$[s^2Y(s) - sy(0) - y'(0)] + 4[sY(s) - y(0)] + 4Y(s) = e^{-4s}.$$

Substituting the initial values $y(0) = 1$ and $y'(0) = 2$ and solving for $Y(s)$ yields

$$Y(s) = \frac{s+6+e^{-4s}}{(s+2)^2} = \frac{s+6}{(s+2)^2} + \frac{e^{-4s}}{(s+2)^2}.$$

Next, we determine a partial fractions decomposition for the first term on the right-hand side:

$$\begin{aligned} \frac{s+6}{(s+2)^2} &= \frac{A}{s+2} + \frac{B}{(s+2)^2} \\ &= \frac{A(s+2)+B}{(s+2)^2} \\ &= \frac{As+(2A+B)}{(s+2)^2}. \end{aligned}$$

Equating numerators on each side according to powers of s , we have $A = 1$ and $2A + B = 6$. Therefore, $A = 1$ and $B = 4$. Hence,

$$\frac{s+6}{(s+2)^2} = \frac{1}{s+2} + \frac{4}{(s+2)^2}.$$

Therefore,

$$\begin{aligned} L^{-1} \left[\frac{s+6}{(s+2)^2} \right] &= L^{-1} \left[\frac{1}{s+2} \right] + 4L^{-1} \left[\frac{1}{(s+2)^2} \right] \\ &= e^{-2t} + 4te^{-2t}. \end{aligned}$$

And since $L^{-1} \left[\frac{1}{(s+2)^2} \right] = te^{-2t}$, we have

$$L^{-1} \left[\frac{e^{-2s}}{(s+2)^2} \right] = u_2(t)(t-2)e^{-2(t-2)}.$$

Therefore, we have

$$y(t) = e^{-2t} + 4te^{-2t} + u_2(t)(t-2)e^{-2(t-2)}.$$

41. Taking the Laplace transform of each differential equation and using the given initial conditions yields:

$$sX_1 - 1 = X_1 + 2X_2 \quad \text{and} \quad sX_2 = 2X_1 + X_2,$$

or equivalently,

$$(s-1)X_1 - 2X_2 = 1 \quad \text{and} \quad -2X_1 + (s-1)X_2 = 0.$$

Solving this algebraic system of equations using Cramer's rule we obtain:

$$\begin{aligned} X_1(s) &= \frac{\begin{vmatrix} 1 & -2 \\ 0 & s-1 \end{vmatrix}}{\begin{vmatrix} s-1 & -2 \\ -2 & s-1 \end{vmatrix}} = \frac{s-1}{(s-1)^2-4} = \frac{s-1}{s^2-2s-3} = \frac{s-1}{(s-3)(s+1)} \\ &= \frac{1}{2(s-3)} + \frac{1}{2(s+1)}, \\ X_2(s) &= \frac{1}{(s-3)(s+1)} \cdot \begin{vmatrix} s-1 & 1 \\ -2 & 0 \end{vmatrix} = \frac{2}{(s-3)(s+1)} = \frac{1}{2(s-3)} - \frac{1}{2(s+1)}. \end{aligned}$$

Taking the inverse Laplace transform of $X_1(s)$ and $X_2(s)$ yields, respectively,

$$x_1(t) = \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \quad \text{and} \quad x_2(t) = \frac{1}{2}e^{3t} - \frac{1}{2}e^{-t}.$$

42. Taking the Laplace transform of the given system yields

$$sX_1 = 2X_2 \quad \text{and} \quad sX_2 - 1 = -2X_1,$$

or equivalently,

$$sX_1 - 2X_2 = 0 \quad \text{and} \quad 2X_1 + sX_2 = 1.$$

Solving this algebraic system of equations using Cramer's rule we obtain:

$$X_1(s) = \frac{\begin{vmatrix} 0 & -2 \\ 1 & s \end{vmatrix}}{\begin{vmatrix} s & -2 \\ 2 & s \end{vmatrix}} = \frac{2}{s^2 + 4},$$

$$X_2(s) = \frac{1}{s^2 + 4} \cdot \begin{vmatrix} s & 0 \\ 2 & 1 \end{vmatrix} = \frac{s}{s^2 + 4}.$$

Taking the inverse Laplace transform of $X_1(s)$ and $X_2(s)$ yields, respectively,

$$x_1(t) = \sin 2t \quad \text{and} \quad x_2(t) = \cos 2t.$$

43. Taking the Laplace transform of the given system yields

$$sX_1 - 1 = -2X_2 \quad \text{and} \quad sX_2 - 1 = 2X_1 + 4X_2,$$

or equivalently,

$$sX_1 + 2X_2 = 1 \quad \text{and} \quad -2X_1 + (s - 4)X_2 = 1.$$

Solving this algebraic system of equations using Cramer's rule we obtain:

$$X_1(s) = \frac{\begin{vmatrix} 1 & 2 \\ 1 & s - 4 \end{vmatrix}}{\begin{vmatrix} s & 2 \\ -2 & s - 4 \end{vmatrix}} = \frac{s - 6}{s^2 - 4s + 4} = \frac{s - 6}{(s - 2)^2}$$

$$= \frac{1}{s - 2} - \frac{4}{(s - 2)^2},$$

$$X_2(s) = \frac{1}{(s - 2)^2} \cdot \begin{vmatrix} s & 1 \\ -2 & 1 \end{vmatrix} = \frac{s + 2}{(s - 2)^2} = \frac{1}{s - 2} + \frac{4}{(s - 2)^2}$$

Taking the inverse Laplace transform of $X_1(s)$ and $X_2(s)$ yields, respectively,

$$x_1(t) = e^{2t} - 4te^{2t} \quad \text{and} \quad x_2(t) = e^{2t} + 4te^{2t}.$$

44. Taking the Laplace transform of the given system yields

$$sX_1 = 2X_1 + 4X_2 + \frac{32}{s^2 + 4} \quad \text{and} \quad sX_2 - 1 = -2X_1 - 2X_2 + \frac{16s}{s^2 + 4},$$

or equivalently,

$$(s-2)X_1 - 4X_2 = \frac{32}{s^2+4} \quad \text{and} \quad 2X_1 + (s+2)X_2 = 1 + \frac{16s}{s^2+4}.$$

Solving this algebraic system of equations using Cramer's rule we obtain:

$$\begin{aligned} X_1(s) &= \frac{\begin{vmatrix} \frac{32}{s^2+4} & -4 \\ 1 + \frac{16s}{s^2+4} & \end{vmatrix}}{\begin{vmatrix} s-2 & -4 \\ 2 & s+2 \end{vmatrix}} = \frac{1}{(s^2+4)^2} [32(s+2) + 64s + 4(s^2+4)] \\ &= \frac{4(s^2 + 24s + 20)}{(s^2+4)^2}, \\ X_2(s) &= \frac{1}{s^2+4} \cdot \begin{vmatrix} s-2 & \frac{32}{s^2+4} \\ 2 & 1 + \frac{16s}{s^2+4} \end{vmatrix} = \frac{1}{(s^2+4)^2} [(s-2)(s^2+16s+4) - 64] \\ &= \frac{s^3 + 14s^2 - 28s - 72}{(s^2+4)^2} \end{aligned}$$

We now determine partial fraction decompositions for the preceding expressions for $X_1(s)$ and $X_2(s)$.

$$\begin{aligned} \frac{4s^2 + 96s + 80}{(s^2+4)^2} &= \frac{As+B}{s^2+4} + \frac{Cs+D}{(s^2+4)^2} \\ &= \frac{(As+B)(s^2+4) + Cs+D}{(s^2+4)^2} \\ &= \frac{As^3 + Bs^2 + (4A+C)s + (4B+D)}{(s^2+4)^2}. \end{aligned}$$

Equating numerators on both sides according to powers of s , we have $A = 0$, $B = 4$, $4A + C = 96$, and $4B + D = 80$. Therefore, $C = 96$ and $D = 64$. Hence,

$$\begin{aligned} X_1(s) &= \frac{4s^2 + 96s + 80}{(s^2+4)^2} = \frac{4}{s^2+4} + \frac{96s+64}{(s^2+4)^2} \\ &= \frac{4}{s^2+4} + \frac{96s}{(s^2+4)^2} + \frac{64}{(s^2+4)^2}. \end{aligned}$$

Taking the inverse Laplace transform yields

$$\begin{aligned} x_1(t) &= 2 \sin 2t + 48(\cos 2t * \sin 2t) + 16(\sin 2t * \sin 2t) \\ &= 2 \sin 2t + 48 \int_0^t \cos 2(t-\tau) \sin 2\tau \, d\tau + 16 \int_0^t \sin 2(t-\tau) \sin 2\tau \, d\tau \\ &= 2 \sin 2t + 48 \left(\cos 2t \int_0^t \cos 2\tau \sin 2\tau \, d\tau + \sin 2t \int_0^t \sin^2 \tau \, d\tau \right) + 16 \left(\sin 2t \int_0^t \cos 2\tau \sin 2\tau \, d\tau - \cos 2t \int_0^t \sin^2 2\tau \, d\tau \right) \\ &= 6 \sin 2t + 24t \sin 2t - 8t \cos 2t \end{aligned}$$

Similarly,

$$\begin{aligned}\frac{s^3 + 14s^2 - 28s - 72}{(s^2 + 4)^2} &= \frac{As + B}{s^2 + 4} + \frac{Cs + D}{(s^2 + 4)^2} \\ &= \frac{(As + B)(s^2 + 4) + Cs + D}{(s^2 + 4)^2} \\ &= \frac{As^3 + Bs^2 + (4A + C)s + (4B + D)}{(s^2 + 4)^2}.\end{aligned}$$

Equating numerators on each side of the equation according to the powers of s , we have $A = 1$, $B = 14$, $C = -32$, and $D = -128$. Therefore,

$$\begin{aligned}X_2(s) &= \frac{s^3 + 14s^2 - 28s - 72}{(s^2 + 4)^2} = \frac{s + 14}{s^2 + 4} - \frac{32(s + 4)}{(s^2 + 4)^2} \\ &= \frac{s}{s^2 + 4} + \frac{14}{s^2 + 4} - \frac{32s}{(s^2 + 4)^2} - \frac{128}{(s^2 + 4)^2}.\end{aligned}$$

Taking the inverse Laplace transform yields

$$\begin{aligned}x_2(t) &= \cos 2t + 7 \sin 2t - 16(\cos 2t * \sin 2t) - 32(\sin 2t * \sin 2t) \\ &= \cos 2t - \sin 2t - 8t \sin 2t + 16t \cos 2t.\end{aligned}$$

45. Taking the Laplace transform of the given integral equation and using the convolution theorem yields

$$X(s) = \frac{2}{s^2} + \frac{1}{s^2 + 1} X(s).$$

That is,

$$X(s) \left(\frac{s^2}{s^2 + 1} \right) = \frac{2}{s^2},$$

so that

$$X(s) = \frac{2(s^2 + 1)}{s^4} = \frac{2}{s^2} + \frac{2}{s^4}.$$

Taking the inverse Laplace transform yields

$$x(t) = 2t + \frac{1}{3}t^3.$$

46. Taking the Laplace transform of the given integral equation and using the convolution theorem yields

$$X(s) = \frac{4}{s^3} + \frac{1}{s^2} X(s).$$

That is,

$$X(s) \left(\frac{s^2 - 1}{s^2} \right) = \frac{4}{s^3},$$

so that

$$X(s) = \frac{4}{s(s^2 - 1)} = -\frac{4}{s} + \frac{2}{s - 1} + \frac{2}{s + 1}.$$

Taking the inverse Laplace transform yields

$$x(t) = -4 + 2e^t + 2e^{-t}.$$

47. Taking the Laplace transform of the given integral equation and using the convolution theorem yields

$$X(s) = \frac{4}{s^3} + \frac{2}{s^2 + 4}X(s).$$

That is,

$$X(s) \left(\frac{s^2 + 2}{s^2 + 4} \right) = \frac{4}{s^3},$$

so that

$$X(s) = \frac{4(s^2 + 4)}{s^3(s^2 + 2)}.$$

Now we use a partial fractions decomposition:

$$\begin{aligned} \frac{4(s^2 + 4)}{s^3(s^2 + 2)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds + E}{s^2 + 2} \\ &= \frac{As^2(s^2 + 2) + Bs(s^2 + 2) + C(s^2 + 2) + (Ds + E)s^3}{s^3(s^2 + 2)} \\ &= \frac{(A + D)s^4 + (B + E)s^3 + (2A + C)s^2 + 2Bs + 2C}{s^3(s^2 + 2)}. \end{aligned}$$

Equating numerators on both sides according to powers of s , we have

$$A + D = 0, \quad B + E = 0, \quad 2A + C = 4, \quad 2B = 0, \quad 2C = 16.$$

Solving this system, we obtain

$$A = -2, \quad B = 0, \quad C = 8, \quad D = 2, \quad E = 0.$$

Therefore,

$$\frac{4(s^2 + 4)}{s^3(s^2 + 2)} = -\frac{2}{s} + \frac{8}{s^3} + \frac{2s}{s^2 + 2}.$$

Hence,

$$\begin{aligned} x(t) &= L^{-1} \left[-\frac{2}{s} + \frac{8}{s^3} + \frac{2s}{s^2 + 2} \right] \\ &= -2L^{-1} \left[\frac{1}{s} \right] + 8L^{-1} \left[\frac{1}{s^3} \right] + 2L^{-1} \left[\frac{s}{s^2 + 2} \right] \\ &= -2 + 4t^2 + 2 \cos \sqrt{2}t. \end{aligned}$$

48. Taking the Laplace transform of the given integral equation and using the convolution theorem yields

$$X(s) = \frac{3}{s} + 4 \left[\frac{s}{s^2 + 1} X(s) \right].$$

That is,

$$X(s) \left(\frac{s^2 - 4s + 1}{s^2 + 1} \right) = \frac{3}{s},$$

so that

$$X(s) = \frac{3(s^2 + 1)}{s(s^2 - 4s + 1)}.$$

Now we use a partial fractions decomposition:

$$\begin{aligned}\frac{3(s^2 + 1)}{s(s^2 - 4s + 1)} &= \frac{A}{s} + \frac{B}{s - (2 + \sqrt{3})} + \frac{C}{s - (2 - \sqrt{3})} \\ &= \frac{A(s^2 - 4s + 1) + Bs(s - (2 - \sqrt{3})) + Cs(s - (2 + \sqrt{3}))}{s(s^2 - 4s + 1)} \\ &= \frac{(A + B + C)s^2 + (-4A - (2 - \sqrt{3})B - (2 + \sqrt{3})C)s + A}{s(s^2 - 4s + 1)}.\end{aligned}$$

Equating numerators on both sides, we see that $A + B + C = 3$, $4A + (2 - \sqrt{3})B + (2 + \sqrt{3})C = 0$, and $A = 3$. From this, we deduce that $B + C = 0$, so we have $12 + 2\sqrt{3}C = 0$. Hence, $C = -\frac{6}{\sqrt{3}}$, and therefore, $B = \frac{6}{\sqrt{3}}$. Hence,

$$\frac{3(s^2 + 1)}{s(s^2 - 4s + 1)} = \frac{3}{s} + \frac{6}{\sqrt{3}(s - (2 + \sqrt{3}))} - \frac{6}{\sqrt{3}(s - (2 - \sqrt{3}))}.$$

Taking the inverse Laplace transform of each side, we obtain

$$\begin{aligned}x(t) &= L^{-1} \left[\frac{3}{s} \right] + \frac{6}{\sqrt{3}} L^{-1} \left[\frac{1}{s - (2 + \sqrt{3})} \right] - \frac{6}{\sqrt{3}} L^{-1} \left[\frac{1}{s - (2 - \sqrt{3})} \right] \\ &= 3 + \frac{6}{\sqrt{3}} e^{(2 + \sqrt{3})t} - \frac{6}{\sqrt{3}} e^{(2 - \sqrt{3})t}.\end{aligned}$$

Solutions to Section 9.1

True-False Review:

1. TRUE. Theorem 9.1.7 addresses this for a rational function $f(x) = \frac{p(x)}{q(x)}$. The radius of convergence of the power series representation of f around the point x_0 is the distance from x_0 to the nearest root of $q(x)$.

2. TRUE. We have

$$\sum_{n=0}^{\infty} (a_n + b_n)x^n = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n,$$

and if each of the latter sums converges at x_1 , say to f and g respectively, then

$$\sum_{n=0}^{\infty} (a_n + b_n)x^n = f + g.$$

3. FALSE. It could be the case, for example, that $b_n = -a_n$ for all $n = 0, 1, 2, \dots$. Then

$$\sum_{n=0}^{\infty} (a_n + b_n)x^n = 0,$$

but the individual summations need not converge.

4. FALSE. This is not necessarily the case, as Example 9.1.3 illustrates. In that example, the radius of convergence is $R = 3$, but the series diverges at the endpoints $x = \pm 3$. The endpoints must be considered separately in general.

5. TRUE. This is part of the statement in Theorem 9.1.6.

6. FALSE. The Taylor series expansion of an infinitely differentiable function f need not converge for all real values of x in the domain. It is only guaranteed to converge for x in the domain that lie within the interval of convergence.

7. TRUE. A polynomial is a rational function $p(x)/q(x)$ where $q(x) = 1$. Since $q(x) = 1$ has no roots, Theorem 9.1.7 guarantees that the power series representation of $p(x)/q(x)$ about any point x_0 has an infinite radius of convergence.

8. FALSE. For example, $f = 0$ has a power series representation centered at $x = 0$ of radius $R_1 = \infty$, so if we choose g to be any function with a power series representation centered at $x = 0$ of radius $R_2 < \infty$, then $fg = 0$ has a power series representation centered at $x = 0$ with radius $R = \infty \neq \min\{R_1, R_2\}$.

9. FALSE. According to the algebra of power series, the coefficient c_n of x^n is

$$c_n = \sum_{k=0}^n a_k b_{n-k}.$$

10. TRUE. This is the definition of a Maclaurin series.

Problems:

1. Using the ratio test, we have $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{2^{2n+2}} \frac{2^{2n}}{1} \right| = \frac{1}{4}$. Therefore, $R = 4$.

2. Using the ratio test, we have $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^2} \cdot n^2 \right| = 1$. Therefore, $R = 1$.

3. Using the ratio test, we have $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2^{n+1}}{n+1} \frac{n}{2^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2n}{n+1} \right| = 2$. Therefore, $R = \frac{1}{2}$.

4. Using the ratio test, we have $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| (n+1)! \frac{1}{n!} \right| = \lim_{n \rightarrow \infty} |n+1| = \infty$. Therefore, $R = 0$.

5. Using the ratio test, we have $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{5^{n+1}}{(n+1)!} \frac{n!}{5^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{5}{n+1} \right| = 0$. Therefore, $R = \infty$.

6. The zero of $q(x) = x + 2$ is -2 . The distance from $x_0 = 0$ to the nearest zero of $q(x)$ is 2; hence, $R = 2$.

7. The zero of $q(x) = x^2 + 1$ are $\pm i$. The distance from $x_0 = 1$ to the nearest zero of $q(x)$ is 1; hence, $R = 1$.

8. The zero of $q(x) = x^2 + 16 = 0$ are $\pm 4i$. The distance from $x_0 = 1$ to the nearest zero of $q(x)$ is $\sqrt{1^2 + 4^2} = \sqrt{17}$; hence, $R = \sqrt{17}$.

9. The zero of $q(x) = x^2 - 2x + 5 = 0$ are $1 \pm 2i$. The distance from $x_0 = 0$ to the nearest zero of $q(x)$ is $\sqrt{1^2 + 2^2} = \sqrt{5}$; hence, $R = \sqrt{5}$.

10. The set of zeros of $q(x) = (x^2 + 4x + 13)(x - 3)$ is $\{3, -2 + 3i, -2 - 3i\}$. The distance from $x_0 = -1$ to $x = 3$ is 4, and the distance from x_0 to $x = -2 \pm 3i$ is $\sqrt{(-2 - [-1])^2 + 3^2} = \sqrt{10}$. Thus, $R = \sqrt{10}$.

11. (a) By Theorem 9.1.6, $f(x) = \frac{1}{x^2 - 1}$ is analytic for all $x \in \mathbb{R}$ such that $x^2 - 1 \neq 0$. Therefore, $f(x)$ is analytic for all x such that $x \neq \pm 1$.

(b) The zeros of $q(x) = x^2 - 1$ are $x = \pm 1$. For $0 \leq x_0 < 1$, $R = 1 - x_0 = 1 - |x_0|$, and for $-1 < x_0 < 0$, $R = |-1 - x_0| = x_0 + 1 = 1 - |x_0|$. For $|x_0| > 1$, $R = |x_0| - 1$. Thus, $R = \begin{cases} 1 - |x_0|, & \text{if } |x_0| < 1, \\ |x_0| - 1, & \text{if } |x_0| > 1. \end{cases}$

12. Replacing n in the first series by $n + 2$ and in the second series by $n + 1$ we obtain:

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-1)a_{n-1}x^{n-2} + \sum_{n=1}^{\infty} na_nx^{n-1} &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n \\ &= \sum_{n=0}^{\infty} a_{n+1}(n+1)[(n+2)+1]x^n = \sum_{n=0}^{\infty} a_{n+1}(n+1)(n+3)x^n. \end{aligned}$$

13. The given equation for determining the coefficients can be written in the equivalent form

$$\sum_{n=1}^{\infty} [n(n+2)a_n + (n-3)a_{n-1}]x^n = 0.$$

Consequently,

$$n(n+2)a_n + (n-3)a_{n-1} = 0, \quad n = 1, 2, 3, \dots$$

so that

$$a_n = \frac{3-n}{n(n+2)}a_{n-1}, \quad n = 1, 2, 3, \dots$$

Substituting for successive values of n , we have:

$$\begin{aligned} n = 1 : \quad & a_1 = \frac{2}{3}a_0, \\ n = 2 : \quad & a_2 = \frac{1}{8}a_1 = \frac{1}{12}a_0, \\ n = 3, 4, \dots : \quad & a_n = 0. \end{aligned}$$

Thus,

$$f(x) = a_0 + \frac{2}{3}a_0x + \frac{1}{12}a_0x^2 = a_0 \left(1 + \frac{2}{3}x + \frac{1}{12}x^2 \right).$$

14. The given equation for determining the coefficients can be written in the equivalent form

$$\sum_{n=0}^{\infty} [(n+2)a_{n+1} - a_n]x^n = 0.$$

Consequently,

$$(n+2)a_{n+1} - a_n = 0, \quad n = 0, 1, 2, \dots$$

so that

$$a_{n+1} = \frac{1}{n+2}a_n, \quad n = 0, 1, 2, \dots$$

Substituting for successive values of n , we have:

$$\begin{aligned} n = 0 : \quad a_1 &= \frac{1}{2}a_0, \\ n = 1 : \quad a_2 &= \frac{1}{3}a_1 = \frac{1}{3!}a_0, \\ n = 2 : \quad a_3 &= \frac{1}{4}a_2 = \frac{1}{4!}a_0. \end{aligned}$$

Continuing in this manner we find:

$$a_n = \frac{1}{(n+1)!}a_0.$$

Thus,

$$\begin{aligned} f(x) &= a_0 \sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^n = \frac{a_0}{x} \sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^{n+1} = \frac{a_0}{x} \sum_{k=1}^{\infty} \frac{1}{k!} x^k \\ &= \frac{a_0}{x} \left(\sum_{k=0}^{\infty} \frac{1}{k!} x^k - 1 \right) = \frac{a_0}{x} (e^x - 1). \end{aligned}$$

15. The given equation for determining the coefficients can be written in the equivalent form

$$\sum_{n=1}^{\infty} [(n+1)(n+2)a_{n+1} - na_{n-1}]x^n = 0.$$

Consequently,

$$(n+1)(n+2)a_{n+1} - na_{n-1} = 0, \quad n = 1, 2, \dots$$

so that

$$a_{n+1} = \frac{n}{(n+1)(n+2)}a_{n-1}, \quad n = 1, 2, \dots$$

For odd n we have:

$$\begin{aligned} n = 1 : \quad a_2 &= \frac{1}{3!}a_0, \\ n = 3 : \quad a_4 &= \frac{3}{5 \cdot 4}a_2 = \frac{3}{5!}a_0, \\ n = 5 : \quad a_6 &= \frac{5}{7 \cdot 6}a_4 = \frac{5 \cdot 3}{7!}a_0. \end{aligned}$$

Continuing in this manner we find:

$$a_{2k} = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{(2k+1)!}a_0, \quad k = 1, 2, \dots$$

For even n we have

$$\begin{aligned} n = 2 : \quad a_3 &= \frac{2}{4 \cdot 3}a_1 = \frac{2^2}{4!}a_1, \\ n = 4 : \quad a_5 &= \frac{4}{6 \cdot 5}a_3 = \frac{2^3 \cdot 2!}{6!}a_1, \\ n = 6 : \quad a_7 &= \frac{6}{8 \cdot 7}a_5 = \frac{2^4 \cdot 3!}{8!}a_1. \end{aligned}$$

Continuing in this manner we find:

$$a_{2k+1} = \frac{2^{k+1}k!}{(2k+2)!}a_1, \quad k = 1, 2, \dots$$

Solutions to Section 9.2

True-False Review:

1. **TRUE.** Since a polynomial is analytic at every x_0 in $\mathbb{R}$, it follows from Definition 9.2.1 that every point of $\mathbb{R}$ is an ordinary point.
2. **FALSE.** According to Theorem 9.2.4 with $x_0 = -3$ and $R = 2$, the radius of convergence of the power series representation of the solution to this differential equation is *at least* 2.
3. **FALSE.** The nearest singularity to $x = 2$ occurs in p with root $x = 1$. Hence, $R = 1$ is the radius of convergence of $p(x) = \frac{1}{x^2-1}$ about $x = 2$. Therefore, by Theorem 9.2.4, the radius of convergence of a power series solution to this differential equation is at least 1 (not at least 2).
4. **TRUE.** This is seen directly by computing $y(x_0)$ and $y'(x_0)$ for the series solution

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

as explained in the statement of Theorem 9.2.4.

5. **TRUE.** The radii of convergence of the power series expansions of p and q are both positive, and thus, Theorem 9.2.4 implies that the general solution to the differential equation can be represented as a power series with a positive radius of convergence.
6. **FALSE.** The two linearly independent solutions to $y'' + (2 - 4x^2)y' - 8xy = 0$ found in Example 9.2.7 both contain common powers of x .
7. **FALSE.** For example, the power series $y_1(x)$ found in Example 9.2.6 is a solution to (9.2.10), but $xy_1(x)$ is not a solution.
8. **FALSE.** The value of a_0 only determines the values $a_0, a_2, a_4, \dots$, but there is no information about the terms $a_1, a_3, a_5, \dots$ unless a_1 is also specified.
9. **TRUE.** The value of a_k depends on a_{k-1} and a_{k-3} , so once a_0, a_1 , and a_2 are specified, we can find $a_3, a_4, a_5, \dots$ uniquely from the recurrence relation.
10. **FALSE.** If the recurrence relation cannot be solved, it simply means that lengthy brute force calculations may be required to determine the solution, not that the solution does not exist.

Problems:

1. The point $x_0 = 0$ is an ordinary point. Both $p(x) = 0$ and $q(x) = -1$ are polynomials, and their power series expansions about 0 are valid for all $x \in \mathbb{R}$. Therefore, by Theorem 9.2.4, we try for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $y'' - y = 0$ requires

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=0}^{\infty} a_k x^k = 0,$$

or equivalently,

$$\sum_{k=0}^{\infty} [(k+2)(k+1) a_{k+2} - a_k] x^k = 0.$$

Therefore, the coefficients must satisfy

$$(k+2)(k+1) a_{k+2} - a_k = 0, \quad k = 0, 1, \dots$$

which gives the recurrence relation

$$a_{k+2} = \frac{1}{(k+2)(k+1)} a_k \quad k = 0, 1, \dots$$

Even k :

$$\begin{aligned} k = 0 : \quad a_2 &= \frac{1}{2} a_0, \\ k = 2 : \quad a_4 &= \frac{1}{4 \cdot 3} a_2 = \frac{1}{4!} a_0, \\ k = 4 : \quad a_6 &= \frac{1}{6 \cdot 5} a_4 = \frac{1}{6!} a_0, \end{aligned}$$

and in general,

$$a_{2n} = \frac{1}{(2n)!} a_0.$$

Odd k :

$$\begin{aligned} k = 1 : \quad a_3 &= \frac{1}{3!} a_1, \\ k = 3 : \quad a_5 &= \frac{1}{5 \cdot 4} a_3 = \frac{1}{5!} a_1, \\ k = 5 : \quad a_7 &= \frac{1}{7 \cdot 6} a_5 = \frac{1}{7!} a_1, \end{aligned}$$

and in general,

$$a_{2n+1} = \frac{1}{(2n+1)!} a_1, \quad n = 1, 2, \dots$$

Thus,

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}.$$

Two linearly independent solutions to the differential equation are

$$y_1(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}.$$

2. The point $x_0 = 0$ is an ordinary point. Both $p(x) = -2x$ and $q(x) = -2$ are polynomials, and their power series expansions about 0 are valid for all $x \in \mathbb{R}$. Therefore, by Theorem 9.2.4 we try for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $y'' - 2xy' - 2y = 0$ requires

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2 \sum_{n=1}^{\infty} n a_n x^n - 2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - 2 \sum_{k=1}^{\infty} k a_k x^k - 2 \sum_{k=0}^{\infty} a_k x^k = 0,$$

or equivalently,

$$\sum_{k=0}^{\infty} [(k+2)(k+1) a_{k+2} - 2(k+1) a_k] x^k = 0.$$

Therefore, the coefficients must satisfy

$$(k+2)(k+1) a_{k+2} - 2(k+1) a_k = 0, \quad k = 0, 1, \dots$$

which gives the recurrence relation

$$a_{k+2} = \frac{2}{k+2} a_k \quad k = 0, 1, \dots$$

Even k :

$$k = 0: \quad a_2 = a_0,$$

$$k = 2: \quad a_4 = \frac{1}{2} a_2 = \frac{1}{2} a_0,$$

$$k = 4: \quad a_6 = \frac{1}{3} a_4 = \frac{1}{2 \cdot 3} a_0,$$

and in general,

$$a_{2n} = \frac{1}{n!} a_0, \quad n = 1, 2, \dots$$

Odd k :

$$\begin{aligned} k = 1 : \quad a_3 &= \frac{2}{3}a_1, \\ k = 3 : \quad a_5 &= \frac{2}{5}a_3 = \frac{2^2}{1 \cdot 3 \cdot 5}a_1, \\ k = 5 : \quad a_7 &= \frac{2}{7}a_5 = \frac{2^3}{1 \cdot 3 \cdot 5 \cdot 7}a_1, \end{aligned}$$

and in general,

$$a_{2n+1} = \frac{2^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} a_1, \quad n = 1, 2, \dots$$

Thus,

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{2^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} x^{2n+1}.$$

Two linearly independent solutions to the differential equation are

$$y_1(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n} \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} \frac{2^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)} x^{2n+1}.$$

3. The point $x_0 = 0$ is an ordinary point. Both $p(x) = 2x$ and $q(x) = 4$ are polynomials, and their power series expansions about 0 are valid for all $x \in \mathbb{R}$. Therefore, by Theorem 9.2.4 we try for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $y'' + 2xy' + 4y = 0$ requires

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=1}^{\infty} n a_n x^n + 4 \sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + 2 \sum_{k=1}^{\infty} k a_k x^k + 4 \sum_{k=0}^{\infty} a_k x^k = 0,$$

or equivalently,

$$\sum_{k=0}^{\infty} [(k+2)(k+1) a_{k+2} + 2k a_k + 4a_k] x^k = 0.$$

Therefore, the coefficients must satisfy

$$(k+2)(k+1) a_{k+2} + 2k a_k + 4a_k = 0, \quad k = 0, 1, \dots$$

which gives the recurrence relation

$$a_{k+2} = -\frac{2}{k+1} a_k, \quad k = 0, 1, \dots$$

Even k :

$$\begin{aligned} k=0: \quad a_2 &= -\frac{2}{1} a_0, \\ k=2: \quad a_4 &= -\frac{2}{3} a_2 = \frac{4}{3} a_0, \\ k=4: \quad a_6 &= -\frac{2}{5} a_4 = -\frac{8}{15} a_0, \end{aligned}$$

and in general,

$$a_{2n} = \frac{(-2)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} a_0, \quad n = 1, 2, \dots$$

Odd k :

$$\begin{aligned} k=1: \quad a_3 &= -a_1, \\ k=3: \quad a_5 &= -\frac{2}{4} a_3 = \frac{1}{2} a_1, \\ k=5: \quad a_7 &= -\frac{2}{6} a_5 = -\frac{1}{6} a_1, \end{aligned}$$

and in general,

$$a_{2n+1} = \frac{(-1)^n}{n!} a_1, \quad n = 1, 2, \dots$$

Thus,

$$y(x) = a_0 \left[1 + \sum_{n=1}^{\infty} \frac{(-2)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} x^{2n} \right] + a_1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+1} \right].$$

Two linearly independent solutions to the differential equation are

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-2)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)} x^{2n} \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+1}.$$

4. The point x_0 is an ordinary point. Both $p(x) = 0$ and $q(x) = x$ are polynomials, and their power series expansions about 0 are valid for all $x \in \mathbb{R}$. Therefore, by Theorem 9.2.4 we try for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $y'' + xy = 0$ requires

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

That is,

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=1}^{\infty} a_{k-1} x^k = 0,$$

or equivalently,

$$2a_2 + \sum_{k=1}^{\infty} [(k+2)(k+1)a_{k+2} + a_{k-1}]x^k = 0.$$

Therefore the coefficients must satisfy

$$a_2 = 0 \quad \text{and} \quad (k+2)(k+1)a_{k+2} + a_{k-1} = 0, \quad k = 1, 2, \dots$$

That is,

$$a_2 = 0 \quad \text{and} \quad a_{k+2} = -\frac{1}{(k+1)(k+2)} a_{k-1}, \quad k = 1, 2, \dots$$

It follows that, for $n = 1, 2, \dots$,

$$a_{3n} = \frac{4 \cdot 7 \cdots (3n-2)(-1)^n}{(3n)!} a_0, \quad a_{3n+1} = \frac{2 \cdot 5 \cdots (3n-1)(-1)^n}{(3n+1)!} a_1, \quad \text{and} \quad a_{3n+2} = 0.$$

Consequently,

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{n=0}^{\infty} a_{3n} x^{3n} + \sum_{n=0}^{\infty} a_{3n+1} x^{3n+1} + \sum_{n=0}^{\infty} a_{3n+2} x^{3n+2} \\ &= a_0 \left[1 + \sum_{n=1}^{\infty} \frac{4 \cdot 7 \cdots (3n-2)(-1)^n}{(3n)!} x^{3n} \right] + a_1 \left[1 + \sum_{n=1}^{\infty} \frac{2 \cdot 5 \cdots (3n-1)(-1)^n}{(3n+1)!} x^{3n+1} \right]. \end{aligned}$$

Thus,

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{4 \cdot 7 \cdots (3n-2)(-1)^n}{(3n)!} x^{3n}$$

and

$$y_2(x) = 1 + \sum_{n=1}^{\infty} \frac{2 \cdot 5 \cdots (3n-1)(-1)^n}{(3n+1)!} x^{3n+1},$$

where $R = \infty$.

5. The point x_0 is an ordinary point. Both $p(x) = -x^2$ and $q(x) = -2x$ are polynomials, and their power series expansions about 0 are valid for all $x \in \mathbb{R}$. Therefore, by Theorem 9.2.4 we try for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $y'' - x^2 y' - 2xy = 0$ requires

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^{n+1} - \sum_{n=0}^{\infty} 2 a_n x^{n+1} = 0.$$

That is,

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k - \sum_{k=2}^{\infty} (k-1)a_{k-1}x^k - \sum_{k=1}^{\infty} 2a_{k-1}x^k = 0,$$

or equivalently,

$$2a_2 + \sum_{k=1}^{\infty} [(k+2)(k+1)a_{k+2} - (k+1)a_{k-1}]x^k = 0.$$

Therefore the coefficients must satisfy

$$a_2 = 0 \quad \text{and} \quad (k+2)(k+1)a_{k+2} - (k+1)a_{k-1} = 0, \quad k = 1, 2, \dots$$

That is,

$$a_2 = 0 \quad \text{and} \quad a_{k+2} = \frac{1}{k+2}a_{k-1}, \quad k = 1, 2, \dots$$

It follows that, for $n = 1, 2, \dots$,

$$a_{3n} = \frac{1}{3 \cdot 6 \cdots (3n)} a_0 = \frac{1}{3^n n!} a_0, \quad a_{3n+1} = \frac{1}{4 \cdot 7 \cdots (3n+1)} a_1, \quad \text{and} \quad a_{3n+2} = 0.$$

Consequently,

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{n=0}^{\infty} a_{3n} x^{3n} + \sum_{n=0}^{\infty} a_{3n+1} x^{3n+1} + \sum_{n=0}^{\infty} a_{3n+2} x^{3n+2} \\ &= a_0 \sum_{n=1}^{\infty} \frac{1}{3^n n!} x^{3n} + a_1 \sum_{n=0}^{\infty} \frac{1}{1 \cdot 4 \cdots (3n+1)} x^{3n+1}. \end{aligned}$$

Thus,

$$y_1(x) = \sum_{n=1}^{\infty} \frac{1}{3^n n!} x^{3n} \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} \frac{1}{1 \cdot 4 \cdots (3n+1)} x^{3n+1},$$

where $R = \infty$.

6. The point $x_0 = 0$ is an ordinary point. Both $p(x) = -x^2$ and $q(x) = -3x$ are polynomials, and their power series expansions about 0 are valid for all $x \in \mathbb{R}$. Therefore, by Theorem 9.2.4 we try for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $y'' - x^2 y' - 3xy = 0$ requires

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^{n+1} - \sum_{n=0}^{\infty} 3 a_n x^{n+1} = 0.$$

That is,

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k - 3a_0x - \sum_{k=2}^{\infty} [(k-1)a_{k-1} + 3a_{k-1}]x^k = 0,$$

or equivalently,

$$2a_2 + [6a_3 - 3a_0]x + \sum_{k=2}^{\infty} [(k+2)(k+1)a_{k+2} - (k-1)a_{k-1} - 3a_{k-1}]x^k = 0.$$

Therefore the coefficients must satisfy:

$$a_2 = 0, \quad 6a_3 - 3a_0 = 0, \quad \text{and} \quad (k+2)(k+1)a_{k+2} - (k-1)a_{k-1} = 0, \quad k = 2, 3, \dots$$

so that

$$a_2 = 0, \quad a_3 = \frac{1}{2}a_0, \quad \text{and} \quad a_{k+2} = \frac{1}{k+1}a_{k-1}, \quad k = 2, 3, \dots$$

Consequently,

$$a_{3n} = \frac{1}{2 \cdot 5 \cdots (3n-1)} a_0, \quad a_{3n+1} = \frac{1}{3 \cdot 6 \cdots (3n)} a_1, \quad n = 1, 2, \dots,$$

and

$$a_{3n+2} = 0, \quad n = 0, 1, 2, \dots$$

Therefore,

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} a_k x^k \\ &= \sum_{n=0}^{\infty} a_{3n} x^{3n} + \sum_{n=0}^{\infty} a_{3n+1} x^{3n+1} + \sum_{n=0}^{\infty} a_{3n+2} x^{3n+2} \\ &= a_0 \left[1 + \sum_{n=1}^{\infty} \frac{1}{2 \cdot 5 \cdots (3n-1)} x^{3n} \right] + a_1 \left[x + \sum_{n=1}^{\infty} \frac{1}{3 \cdot 6 \cdots (3n)} x^{3n+1} \right]. \end{aligned}$$

Consequently,

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{2 \cdot 5 \cdots (3n-1)} x^{3n} \quad \text{and} \quad y_2(x) = x + \sum_{n=1}^{\infty} \frac{1}{3 \cdot 6 \cdots (3n)} x^{3n+1},$$

where $R = \infty$.

7. The point $x_0 = 0$ is an ordinary point. Both $p(x) = x$ and $q(x) = 3$ are polynomials, and their power series expansions about 0 are valid for all $x \in \mathbb{R}$. Therefore, by Theorem 9.2.4 we try for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $y'' + xy' + 3y = 0$ requires

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + 3 \sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k + \sum_{k=1}^{\infty} k a_k x^k + 3 \sum_{k=0}^{\infty} a_k x^k = 0,$$

or equivalently,

$$\sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2} + (k+3)a_k]x^k = 0.$$

Therefore the coefficients must satisfy:

$$(k+2)(k+1)a_{k+2} + (k+3)a_k = 0, \quad k = 1, 2, \dots,$$

That is,

$$a_{k+2} = -\frac{(k+3)}{(k+2)(k+1)}a_k, \quad k = 0, 1, \dots$$

Even k :

$$k = 0: \quad a_2 = -\frac{3}{2!}a_0,$$

$$k = 2: \quad a_4 = -\frac{5}{4 \cdot 3}a_2 = \frac{5 \cdot 3}{4!}a_0,$$

$$k = 4: \quad a_6 = -\frac{7}{6 \cdot 5}a_4 = -\frac{7 \cdot 5 \cdot 3}{6!}a_0,$$

and in general,

$$a_{2n} = \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)!} a_0 \quad n = 1, 2, \dots$$

Odd k :

$$k = 1: \quad a_3 = -\frac{4}{3 \cdot 2}a_1,$$

$$k = 3: \quad a_5 = -\frac{6}{5 \cdot 4}a_3 = \frac{4 \cdot 6}{5!}a_1,$$

$$k = 5: \quad a_7 = -\frac{8}{7 \cdot 6}a_5 = -\frac{4 \cdot 6 \cdot 8}{7!}a_1,$$

and in general,

$$a_{2n+1} = \frac{(-2)^n (n+1)!}{(2n+1)!} a_1, \quad n = 1, 2, \dots$$

Therefore,

$$y(x) = a_0 \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)!} x^{2n} \right] + a_1 \sum_{n=0}^{\infty} \frac{(-2)^n (n+1)!}{(2n+1)!} x^{2n+1}.$$

Consequently,

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)!} x^{2n} \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} \frac{(-2)^n (n+1)!}{(2n+1)!} x^{2n+1},$$

where $R = \infty$.

8. The point $x_0 = 0$ is an ordinary point. Both $p(x) = 2x^2$ and $q(x) = 2x$ are polynomials, and their power series expansions about 0 are valid for all $x \in \mathbb{R}$. Therefore, by Theorem 9.2.4 we try for a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $y'' + 2x^2 y' + 2xy = 0$ requires

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=1}^{\infty} n a_n x^{n+1} + 2 \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

That is,

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + 2 \sum_{k=1}^{\infty} (k-1) a_{k-1} x^k + 2 \sum_{k=1}^{\infty} a_{k-1} x^k = 0,$$

or equivalently,

$$2a_2 + \sum_{k=1}^{\infty} [(k+2)(k+1) a_{k+2} + 2k a_{k-1}] x^k = 0.$$

Therefore, the coefficients must satisfy:

$$2a_2 = 0 \quad \text{and} \quad (k+2)(k+1) a_{k+2} + 2k a_{k-1} = 0, \quad k = 1, 2, \dots$$

so that

$$a_2 = 0 \quad \text{and} \quad a_{k+2} = -\frac{2k}{(k+2)(k+1)} a_{k-1}, \quad k = 1, 2, \dots$$

We now distinguish three cases:

Case 1: Since $a_2 = 0$ it follows that

$$a_5 = 0, \quad a_8 = 0, \quad a_{11} = 0, \quad \dots$$

and in general,

$$a_{3n+2} = 0.$$

Case 2: $k = 1, 4, 7, \dots$:

$$\begin{aligned} k = 1: \quad a_3 &= -\frac{2}{3!} a_0, \\ k = 4: \quad a_6 &= -\frac{8}{6 \cdot 5} a_3 = -\frac{2^2 \cdot 4^2}{6!} a_0, \\ k = 7: \quad a_9 &= -\frac{14}{9 \cdot 8} a_6 = -\frac{2^3 \cdot 4^2 \cdot 7^2}{9!} a_0, \end{aligned}$$

and in general,

$$a_{3n} = \frac{(-1)^n 2^n \cdot 4^2 \cdot 7^2 \cdots (3n-2)^2}{(3n)!} a_0, \quad n = 1, 2, \dots$$

Case 3: $k = 2, 5, 8, \dots$:

$$\begin{aligned} k = 2: \quad a_4 &= -\frac{2^3}{4!}a_1, \\ k = 5: \quad a_7 &= \frac{2^4 \cdot 5^2}{7!}a_1, \\ k = 8: \quad a_{10} &= -\frac{2^5 \cdot 5^2 \cdot 8^2}{10!}a_1, \end{aligned}$$

and in general,

$$a_{3n+1} = \frac{(-1)^n 2^n \cdot 2^2 \cdot 5^2 \cdot 8^2 \cdots (3n-1)^2}{(3n+1)!} a_1, \quad n = 1, 2, \dots$$

Thus,

$$\begin{aligned} y(x) = a_0 &\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n \cdot 4^2 \cdot 7^2 \cdots (3n-2)^2 a_0}{(3n)!} x^{3n} \right] \\ &+ a_1 \left[x + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n \cdot 2^2 \cdot 5^2 \cdot 8^2 \cdots (3n-1)^2 a_1}{(3n+1)!} x^{3n+1} \right], \end{aligned}$$

so that two linearly independent power series solutions centered at $x = 0$ to the differential equation are given by

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n \cdot 4^2 \cdot 7^2 \cdots (3n-2)^2 a_0}{(3n)!} x^{3n},$$

and

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n \cdot 2^2 \cdot 5^2 \cdot 8^2 \cdots (3n-1)^2 a_1}{(3n+1)!} x^{3n+1},$$

where $R = \infty$.

9. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $(1+x^2)y'' + 4xy' + 2y = 0$ requires

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0.$$

That is,

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=2}^{\infty} k(k-1) a_k x^k + \sum_{k=1}^{\infty} 4k a_k x^k + \sum_{k=0}^{\infty} 2a_k x^k = 0,$$

or equivalently,

$$\sum_{k=0}^{\infty} [(k+2)(k+1) a_{k+2} + (k+2)(k+1) a_k] x^k = 0.$$

Therefore the coefficients must satisfy

$$a_{k+2} = -a_k, \quad k = 0, 1, \dots$$

Even k :

$$k = 0 : \quad a_2 = -a_0,$$

$$k = 2 : \quad a_4 = -a_2 = a_0,$$

$$k = 4 : \quad a_6 = -a_4 = -a_0,$$

and in general,

$$a_{2n} = (-1)^n a_0, \quad n = 1, 2, \dots$$

Odd k :

$$k = 1 : \quad a_3 = -a_1,$$

$$k = 3 : \quad a_5 = -a_3 = a_1,$$

$$k = 5 : \quad a_7 = -a_5 = -a_1,$$

and in general,

$$a_{2n+1} = (-1)^n a_1, \quad n = 1, 2, \dots$$

Therefore,

$$y(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n} + \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} = a_0 \sum_{n=0}^{\infty} (-1)^n x^{2n} + a_1 \sum_{n=0}^{\infty} (-1)^n x^{2n+1}.$$

Consequently, two linearly independent power series solutions centered at $x = 0$ to the differential equation are

$$y_1(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n+1}.$$

Since $p(x) = \frac{4x}{1+x^2}$ and $q(x) = \frac{2}{1+x^2}$ we have by Theorem 9.2.4 that $R = 1$ (lower bound).

10. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $(x^2 - 3)y'' - 3xy' - 5y = 0$ requires

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n - 3 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 3 \sum_{n=1}^{\infty} n a_n x^n - 5 \sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$\sum_{k=0}^{\infty} [k(k-1)a_k - 3(k+2)(k+1)a_{k+2} - 3ka_k - 5a_k]x^k = 0$$

so that

$$(k+1)[(k-5)a_k - 3(k+2)a_{k+2}] = 0, \quad k = 0, 1, \dots$$

Therefore the coefficients must satisfy

$$a_{k+2} = \frac{k-5}{3(k+2)}a_k, \quad k = 0, 1, \dots$$

Thus,

$$a_{2n} = \frac{(-5)(-3)(-1)\cdots(2n-7)}{3^n(2 \cdot 4 \cdot 6 \cdots 2n)}a_0, \quad n = 1, 2, \dots,$$

whereas

$$a_3 = -\frac{4}{9}a_1, \quad a_5 = -\frac{2}{15}a_3 = \frac{8}{135}a_1, \quad a_{2k+1} = 0, \quad k = 3, 4, \dots$$

Hence, two linearly independent power series solutions to the given differential equation centered at $x = 0$ are

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-5)(-3)(-1)\cdots(2n-7)}{3^n(2 \cdot 4 \cdot 6 \cdots 2n)}x^{2n} \quad \text{and} \quad y_2(x) = x - \frac{4}{9}x^3 + \frac{8}{135}x^5.$$

Since $p(x) = -\frac{3x}{x^2-3}$ and $q(x) = -\frac{5}{x^2-3}$ we have by Theorem 9.2.4 that $R = \sqrt{3}$ (lower bound).

11. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $(x^2 - 1)y'' - 6xy' + 12y = 0$ requires

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 6 \sum_{n=1}^{\infty} n a_n x^n + 12 \sum_{n=0}^{\infty} a_n x^n = 0,$$

or equivalently,

$$\sum_{k=0}^{\infty} k(k-1) a_k x^k - \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - 6 \sum_{k=0}^{\infty} k a_k x^k + 12 \sum_{k=0}^{\infty} a_k x^k = 0.$$

Therefore the coefficients must satisfy

$$(k-3)(k-4)a_k - a_{k+2}(k+2)(k+1) = 0, \quad k = 0, 1, \dots,$$

so that

$$a_{k+2} = \frac{(k-3)(k-4)}{(k+2)(k+1)}a_k, \quad k = 0, 1, \dots$$

From the preceding recurrence relation we see that the even coefficients satisfy

$$a_2 = 6a_0, \quad a_4 = a_0, \quad a_{2k} = 0, \quad k = 3, 4, \dots$$

whereas the odd coefficients satisfy

$$a_3 = a_1, \quad a_{2k+1} = 0, \quad k = 2, 3, \dots$$

Hence,

$$y(x) = a_0(1 + 6x^2 + x^4) + a_1(x + x^3)$$

so that two linearly independent power series solutions to the given differential equation centered at $x = 0$ are

$$y_1(x) = x^4 + 6x^2 + 1 \quad \text{and} \quad y_2(x) = x^3 + x.$$

Moreover, $R = \infty$.

12. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $(1 - 4x^2)y'' - 20xy' - 16y = 0$ requires

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 4 \sum_{n=2}^{\infty} n(n-1) a_n x^n - 20 \sum_{n=1}^{\infty} n a_n x^n - 16 \sum_{n=0}^{\infty} a_n x^n = 0,$$

or equivalently,

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - 4 \sum_{k=0}^{\infty} k(k-1) a_k x^k - 20 \sum_{k=0}^{\infty} k a_k x^k - 16 \sum_{k=0}^{\infty} a_k x^k = 0.$$

Therefore the coefficients must satisfy

$$(k+2)(k+1) a_{k+2} - 4(k+2)^2 a_k = 0, \quad k = 0, 1, \dots,$$

so that

$$a_{k+2} = \frac{4(k+2)}{k+1} a_k, \quad k = 0, 1, \dots$$

It follows from the preceding relation that

$$a_{2n} = \frac{4^n (2 \cdot 4 \cdot 6 \cdots 2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} a_0, \quad n = 1, 2, \dots,$$

and

$$a_{2n+1} = \frac{4^n (1 \cdot 3 \cdot 5 \cdots (2n-1))}{2 \cdot 4 \cdot 6 \cdots 2n} a_1, \quad n = 1, 2, \dots$$

Thus,

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{4^n (2 \cdot 4 \cdot 6 \cdots 2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} x^{2n} \quad \text{and} \quad y_2(x) = x + \sum_{n=1}^{\infty} \frac{4^n (1 \cdot 3 \cdot 5 \cdots (2n-1))}{2 \cdot 4 \cdot 6 \cdots 2n} x^{2n+1}.$$

Since $p(x) = -\frac{20x}{1-4x^2}$ and $q(x) = -\frac{16}{1-4x^2}$ we have by Theorem 9.2.4 that $R = \frac{1}{2}$ (lower bound).

13. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $y'' + xy' + (2+x)y = 0$ requires

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2 a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0,$$

or equivalently,

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} 2 a_k x^k + \sum_{k=1}^{\infty} a_{k-1} x^k = 0,$$

which can be written as

$$2a_2 + 2a_0 + \sum_{k=1}^{\infty} [(k+2)(k+1) a_{k+2} + (k+2) a_k + a_{k-1}] x^k = 0.$$

Therefore the coefficients must satisfy

$$2a_2 + 2a_0 = 0$$

and

$$(k+2)(k+1) a_{k+2} + (k+2) a_k + a_{k-1} = 0, \quad k = 1, 2, \dots$$

Hence,

$$a_2 = -a_0, \quad \text{and} \quad a_{k+2} = -\frac{(k+2) a_k + a_{k-1}}{(k+2)(k+1)}, \quad k = 1, 2, \dots$$

Substituting $k = 1, 2, 3$ yields

$$a_3 = -\frac{1}{6}(3a_1 + a_0), \quad a_4 = -\frac{1}{12}(-4a_0 + a_1), \quad \text{and} \quad a_5 = \frac{1}{8}a_1 + \frac{11}{120}a_0,$$

respectively. Thus,

$$y_1(x) = 1 - x^2 - \frac{1}{6}x^3 + \frac{1}{3}x^4 + \frac{11}{120}x^5 + \dots \quad \text{and} \quad y_2(x) = x - \frac{1}{2}x^3 - \frac{1}{12}x^4 + \frac{1}{8}x^5 + \dots$$

Since $p(x) = x$ and $q(x) = 2 + x$, $R = \infty$.

14. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $y'' + 2y' + 4xy = 0$ requires

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=1}^{\infty} n a_n x^{n-1} + 4 \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

That is,

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k + 2 \sum_{k=0}^{\infty} (k+1)a_{k+1}x^k + 4 \sum_{k=1}^{\infty} a_{k-1}x^k = 0,$$

or equivalently,

$$2a_2 + 2a_1 + \sum_{k=1}^{\infty} [(k+2)(k+1)a_{k+2} + 2(k+1)a_{k+1} + 4a_{k-1}]x^k = 0.$$

Therefore the coefficients must satisfy

$$2a_2 + 2a_1 = 0$$

and

$$(k+1)(k+2)a_{k+2} + 2(k+1)a_{k+1} + 4a_{k-1} = 0, \quad k = 1, 2, \dots$$

Hence,

$$a_2 = -a_1, \quad \text{and} \quad a_{k+2} = -\frac{4a_{k-1} + 2(k+1)a_{k+1}}{(k+1)(k+2)}, \quad k = 1, 2, \dots$$

Substituting $k = 1, 2, 3$ yields:

$$\begin{aligned} a_3 &= -\frac{2}{3}(a_0 + a_2) = -\frac{2}{3}(a_0 - a_1), \\ a_4 &= -\frac{1}{6}(2a_1 + 3a_3) = \frac{1}{3}(a_0 - 2a_1), \\ a_5 &= -\frac{1}{5}(a_2 + 2a_4) = -\frac{1}{15}(2a_0 - 7a_1), \end{aligned}$$

respectively. Thus,

$$\begin{aligned} y(x) &= a_0 + a_1x - a_1x^2 - \frac{2}{3}(a_0 - a_1)x^3 + \frac{1}{3}(a_0 - 2a_1)x^4 - \frac{1}{15}(2a_0 - 7a_1)x^5 + \dots \\ &= a_0 \left(1 - \frac{2}{3}x^3 + \frac{1}{3}x^4 - \frac{2}{15}x^5 + \dots \right) + a_1 \left(x - x^2 + \frac{2}{3}x^3 - \frac{2}{3}x^4 + \frac{7}{15}x^5 + \dots \right). \end{aligned}$$

Therefore,

$$y_1(x) = 1 - \frac{2}{3}x^3 + \frac{1}{3}x^4 - \frac{2}{15}x^5 + \dots \quad \text{and} \quad y_2(x) = x - x^2 + \frac{2}{3}x^3 - \frac{2}{3}x^4 + \frac{7}{15}x^5 + \dots$$

Since $p(x) = 2$ and $q(x) = 4x$, it follows that $R = \infty$.

15. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $y'' - e^x y = 0$ requires

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) \sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) (a_0 + a_1 x + a_2 x^2 + \cdots) = 0,$$

or equivalently,

$$2a_2 - a_0 + (6a_3 - a_0 - a_1)x + \left(12a_4 - \frac{1}{2}a_0 - a_1 - a_2\right)x^2 + \left(20a_5 - \frac{1}{6}a_0 - \frac{1}{2}a_1 - a_2 - a_3\right)x^3 + \cdots = 0.$$

Therefore the coefficients must satisfy:

$$2a_2 - a_0 = 0, \quad 6a_3 - a_0 - a_1 = 0, \quad 12a_4 - \frac{1}{2}a_0 - a_1 - a_2 = 0, \quad 20a_5 - \frac{1}{6}a_0 - \frac{1}{2}a_1 - a_2 - a_3 = 0,$$

so that

$$a_2 = \frac{1}{2}a_0, \quad a_3 = \frac{1}{6}(a_0 + a_1), \quad a_4 = \frac{1}{12}(a_0 + a_1), \quad a_5 = \frac{1}{120}(5a_0 + 4a_1).$$

Hence,

$$y(x) = a_0 \left(1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{24} + \cdots\right) + a_1 \left(x + \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{30} + \cdots\right),$$

so that

$$y_1(x) = 1 + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{24} + \cdots \quad \text{and} \quad y_2(x) = x + \frac{x^3}{6} + \frac{x^4}{12} + \frac{x^5}{30} + \cdots.$$

Since $p(x) = 0$ and $q(x) = -e^x$, it follows that $R = \infty$.

16. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $y'' + \sin x y' + y = 0$ requires

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sin x \sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0,$$

or equivalently,

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k + \sum_{k=0}^{\infty} a_k x^k = 0.$$

This can be written as

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=0}^{\infty} (k+1) a_{k+1} \left(x^{k+1} - \frac{1}{3!} x^{k+3} + \frac{1}{5!} x^{k+5} - \cdots\right) + \sum_{k=0}^{\infty} a_k x^k = 0.$$

Therefore

$$2a_2 + a_0 + (6a_3 + 2a_1)x + (12a_4 + 3a_2)x^2 + \left(20a_5 + 4a_3 - \frac{1}{3!}a_1\right)x^3 + \cdots = 0.$$

Consequently,

$$2a_2 + a_0 = 0, \quad 6a_3 + 2a_1 = 0, \quad 12a_4 + 3a_2 = 0, \quad 20a_5 + 4a_3 - \frac{1}{3!}a_1 = 0, \quad \dots,$$

so that

$$a_2 = -\frac{1}{2}a_0, \quad a_3 = -\frac{1}{3}a_1, \quad a_4 = \frac{1}{8}a_0, \quad a_5 = \frac{3}{40}a_1, \quad \dots$$

Hence,

$$y(x) = a_0 \left(1 - \frac{x^2}{2} + \frac{x^4}{8} + \cdots\right) + a_1 \left(x - \frac{x^3}{3} + \frac{3x^5}{40} + \cdots\right).$$

Consequently,

$$y_1(x) = 1 - \frac{x^2}{2} + \frac{x^4}{8} + \cdots \quad \text{and} \quad y_2(x) = x - \frac{x^3}{3} + \frac{3x^5}{40} + \cdots.$$

Since $p(x) = \sin x$ and $q(x) = 1$, it follows that $R = \infty$.

17. (a) Since $p(x) = \frac{1-x}{x}$ and $q(x) = -1$, we see that $x = 0$ is not an ordinary point.

(b) The change of variables $z = x - 1$ transforms the given differential equation into

$$(z+1)y'' - zy' - (z+1)y = 0,$$

where the prime symbol here denotes differentiation with respect to z . Let

$$y(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then

$$y'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \quad \text{and} \quad y''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}.$$

Inserting these results into the differential equation $(z+1)y'' - zy' - (z+1)y = 0$ requires

$$(z+1) \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} - z \sum_{n=1}^{\infty} n a_n z^{n-1} - (z+1) \sum_{n=0}^{\infty} a_n z^n = 0.$$

That is,

$$\sum_{n=2}^{\infty} n(n-1) a_n z^{n-1} + \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} - \sum_{n=0}^{\infty} (n+1) a_n z^n - \sum_{n=0}^{\infty} a_n z^{n+1} = 0,$$

or equivalently,

$$2a_2 - a_0 + \sum_{k=1}^{\infty} [(k+1)k a_{k+1} + (k+2)(k+1) a_{k+2} - (k+1) a_k - a_{k-1}] z^k = 0.$$

Therefore the coefficients must satisfy

$$2a_2 - a_0 = 0, \quad a_{k+2} = \frac{(k+1)a_k + a_{k-1} - k(k+1)a_{k+1}}{(k+2)(k+1)}, \quad k = 1, 2, \dots$$

so that

$$a_2 = \frac{1}{2}a_0, \quad a_3 = \frac{1}{3}a_1, \quad a_4 = \frac{1}{24}(3a_0 - 2a_1), \quad a_5 = \frac{1}{60}(7a_1 - 3a_0), \quad \dots$$

Consequently,

$$y(z) = a_0 \left(1 + \frac{z^2}{2} + \frac{z^4}{8} - \frac{z^5}{20} + \dots \right) + a_1 \left(z + \frac{z^3}{3} - \frac{z^4}{12} + \frac{7z^5}{60} + \dots \right).$$

Now substitute $z = x - 1$ and write only the first three terms of each series:

$$y(x) = a_0 \left[1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{8} + \dots \right] + a_1 \left[(x-1) + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{12} + \dots \right].$$

The radius of convergence is at least one.

18. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $(1 + 2x^2)y'' + 7xy' + 2y = 0$ requires

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=2}^{\infty} n(n-1) a_n x^n + 7 \sum_{n=1}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$\sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2} + 2k(k-1)a_k + 7ka_k + 2a_k]x^k = 0.$$

Therefore the coefficients must satisfy

$$(k+2)[(k+1)a_{k+2} + (2k+1)a_k] = 0, \quad k = 0, 1, \dots$$

so that

$$a_{k+2} = -\frac{(2k+1)}{k+1}a_k, \quad k = 0, 1, \dots$$

Even k :

$$\begin{aligned} k = 0 : a_2 &= -a_0, \\ k = 2 : a_4 &= -\frac{5}{3}a_2 = \frac{1 \cdot 5}{1 \cdot 3}a_0, \\ k = 4 : a_6 &= -\frac{9}{5}a_4 = -\frac{1 \cdot 5 \cdot 9}{1 \cdot 3 \cdot 5}a_0, \\ k = 6 : a_8 &= -\frac{13}{7}a_6 = \frac{1 \cdot 5 \cdot 9 \cdot 13}{1 \cdot 3 \cdot 5 \cdot 7}a_0, \end{aligned}$$

and in general

$$a_{2n} = (-1)^n \frac{1 \cdot 5 \cdots (4n-3)}{1 \cdot 3 \cdots (2n-1)} a_0, \quad n = 1, 2, \dots$$

Odd k :

$$\begin{aligned} k = 1 : a_3 &= -\frac{3}{2}a_1, \\ k = 3 : a_5 &= -\frac{7}{4}a_3 = \frac{3 \cdot 7}{2 \cdot 4}a_1, \\ k = 5 : a_7 &= -\frac{11}{6}a_5 = -\frac{3 \cdot 7 \cdot 11}{2 \cdot 4 \cdot 6}a_1, \\ k = 7 : a_9 &= -\frac{15}{8}a_7 = \frac{3 \cdot 7 \cdot 11 \cdot 15}{2 \cdot 4 \cdot 6 \cdot 8}a_1, \end{aligned}$$

and in general

$$a_{2n+1} = (-1)^n \frac{3 \cdot 7 \cdots (4n-1)}{2 \cdot 4 \cdots (2n)} a_1 = (-1)^n \frac{3 \cdot 7 \cdots (4n-1)}{2^n \cdot n!} a_1, \quad n = 1, 2, \dots$$

Thus, the general solution to the differential equation is

$$y(x) = a_0 \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 5 \cdots (4n-3)}{1 \cdot 3 \cdots (2n-1)} x^{2n} \right] + a_1 \left[x + \sum_{n=1}^{\infty} (-1)^n \frac{3 \cdot 7 \cdots (4n-1)}{2^n \cdot n!} x^{2n+1} \right].$$

Imposing the given initial conditions $y(0) = 0$, $y'(0) = 1$, yields $a_0 = 0$, $a_1 = 1$, so that the solution to the initial value problem is

$$y(x) = x + \sum_{n=1}^{\infty} (-1)^n \frac{3 \cdot 7 \cdots (4n-1)}{2^n \cdot n!} x^{2n+1}.$$

19. (a) Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $4y'' + xy' + 4y = 0$ requires

$$4 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + 4 \sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$4 \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=0}^{\infty} k a_k x^k + 4 \sum_{k=0}^{\infty} a_k x^k = 0,$$

or equivalently,

$$\sum_{k=0}^{\infty} [4(k+2)(k+1) a_{k+2} + (k+4) a_k] x^k = 0.$$

Therefore, the coefficients must satisfy:

$$a_{k+2} = -\frac{(k+4)}{4(k+2)(k+1)} a_k, \quad k = 0, 1, \dots$$

Even k :

$$\begin{aligned} k=0 : a_2 &= -\frac{1}{2}a_0, \\ k=2 : a_4 &= -\frac{6}{4 \cdot 4 \cdot 3}a_2 = \frac{3!}{4 \cdot 4 \cdot 3 \cdot 2}a_0 = \frac{3!}{2^2 \cdot 4!}a_0, \\ k=4 : a_6 &= -\frac{8}{4 \cdot 6 \cdot 5}a_4 = -\frac{4!}{2^3 \cdot 6!}a_0, \\ k=6 : a_8 &= -\frac{10}{4 \cdot 8 \cdot 7}a_6 = \frac{5!}{2^4 \cdot 8!}a_0, \end{aligned}$$

and in general

$$a_{2n} = (-1)^n \frac{(n+1)!}{2^n(2n)!}a_0, \quad n = 1, 2, \dots$$

Odd k :

$$\begin{aligned} k=1 : a_3 &= -\frac{5}{4 \cdot 3 \cdot 2}a_1 = -\frac{5}{4!}a_1, \\ k=3 : a_5 &= -\frac{7}{4 \cdot 5 \cdot 4}a_3 = \frac{5 \cdot 7}{4^2 \cdot 5!}a_1, \\ k=5 : a_7 &= -\frac{9}{4 \cdot 7 \cdot 6}a_5 = -\frac{5 \cdot 7 \cdot 9}{4^3 \cdot 7!}a_1, \end{aligned}$$

and in general

$$a_{2n+1} = \frac{(-1)^n [5 \cdot 7 \cdots (2n+3)]}{4^n(2n+1)!}a_1, \quad n = 1, 2, \dots$$

The general solution to the given differential equation is thus

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{2^n(2n)!} x^{2n} + a_1 \left\{ x + \sum_{n=1}^{\infty} \frac{(-1)^n [5 \cdot 7 \cdots (2n+3)] a_1}{4^n(2n+1)!} x^{2n+1} \right\}.$$

Imposing the initial conditions $y(0) = 1$, $y'(0) = 0$ yields $a_0 = 1$, $a_1 = 0$ so that the solution to the given initial value problem is

$$y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{2^n(2n)!} x^{2n}.$$

(b) A polynomial must be found that approximates the solution with an error less than 10^{-5} . Recall that the numerical error made in truncating a convergent alternating series at a particular term is less than the absolute value of the next term. Letting $x = 1$ in the series solution obtained in part (a) above yields

$$y(x) = 1 - \frac{1}{2} + \frac{1}{16} - \frac{1}{240} + \frac{1}{5376} - \frac{1}{161280} + \cdots$$

The coefficient of x^{10} is $\frac{1}{161280} = 6.3 \times 10^{-6} < 10^{-5}$. Thus, the desired polynomial is

$$y(x) = 1 - \frac{x^2}{2} + \frac{x^4}{16} - \frac{x^6}{240} + \frac{x^8}{5376}.$$

20. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $y'' + 2x^2 y' + xy = 2 \cos x$ requires

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

That is,

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=1}^{\infty} (2k-1) a_{k-1} x^k = 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k},$$

or equivalently,

$$2a_2 + \sum_{k=1}^{\infty} [(k+2)(k+1) a_{k+2} + (2k-1) a_{k-1}] x^k = 2 + 2 \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}.$$

This can be written as

$$2a_2 - 2 + \sum_{k=1}^{\infty} \left\{ [(k+2)(k+1) a_{k+2} + (2k-1) a_{k-1}] x^k - 2 \frac{(-1)^k}{(2k)!} x^{2k} \right\} = 0.$$

Therefore the coefficients must satisfy

$$\begin{aligned} a_2 &= 1, \\ a_{2l+3} &= -\frac{4l+1}{(2l+3)(2l+2)} a_{2l}, \quad l = 0, 1, \dots, \\ a_{2m+2} &= \frac{1}{(2m+2)(2m+1)} \left[2 \frac{(-1)^m}{(2m)!} - (4m-1) a_{2m-1} \right], \quad m = 1, 2, \dots \end{aligned}$$

From these relations we easily compute:

$$a_2 = 1, \quad a_3 = -\frac{1}{6} a_0, \quad a_4 = -\frac{1}{12} (1 + 3a_1), \quad a_5 = -\frac{1}{4}, \quad a_6 = \frac{1}{360} + \frac{7}{180} a_0, \quad \dots$$

Consequently,

$$y(x) = a_0 \left(1 - \frac{x^3}{6} + \frac{7}{180} x^6 - \dots \right) + a_1 \left(x - \frac{x^4}{4} + \dots \right) + \left(x^2 - \frac{x^4}{12} - \frac{x^5}{4} + \frac{x^6}{360} + \dots \right).$$

The complementary function is

$$y_c(x) = a_0 \left(1 - \frac{x^3}{6} + \frac{7}{180} x^6 - \dots \right) + a_1 \left(x - \frac{x^4}{4} + \dots \right),$$

and a particular solution is

$$y_p(x) = x^2 - \frac{x^4}{12} - \frac{x^5}{4} + \frac{x^6}{360} + \dots$$

21. The point $x_0 = 0$ is an ordinary point. Both $p(x) = x$ and $q(x) = -4$ are polynomials, and their power series expansion about 0 are valid for all $x \in \mathbb{R}$. Therefore, by Theorem 9.2.4 we let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $y'' + xy' - 4y = 6e^x$ requires

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n - 4 \sum_{n=0}^{\infty} a_n x^n = 6 \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

That is,

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=1}^{\infty} k a_k x^k - 4 \sum_{k=0}^{\infty} a_k x^k - 6 \sum_{k=0}^{\infty} \frac{x^k}{k!} = 0,$$

or equivalently,

$$\sum_{k=0}^{\infty} \left[(k+2)(k+1) a_{k+2} + (k-4) a_k - \frac{6}{k!} \right] x^k = 0.$$

Therefore the coefficients must satisfy

$$(k+2)(k+1) a_{k+2} + (k-4) a_k - \frac{6}{k!} = 0, \quad k = 0, 1, \dots$$

so that

$$a_{k+2} = \frac{1}{(k+2)(k+1)} \left[\frac{6}{k!} - (k-4) a_k \right], \quad k = 0, 1, \dots$$

From this relation we easily compute:

$$a_2 = 3 + 2a_0, \quad a_3 = 1 + \frac{1}{2}a_1, \quad a_4 = \frac{3}{4} + \frac{1}{3}a_0, \quad a_5 = \frac{1}{10} + \frac{1}{40}a_1, \quad a_6 = \frac{1}{120}, \quad \dots$$

Thus,

$$y(x) = a_0 \left(1 + 2x^2 + \frac{x^4}{3} + \dots \right) + a_1 \left(x + \frac{x^3}{2} + \frac{x^5}{40} + \dots \right) + \left(3x^2 + x^3 + \frac{3}{4}x^4 + \frac{x^5}{10} + \frac{x^6}{120} + \dots \right).$$

The complementary function is

$$y_c(x) = a_0 \left(1 + 2x^2 + \frac{x^4}{3} + \dots \right) + a_1 \left(x + \frac{x^3}{2} + \frac{x^5}{40} + \dots \right),$$

and a particular solution is

$$y_p(x) = 3x^2 + x^3 + \frac{3}{4}x^4 + \frac{x^5}{10} + \frac{x^6}{120} + \dots$$

Solutions to Section 9.3

True-False Review:

1. **FALSE.** Theorem 9.2.4 guarantees that $R = 1$ is a *lower bound* on the radius of convergence of the power series solutions to Legendre's equation about $x = 0$. The radius of convergence could be greater than 1.
2. **TRUE.** Relative to the inner product

$$\langle p, q \rangle = \int_{-1}^1 p(x)q(x)dx,$$

$\{P_0, P_1, \dots, P_N\}$ is an orthogonal set (Theorem 9.3.4), and since each P_i is of different degree, ranging from 0 to N , $\{P_0, P_1, \dots, P_N\}$ is linearly independent.

3. **TRUE.** If α is a positive even integer or a negative odd integer or zero, then eventually the coefficients of (9.3.3) become zero, whereas if α is a negative even integer or a positive odd integer, then eventually the coefficients of (9.3.4) become zero. Therefore, for any integer α , either (9.3.3) or (9.3.4) contains only finitely many nonzero terms.
4. **TRUE.** The Legendre polynomials correspond to polynomial solutions to (9.3.3) and (9.3.4), and we see directly from their form that these solutions contain terms with all odd powers of x or with all even powers of x .

Problems:

1. From Equations (9.3.3) and (9.3.4) two linearly independent solutions to Legendre's equation are

$$y_1(x) = a_0 \left[1 - \frac{\alpha(\alpha+1)}{2}x^2 + \frac{(\alpha-2)\alpha(\alpha+1)(\alpha+3)}{4!}x^4 - \frac{(\alpha-4)(\alpha-2)\alpha(\alpha+1)(\alpha+3)(\alpha+5)}{6!}x^6 + \dots \right], \quad (0.0.55)$$

and

$$y_2(x) = a_1 \left[x - \frac{(\alpha-1)(\alpha+2)}{3!}x^3 + \frac{(\alpha-3)(\alpha-1)(\alpha+2)(\alpha+4)}{5!}x^5 + \dots \right]. \quad (0.0.56)$$

Setting $\alpha = 3$ in Equation (0.0.56) yields

$$y_2(x) = a_1 \left(x - \frac{5}{3}x^3 \right).$$

To obtain the corresponding Legendre polynomial we must choose a_1 so that $y_2(1) = 1$. This requires

$$a_1 \left(1 - \frac{5}{3} \right) = 1$$

so that $a_1 = -\frac{3}{2}$. Consequently,

$$P_3(x) = -\frac{3}{2}x \left(1 - \frac{5}{3}x^2 \right) = \frac{1}{2}x \left(5x^2 - 3 \right).$$

Similarly, setting $\alpha = 4$ in Equation (0.055) yields

$$y_1(x) = a_0 \left(1 - 10x^2 + \frac{35}{3}x^4 \right).$$

To obtain the corresponding Legendre polynomial we must choose a_0 so that $y_1(1) = 1$. This requires

$$a_0 \left(1 - 10 + \frac{35}{3} \right) = 1,$$

so that $a_0 = \frac{3}{8}$. Therefore,

$$P_4(x) = \frac{3}{8} \left(1 - 10x^2 + \frac{35}{3}x^4 \right) = \frac{1}{8} (35x^4 - 30x^2 + 3).$$

2. Starting with $P_0(x) = 1$, and $P_1(x) = x$ we use the recurrence relation

$$(n+1)P_{n+1}(x) + nP_{n-1}(x) = (2n+1)xP_n(x), \quad n = 1, 2, 3, \quad (0.057)$$

to generate $P_2(x), P_3(x), P_4(x)$:

n = 1: Setting $n = 1$ in (0.057) yields

$$2P_2(x) + P_0(x) = 3xP_1(x),$$

so that

$$P_2(x) = \frac{1}{2} [3xP_1(x) - P_0(x)].$$

That is,

$$P_2(x) = \frac{1}{2} (3x^2 - 1).$$

n = 2: Setting $n = 2$ in (0.057) yields

$$3P_3(x) + 2P_1(x) = 5xP_2(x),$$

so that

$$P_3(x) = \frac{1}{3} \left[5xP_2(x) - 2P_1(x) \right].$$

Substituting the expression for $P_2(x)$ obtained above and simplifying, we obtain

$$P_3(x) = \frac{1}{2} x (5x^2 - 3).$$

n=3: Setting $n = 3$ in (0.057) yields

$$4P_4(x) + 3P_2(x) = 7xP_3(x),$$

so that

$$P_4(x) = \frac{1}{4} \left[7xP_3(x) + 3P_2(x) \right].$$

Substituting the expressions for $P_2(x)$ and $P_3(x)$ obtained above and simplifying, we obtain

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3).$$

3. From Rodrigues' Formula,

$$P_3(x) = \frac{1}{2^3 \cdot 3!} \frac{d^3}{dx^3} (x^2 - 1)^3 = \frac{1}{48} \frac{d^3}{dx^3} (x^6 - 3x^4 + 3x^2 - 1).$$

Computing the derivatives yields

$$P_3(x) = \frac{1}{48}(120x^3 - 72x) = \frac{1}{2}x(5x^2 - 3).$$

4. We have

$$x^3 + 2x = a_0P_0 + a_1P_1 + a_2P_2 + a_3P_3,$$

where

$$a_j = \frac{2j+1}{2} \int_{-1}^1 (x^3 + 2x)P_j(x)dx, \quad j = 0, 1, 2, 3,$$

and

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}x(5x^2 - 3).$$

Thus, we obtain the following:

$$\begin{aligned} a_0 &= \frac{1}{2} \int_{-1}^1 (x^3 + 2x) \cdot 1 \, dx = 0, \\ a_1 &= \frac{3}{2} \int_{-1}^1 (x^3 + 2x) \cdot x \, dx = \frac{3}{2} \int_{-1}^1 (x^4 + 2x^2) \, dx = \frac{13}{5}, \\ a_2 &= \frac{5}{2} \int_{-1}^1 (x^3 + 2x) \cdot \frac{1}{2}(3x^2 - 1) \, dx = \frac{5}{4} \int_{-1}^1 (3x^5 + 5x^3 - 2x) \, dx = 0, \\ a_3 &= \frac{7}{2} \int_{-1}^1 (x^3 + 2x) \cdot \frac{1}{2}x(5x^2 - 3) \, dx = \frac{7}{4} \int_{-1}^1 (5x^6 + 7x^4 - 6x^2) \, dx = \frac{2}{5}. \end{aligned}$$

Consequently,

$$x^3 + 2x = \frac{13}{5}P_1(x) + \frac{2}{5}P_3(x).$$

5. Since $p(x)$ has degree 3, no Legendre polynomials of degree 4 or more can occur in the requested linear combination. Therefore, we can write

$$2x^3 + x^2 + 5 = a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + a_3P_3(x),$$

where

$$a_j = \frac{2j+1}{2} \int_{-1}^1 (2x^3 + x^2 + 5)P_j(x)dx, \quad j = 0, 1, 2, 3,$$

and

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}x(5x^2 - 3).$$

Thus, we obtain the following:

$$\begin{aligned} a_0 &= \frac{1}{2} \int_{-1}^1 (2x^3 + x^2 + 5) \cdot 1 \, dx = \frac{16}{3}, \\ a_1 &= \frac{3}{2} \int_{-1}^1 (2x^3 + x^2 + 5) \cdot x \, dx = \frac{3}{2} \int_{-1}^1 (2x^4 + x^3 + 5x) \, dx = \frac{6}{5}, \\ a_2 &= \frac{5}{2} \int_{-1}^1 (2x^3 + x^2 + 5) \cdot \frac{1}{2}(3x^2 - 1) \, dx = \frac{5}{4} \int_{-1}^1 (6x^5 + 3x^4 - 2x^3 + 14x^2 - 5) \, dx = \frac{2}{3}, \\ a_3 &= \frac{7}{2} \int_{-1}^1 (2x^3 + x^2 + 5) \cdot \frac{1}{2}x(5x^2 - 3) \, dx = \frac{7}{4} \int_{-1}^1 (10x^6 + 5x^5 - 6x^4 + 22x^3 - 15x) \, dx = \frac{4}{5}. \end{aligned}$$

Consequently,

$$2x^3 + x^2 + 5 = \frac{16}{3}P_0(x) + \frac{6}{5}P_1(x) + \frac{2}{3}P_2(x) + \frac{4}{5}P_3(x).$$

6. We can write

$$Q(x) = \sum_{i=0}^{N-1} a_i P_i(x)$$

for some constants a_i , where $i = 0, 1, \dots, N-1$. Thus

$$\int_{-1}^1 Q(x)P_N(x)dx = \sum_{i=0}^{N-1} a_i \int_{-1}^1 P_i(x)P_N(x)dx = 0,$$

since the Legendre polynomial are orthogonal on $[-1,1]$ and $i < N$ for each i in the summation.

7. Applying the chain rule we have

$$\frac{dY}{d\phi} = \frac{dY}{dx} \frac{dx}{d\phi} = \frac{dY}{dx}(-\sin \phi)$$

so that

$$\begin{aligned} \frac{d^2Y}{d\phi^2} &= (-\sin \phi) \frac{d}{d\phi} \left(\frac{dY}{dx} \right) + \frac{dY}{dx}(-\cos \phi) \\ &= (-\sin \phi) \frac{d}{dx} \left(\frac{dY}{dx} \right) \frac{dx}{d\phi} - \cos \phi \frac{dY}{dx} \\ &= \sin^2 \phi \frac{d^2Y}{dx^2} - \cos \phi \frac{dY}{dx}. \end{aligned}$$

Substituting these results into the given differential equation yields

$$\sin^2 \phi \frac{d^2Y}{dx^2} - \cos \phi \frac{dY}{dx} + \cot \phi \frac{dY}{dx}(-\sin \phi) + \alpha(\alpha + 1)Y = 0.$$

That is, since $x = \cos \phi$,

$$(1 - x^2) \frac{d^2Y}{dx^2} - 2x \frac{dY}{dx} + \alpha(\alpha + 1)Y = 0$$

which is Legendre's equation.

8. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Inserting these results into the differential equation $y'' - 2xy' + 2\alpha y = 0$ requires

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2 \sum_{n=1}^{\infty} n a_n x^n + 2\alpha \sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$\sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2} - 2ka_k + 2\alpha a_k] x^k = 0,$$

so that the coefficients must satisfy

$$(k+2)(k+1)a_{k+2} - 2ka_k + 2\alpha a_k = 0, \quad k = 0, 1, \dots$$

Consequently,

$$a_{k+2} = \frac{2(k-\alpha)}{(k+2)(k+1)} a_k, \quad k = 0, 1, \dots$$

From the preceding recurrence relation it is straightforward to derive:

$$a_{2n} = \frac{2^n(-\alpha) \cdots (2n-2-\alpha)}{(2n)!} a_0, \quad n = 1, 2, \dots$$

and

$$a_{2n+1} = \frac{2^n(1-\alpha) \cdots (2n-1-\alpha)}{(2n+1)!} a_1, \quad n = 1, 2, \dots$$

Hence,

$$y(x) = a_0 \left[1 + \sum_{n=1}^{\infty} \frac{2^n(-\alpha) \cdots (2n-2-\alpha)}{(2n)!} x^{2n} \right] + a_1 \left[x + \sum_{n=1}^{\infty} \frac{2^n(1-\alpha) \cdots (2n-1-\alpha)}{(2n+1)!} x^{2n+1} \right],$$

so that

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{2^n(-\alpha) \cdots (2n-2-\alpha)}{(2n)!} x^{2n},$$

and

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{2^n(1-\alpha) \cdots (2n-1-\alpha)}{(2n+1)!} x^{2n+1}$$

are two linearly independent series solutions to Hermite's equation centered at $x = 0$.

9. From Problem 8 we have

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{2^n(-\alpha) \cdots (2n-2-\alpha)}{(2n)!} x^{2n},$$

and

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{2^n(1-\alpha)\cdots(2n-1-\alpha)}{(2n+1)!} x^{2n+1}.$$

Notice that if $\alpha = N$, where N is a nonnegative even integer, then $y_1(x)$ is a polynomial of degree N . Similarly, if $\alpha = M$, where M is an odd positive integer, $y_2(x)$ is a polynomial of degree M . The polynomial solutions corresponding to $\alpha = 0, 1, 2$, and 3 are as follows:

$$\begin{aligned}\alpha = 0 : & \quad y_1(x) = 1, \\ \alpha = 1 : & \quad y_2(x) = x, \\ \alpha = 2 : & \quad y_1(x) = 1 - 2x^2, \\ \alpha = 3 : & \quad y_2(x) = \frac{1}{3}x(3 - 2x^2).\end{aligned}$$

10. (a) The Hermite polynomials satisfy

$$H_N'' - 2xH_N' + 2NH_N = 0,$$

where N is a nonnegative integer. Multiplying this equation by e^{-x^2} yields

$$e^{-x^2}H_N'' - 2e^{-x^2}xH_N' + 2Ne^{-x^2}H_N = 0$$

which can be written in the equivalent form

$$\left(e^{-x^2}H_N'\right)' + 2Ne^{-x^2}H_N = 0.$$

(b) Let $H_M(x)$ and $H_N(x)$ be Hermite polynomials. Then from part (a)

$$\left(e^{-x^2}H_M'\right)' + 2Me^{-x^2}H_M = 0$$

and

$$\left(e^{-x^2}H_N'\right)' + 2Ne^{-x^2}H_N = 0.$$

Multiplying the first equation by H_N , the second equation by H_M and subtracting the resulting equations yields:

$$H_M \left(e^{-x^2}H_N'\right)' - H_N \left(e^{-x^2}H_M'\right)' + 2(N-M)e^{-x^2}H_MH_N = 0.$$

We now integrate over $(-\infty, \infty)$ and use integration by parts on the first two terms to obtain

$$\lim_{k \rightarrow \infty} \left[e^{-x^2}(H_MH_N' - H_M'H_N) \right]_{-k}^k + 2(N-M) \int_{-\infty}^{\infty} e^{-x^2}H_N(x)H_M(x)dx = 0. \quad (0.058)$$

Since H_M and H_N are polynomials, we have

$$\lim_{k \rightarrow \pm\infty} e^{-x^2}H_MH_N' = 0, \quad \text{and} \quad \lim_{k \rightarrow \pm\infty} e^{-x^2}H_M'H_N = 0$$

so that (0.058) reduces to

$$(N-M) \int_{-\infty}^{\infty} e^{-x^2}H_N(x)H_M(x)dx = 0.$$

Consequently,

$$\int_{-\infty}^{\infty} e^{-x^2} H_N(x) H_M(x) dx = 0$$

whenever $M \neq N$.

(c) Multiplying

$$p(x) = \sum_{k=1}^N a_k H_k(x)$$

by $e^{-x^2} H_j(x)$, where $j = 1, 2, \dots, N$, and integrating over $(-\infty, \infty)$ yields

$$\int_{-\infty}^{\infty} e^{-x^2} H_j(x) p(x) dx = \int_{-\infty}^{\infty} e^{-x^2} H_j(x) \sum_{k=1}^N a_k H_k(x) dx.$$

That is,

$$\int_{-\infty}^{\infty} e^{-x^2} H_j(x) p(x) dx = \sum_{k=1}^N a_k \int_{-\infty}^{\infty} e^{-x^2} H_j(x) H_k(x) dx. \quad (0.0.59)$$

However, we have already shown in part (b) that

$$\int_{-\infty}^{\infty} e^{-x^2} H_j(x) H_k(x) dx = 0, \quad \text{whenever } j \neq k.$$

Therefore, (0.0.59) reduces to

$$\int_{-\infty}^{\infty} e^{-x^2} H_j(x) p(x) dx = a_j \int_{-\infty}^{\infty} e^{-x^2} H_j^2(x) dx = a_j 2^j j! \sqrt{\pi},$$

where we have used $\int_{-\infty}^{\infty} e^{-x^2} H_j^2(x) dx = 2^j j! \sqrt{\pi}$. Consequently,

$$a_j = \frac{1}{2^j j! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_j(x) p(x) dx.$$

11. We can write

$$3x^3 - 1 = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + a_3 P_3(x),$$

where

$$a_n = \frac{2n+1}{2} \int_{-1}^1 (3x^3 - 1) P_n(x) dx, \quad n = 0, 1, 2, 3.$$

Evaluating these integrals yields $a_0 = -1$, $a_1 = \frac{9}{5}$, $a_2 = 0$, $a_3 = \frac{6}{5}$. Hence,

$$3x^3 - 1 = -P_0(x) + \frac{9}{5} P_1(x) + \frac{6}{5} P_3(x).$$

12. The coefficients in the Legendre series expansion of $f(x) = \cos \pi x$ are

$$a_n = \frac{2n+1}{2} \int_{-1}^1 \cos \pi x P_n(x) dx, \quad n = 0, 1, \dots$$

Using some form of technology it is easy to obtain:

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = -\frac{15}{\pi^2}, \quad a_3 = 0, \quad a_4 = -\frac{45}{\pi^4}(2\pi^2 - 21).$$

Hence, the first four terms in the Legendre series expansion of $f(x) = \cos \pi x$ are

$$0 \cdot P_0(x) + 0 \cdot P_1(x) - \frac{15}{\pi^2}P_2(x) + 0 \cdot P_3(x) - \frac{45}{\pi^4}(2\pi^2 - 21)P_4(x).$$

Hence,

$$S_1 = 0, \quad S_3 = -\frac{15}{\pi^2}P_2(x), \quad S_5 = -\frac{15}{\pi^2}P_2(x) - \frac{45}{\pi^4}(2\pi^2 - 21)P_4(x).$$

In the accompanying figure, we have sketched $f(x)$ and the two approximations S_3 and S_5 .

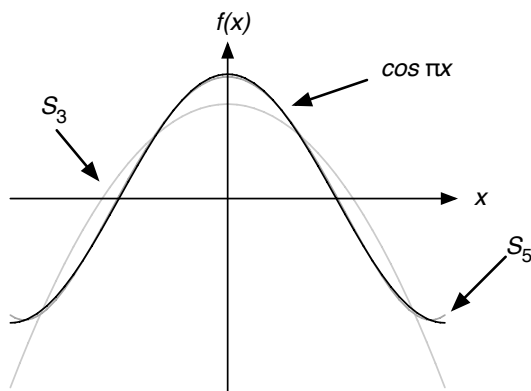


Figure 153: Figure for Problem 12

13. The coefficients in the Legendre series expansion of $f(x) = x(1 - x^2)e^x$ are

$$a_n = \frac{2n+1}{2} \int_{-1}^1 x(1-x^2)e^x P_n(x) dx, \quad n = 0, 1, \dots$$

Using some form of technology it is easy to obtain:

$$\begin{aligned} a_0 &= \frac{1}{e}(e^2 - 7), \quad a_1 = \frac{6}{e}(15 - 2e^2), \quad a_2 = \frac{5}{e}(31e^2 - 229), \\ a_3 &= \frac{2}{e}(8120 - 1099e^2), \quad a_4 = \frac{1}{e}(24929e^2 - 258093). \end{aligned}$$

Hence, the first four terms in the Legendre series expansion of $f(x) = x(1 - x^2)e^x$ are

$$\begin{aligned} \frac{1}{e}(e^2 - 7)P_0(x) + \frac{6}{e}(15 - 2e^2)P_1(x) + \frac{5}{e}(31e^2 - 229)P_2(x) \\ + \frac{2}{e}(8120 - 1099e^2)P_3(x) + \frac{1}{e}(24929e^2 - 258093)P_4(x). \end{aligned}$$

Therefore,

$$\begin{aligned}
 S_1 &= \frac{1}{e}(e^2 - 7)P_0(x), \\
 S_3 &= \frac{1}{e}(e^2 - 7)P_0(x) + \frac{6}{e}(15 - 2e^2)P_1(x) + \frac{5}{e}(31e^2 - 229)P_2(x), \\
 S_5 &= \frac{1}{e}(e^2 - 7)P_0(x) + \frac{6}{e}(15 - 2e^2)P_1(x) + \frac{5}{e}(31e^2 - 229)P_2(x) \\
 &\quad + \frac{2}{e}(8120 - 1099e^2)P_3(x) + \frac{1}{e}(24929e^2 - 258093)P_4(x).
 \end{aligned}$$

In the accompanying figure, we have sketched $f(x)$ and the three approximations S_1 , S_3 and S_5 .

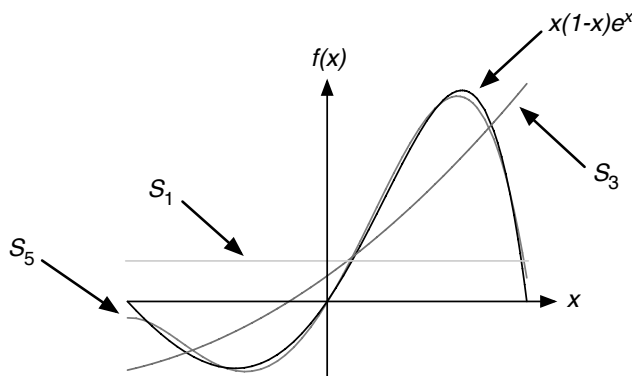


Figure 154: Figure for Problem 13

Solutions to Section 9.4

True-False Review:

1. **FALSE.** It is required that $(x - x_0)^2 Q(x)$ be analytic at $x = x_0$, not that $(x - x_0)Q(x)$ be analytic at $x = x_0$.
2. **FALSE.** It is necessary to assume that r_1 and r_2 are distinct and do not differ by an integer in order to draw this conclusion.
3. **TRUE.** This is well-illustrated by Examples 9.4.5 and 9.4.6. The values of $a_0, a_1, a_2, \dots$ in the Frobenius series solution are obtained by directly substituting it into both sides of the differential equation and matching up the coefficients of x resulting on each side of the differential equation.
4. **TRUE.** Example 9.4.2 is a fine illustration of this.
5. **TRUE.** For instance, the only singular point of $y'' + \frac{1}{x^2}y' + y = 0$ is $x_0 = 0$, and it is an irregular singular point.

Problems:

1. For the given differential equation

$$P(x) = \frac{1}{1-x} \quad \text{and} \quad Q(x) = x.$$

Therefore $x = 1$ is the only singular point. Further, since

$$p(x) = (x-1)\frac{1}{1-x} = -1 \quad \text{and} \quad q(x) = (x-1)^2x$$

are both analytic at $x = 1$, it follows that $x = 1$ is a regular singular point.

2. For the given differential equation

$$P(x) = \frac{1}{x(1-x^2)^2} \quad \text{and} \quad Q(x) = \frac{1}{x^2}.$$

Therefore the only singular points of the differential equation are $x = 0, -1, 1$.

For $x = 0$: Both

$$p(x) = \frac{1}{(1-x^2)^2} \quad \text{and} \quad q(x) = 1$$

are analytic at $x = 0$. Therefore, $x = 0$ is a regular singular point.

For $x = 1$: Since

$$p(x) = \frac{x-1}{x(1-x^2)^2} \quad \text{and} \quad q(x) = \frac{(x-1)^2}{x^2}$$

are not both analytic at $x = 1$ it follows that $x = 1$ is an irregular singular point.

For $x = -1$: Since

$$p(x) = \frac{x+1}{x(1-x^2)^2} \quad \text{and} \quad q(x) = \frac{(x+1)^2}{x^2}$$

are not both analytic at $x = -1$ it follows that $x = -1$ is an irregular singular point.

3. For the given differential equation

$$P(x) = \frac{e^x}{x-2} \quad \text{and} \quad Q(x) = \frac{4}{x(x-2)^2}.$$

Therefore $x = 0$ and $x = 2$ are the only singular points.

For $x = 0$: Since

$$p(x) = x \cdot \frac{e^x}{x-2} \quad \text{and} \quad q(x) = x^2 \cdot \frac{4}{x(x-2)^2} = \frac{4x}{(x-2)^2}$$

are both analytic at $x = 0$ it follows that $x = 0$ is a regular singular point.

For $x = 2$: Since

$$p(x) = e^x \quad \text{and} \quad q(x) = \frac{4}{x}$$

are both analytic at $x = 2$ it follows that $x = 2$ is a regular singular point.

4. For the given differential equation

$$P(x) = \frac{2}{x(x-3)} \quad \text{and} \quad Q(x) = -\frac{1}{x^3(x+3)}.$$

Therefore $x = -3, 0, 3$ are the only singular points.

For $x = -3$: Since

$$p(x) = \frac{2(x+3)}{x(x-3)} \quad \text{and} \quad q(x) = -\frac{(x+3)}{x^3}$$

are both analytic at $x = -3$ it follows that $x = -3$ is a regular singular point.

For $x = 0$: Since

$$p(x) = \frac{2}{x-3} \quad \text{and} \quad q(x) = -\frac{1}{x(x+3)}$$

are not both analytic at $x = 0$ it follows that $x = 0$ is an irregular singular point.

For $x = 3$: Since

$$p(x) = \frac{2}{x} \quad \text{and} \quad q(x) = -\frac{(x-3)^2}{x^3(x+3)}$$

are both analytic at $x = 3$ it follows that $x = 3$ is a regular singular point.

5. In this case

$$p(x) = 1 - x \quad \text{and} \quad q(x) = -7,$$

so that

$$p_0 = p(0) = 1 \quad \text{and} \quad q_0 = q(0) = -7.$$

Therefore the indicial equation is

$$r(r-1) + r - 7 = 0.$$

That is,

$$r^2 - 7 = 0,$$

with roots $r = \pm\sqrt{7}$.

6. In this case

$$p(x) = \frac{1}{4}e^x \quad \text{and} \quad q(x) = -\frac{1}{4},$$

so that

$$p_0 = p(0) = \frac{1}{4} \quad \text{and} \quad q_0 = q(0) = -\frac{1}{4}.$$

Therefore the indicial equation is

$$r(r-1) + \frac{1}{4}r - \frac{1}{4} = 0.$$

That is,

$$4r^2 - 3r - 1 = (4r+1)(r-1) = 0,$$

with roots $r = -\frac{1}{4}, 1$.

7. In this case

$$p(x) = -\frac{x}{4} \quad \text{and} \quad q(x) = \frac{x}{2},$$

so that

$$p_0 = p(0) = 0 \quad \text{and} \quad q_0 = q(0) = 0.$$

Therefore the indicial equation is

$$r(r-1) = 0,$$

with roots $r = 0, 1$.

8. In this case

$$p(x) = -\cos x \quad \text{and} \quad q(x) = 5e^{2x},$$

so that

$$p_0 = p(0) = -1 \quad \text{and} \quad q_0 = q(0) = 5.$$

Therefore the indicial equation is

$$r(r-1) - r + 5 = 0.$$

That is,

$$r^2 - 2r + 5 = 0,$$

with roots $r = 1 \pm 2i$.

9. The indicial equation here is $r(r-1) + \frac{3}{4}r = 0$, which can be written $r^2 - \frac{1}{4}r = 0$, with roots $r = 0$ and $r = \frac{1}{4}$ that do not differ by an integer. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

into the given differential equation and simplifying yields:

$$\sum_{n=0}^{\infty} 4(n+r)(n+r-1)a_n x^n + \sum_{n=0}^{\infty} 3(n+r)a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

That is,

$$\sum_{n=0}^{\infty} (n+r)(4n+4r-1)a_n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

When $n = 0$, we obtain $r(4r-1) = 0$, with roots $r = \frac{1}{4}, 0$.

The remaining coefficients must satisfy

$$(n+r)(4n+4r-1)a_n + a_{n-1} = 0, \quad n = 1, 2, \dots \quad (0.0.60)$$

Case 1: $r = \frac{1}{4}$: Equation (0.0.60) implies that

$$a_n = -\frac{1}{n(4n+1)}a_{n-1}, \quad n = 1, 2, \dots$$

Substituting $n = 1, 2, 3, \dots$ into this recurrence relation we find

$$a_n = \frac{(-1)^n}{n![5 \cdot 9 \cdots (4n+1)]}a_0, \quad n = 1, 2, \dots$$

Hence,

$$y_1(x) = x^{1/4} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n![5 \cdot 9 \cdots (4n+1)]} \right\}.$$

Case 2: $r = 0$: Equation (0.0.60) implies that

$$a_n = -\frac{1}{n(4n-1)}a_{n-1}, \quad n = 1, 2, \dots$$

Substituting $n = 1, 2, 3, \dots$ into this recurrence relation yields

$$a_n = \frac{(-1)^n}{n![3 \cdot 7 \cdots (4n-1)]}a_0, \quad n = 1, 2, \dots$$

Hence,

$$y_2(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n![3 \cdot 7 \cdots (4n-1)]}.$$

10. The indicial equation here is $r(r-1) + \frac{1}{6}r + \frac{1}{6} = 0$, which can be written $r^2 - \frac{5}{6}r + \frac{1}{6} = 0$, with roots $r = \frac{1}{2}$ and $r = \frac{1}{3}$ that do not differ by an integer. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

into the given differential equation and simplifying yields:

$$\sum_{n=0}^{\infty} [6(n+r)(n+r-1) + (n+r) + 1]a_n x^n + \sum_{n=1}^{\infty} 6[3(n+r) - 1]a_{n-1} x^n = 0.$$

When $n = 0$, we obtain $6r(r-1) + r + 1 = 0$, with roots $r = \frac{1}{2}, \frac{1}{3}$.

The remaining coefficients must satisfy

$$[6(n+r)(n+r-1) + (n+r) + 1]a_n + 6[3(n+r) - 1]a_{n-1} = 0, \quad n = 1, 2, \dots \quad (0.0.61)$$

Case 1: $r = \frac{1}{2}$: Equation (0.0.61) implies that

$$(6n^2 + n)a_n + (18n + 3)a_{n-1} = 0, \quad n = 1, 2, \dots$$

so that

$$a_n = -\frac{3}{n}a_{n-1}, \quad n = 1, 2, \dots$$

Substituting $n = 1, 2, 3, \dots$ into this recurrence relation yields

$$a_n = \frac{(-3)^n}{n!}a_0, \quad n = 1, 2, \dots$$

Hence,

$$y_1(x) = x^{1/2} \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} x^n = x^{1/2} e^{-3x}.$$

Case 2: $r = \frac{1}{3}$: Equation (0.0.61) implies that

$$(6n^2 - n)a_n + (18n)a_{n-1} = 0, \quad n = 1, 2, \dots$$

so that

$$a_n = -\frac{18}{6n-1}a_{n-1}, \quad n = 1, 2, \dots$$

Substituting $n = 1, 2, 3, \dots$ into this recurrence relation yields

$$a_n = \frac{(-18)^n}{5 \cdot 11 \cdots (6n-1)}a_0, \quad n = 1, 2, \dots$$

Hence,

$$y_2(x) = x^{1/3} \left[1 + \sum_{n=1}^{\infty} \frac{(-18)^n}{5 \cdot 11 \cdots (6n-1)} x^n \right].$$

11. The indicial equation here is $r(r-1) + r - 2 = 0$, which can be written $r^2 - 2 = 0$, with roots $r = \pm\sqrt{2}$, that do not differ by an integer. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

into the given differential equation and simplifying yields:

$$\sum_{n=0}^{\infty} [(n+r)^2 - 2] a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

When $n = 0$, we obtain $r^2 - 2 = 0$, with roots $r = \pm\sqrt{2}$.

The remaining coefficients must satisfy

$$[(n+r)^2 - 2] a_n - a_{n-1} = 0, \quad n = 1, 2, \dots \quad (0.0.62)$$

Case 1: $r = \sqrt{2}$: Equation (0.0.62) implies that

$$a_n = \frac{1}{n(n+2\sqrt{2})} a_{n-1}, \quad n = 1, 2, \dots$$

Substituting $n = 1, 2, 3, \dots$ into this recurrence relation and simplifying yields

$$a_n = \frac{1}{n!(1+2\sqrt{2})(2+2\sqrt{2}) \cdots (n+2\sqrt{2})} a_0, \quad n = 1, 2, \dots$$

Hence,

$$y_1(x) = x^{\sqrt{2}} \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!(1+2\sqrt{2})(2+2\sqrt{2}) \cdots (n+2\sqrt{2})} \right].$$

Case 2: $r = -\sqrt{2}$: Equation (0.0.62) implies that

$$a_n = \frac{1}{n(n-2\sqrt{2})} a_{n-1}, \quad n = 1, 2, \dots$$

Substituting $n = 1, 2, 3, \dots$ into this recurrence relation and simplifying we find:

$$a_n = \frac{1}{n!(1-2\sqrt{2})(2-2\sqrt{2}) \cdots (n-2\sqrt{2})} a_0, \quad n = 1, 2, \dots$$

Hence,

$$y_2(x) = x^{-\sqrt{2}} \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{n!(1-2\sqrt{2})(2-2\sqrt{2}) \cdots (n-2\sqrt{2})} \right].$$

12. The indicial equation here is $r(r-1) + \frac{1}{2}r = 0$, which can be written $r^2 - \frac{1}{2}r = 0$, with roots $r = 0$ and $r = \frac{1}{2}$ that do not differ by an integer. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

into the given differential equation and simplifying yields:

$$\sum_{n=0}^{\infty} [2(n+r)(n+r-1) + n+r] a_n x^n - \sum_{n=2}^{\infty} 2a_{n-2} x^n = 0.$$

When $n = 0$, we obtain $r(2r-1)a_0 = 0$, with roots $r = 0, \frac{1}{2}$. When $n = 1$, we obtain

$$(2r^2 + 3r + 1)a_1 = 0$$

which implies that $a_1 = 0$, since the only values r can assume are 0 or $\frac{1}{2}$. The remaining coefficients must satisfy

$$[2(n+r)(n+r-1) + n+r]a_n - 2a_{n-2} = 0, \quad n = 2, 3, \dots \quad (0.0.63)$$

Case 1: $r = 0$: Equation (0.0.63) implies that

$$a_n = \frac{2}{n(2n-1)} a_{n-2}, \quad n = 2, 3, \dots \quad (0.0.64)$$

Since $a_1 = 0$, it follows that

$$a_{2n+1} = 0, \quad n = 0, 1, \dots,$$

whereas, from Equation (0.0.64) the even coefficients must satisfy:

$$a_{2k} = \frac{1}{k(4k-1)} a_{2k-2}, \quad k = 1, 2, \dots,$$

from which we obtain

$$a_{2k} = \frac{1}{k![3 \cdot 7 \cdots (4k-1)]} a_0, \quad k = 1, 2, \dots$$

Hence,

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n![3 \cdot 7 \cdots (4n-1)]}.$$

Case 2: $r = \frac{1}{2}$: Equation (0.0.63) implies that

$$a_n = \frac{2}{n(2n+1)} a_{n-2} = 0, \quad n = 1, 2, \dots \quad (0.0.65)$$

Once more, since $a_1 = 0$, it follows that

$$a_{2n+1} = 0, \quad n = 0, 1, \dots,$$

whereas, from Equation (0.0.65), the even coefficients must satisfy

$$a_{2k} = \frac{1}{k(4k+1)} a_{2k-2} = 0, \quad k = 1, 2, \dots,$$

from which we obtain

$$a_{2k} = \frac{1}{k![5 \cdot 9 \cdots (4k+1)]} a_0, \quad k = 1, 2, \dots$$

Hence,

$$y_2(x) = x^{1/2} \left\{ 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{n![5 \cdot 9 \cdots (4n+1)]} \right\}.$$

13. The indicial equation here is $r(r-1) - \frac{8}{3}r + 2 = 0$, which can be written $r^2 - \frac{11}{3}r + 2 = 0$, with roots $r = \frac{2}{3}$ and $r = 3$ that do not differ by an integer. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

into the given differential equation and simplifying yields:

$$\sum_{n=0}^{\infty} [3(n+r)(n+r-1) - 8(n+r) + 6] a_n x^n - \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^n = 0.$$

When $n = 0$, we obtain $3r^2 - 11r + 6 = (3r-2)(r-3) = 0$, with roots $r = \frac{2}{3}, 3$. The remaining coefficients must satisfy

$$[3(n+r)(n+r-1) - 8(n+r) + 6] a_n - (n+r-1) a_{n-1} = 0, \quad n = 1, 2, \dots \quad (0.0.66)$$

Case 1: $r = \frac{2}{3}$: Equation (0.0.66) implies that

$$a_n = \frac{3n-1}{3n(3n-7)} a_{n-1}, \quad n = 1, 2, \dots$$

Substituting $n = 1, 2, 3, \dots$ into this recurrence relation we find

$$a_n = \frac{(3n-4)(3n-1)}{4 \cdot n! \cdot 3^n} a_0, \quad n = 1, 2, \dots$$

Hence,

$$y_1(x) = x^{2/3} \sum_{n=0}^{\infty} \frac{(3n-4)(3n-1)}{4 \cdot n! \cdot 3^n} x^n.$$

Case 2: $r = 3$: Equation (0.0.66) implies that

$$a_n = \frac{n+2}{n(3n+7)} a_{n-1}, \quad n = 1, 2, \dots$$

Substituting $n = 1, 2, 3, \dots$ into this recurrence relation yields

$$a_n = \frac{(n+1)(n+2)}{10 \cdot 13 \cdots (3n+7)} a_0, \quad n = 1, 2, \dots$$

Hence,

$$y_2(x) = x^3 \left[1 + \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{10 \cdot 13 \cdots (3n+7)} x^n \right].$$

14. The indicial equation here is $r(r-1) - \frac{1}{2}r - 1 = 0$, which can be written $r^2 - \frac{3}{2}r - 1 = 0$, with roots $r = 2$ and $r = -\frac{1}{2}$ that do not differ by an integer. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

into the given differential equation and simplifying yields:

$$\sum_{n=0}^{\infty} [2(n+r)(n+r-1) - (n+r) - 2] a_n x^n - \sum_{n=1}^{\infty} [2(n+r-1) - 8] a_{n-1} x^n = 0.$$

When $n = 0$, we obtain $2r(r-1) - r - 2 = 0$. That is,

$$2r^2 - 3r - 2 = (2r+1)(r-2) = 0,$$

with roots $r = 2, -\frac{1}{2}$. The remaining coefficients must satisfy

$$[(n+r)(2n+2r-3) - 2]a_n - [2(n+r-1) - 8]a_{n-1} = 0, \quad n = 1, 2, \dots \quad (0.0.67)$$

Case 1: $r = 2$: Equation (0.0.67) implies that

$$[(n+2)(2n+1) - 2]a_n - [2(n+1) - 8]a_{n-1} = 0, \quad n = 1, 2, \dots$$

so that

$$a_n = \frac{2(n-3)}{n(2n+5)} a_{n-1}, \quad n = 1, 2, \dots$$

Hence,

$$a_1 = -\frac{4}{7}a_0, \quad a_2 = \frac{4}{63}a_0, \quad a_n = 0, \quad n = 3, 4, \dots,$$

so that

$$y_1(x) = x^2 \left(1 - \frac{4}{7}x + \frac{4}{63}x^2 \right).$$

Case 2: $r = -\frac{1}{2}$: Equation (0.0.67) implies that

$$a_n = \frac{2n-11}{n(2n-5)} a_{n-1}, \quad n = 1, 2, \dots,$$

Substituting $n = 1, 2, 3, \dots$ into this recurrence relation yields

$$a_n = -\frac{315}{n!(2n-5)(2n-7)(2n-9)} a_0, \quad n = 1, 2, \dots$$

Hence,

$$y_2(x) = x^{-1/2} \left[1 - \sum_{n=1}^{\infty} \frac{315}{n!(2n-5)(2n-7)(2n-9)} x^n \right].$$

15. The indicial equation here is $r(r-1) + r - 5 = 0$, which can be written $r^2 - 5 = 0$, with roots $r = \pm\sqrt{5}$ that do not differ by an integer. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

into the given differential equation and simplifying yields:

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + (n+r) - 5] a_n x^n - \sum_{n=1}^{\infty} [(n+r-1) + 1] a_{n-1} x^n = 0.$$

When $n = 0$, we obtain $r(r-1) + r - 5 = 0$. That is

$$r^2 - 5 = 0,$$

with roots $r = \pm\sqrt{5}$. The remaining coefficients must satisfy

$$[(n+r)^2 - 5]a_n - (n+r)a_{n-1} = 0, \quad n = 1, 2, \dots \quad (0.0.68)$$

Case 1: $r = \sqrt{5}$: Equation (0.0.68) implies that

$$a_n = \frac{n + \sqrt{5}}{n(n + 2\sqrt{5})} a_{n-1}, \quad n = 1, 2, \dots$$

Substituting $n = 1, 2, 3, \dots$ into this recurrence relation we find

$$a_n = \frac{(1 + \sqrt{5})(2 + \sqrt{5}) \cdots (n + \sqrt{5})}{n!(1 + 2\sqrt{5})(2 + 2\sqrt{5}) \cdots (n + 2\sqrt{5})} a_0, \quad n = 1, 2, \dots$$

Hence,

$$y_1(x) = x^{\sqrt{5}} \left[1 + \sum_{n=1}^{\infty} \frac{(1 + \sqrt{5})(2 + \sqrt{5}) \cdots (n + \sqrt{5})}{n!(1 + 2\sqrt{5})(2 + 2\sqrt{5}) \cdots (n + 2\sqrt{5})} x^n \right].$$

Case 2: $r = -\sqrt{5}$: Equation (0.0.68) implies that

$$a_n = \frac{n - \sqrt{5}}{n^2 - 2n\sqrt{5}} a_{n-1}, \quad n = 1, 2, \dots$$

Substituting $n = 1, 2, 3, \dots$ into this recurrence relation yields

$$a_n = \frac{(1 - \sqrt{5})(2 - \sqrt{5}) \cdots (n - \sqrt{5})}{n!(1 - 2\sqrt{5})(2 - 2\sqrt{5}) \cdots (n - 2\sqrt{5})} a_0, \quad n = 1, 2, \dots$$

Hence,

$$y_2(x) = x^{-1/5} \left[1 + \sum_{n=1}^{\infty} \frac{(1 - \sqrt{5})(2 - \sqrt{5}) \cdots (n - \sqrt{5})}{n!(1 - 2\sqrt{5})(2 - 2\sqrt{5}) \cdots (n - 2\sqrt{5})} x^n \right].$$

16. The indicial equation here is $r(r-1) + \frac{7}{3}r + \frac{1}{3} = 0$, which can be written $r^2 + \frac{4}{3}r + \frac{1}{3} = 0$, with roots $r = -1$ and $r = -\frac{1}{3}$ that do not differ by an integer. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

into the given differential equation and simplifying yields:

$$\sum_{n=0}^{\infty} [3(n+r)(n+r-1) + 7(n+r) + 1] a_n x^n + \sum_{n=1}^{\infty} [3(n+r-1) + 6] a_{n-1} x^n = 0.$$

When $n = 0$, we obtain $3r(r-1) + 7r + 1 = 0$. That is,

$$3r^2 + 4r + 1 = (3r+1)(r+1) = 0,$$

with roots $r = -1, -\frac{1}{3}$. The remaining coefficients must satisfy

$$[(n+r)(3n+3r+4) + 1] a_n + 3(n+r+1) a_{n-1} = 0, \quad n = 1, 2, \dots \quad (0.0.69)$$

Case 1: $r = -1$: Equation (0.0.69) implies that

$$a_n = -\frac{3}{(3n-2)} a_{n-1}, \quad n = 1, 2, \dots$$

Substituting $n = 1, 2, 3, \dots$ into this recurrence relation we find

$$a_n = \frac{(-3)^n}{1 \cdot 4 \cdots (3n-2)} a_0, \quad n = 1, 2, \dots$$

Hence,

$$y_1(x) = x^{-1} \left[1 + \sum_{n=1}^{\infty} \frac{(-3)^n}{1 \cdot 4 \cdots (3n-2)} x^n \right].$$

Case 2: $r = -\frac{1}{3}$: Equation (0.0.69) implies that

$$a_n = -\frac{1}{n} a_{n-1}, \quad n = 1, 2, \dots$$

Substituting $n = 1, 2, 3, \dots$ into this recurrence relation yields

$$a_n = \frac{(-1)^n}{n!} a_0, \quad n = 1, 2, \dots$$

Hence,

$$y_2(x) = x^{-1/3} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = x^{-1/3} e^{-x}.$$

17. (a) In this case

$$p(x) = 1 \quad \text{and} \quad q(x) = 1 - x$$

so that

$$p_0 = p(0) = 1 \quad \text{and} \quad q_0 = q(0) = 1.$$

Therefore the indicial equation is

$$r(r-1) + r + 1 = 0.$$

That is,

$$r^2 + 1 = 0,$$

with roots $r = \pm i$.

(b) Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

into the given differential equation and simplifying yields:

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + (n+r) + 1] a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

When $n = 0$, we obtain the indicial equation already found in (a). The remaining coefficients must satisfy

$$[(n+r)(n+r-1) + (n+r) + 1] a_n - a_{n-1} = 0, \quad n = 1, 2, \dots$$

Letting $r = i$ in the last equation and simplifying gives

$$a_n = \frac{1}{n(n+2i)} a_{n-1} = \frac{n-2i}{n(n^2+4)} a_{n-1}, \quad n = 1, 2, \dots$$

Substituting $n = 1, 2, 3$ into this recurrence relation we find

$$a_1 = \frac{1-2i}{5} a_0, \quad a_2 = -\frac{1}{40} (1+3i) a_0, \quad a_3 = -\frac{1}{1560} (9+7i) a_0.$$

Hence, a complex-valued solution to the given differential equation is

$$\begin{aligned} Y(x) &= x^i \left[\left(1 + \frac{1}{5}x - \frac{1}{40}x^2 - \frac{9}{1560}x^3 + \dots \right) + i \left(-\frac{2}{5}x - \frac{3}{40}x^2 - \frac{7}{1560}x^3 + \dots \right) \right] \\ &= [\cos(\ln x) + i \sin(\ln x)] \cdot \left[\left(1 + \frac{1}{5}x - \frac{1}{40}x^2 - \frac{9}{1560}x^3 + \dots \right) + i \left(-\frac{2}{5}x - \frac{3}{40}x^2 - \frac{7}{1560}x^3 + \dots \right) \right]. \end{aligned}$$

(c) Extracting the real and imaginary parts of the complex-valued solution obtained in (b) yields the following two real-valued solutions

$$\begin{aligned} y_1(x) &= \left(1 + \frac{1}{5}x - \frac{1}{40}x^2 - \frac{9}{1560}x^3 + \dots \right) \cos(\ln x) + \left(\frac{2}{5}x + \frac{3}{40}x^2 + \frac{7}{1560}x^3 + \dots \right) \sin(\ln x), \\ y_2(x) &= \left(1 + \frac{1}{5}x - \frac{1}{40}x^2 - \frac{9}{1560}x^3 + \dots \right) \sin(\ln x) + \left(-\frac{2}{5}x - \frac{3}{40}x^2 - \frac{7}{1560}x^3 + \dots \right) \cos(\ln x), \end{aligned}$$

respectively.

18. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

into the given differential equation and simplifying yields:

$$\sum_{n=0}^{\infty} [3(n+r)(n+r-1) + n+r] a_n x^n + \sum_{n=2}^{\infty} 3(n+r-2) a_{n-2} x^n - \sum_{n=1}^{\infty} 2a_{n-1} x^n = 0.$$

When $n = 0$, we obtain the indicial equation

$$3r(r-1) + r = 0.$$

That is,

$$r(3r-2) = 0,$$

with roots $r = 0, \frac{2}{3}$. When $n = 1$, we have

$$(3r+1)(r+1)a_1 = 2a_0, \quad (0.0.70)$$

and the remaining coefficients must satisfy

$$(n+r)(3n+3r-2)a_n = 2a_{n-1} - 3(2-n-r)a_{n-2}, \quad n = 2, 3, \dots \quad (0.0.71)$$

Case 1: $r = 0$: Equation (0.0.70) implies that

$$a_1 = 2a_0,$$

and Equation (0.0.71) requires

$$a_n = \frac{2a_{n-1} - 3(2-n)a_{n-2}}{n(3n-2)}, \quad n = 2, 3, \dots$$

Substituting $n = 2, 3, 4$ into this recurrence relation yields

$$a_2 = \frac{a_0}{2}, \quad a_3 = \frac{a_0}{3}, \quad a_4 = \frac{11}{120}a_0.$$

Hence,

$$y_1(x) = 1 + 2x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{11}{120}x^4 + \dots$$

Case 2: $r = \frac{2}{3}$: Equation (0.0.70) implies that

$$a_1 = \frac{2}{5}a_0,$$

and Equation (0.0.71) requires

$$a_n = \frac{2a_{n-1} + (3n-4)a_{n-2}}{n(3n+2)}, \quad n = 2, 3, \dots$$

Substituting $n = 2, 3, 4$ into this recurrence relation yields

$$a_2 = \frac{7}{40}a_0, \quad a_3 = \frac{47}{660}a_0, \quad a_4 = \frac{509}{18480}a_0.$$

Hence,

$$y_2(x) = x^{2/3} \left(1 + \frac{2}{5}x + \frac{7}{40}x^2 + \frac{47}{660}x^3 + \frac{509}{18480}x^4 + \dots \right).$$

19. (a) The indicial equation is $r(r-1) + \frac{1}{4} = (r - \frac{1}{2})^2 = 0$, with root $r = \frac{1}{2}$, with multiplicity 2. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

into the given differential equation and simplifying yields:

$$\sum_{n=0}^{\infty} [4(n+r)(n+r-1)+1]a_n x^n - \sum_{n=1}^{\infty} [4(n+r-1)-2]a_{n-1} x^n = 0.$$

When $n = 0$, we obtain $4r^2 - 4r + 1 = 0$. That is,

$$(2r - 1)^2 = 0,$$

which has the repeated root $r = \frac{1}{2}$. The remaining coefficients must satisfy

$$[4(n+r)(n+r-1)+1]a_n - [4(n+r-1)-2]a_{n-1} = 0, \quad n = 1, 2, \dots \quad (0.0.72)$$

Setting $r = \frac{1}{2}$ in Equation (0.0.72) yields

$$a_n = \frac{(n-1)}{n^2} a_{n-1}, \quad n = 1, 2, \dots,$$

which implies that

$$a_n = 0, \quad n = 1, 2, \dots$$

Consequently,

$$y_1(x) = x^{1/2}.$$

(b) Let

$$y_2(x) = y_1(x)u(x) = x^{1/2}u(x).$$

Then

$$\begin{aligned} y_2'(x) &= x^{1/2}u' + \frac{1}{2}x^{-1/2}u, \\ y_2''(x) &= x^{1/2}u'' + x^{-1/2}u' - \frac{1}{4}x^{-3/2}u. \end{aligned}$$

Inserting these expressions into the original differential equation gives

$$4x^2 \left(x^{1/2}u'' + x^{-1/2}u' - \frac{1}{4}x^{-3/2}u \right) - 4x^2 \left(x^{1/2}u' + \frac{1}{2}x^{-1/2}u \right) + (1+2x)x^{1/2}u = 0.$$

That is, upon simplification,

$$u'' + \left(\frac{1}{x} - 1 \right) u' = 0,$$

or equivalently

$$\frac{u''}{u'} = 1 - \frac{1}{x}$$

which can be integrated directly to obtain

$$\ln u' = x - \ln x + c.$$

Exponentiating both sides of this equation, and setting the integration constant to 1, yields

$$u' = \frac{1}{x} e^x,$$

which we write as

$$u' = \frac{1}{x} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n-1}}{n!},$$

or equivalently,

$$u' = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}.$$

Therefore,

$$\begin{aligned} u(x) &= \int \left[\frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \right] dx + c \\ &= \ln x + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}, \end{aligned}$$

where we have set the integration constant to zero. Consequently,

$$y_2(x) = x^{1/2}u(x) = x^{1/2} \left(\ln x + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!} \right).$$

20. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

into the given differential equation and simplifying yields:

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + 3(n+r) + 1] a_n x^n - \sum_{n=1}^{\infty} [2(n+r-1) + 2] a_{n-1} x^n.$$

When $n = 0$, we obtain the indicial equation

$$r(r-1) + 3r + 1 = 0.$$

That is

$$(r+1)^2 = 0,$$

which has the repeated root $r = -1$. The remaining coefficients must satisfy

$$[(n+r)(n+r-1) + 3(n+r) + 1] a_n - [2(n+r-1) + 2] a_{n-1} = 0, \quad n = 1, 2, \dots \quad (0.0.73)$$

Setting $r = -1$ in Equation (0.0.73) and simplifying yields

$$a_n = \frac{2(n-1)}{n^2} a_{n-1}, \quad n = 1, 2, \dots,$$

which implies that

$$a_n = 0, \quad n = 1, 2, \dots$$

Consequently,

$$y_1(x) = x^{-1}.$$

We now use the reduction of order method to obtain a second linearly independent solution of the form

$$y_2(x) = y_1(x)u(x) = x^{-1}u(x).$$

Then

$$\begin{aligned} y_2'(x) &= x^{-1}u' - x^{-2}u, \\ y_2''(x) &= x^{-1}u'' - 2x^{-2}u' + 2x^{-3}u. \end{aligned}$$

Inserting these expressions into the original differential equation gives

$$x^2(x^{-1}u'' - 2x^{-2}u' + 2x^{-3}u) + x(3 - 2x)(x^{-1}u' - x^{-2}u) + (1 - 2x)x^{-1}u = 0.$$

That is, upon simplification,

$$u'' + \frac{1}{x}(1 - 2x)u' = 0,$$

or equivalently

$$\frac{u''}{u'} = 2 - \frac{1}{x}$$

which can be integrated directly to obtain

$$\ln u' = 2x - \ln x + c.$$

Exponentiating both sides of this equation, and setting the integration constant to 1, yields

$$u' = \frac{1}{x}e^{2x},$$

which we write as

$$u' = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^{n-1}}{n!},$$

or equivalently,

$$u' = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^n x^{n-1}}{n!}.$$

Therefore,

$$\begin{aligned} u(x) &= \int \left[\frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^n x^{n-1}}{n!} \right] dx + c \\ &= \ln x + \sum_{n=1}^{\infty} \frac{(2x)^n}{n \cdot n!}, \end{aligned}$$

where we have set the integration constant to zero. Consequently,

$$y_2(x) = x^{-1}u(x) = x^{-1} \left[\ln x + \sum_{n=1}^{\infty} \frac{(2x)^n}{n \cdot n!} \right].$$

Solutions to Section 9.5

True-False Review:

- 1. FALSE.** The indicial equation (9.5.4) in this case reads $r(r-1) - r + 1 = 0$, or $r^2 - 2r + 1 = 0$, and the roots of this equation are $r = 1, 1$, so the roots are not distinct.
- 2. TRUE.** The indicial equation (9.5.4) in this case reads $r(r-1) + (1-2\sqrt{5})r + \frac{19}{4} = 0$, or $r^2 - 2\sqrt{5}r + \frac{19}{4} = 0$. The quadratic formula quickly yields the roots $r = \sqrt{5} \pm \frac{1}{2}$, which are distinct and differ by 1.
- 3. FALSE.** The indicial equation (9.5.4) in this case reads $r(r-1) + 9r + 25 = 0$, or $r^2 + 8r + 25 = 0$. The quadratic formula quickly yields the roots $r = -4 \pm 3i$, which do not differ by an integer.
- 4. TRUE.** The indicial equation (9.5.4) in this case reads $r(r-1) + 4r - \frac{7}{4} = 0$, or $r^2 + 3r - \frac{7}{4} = 0$. The quadratic formula quickly yields the roots $r = -\frac{7}{2}$ and $r = \frac{1}{2}$, which are distinct and differ by 4.
- 5. TRUE.** The indicial equation (9.5.4) in this case reads $r(r-1) = 0$, with roots $r = 0$ and $r = 1$. They differ by 1.
- 6. FALSE.** As indicated in Theorem 9.5.1, if the roots of the indicial equation do not differ by an integer, then two linearly independent solutions to $x^2y'' + xp(x)y' + q(x)y = 0$ can be found as in Equations (9.5.13) and (9.5.14).

Problems:

- 1.** In this case

$$p(x) = \frac{x}{2} \quad \text{and} \quad q(x) = \frac{1}{4}$$

so that

$$p_0 = p(0) = 0 \quad \text{and} \quad q_0 = q(0) = \frac{1}{4}.$$

Therefore the indicial equation is

$$4r^2 - 4r + 1 = 0.$$

That is,

$$(2r - 1)^2 = 0,$$

with the repeated root $r = \frac{1}{2}$. Hence,

$$y_1(x) = x^{1/2} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = y_1 \ln x + x^{1/2} \sum_{n=1}^{\infty} b_n x^n.$$

- 2.** In this case

$$p(x) = \cos x \quad \text{and} \quad q(x) = -2e^x$$

so that

$$p_0 = p(0) = 1 \quad \text{and} \quad q_0 = q(0) = -2.$$

Therefore the indicial equation is

$$r(r-1) + r - 2 = 0.$$

That is,

$$r^2 - 2 = 0,$$

with roots $r = \pm\sqrt{2}$. Hence,

$$y_1(x) = x^{\sqrt{2}} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-\sqrt{2}} \sum_{n=0}^{\infty} b_n x^n.$$

3. In this case

$$p(x) = x \quad \text{and} \quad q(x) = -2 - x$$

so that

$$p_0 = p(0) = 0 \quad \text{and} \quad q_0 = q(0) = -2.$$

Therefore the indicial equation is

$$r(r-1) - 2 = 0.$$

That is,

$$(r+1)(r-2) = 0,$$

with roots $r = 2, -1$. Hence,

$$y_1(x) = x^2 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = A y_1(x) \ln x + x^{-1} \sum_{n=0}^{\infty} b_n x^n.$$

To determine whether A is nonzero we substitute

$$y(x) = x^{-1} \sum_{n=0}^{\infty} a_n x^n$$

into the given differential equation. The resulting recurrence relation is

$$n(n-3)a_n = -(n-3)a_{n-1}, \quad n = 1, 2, \dots$$

Thus, when $n = 3$, we obtain

$$0 \cdot a_3 = 0,$$

so that a second Frobenius solution exists. Consequently, $A = 0$ and

$$y_2(x) = x^{-1} \sum_{n=0}^{\infty} b_n x^n.$$

4. In this case

$$p(x) = 2x \quad \text{and} \quad q(x) = x - \frac{3}{4}$$

so that

$$p_0 = p(0) = 0 \quad \text{and} \quad q_0 = q(0) = -\frac{3}{4}.$$

Therefore the indicial equation is

$$r(r-1) - \frac{3}{4} = 0.$$

That is,

$$4r^2 - 4r - 3 = (2r-3)(2r+1) = 0,$$

with roots $r = \frac{3}{2}, -\frac{1}{2}$. Hence,

$$y_1(x) = x^{3/2} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = A y_1(x) \ln x + x^{-1/2} \sum_{n=0}^{\infty} b_n x^n.$$

To determine whether A is nonzero we substitute

$$y(x) = x^{-1/2} \sum_{n=0}^{\infty} a_n x^n$$

into the given differential equation. The resulting recurrence relation is

$$n(n-2)a_n = -2(n-1)a_{n-1}, \quad n = 1, 2, \dots$$

This implies that $a_1 = 0$, and when $n = 2$ the relation holds no matter how we choose a_2 . For simplicity, we choose $a_2 = 0$ which then gives

$$a_n = 0, \quad n = 3, 4, \dots$$

It follows that a second Frobenius solution exists. Consequently, $A = 0$ and

$$y_2(x) = x^{-1/2} \sum_{n=0}^{\infty} b_n x^n.$$

5. In this case

$$p(x) = 1 \quad \text{and} \quad q(x) = 2x - 1$$

so that

$$p_0 = p(0) = 1 \quad \text{and} \quad q_0 = q(0) = -1.$$

Therefore the indicial equation is

$$r(r-1) + r - 1 = 0.$$

That is,

$$r^2 - 1 = 0,$$

with roots $r = \pm 1$. Hence,

$$y_1(x) = x \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = A y_1(x) \ln x + x^{-1} \sum_{n=0}^{\infty} b_n x^n.$$

To determine whether A is nonzero we substitute

$$y(x) = x^{-1} \sum_{n=0}^{\infty} a_n x^n$$

into the given differential equation. The resulting recurrence relation is

$$n(n-2)a_n = 2a_{n-1}, \quad n = 1, 2, \dots$$

Inserting the values $n = 1$ and $n = 2$ yields, respectively,

$$a_1 = -2a_0, \quad a_1 = 0.$$

Since, by assumption, $a_0 \neq 0$ we have a contradiction. Consequently, the recurrence relation is inconsistent and so the second solution is not of the Frobenius form. Therefore, $A \neq 0$.

6. In this case

$$p(x) = x^2 \quad \text{and} \quad q(x) = -2 - x$$

so that

$$p_0 = p(0) = 0 \quad \text{and} \quad q_0 = q(0) = -2.$$

Therefore the indicial equation is

$$r(r-1) - 2 = 0.$$

That is,

$$r^2 - r - 2 = (r-2)(r+1) = 0,$$

with roots $r = 2, -1$. Hence,

$$y_1(x) = x^2 \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = A y_1(x) \ln x + x^{-1} \sum_{n=0}^{\infty} b_n x^n.$$

To determine whether A is nonzero we substitute

$$y(x) = x^{-1} \sum_{n=0}^{\infty} a_n x^n$$

into the given differential equation to obtain

$$\sum_{n=1}^{\infty} n(n-3)a_n x^n - \sum_{n=1}^{\infty} 2a_{n-1} x^n + \sum_{n=2}^{\infty} (n-3)a_{n-2} x^n = 0.$$

Therefore, $a_1 = -a_0$, and the general recurrence relation is

$$n(n-3)a_n = 2a_{n-1} - (n-3)a_{n-2}, \quad n = 2, 3, \dots$$

When $n = 2$, this gives

$$-2a_2 = 2a_1 + a_0,$$

so that

$$a_2 = \frac{1}{2}a_0.$$

Since $a_0 \neq 0$, it follows that $a_2 \neq 0$. However, when $n = 3$ the recurrence relation requires

$$0 = 2a_2,$$

which is a contradiction. Consequently, the recurrence relation is inconsistent and so the second solution is not of the Frobenius form. Therefore, $A \neq 0$.

7. In this case

$$p(x) = \frac{7e^x}{x^2 + 1} \quad \text{and} \quad q(x) = \frac{9(1 + \tan x)}{x^2 + 1}$$

so that

$$p_0 = p(0) = 7 \quad \text{and} \quad q_0 = q(0) = 9.$$

Therefore the indicial equation is

$$r(r-1) + 7r + 9 = 0.$$

That is,

$$r^2 + 6r + 9 = (r+3)^2 = 0,$$

with the repeated root $r = -3$. Hence,

$$y_1(x) = x^{-3} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = y_1 \ln x + x^{-3} \sum_{n=1}^{\infty} b_n x^n.$$

8. In this case

$$p(x) = 1 - 2x \quad \text{and} \quad q(x) = 2(\alpha - 1)x - \alpha^2$$

so that

$$p_0 = p(0) = 1 \quad \text{and} \quad q_0 = q(0) = -\alpha^2.$$

Therefore the indicial equation is

$$r(r - 1) + r - \alpha^2 = 0.$$

That is,

$$r^2 - \alpha^2 = 0,$$

with roots $r = \pm\alpha$.

Case 1: $\alpha = 0$ This is the case of a repeated root and so the differential equation has only one linearly independent Frobenius series solution.

Case 2: $2\alpha \neq N$ for any integer N In this case the general theory tells us that the differential equation has two linearly independent Frobenius series solutions.

Case 3: $2\alpha = N$ for some integer $N \neq 0$ In this case we check for consistency of the recurrence relation that arises when we use the smallest root of the indicial equation. If $\alpha > 0$ the smallest root is $-\alpha$. Substituting

$$y(x) = x^{-\alpha} \sum_{n=0}^{\infty} a_n x^n$$

into the given differential equation and simplifying yields the recurrence relation

$$n(n - 2\alpha)a_n = 2(n - 2\alpha)a_{n-1}, \quad n = 1, 2, \dots$$

Since this is consistent when $n = 2\alpha$ it follows that a second linearly independent Frobenius series solution exists. Finally, if $\alpha < 0$, then $-\alpha > 0$ and so the smallest root of the indicial equation is α . Substituting

$$y(x) = x^{\alpha} \sum_{n=0}^{\infty} a_n x^n$$

into the given differential equation and simplifying yields the recurrence relation

$$n(n + 2\alpha)a_n = 2(n + 2\alpha)a_{n-1}, \quad n = 1, 2, \dots$$

This is consistent when $n = -2\alpha$ and so a second linearly independent Frobenius series solution exists in this case also.

9. (a) If b is not an integer, then the roots of the indicial equation are $r_1 = b$ and $r_2 = -1$ where $r_1 - r_2$ is not an integer. Thus, the differential equation has two linearly independent Frobenius series solutions of the form

$$y_1(x) = x^b \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = x^{-1} \sum_{n=0}^{\infty} b_n x^n.$$

(b) If $b = -1$ then the roots are equal so that the differential equation has two linearly independent solutions of the form

$$y_1(x) = x^{-1} \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = y_1 \ln x + x^{-1} \sum_{n=1}^{\infty} b_n x^n.$$

(c) If $b = N$, a nonnegative integer, then the roots of the indicial equation are $r_1 = N$ and $r_2 = -1$, so that $r_1 - r_2 = N + 1$ a positive integer. Hence, the differential equation has two linearly independent solutions of the form

$$y_1(x) = x^N \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2(x) = A y_1 \ln x + x^{-1} \sum_{n=0}^{\infty} b_n x^n.$$

We now determine whether or not $A = 0$. Substituting

$$y(x) = x^{-1} \sum_{n=0}^{\infty} a_n x^n$$

into the given differential equation yields the recurrence relation

$$n(n-1-N)a_n = (2-n-\gamma)a_{n-1}, \quad n = 1, 2, \dots$$

For $n = 1, 2, \dots, N$ this gives

$$a_n = \frac{(\gamma-1)\gamma \cdots (\gamma+n-2)}{N(N-1) \cdots (N-n+1) \cdot n!} a_0,$$

and when $n = r_1 - r_2 = N + 1$ consistency requires

$$(N+1) \cdot 0 \cdot a_{N+1} = -\frac{(\gamma-1)\gamma \cdots (\gamma+N-1)}{N(N-1) \cdots (N-n+1) \cdot n!} a_0.$$

Consequently, a second Frobenius series solution will exist only if the right-hand side of the preceding equality is zero from which we can conclude that $A = 0$ (a Frobenius series solution exists) if $\gamma = 1, 0, -1, \dots, (1-N)$; otherwise, $A \neq 0$.

10. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1)-2]a_n x^n + \sum_{n=1}^{\infty} (n+r-1)^2 a_{n-1} x^n = 0.$$

When $n = 0$, we obtain the indicial equation

$$r(r-1)-2=0.$$

That is,

$$r^2 - r - 2 = (r-2)(r+1) = 0,$$

with roots

$$r = r_1 = 2 \quad \text{and} \quad r = r_2 = -1.$$

For $r = 2$: The recurrence relation is

$$a_n = -\frac{(n+1)^2}{n(n+3)}a_{n-1}, \quad n = 1, 2, \dots,$$

which implies that

$$a_n = \frac{(-1)^n 3!(n+1)}{(n+2)(n+3)}a_0, \quad n = 1, 2, \dots,$$

so that

$$y_1(x) = x^2 \left[1 + 6 \sum_{n=1}^{\infty} \frac{(-1)^n (n+1)}{(n+2)(n+3)} x^n \right].$$

For $r = -1$: The recurrence relation is

$$n(n-3)a_n = -(n-2)^2 a_{n-1}, \quad n = 1, 2, \dots,$$

which implies that

$$a_1 = \frac{1}{2}a_0, \quad a_2 = 0,$$

so that the consistency requirement when $n = r_1 - r_2 = 3$ is:

$$3 \cdot 0 \cdot a_3 = 0,$$

which will be satisfied for any choice of a_3 . For simplicity we choose $a_3 = 0$ in which case all of the remaining coefficients are also zero. Therefore a second linearly independent solution to the given differential equation is

$$y_2(x) = x^{-1} \left(1 + \frac{x}{2} \right).$$

11. (a) Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + 3(n+r) + 1] a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

When $n = 0$, we obtain the indicial equation

$$r(r-1) + 3r + 1 = 0.$$

That is,

$$r^2 + 2r + 1 = (r+1)^2 = 0,$$

with the repeated root $r = -1$.

(b) When $r = -1$, we obtain the recurrence relation

$$a_n = \frac{1}{n^2} a_{n-1}, \quad n = 1, 2, \dots,$$

or equivalently,

$$a_n = \frac{1}{(n!)^2} a_0, \quad n = 1, 2, \dots$$

Consequently,

$$y_1(x) = x^{-1} \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}.$$

(c) Substituting

$$y_2(x) = y_1(x) \ln x + x^{-1} \sum_{n=1}^{\infty} b_n x^n$$

into the given differential equation gives

$$\begin{aligned} x^2 y_1'' \ln x + 2xy_1' - y_1 + \sum_{n=1}^{\infty} (n-1)(n-2)b_n x^{n-1} + 3xy_1' \ln x + 3y_1 + 3 \sum_{n=1}^{\infty} (n-1)b_n x^{n-1} \\ + y_1 \ln x + \sum_{n=1}^{\infty} b_n x^{n-1} - xy_1 \ln x - \sum_{n=1}^{\infty} b_n x^n = 0, \end{aligned}$$

which simplifies to

$$2xy_1' + 2y_1 + \sum_{n=1}^{\infty} n^2 b_n x^{n-1} - \sum_{n=2}^{\infty} b_{n-1} x^{n-1} = 0.$$

Substituting

$$y_1(x) = x^{-1} \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

into the last equation gives

$$2 \sum_{n=0}^{\infty} \frac{(n-1)}{(n!)^2} x^{n-1} + 2 \sum_{n=0}^{\infty} \frac{1}{(n!)^2} x^{n-1} + \sum_{n=1}^{\infty} n^2 b_n x^{n-1} - \sum_{n=2}^{\infty} b_{n-1} x^{n-1} = 0.$$

The terms corresponding to $n = 0$ cancel, and the $n = 1$ terms require

$$2 + b_1 = 0,$$

so that $b_1 = -2$. For $n \geq 2$ we obtain the recurrence relation

$$b_n = \frac{1}{n^2} \left[b_{n-1} - \frac{2}{n! \cdot (n-1)!} \right], \quad n = 2, 3, \dots$$

Evaluating the last expression for $n = 2, 3$ yields

$$b_2 = -\frac{3}{4}, \quad b_3 = -\frac{11}{108}.$$

Thus,

$$y_2(x) = y_1 \ln x + x^{-1} \left(-2x - \frac{3}{4}x^2 - \frac{11}{108}x^3 + \dots \right)$$

or equivalently,

$$y_2(x) = y_1 \ln x - \left(2 + \frac{3}{4}x + \frac{11}{108}x^2 + \dots \right).$$

12. (a) Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

When $n = 0$, we obtain the indicial equation

$$r(r-1) = 0,$$

with roots

$$r = r_1 = 1 \quad \text{and} \quad r = r_2 = 0.$$

When $r = 1$ we obtain the recurrence relation

$$a_n = \frac{1}{n(n+1)} a_{n-1}, \quad n = 1, 2, \dots,$$

which implies

$$a_n = \frac{1}{n!(n+1)!} a_0, \quad n = 1, 2, \dots$$

Thus,

$$y_1(x) = x \sum_{n=0}^{\infty} \frac{x^n}{n!(n+1)!}.$$

(b) When $r = 0$, we obtain the recurrence relation

$$n(n-1)a_n = a_{n-1}.$$

When $n = 1$, consistency requires that $a_0 = 0$, which contradicts our assumption that $a_0 \neq 0$. Hence, there does not exist a second linearly independent Frobenius series solution.

(c) Let

$$y_2(x) = Ay_1 \ln x + \sum_{n=0}^{\infty} b_n x^n.$$

Then

$$\begin{aligned} y_2'(x) &= A(y_1' \ln x + x^{-1}y_1) + \sum_{n=0}^{\infty} nb_n x^{n-1}, \\ y_2''(x) &= A(y_1'' \ln x + 2x^{-1}y_1' - x^{-2}y_1) + \sum_{n=0}^{\infty} n(n-1)b_n x^{n-2}. \end{aligned}$$

Substituting these results into the given differential equation and simplifying yields

$$Ay_1'' x \ln x + 2Ay_1' - Ax^{-1}y_1 + \sum_{n=0}^{\infty} n(n-1)b_n x^{n-1} - Ay_1 \ln x - \sum_{n=0}^{\infty} b_n x^n = 0,$$

or equivalently (using the fact that $xy_1'' - y_1 = 0$),

$$2Ay_1' - Ax^{-1}y_1 + \sum_{n=0}^{\infty} n(n-1)b_n x^{n-1} - \sum_{n=1}^{\infty} b_{n-1} x^{n-1} = 0.$$

Inserting

$$y_1(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!(n+1)!}$$

into the last equation yields

$$2A \sum_{n=0}^{\infty} \frac{1}{(n!)^2} x^n - A \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!} x^n + \sum_{n=-1}^{\infty} n(n+1)b_{n+1} x^n - \sum_{n=0}^{\infty} b_n x^n = 0.$$

The $n = -1$ term cancels and the $n = 0$ terms yield $b_0 = A$. For $n > 0$, we obtain

$$b_{n+1} = \frac{b_n}{n(n+1)} - \frac{(2n+1)A}{n[(n+1)!]^2}, \quad n = 1, 2, \dots$$

Setting $b_0 = 1 = A$ and $b_1 = 0$ (without loss of generality) it follows that

$$b_2 = -\frac{3}{4}, \quad \text{and} \quad b_3 = -\frac{7}{36},$$

so that

$$y_2(x) = y_1 \ln x + \left(1 - \frac{3}{4}x^2 - \frac{7}{36}x^3 + \dots\right).$$

13. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + n+r-1] a_n x^n - \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^n = 0.$$

When $n = 0$, we obtain the indicial equation

$$r(r-1) + r - 1 = 0.$$

That is,

$$r^2 - 1 = 0,$$

with roots

$$r = r_1 = 1, \quad r = r_2 = -1.$$

For $r = 1$: We obtain the recurrence relation

$$a_n = \frac{1}{(n+2)} a_{n-1}, \quad n = 1, 2, \dots,$$

which implies

$$a_n = \frac{2}{(n+2)!} a_0, \quad n = 1, 2, \dots$$

Consequently,

$$y_1(x) = x \sum_{n=0}^{\infty} \frac{2}{(n+2)!} x^n$$

or equivalently,

$$y_1(x) = 2x^{-1}(e^x - x - 1).$$

Note, however, that since the solution set to this differential equation is linear, any multiple of the given solution $y_1(x)$ can also be taken. For instance, $y_1(x) = x^{-1}(e^x - x - 1)$ is also acceptable.

For $r = -1$: We obtain the recurrence relation

$$(n-2)na_n = (n-2)a_{n-1}, \quad n = 1, 2, \dots$$

Thus, $a_1 = a_0$ and consistency when $n = 2$ requires

$$0 \cdot a_2 = 0 \cdot a_1,$$

which is satisfied by any choice of a_2 . We set $a_2 = 0$, in which case all of the subsequent coefficients are also zero. Consequently, if we take $a_0 = a_1 = 1$, then we have

$$y_2(x) = x^{-1}(1 + x).$$

14. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + 6(n+r) + 6] a_n x^n + \sum_{n=2}^{\infty} (n+r-2) a_{n-2} x^n = 0.$$

When $n = 0$, we obtain the indicial equation

$$r^2 + 5r + 6 = (r+2)(r+3) = 0,$$

with roots

$$r = r_1 = -2, \quad r = r_2 = -3.$$

When $n = 1$, we obtain

$$[r(r+1) + 6(r+1) + 6] a_1 = 0, \tag{0.0.74}$$

and when $n \geq 2$ we obtain

$$[(n+r)(n+r+5) + 6] a_n = -(n+r-2) a_{n-2}, \quad n = 2, 3, \dots \tag{0.0.75}$$

For $r = -2$, Equation (0.0.74) implies that $a_1 = 0$ and from Equation (0.0.75)

$$a_n = -\frac{(n-4)}{n(n+1)} a_{n-2}, \quad n = 2, 3, \dots$$

It follows that all odd coefficients are zero, and the even coefficients must satisfy

$$a_{2k} = -\frac{(k-2)}{k(2k+1)}a_{2k-2}, \quad k = 1, 2, \dots,$$

which gives

$$a_2 = \frac{1}{3}a_0, \quad a_{2n} = 0, \quad n = 2, 3, \dots$$

Consequently,

$$y_1(x) = x^{-2} \left(1 + \frac{1}{3}x^2 \right)$$

For $r = -3$, Equation (0.0.74) requires

$$0 \cdot a_1 = 0$$

which is satisfied for any value of a_1 . We choose $a_1 = 0$. From Equation (0.0.75) the remaining coefficients must satisfy

$$a_n = -\frac{(n-5)}{n(n-1)}a_{n-2}, \quad n = 2, 3, \dots$$

It follows that all of the odd coefficients are zero, and the even coefficients are given by

$$a_{2k} = -\frac{(2k-5)}{2k(2k-1)}a_{2k-2}, \quad k = 1, 2, \dots,$$

so that

$$a_{2k} = \frac{(-1)^k(-3)(-1)\cdots(2k-5)}{2^k k! [1 \cdot 3 \cdots (2k-1)]} a_0.$$

Hence, a second linearly independent solution to the given differential equation is

$$y_2(x) = x^{-3} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n(-3)(-1)\cdots(2n-5)}{2^n n! [1 \cdot 3 \cdots (2n-1)]} x^{2n} \right].$$

15. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + n+r] a_n x^n - 2 \sum_{n=1}^{\infty} a_{n-1} x^n = 0. \quad (0.0.76)$$

When $n = 0$, we obtain the indicial equation

$$r^2 = 0,$$

which has the repeated root $r = 0$. Inserting this value of r into Equation (0.0.76) leads to the recurrence relation

$$a_n = \frac{2}{n^2} a_{n-1}, \quad n = 1, 2, \dots$$

so that

$$a_n = \frac{2^n}{(n!)^2} a_0.$$

Consequently,

$$y_1(x) = \sum_{n=0}^{\infty} \frac{2^n}{(n!)^2} x^n.$$

Now substitute

$$y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} b_n x^n$$

into the given differential equation to obtain

$$\begin{aligned} x \left[y_1'' \ln x + 2y_1' x^{-1} - y_1 x^{-2} + \sum_{n=1}^{\infty} n(n-1)b_n x^{n-2} \right] + \left[y' \ln x + y_1 x^{-1} + \sum_{n=1}^{\infty} n b_n x^{n-1} \right] \\ - 2 \left[y_1 \ln x + \sum_{n=1}^{\infty} b_n x^n \right] = 0, \end{aligned}$$

which simplifies to

$$2y_1' - 2 \sum_{n=1}^{\infty} b_n x^n + \sum_{n=1}^{\infty} n^2 b_n x^{n-1} = 0.$$

Multiplying this equation through by x , this becomes

$$2xy_1' - 2 \sum_{n=1}^{\infty} b_n x^{n+1} + \sum_{n=1}^{\infty} n^2 b_n x^n = 0.$$

Substituting

$$y_1(x) = \sum_{n=0}^{\infty} \frac{2^n}{(n!)^2} x^n$$

into the preceding equation and simplifying yields

$$2 \sum_{n=1}^{\infty} \frac{n 2^n}{(n!)^2} x^n + \sum_{n=1}^{\infty} n^2 b_n x^n - 2 \sum_{n=2}^{\infty} b_{n-1} x^n = 0.$$

Consequently, by examining the $n = 1$ terms, we have

$$4 + b_1 = 0 \implies b_1 = -4,$$

and

$$b_n = \frac{1}{n^2} \left[2b_{n-1} - \frac{2^{n+1}}{n!(n-1)!} \right], \quad n = 2, 3, \dots$$

Therefore,

$$b_2 = -3, \quad \text{and} \quad b_3 = -\frac{22}{27}.$$

Hence, a second linearly independent solution to the given differential equation is

$$y_2(x) = y_1(x) \ln x - x \left(4 + 3x + \frac{22}{27} x^2 + \dots \right).$$

16. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} [4(n+r)(n+r-1) + 1] a_n x^n - \sum_{n=1}^{\infty} 4a_{n-1} x^n = 0. \quad (0.0.77)$$

When $n = 0$, we obtain the indicial equation

$$4r(r-1) + 1 = 4r^2 - 4r + 1 = (2r-1)^2 = 0,$$

which has the repeated root

$$r = \frac{1}{2}.$$

Inserting this value for r into Equation (0.0.77) leads to the recurrence relation

$$a_n = \frac{1}{n^2} a_{n-1}, \quad n = 1, 2, \dots,$$

which implies

$$a_n = \frac{1}{(n!)^2} a_0, \quad n = 1, 2, \dots$$

Thus,

$$y_1(x) = x^{1/2} \left[1 + \sum_{n=1}^{\infty} \frac{x^n}{(n!)^2} \right],$$

or equivalently,

$$y_1(x) = x^{1/2} \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}.$$

Now substitute

$$y_2(x) = y_1(x) \ln x + x^{1/2} \sum_{n=1}^{\infty} b_n x^n$$

into the given differential equation to obtain

$$4x^2 \left[y_1'' \ln x + 2x^{-1} y_1' - x^{-2} y_1 + \sum_{n=1}^{\infty} (n^2 - 1/4) b_n x^{n-3/2} \right] + (1 - 4x) \left(y_1 \ln x + x^{1/2} \sum_{n=1}^{\infty} b_n x^n \right) = 0,$$

which simplifies to

$$2xy_1' - y_1 + \sum_{n=1}^{\infty} n^2 b_n x^{n+1/2} - \sum_{n=1}^{\infty} b_n x^{n+3/2} = 0.$$

Substituting

$$y_1(x) = \sum_{n=0}^{\infty} \frac{x^{n+1/2}}{(n!)^2}$$

into the preceding equation and simplifying yields

$$\sum_{n=0}^{\infty} \frac{2nx^n}{(n!)^2} + \sum_{n=1}^{\infty} n^2 b_n x^n - \sum_{n=2}^{\infty} b_{n-1} x^n = 0,$$

which, according to the $n = 1$ terms, requires

$$2 + b_1 = 0,$$

and

$$b_n = \frac{1}{n^2} b_{n-1} - \frac{2}{n(n!)^2}, \quad n = 2, 3, \dots$$

Therefore,

$$b_1 = -2, \quad b_2 = -\frac{3}{4}, \quad b_3 = -\frac{11}{108}, \quad \dots$$

Consequently,

$$y_2(x) = y_1 \ln x - x^{3/2} \left(2 + \frac{3}{4}x + \frac{11}{108}x^2 + \dots \right).$$

17. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) - 3(n+r) + 4] a_n x^n - \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^n = 0. \quad (0.0.78)$$

When $n = 0$, we obtain the indicial equation

$$r(r-1) - 3r + 4 = r^2 - 4r + 4 = (r-2)^2 = 0,$$

which has the repeated root

$$r = 2.$$

Inserting this value for r into Equation (0.0.78) leads to the recurrence relation

$$a_n = \frac{(n+1)}{n^2} a_{n-1}, \quad n = 1, 2, \dots,$$

which implies

$$a_n = \frac{(n+1)}{n!} a_0, \quad n = 1, 2, \dots$$

Thus,

$$y_1(x) = x^2 \sum_{n=0}^{\infty} \frac{(n+1)}{n!} x^n.$$

Now substitute

$$y_2(x) = y_1(x) \ln x + x^2 \sum_{n=1}^{\infty} b_n x^n$$

into the given differential equation to obtain

$$x^2 \left[y_1'' \ln x + 2x^{-1}y_1' - x^{-2}y_1 + \sum_{n=1}^{\infty} (n+2)(n+1)b_n x^n \right] - x^2 \left[y_1' \ln x + x^{-1}y_1 + \sum_{n=1}^{\infty} (n+2)b_n x^{n+1} \right] \\ - 3x \left[y_1' \ln x + x^{-1}y_1 + \sum_{n=1}^{\infty} (n+2)b_n x^{n+1} \right] + 4 \left(y_1 \ln x + x^2 \sum_{n=1}^{\infty} b_n x^n \right) = 0,$$

which simplifies to

$$2xy_1' - 4y_1 - xy_1 + \sum_{n=1}^{\infty} n^2 b_n x^{n+2} - \sum_{n=2}^{\infty} (n+1)b_{n-1} x^{n+2} = 0.$$

Substituting

$$y_1(x) = x^2 \sum_{n=0}^{\infty} \frac{(n+1)}{n!} x^n$$

into the preceding equation and simplifying yields

$$\sum_{n=1}^{\infty} \frac{n+2}{(n-1)!} x^n + \sum_{n=1}^{\infty} n^2 b_n x^n - \sum_{n=2}^{\infty} (n+1)b_{n-1} x^n = 0,$$

which requires

$$3 + b_1 = 0,$$

and

$$b_n = \frac{1}{n^2} \left[(n+1)b_{n-1} - \frac{n+2}{(n-1)!} \right], \quad n = 2, 3, \dots$$

Therefore,

$$b_1 = -3, \quad b_2 = -\frac{13}{4}, \quad b_3 = -\frac{31}{18}, \quad \dots$$

Consequently,

$$y_2(x) = y_1(x) \ln x - x^3 \left(3 + \frac{13}{4}x + \frac{31}{18}x^2 + \dots \right).$$

18. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

into the given differential equation and simplifying yields:

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + n+r-1] a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0. \quad (0.0.79)$$

When $n = 0$, we obtain the indicial equation

$$r(r-1) + r - 1 = 0.$$

That is,

$$r^2 - 1 = 0,$$

with roots $r = \pm 1$. Inserting $r = 1$ into Equation (0.0.79) leads to the recurrence relation

$$a_n = \frac{1}{n(n+2)}a_{n-1}, \quad n = 1, 2, \dots,$$

which implies that

$$a_n = \frac{2}{n!(n+2)!}a_0, \quad n = 1, 2, \dots$$

Consequently, one solution to the given differential equation is

$$y_1(x) = \sum_{n=0}^{\infty} \frac{1}{n!(n+2)!}x^n,$$

where we have set $a_0 = 1/2$ without loss of generality. Inserting $r = -1$ into Equation (0.0.79) leads to the recurrence relation

$$(n-2)na_n = a_{n-1}, \quad n = 1, 2, \dots,$$

which requires that $a_1 = -a_0$, and $0 \cdot a_2 = -a_0$. Since $a_0 \neq 0$, this implies that a second Frobenius series solution does not exist. Thus, we let

$$y_2(x) = Ay_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n-1}.$$

Substituting into the given differential equation and simplifying yields

$$2Axy_1' + \sum_{n=0}^{\infty} n(n-2)b_n x^{n-1} - \sum_{n=1}^{\infty} b_{n-1} x^{n-1} = 0.$$

Inserting

$$y_1(x) = \sum_{n=0}^{\infty} \frac{1}{n!(n+2)!}x^n$$

into the preceding equation and simplifying we find

$$2A \sum_{n=1}^{\infty} \frac{1}{(n-1)!(n+2)!}x^n + \sum_{n=0}^{\infty} n(n-2)b_n x^{n-1} - \sum_{n=1}^{\infty} b_{n-1} x^{n-1} = 0,$$

which, upon replacing n by $n+1$ in the middle summation, can be written in the equivalent form

$$2A \sum_{n=1}^{\infty} \frac{1}{(n-1)!(n+2)!}x^n + \sum_{n=0}^{\infty} (n^2-1)b_{n+1}x^n - \sum_{n=0}^{\infty} b_n x^n = 0.$$

This requires

$$-b_1 - b_0 = 0, \quad b_1 = \frac{1}{3}A, \quad b_{n+1} = \frac{1}{n^2-1} \left[b_n - \frac{2A}{(n-1)!(n+2)!} \right], \quad n = 2, 3, \dots$$

We choose $b_0 = -1$ in which case $b_1 = 1$, and $A = 3$. Finally, we choose $b_2 = 0$. Then,

$$b_3 = -1/12, \quad b_4 = -13/960, \quad \dots$$

so a second linearly independent solution to the given differential equation is

$$y_2(x) = 3y_1(x) \ln x - x^{-1} \left(1 - x + \frac{1}{12}x^3 + \frac{13}{960}x^4 + \cdots \right).$$

19. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1)-2]a_n x^n - \sum_{n=1}^{\infty} (n+r+2)a_{n-1} x^n = 0. \quad (0.0.80)$$

When $n = 0$ we obtain the indicial equation

$$r(r-1) - 2 = r^2 - r - 2 = 0,$$

with roots $r = 2, -1$. Inserting $r = 2$ into Equation (0.0.80) and simplifying yields the recurrence relation

$$a_n = \frac{n+4}{n(n+3)} a_{n-1}, \quad n = 1, 2, \dots,$$

which implies that

$$a_n = \frac{n+4}{4(n!)} a_0, \quad n = 1, 2, \dots$$

Thus, one solution to the given differential equation is

$$y_1(x) = x^2 \left(1 + \sum_{n=1}^{\infty} \frac{n+4}{4(n!)} x^n \right).$$

We now insert $r = -1$ into Equation (0.0.80) and simplify to obtain the recurrence relation

$$n(n-3)a_n = (n+1)a_{n-1}, \quad n = 1, 2, \dots$$

This implies that

$$a_1 = -a_0 \quad \text{and} \quad a_2 = \frac{3}{2}a_1,$$

so that the consistency condition for the existence of a second Frobenius series solution is

$$0 \cdot a_3 = 4a_2 = 6a_0,$$

which cannot hold since $a_0 \neq 0$. Consequently, we must seek a second solution of the form

$$y_2(x) = Ay_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n-1}.$$

Substituting this expression for y_2 into the given differential equation and simplifying yields

$$2Axy_1' - Ay_1 - Axy_1 + \sum_{n=0}^{\infty} n(n-3)b_n x^{n-1} - \sum_{n=0}^{\infty} (n+2)b_n x^n = 0.$$

Now insert

$$y_1(x) = x^2 \left[1 + \sum_{n=1}^{\infty} \frac{(n+4)}{4(n!)} x^n \right]$$

into the preceding equation and simplify to obtain

$$A \left[3x^2 - x^3 + \sum_{n=1}^{\infty} \frac{(n+4)(2n+3)}{4(n!)} x^{n+2} - \sum_{n=1}^{\infty} \frac{(n+4)}{4(n!)} x^{n+3} \right] + \sum_{n=0}^{\infty} n(n-3)b_n x^{n-1} - \sum_{n=0}^{\infty} (n+2)b_n x^n = 0,$$

which can be written in the equivalent form

$$A \left[3x^2 - x^3 + \sum_{n=3}^{\infty} \frac{(n+2)(2n-1)}{4 \cdot (n-2)!} x^n - \sum_{n=4}^{\infty} \frac{(n+1)}{4 \cdot (n-3)!} x^n \right] + \sum_{n=0}^{\infty} (n+1)(n-2)b_{n+1} x^n - \sum_{n=0}^{\infty} (n+2)b_n x^n = 0.$$

Letting $n = 0, 1, 2, 3$ we find that

$$b_1 = -b_0, \quad b_2 = \frac{3}{2}b_0, \quad A = 2b_0, \quad b_4 = -\frac{21}{8}b_0,$$

where we have set $b_3 = 0$ without loss of generality. The general recurrence relation is

$$b_{n+1} = \frac{1}{(n+1)(n-2)} \left[(n+2)b_n - \frac{An(n+4)}{4 \cdot (n-2)!} \right], \quad n = 4, 5, \dots$$

Consequently, a second linearly independent solution to the given differential equation is

$$y_2(x) = 2y_1(x) \ln x + x^{-1} \left(1 - x + \frac{3}{2}x^2 - \frac{21}{8}x^4 + \dots \right).$$

20. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) - 2]a_n x^n - \sum_{n=1}^{\infty} (n+r-1)a_{n-1} x^n = 0. \quad (0.0.81)$$

When $n = 0$, we obtain the indicial equation

$$r(r-1) - 2 = r^2 - r - 2 = 0,$$

with roots $r = 2, -1$. Inserting $r = 2$ into Equation (0.0.81) and simplifying yields the recurrence relation

$$a_n = \frac{(n+1)}{n(n+3)} a_{n-1}, \quad n = 1, 2, \dots,$$

which implies that

$$a_n = \frac{6(n+1)}{(n+3)!} a_0, \quad n = 1, 2, \dots$$

The choice $a_0 = \frac{1}{6}$ yields the solution

$$y_1(x) = x^2 \sum_{n=0}^{\infty} \frac{(n+1)}{(n+3)!} x^n.$$

Inserting $r = -1$ into Equation (0.0.81) and simplifying yields the recurrence relation

$$n(n-3)a_n = (n-2)a_{n-1}, \quad n = 1, 2, \dots$$

This implies that

$$a_1 = \frac{1}{2}a_0, \quad \text{and} \quad a_2 = 0,$$

so that the consistency condition for the existence of a second Frobenius series solution is

$$3 \cdot 0 \cdot a_3 = 0$$

which is satisfied for any choice of a_3 . We choose $a_3 = 0$ in which case all subsequent coefficients are also zero. Therefore, a second linearly independent solution to the given differential equation is

$$y_2(x) = x^{-1} \left(1 + \frac{1}{2}x \right).$$

21. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} [4(n+r)(n+r-1) + 4(n+r) - 9] a_n x^n - \sum_{n=1}^{\infty} [4(n+r-1) - 2] a_{n-1} x^n = 0. \quad (0.0.82)$$

When $n = 0$ we obtain the indicial equation

$$4r(r-1) + 4r - 9 = 4r^2 - 9 = 0,$$

with roots $r = \pm \frac{3}{2}$. Inserting $r = \frac{3}{2}$ into Equation (0.0.82) and simplifying yields the recurrence relation

$$a_n = \frac{1}{n+3} a_{n-1}, \quad n = 1, 2, \dots$$

which implies that

$$a_n = \frac{3!}{(n+3)!} a_0, \quad n = 1, 2, \dots$$

Choosing $a_0 = \frac{1}{6}$ gives

$$y_1(x) = x^{3/2} \sum_{n=0}^{\infty} \frac{1}{(n+3)!} x^n,$$

or equivalently,

$$y_1(x) = x^{-3/2} \sum_{n=0}^{\infty} \frac{1}{(n+3)!} x^{n+3} = x^{-3/2} \left(e^x - 1 - x - \frac{1}{2}x^2 \right).$$

Inserting $r = -\frac{3}{2}$ into Equation (0.0.82) and simplifying yields the recurrence relation

$$n(n-3)a_n = (n-3)a_{n-1}, \quad n = 1, 2, \dots$$

This implies that

$$a_1 = a_0, \quad a_2 = \frac{1}{2}a_0,$$

and the consistency condition for the existence of a second Frobenius solution is

$$3 \cdot 0 \cdot a_3 = 0 \cdot a_2,$$

which is satisfied for any choice of a_3 . We choose $a_3 = 0$ in which case all subsequent coefficients are also zero. Consequently, a second linearly independent solution to the given differential equation is

$$y_2(x) = x^{-3/2} \left(1 + x + \frac{1}{2}x^2 \right).$$

Comparing this solution with $y_1(x)$ we see that

$$y(x) = y_1(x) + y_2(x) = x^{-3/2}e^x$$

is also a (linearly dependent) solution to the given differential equation.

22. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + 5(n+r) + 4] a_n x^n - \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^n = 0. \quad (0.0.83)$$

When $n = 0$, we obtain the indicial equation

$$r^2 + 4r + 4 = 0$$

with the repeated root $r = -2$. Inserting this value of r into Equation (0.0.83) and simplifying gives the recurrence relation

$$a_n = \frac{(n-3)}{n^2} a_{n-1}, \quad n = 1, 2, \dots,$$

which implies that

$$a_1 = -2a_0, \quad a_2 = \frac{1}{2}a_0, \quad a_n = 0, \quad n = 3, 4, \dots$$

Hence,

$$y_1(x) = x^{-2} \left(1 - 2x + \frac{1}{2}x^2 \right).$$

To find a second linearly independent solution, we substitute

$$y_2(x) = y_1(x) \ln x + x^{-2} \sum_{n=1}^{\infty} b_n x^n$$

into the given differential equation. This gives, after simplification,

$$2xy_1' + 4y_1 - xy_1 + \sum_{n=1}^{\infty} n^2 b_n x^{n-2} - \sum_{n=1}^{\infty} (n-2)b_n x^{n-1} = 0.$$

Inserting

$$y_1(x) = x^{-2} \left(1 - 2x + \frac{1}{2}x^2 \right)$$

into the preceding equation and simplifying yields

$$(b_1 - 5)x^{-1} + (4 + 4b_2 + b_1) + \left(9b_3 - \frac{1}{2} \right) x + \sum_{n=4}^{\infty} [n^2 b_n - (n-3)b_{n-1}]x^{n-2} = 0.$$

Consequently,

$$b_1 = 5, \quad b_2 = -\frac{9}{4}, \quad b_3 = \frac{1}{18},$$

and

$$b_n = \frac{(n-3)}{n^2} b_{n-1}, \quad n = 4, 5, \dots$$

The preceding recurrence relation implies that

$$b_n = \frac{2}{n(n-1)(n-2)n!}, \quad n = 4, 5, \dots$$

Thus, a second linearly independent solution to the given differential equation is

$$\begin{aligned} y_2(x) &= y_1(x) \ln(x) + x^{-2} \left[5x - \frac{9}{4}x^2 + \frac{1}{18}x^3 + 2 \sum_{n=4}^{\infty} \frac{1}{n(n-1)(n-2)n!} x^n \right] \\ &= x^{-2} \left[\left(1 - 2x + \frac{1}{2}x^2 \right) \ln(x) + 5x - \frac{9}{4}x^2 + \frac{1}{18}x^3 + 2 \sum_{n=4}^{\infty} \frac{1}{n(n-1)(n-2)n!} x^n \right]. \end{aligned}$$

23. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) - (n+r) + 1] a_n x^n + \sum_{n=1}^{\infty} [(n+r-1) - 1] a_{n-1} x^n = 0. \quad (0.0.84)$$

When $n = 0$, we obtain the indicial equation

$$r^2 - 2r + 1 = (r-1)^2 = 0,$$

which has the repeated root $r = 1$. Inserting $r = 1$ into Equation (0.0.84) and simplifying yields

$$a_n = -\frac{(n-1)}{n^2} a_{n-1}, \quad n = 1, 2, \dots,$$

so that $a_1 = a_2 = \cdots = 0$. Consequently,

$$y_1(x) = x.$$

To obtain a second linearly independent solution we substitute

$$y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} b_n x^{n+1} = x \ln x + \sum_{n=1}^{\infty} b_n x^{n+1}$$

into the given differential equation and simplify to obtain

$$(1 + b_1)x^2 + \sum_{n=2}^{\infty} [n^2 b_n + (n-1)b_{n-1}]x^{n+1} = 0.$$

Hence,

$$b_1 = -1 \quad \text{and} \quad b_n = -\frac{(n-1)}{n^2} b_{n-1}, \quad n = 2, 3, \dots,$$

from which we find

$$b_n = \frac{(-1)^n}{n(n!)}, \quad n = 2, 3, \dots$$

Therefore, a second linearly independent solution to the given differential equation is

$$y_2(x) = x \ln x + \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n!)} x^{n+1}.$$

24. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + 4(n+r) + 2] a_n x^n + \sum_{n=1}^{\infty} [2(n+r-1) + 2] a_{n-1} x^n = 0. \quad (0.0.85)$$

When $n = 0$, we obtain the indicial equation

$$r(r-1) + 4r + 2 = r^2 + 3r + 2 = (r+1)(r+2) = 0,$$

with roots $r = -1, -2$. Inserting $r = -1$ into Equation (0.0.85) and simplifying yields

$$a_n = -\frac{2(n-1)}{n(n+1)} a_{n-1}, \quad n = 1, 2, \dots,$$

so that

$$a_1 = a_2 = \cdots = 0.$$

Consequently,

$$y_1(x) = x^{-1}.$$

Inserting $r = -2$ into Equation (0.0.85) and simplifying yields

$$n(n-1)a_n = 2(n-2)a_{n-1}, \quad n = 1, 2, \dots$$

When $n = 1$, this requires

$$0 = -2a_0,$$

which is inconsistent since $a_0 \neq 0$. Consequently, we substitute

$$y_2(x) = Ay_1(x) \ln x + x^{-2} \sum_{n=0}^{\infty} b_n x^n = Ax^{-1} \ln x + x^{-2} \sum_{n=0}^{\infty} b_n x^n$$

into the given differential equation to determine a second linearly independent solution. After simplifying we obtain

$$(A - 2b_0)x^{-1} + 2(A + b_2) + \sum_{n=3}^{\infty} [n(n-1)b_n + 2(n-2)b_{n-1}]x^{n-2} = 0.$$

Hence,

$$A = 2b_0, \quad b_2 = -2b_0, \quad b_n = -\frac{2(n-2)}{n(n-1)}b_{n-1}, \quad n = 3, 4, \dots,$$

from which we find

$$b_n = \frac{(-1)^{n+1} \cdot 2^n}{n! \cdot (n-1)} b_0, \quad n = 3, 4, \dots$$

Setting $b_0 = 1$, and $b_1 = 0$ (without loss of generality) yields

$$y_2(x) = 2x^{-1} \ln x + x^{-2} \left[1 + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 2^n}{n! \cdot (n-1)} x^n \right].$$

25. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} [4(n+r)(n+r-1) - 3]a_n x^n - \sum_{n=1}^{\infty} 4a_{n-1} x^n = 0. \quad (0.0.86)$$

When $n = 0$, we obtain the indicial equation

$$4r(r-1) - 3 = 4r^2 - 4r - 3 = (2r-3)(2r+1) = 0,$$

with roots $r = \frac{3}{2}, -\frac{1}{2}$. Inserting $r = \frac{3}{2}$ into Equation (0.0.86) and simplifying yields

$$a_n = \frac{1}{n(n+2)} a_{n-1}, \quad n = 1, 2, \dots,$$

which implies that

$$a_n = \frac{2}{n!(n+2)!} a_0, \quad n = 1, 2, \dots$$

Consequently,

$$y_1(x) = x^{3/2} \sum_{n=0}^{\infty} \frac{1}{n!(n+2)!} x^n$$

where we have set $a_0 = \frac{1}{2}$ without loss of generality. Inserting $r = -\frac{1}{2}$ into Equation (0.0.86) and simplifying yields

$$n(n-2)a_n = a_{n-1}, \quad n = 1, 2, \dots$$

Therefore, $a_1 = -a_0$ so that the consistency condition when $n = 2$ is

$$0 \cdot a_2 = -a_0$$

which cannot be satisfied since $a_0 \neq 0$. Consequently, there is not a second linearly independent Frobenius series solution. Hence we let

$$y_2(x) = Ay_1(x) \ln x + x^{-1/2} \sum_{n=0}^{\infty} b_n x^n.$$

Substituting this expression for $y_2(x)$ into the given differential equation and simplifying yields

$$2Axy_1' - Ay_1 + \sum_{n=0}^{\infty} n(n-2)b_n x^{n-1/2} - \sum_{n=1}^{\infty} b_{n-1} x^{n-1/2} = 0.$$

Inserting

$$y_1(x) = x^{3/2} \sum_{n=0}^{\infty} \frac{1}{n!(n+2)!} x^n$$

into the preceding equation and simplifying gives

$$-(b_1 + b_0) + \sum_{n=2}^{\infty} \left[n(n-2)b_n - b_{n-1} + \frac{2A(n-1)}{n!(n-2)!} \right] x^n = 0.$$

Hence,

$$b_1 = -b_0, \quad A = -b_0, \quad b_n = \frac{1}{n(n-2)} \left[b_{n-1} + \frac{2b_0(n-1)}{n!(n-2)!} \right], \quad n = 3, 4, \dots$$

Without loss of generality we let $b_0 = -1$ and $b_2 = 0$. Then,

$$b_1 = 1, \quad A = 1, \quad b_3 = -\frac{2}{9}, \quad b_4 = -\frac{25}{576}, \quad \dots,$$

so that

$$y_2(x) = y_1(x) \ln x - x^{-1/2} \left(1 - x + \frac{2}{9}x^3 + \frac{25}{576}x^4 + \dots \right).$$

26. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} [4(n+r)(n+r-1) + 4(n+r) - 1] a_n x^n + \sum_{n=1}^{\infty} [8(n+r-1) + 4] a_{n-1} x^n = 0. \quad (0.0.87)$$

When $n = 0$, we obtain the indicial equation

$$4r(r-1) + 4r - 1 = 4r^2 - 1 = (2r-1)^2 = 0,$$

with roots $r = \pm \frac{1}{2}$. Inserting $r = \frac{1}{2}$ into Equation (0.0.87) and simplifying leads to the recurrence relation

$$a_n = -\frac{2}{n+1}a_{n-1}, \quad n = 1, 2, \dots,$$

which implies

$$a_n = \frac{(-2)^n}{(n+1)!}a_0, \quad n = 1, 2, \dots$$

Hence,

$$y_1(x) = x^{1/2} \sum_{n=0}^{\infty} \frac{(-2)^n}{(n+1)!} x^n.$$

Inserting $r = -\frac{1}{2}$ into Equation (0.0.87) and simplifying yields

$$4n(n-1)a_n = -8(n-1)a_{n-1}, \quad n = 1, 2, \dots \quad (0.0.88)$$

When $n = 1$, we obtain the consistency condition

$$4(0)a_1 = -8(0)a_0,$$

which is satisfied for any choice of a_1 . We let $a_1 = 0$, in which case from Equation (0.0.88), all subsequent coefficients are also zero. Consequently,

$$y_2(x) = x^{-1/2}.$$

27. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^n - \sum_{n=1}^{\infty} [(n+r-1)-1]a_{n-1} x^n = 0. \quad (0.0.89)$$

When $n = 0$, we obtain the indicial equation

$$r(r-1) = 0,$$

with roots $r = 0, 1$. Inserting $r = 1$ into Equation (0.0.89) and simplifying leads to the recurrence relation

$$a_n = \frac{(n-1)}{n(n+1)}a_{n-1}, \quad n = 1, 2, \dots,$$

which implies that

$$a_n = 0, \quad n = 1, 2, \dots$$

Consequently,

$$y_1(x) = x.$$

We now use the reduction of order technique to find a second solution. Substituting

$$y_2(x) = xu(x)$$

into the given differential equation and simplifying we find

$$u'' + (2x^{-1} - 1)u' = 0,$$

or equivalently,

$$\frac{u''}{u'} = 1 - 2x^{-1},$$

which can be integrated directly to obtain

$$\ln u' = x - 2 \ln x$$

where we have set the integration constant to zero. Exponentiating the preceding equation gives

$$u' = x^{-2}e^x = x^{-2} \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n-2},$$

or equivalently,

$$u' = x^{-2} + x^{-1} + \sum_{n=2}^{\infty} \frac{1}{n!} x^{n-2},$$

which upon integration yields

$$u(x) = -x^{-1} + \ln x + \sum_{n=2}^{\infty} \frac{1}{n!(n-1)} x^{n-1},$$

where once more we have set the integration constant to zero. Thus, a second linearly independent solution to the given differential equation is

$$y_2(x) = x \ln x - 1 + \sum_{n=2}^{\infty} \frac{1}{n!(n-1)} x^n.$$

28. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + 4(n+r) + 2] a_n x^n + \sum_{n=1}^{\infty} [(n+r-1) + 1] a_{n-1} x^n = 0. \quad (0.0.90)$$

When $n = 0$, we obtain the indicial equation

$$r(r-1) + 4r + 2 = r^2 + 3r + 2 = 0,$$

with roots $r = -1, -2$. Inserting $r = -1$ into Equation (0.0.90) and simplifying leads to the recurrence relation

$$a_n = -\frac{(n-1)}{n(n+1)} a_{n-1}, \quad n = 1, 2, \dots,$$

which implies that

$$a_n = 0, \quad n = 1, 2, \dots$$

Hence,

$$y_1(x) = x^{-1}.$$

We now use the reduction of order technique to find a second linearly independent solution. Substituting

$$y_2(x) = x^{-1}u(x)$$

into the given differential equation and simplifying yields

$$xu'' + (x + 2)u' = 0.$$

That is,

$$\frac{u''}{u'} = -(2x^{-1} + 1)$$

which can be integrated to obtain

$$\ln u' = -2 \ln x - x$$

where we have set the integration constant to zero. Exponentiating the preceding equation gives

$$u' = x^{-2}e^{-x} = x^{-2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

which can be written as

$$u' = x^{-2} - \frac{1}{x} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} x^{n-2}.$$

Integrating yields

$$u(x) = -x^{-1} - \ln x + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!(n-1)} x^{n-1},$$

where we have set the integration constant to zero. Thus

$$y_2(x) = -x^{-2} - x^{-1} \ln x + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!(n-1)} x^{n-2}$$

or equivalently (after factoring out a negative sign),

$$y_2(x) = x^{-1} \ln x + x^{-2} \left[1 + \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n!(n-1)} x^n \right].$$

29. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} (n+r)^2 a_n x^n - \sum_{n=1}^{\infty} [(n+r-1) - N] a_{n-1} x^n = 0. \quad (0.0.91)$$

When $n = 0$, we obtain the indicial equation

$$r^2 = 0$$

with the repeated root $r = 0$. Inserting $r = 0$ into Equation (0.0.91) and simplifying leads to the recurrence relation

$$a_n = \frac{(n-1-N)}{n^2} a_{n-1}, \quad n = 1, 2, \dots,$$

which implies that

$$a_n = \frac{(-1)^n N(N-1) \cdots (N-n+1)}{(n!)^2} a_0, \quad n = 1, 2, \dots$$

Consequently,

$$a_n = 0, \quad n = N+1, N+2, \dots$$

so that

$$y_1(x) = 1 + \sum_{n=1}^N \frac{(-1)^n N(N-1) \cdots (N-n+1)}{(n!)^2} x^n,$$

which is a polynomial of degree N .

30. (a) Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} [(n+r)(n+r+2N) + N^2] a_n x^n + \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^n = 0. \quad (0.0.92)$$

When $n = 0$, we obtain the indicial equation

$$(r+N)^2 = 0,$$

which has the repeated root $r = -N$. Consequently, there is only one Frobenius series solution. Inserting $r = -N$ into Equation (0.0.92) and simplifying leads to the recurrence relation

$$a_n = \frac{N-(n-1)}{n^2} a_{n-1}, \quad n = 1, 2, \dots,$$

which implies that

$$a_n = 0, \quad n = N+1, N+2, \dots$$

Consequently the Frobenius series solution terminates after $N+1$ terms.

(b) Let $Y(x) = x^N y(x)$. That is, $y(x) = x^{-N} Y(x)$. Differentiating this expression yields

$$\begin{aligned} y'(x) &= x^{-N} Y'(x) - N x^{-(N+1)} Y(x), \\ y''(x) &= x^{-N} Y''(x) - 2N x^{-(N+1)} Y'(x) + N(N+1) x^{-(N+2)} Y(x). \end{aligned}$$

Inserting these expressions into the given differential equation yields

$$x^2 \left[x^{-N} Y'' - 2N x^{-(N+1)} Y' + N(N+1) x^{-(N+2)} Y \right] + x(1+2N-x)(x^{-N} Y' - N x^{-(N+1)} Y) + N^2 (x^{-N} Y) = 0.$$

Combining terms and dividing by x^{-N} gives

$$x^2 Y'' + x(1-x) Y' + N x Y = 0,$$

which is the Laguerre equation.

Solutions to Section 9.6

True-False Review:

1. FALSE. The requirement that guarantees the existence of two linearly independent Frobenius series solutions is that $2p$ is not an integer, not that p is a positive noninteger.

2. TRUE. Since the roots of the indicial equation are $r = \pm p$, it follows from the general Frobenius theory that, provided $2p$ is not equal to an integer, we can get two linearly independent Frobenius series solutions to Bessel's equation of order p , written as J_p and J_{-p} in (9.6.18). The variation appearing in (9.6.20) also applies when p is not a positive integer, which holds in particular if $2p$ is not an integer.

3. FALSE. The gamma function is not defined for $p = 0, -1, -2, \dots$.

4. TRUE. From Figure 9.6.2, the values of p such that $\Gamma(p) < 0$ occur in the intervals $(-1, 0)$, $(-3, -2)$, $(-5, -4)$, and so on. In these intervals, the greatest integer less than or equal to p are $-1, -3, -5, \dots$, which are the odd, negative integers.

5. FALSE. Although it is possible via Property 3 in Equation (9.6.27) to express $J_p(x)$ in terms of $J_{p-1}(x)$ and $J_{p-2}(x)$ as

$$J_p(x) = 2x^{-1}(p-1)J_{p-1}(x) - J_{p-2}(x),$$

the expression on the right-hand side is not a *linear combination*.

6. FALSE. The correct formula for $J'_p(x)$ can be computed directly from Equation (9.6.22). A valid expression for $J'_p(x)$ is also given in Property 4 (Equation (9.6.28)).

7. FALSE. From Equation (9.6.31), we see that $J_p(\lambda_n x)$ and $J_p(\lambda_m x)$ are orthogonal on $(0, 1)$ relative to the *weight function* $w(x) = x$:

$$\int_0^1 x J_p(\lambda_m x) J_p(\lambda_n x) dx = 0.$$

Problems:

1. Let $p = \frac{1}{2}m$, where m is an odd, positive integer. Then the roots of the indicial equation for Bessel's equation of order p , $r = \pm p$, differ by the integer m , and so we must show that when $r = -p$ a second Frobenius series *can* be obtained. Inserting $r = -p$ into Equations (9.6.4) and (9.6.5) of the text yields

$$(1 - 2p)a_1 = 0, \tag{0.0.93}$$

and

$$n(n - 2p)a_n = -a_{n-2}, \tag{0.0.94}$$

respectively. We consider two cases:

Case 1: $p = 1/2$: In this case, Equation (0.0.93) implies that

$$0 \cdot a_1 = 0,$$

in which case a_1 can be assigned an arbitrary value and the resulting coefficients in the Frobenius series solution can be obtained from (0.0.94).

Case 2: $p = (2k + 1)/2$, where k is a positive integer: In this case, Equation (0.0.93) implies that

$$a_1 = 0,$$

in which case, from Equation (0.0.94),

$$a_3 = a_5 = \cdots = a_{2k-1} = 0.$$

Setting $n = 2p = 2k + 1$ in Equation (0.0.94) yields the consistency condition

$$(2k + 1) \cdot 0 \cdot a_{2k+1} = -a_{2k-1} = 0,$$

which is satisfied for all values of a_{2k+1} . Consequently we can assign an arbitrary value to a_{2k+1} and then the remaining coefficients in a second Frobenius series solution can be obtained from (0.0.94).

2. The given differential equation is a Bessel equation of order $p = \frac{3}{2}$. Consequently, two linearly independent solutions are given by

$$y_1(x) = J_{3/2}(x) \quad \text{and} \quad y_2(x) = J_{-3/2}(x).$$

3. According to Lemma 9.6.2,

$$\Gamma(p + 1) = p\Gamma(p), \quad p > 0.$$

Therefore,

$$\Gamma(p + 1) \cdot (p + 1) = \Gamma(p + 2), \quad \Gamma(p + 2) \cdot (p + 2) = \Gamma(p + 3), \quad \dots, \quad \Gamma(p + k) \cdot (p + k) = \Gamma(p + k + 1),$$

so that

$$\Gamma(p + 1) [(p + 1)(p + 2) \cdots (p + k)] = \Gamma(p + 2) [(p + 2) \cdots (p + k)] = \cdots = \Gamma(p + k) \cdot (p + k) = \Gamma(p + k + 1).$$

4. (a) We have

$$\Gamma(1/2) = \int_0^\infty t^{-1/2} e^{-t} dt.$$

Let $t = x^2$ in which case $dt = 2x dx$ so that $t^{-1/2} dt = 2 dx$. Substituting these results into the preceding integral yields

$$\Gamma(1/2) = 2 \int_0^\infty e^{-x^2} dx.$$

(b) From the result of (a) we can write

$$[\Gamma(1/2)]^2 = \left[2 \int_0^\infty e^{-x^2} dx \right] \left[2 \int_0^\infty e^{-y^2} dy \right] = 4 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy.$$

(c) From (b) we have

$$[\Gamma(1/2)]^2 = 4 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy.$$

Changing to polar coordinates (r, θ) where $x = r \cos \theta$, $y = r \sin \theta$, yields

$$[\Gamma(1/2)]^2 = 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^\infty e^{-r^2} r dr d\theta = 4 \int_{\theta=0}^{\pi/2} \left[-\frac{1}{2} e^{-r^2} \right]_0^\infty d\theta = \pi,$$

so that

$$\Gamma(1/2) = \sqrt{\pi}.$$

(d) Using

$$\Gamma(p+1) = p\Gamma(p), \quad p > 0, \quad (0.0.95)$$

with $p = 1/2$ yields

$$\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}.$$

Now use

$$\Gamma(p) = \frac{\Gamma(p+1)}{p}, \quad p < 0$$

with $p = -1/2$ to obtain

$$\Gamma(-1/2) = \frac{\Gamma(1/2)}{-1/2} = -2\sqrt{\pi}.$$

5. We have

$$\begin{aligned} \frac{d}{dx}[x^{-p}J_p(x)] &= \frac{d}{dx} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{p+2k} k! \Gamma(p+k+1)} = x^{-p} \sum_{k=1}^{\infty} \frac{(-1)^k x^{p+2k-1}}{2^{p+2k-1} (k-1)! \Gamma(p+k+1)} \\ &= x^{-p} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{p+2n+1}}{2^{p+2n+1} n! \Gamma(p+n+2)} = -x^{-p} J_{n+1}(x). \end{aligned}$$

6. We have

$$J_{1/2}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(\frac{1}{2}+k+1)} \left(\frac{x}{2}\right)^{2k+1/2}. \quad (0.0.96)$$

But, from Problem 3, for $p > 0$

$$\Gamma(p+k+1) = \Gamma(p+1) [(p+1)(p+2)\cdots(p+k)] = \Gamma(p) [p(p+1)\cdots(p+k)],$$

so that, when $p = 1/2$,

$$\begin{aligned} \Gamma\left(\frac{1}{2}+k+1\right) &= \Gamma\left(\frac{1}{2}\right) \left[\frac{1}{2} \cdot \frac{3}{2} \cdots \frac{(2k+1)}{2}\right] = \Gamma(1/2) \cdot \frac{1 \cdot 3 \cdots (2k+1)}{2^{k+1}} \\ &= \sqrt{\pi} \cdot \frac{1 \cdot 3 \cdots (2k+1)}{2^{k+1}}. \end{aligned}$$

Inserting this expression for $\Gamma(1/2+k+1)$ into Equation (0.0.96) yields

$$J_{1/2}(x) = \frac{1}{\sqrt{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k 2^{k+1}}{k! [1 \cdot 3 \cdots (2k+1)] 2^{2k+1/2}} x^{2k+1} = \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! [1 \cdot 3 \cdots (2k+1)] 2^k} x^{2k+1}.$$

But,

$$2^k k! = 2 \cdot 4 \cdot 6 \cdots (2k),$$

so that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sqrt{\frac{2}{\pi x}} \sin x.$$

7. By Property 3 of the Bessel functions,

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x).$$

Inserting $p = 1/2$ into this equality yields

$$J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right) = \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x - x \cos x}{x} \right).$$

Similarly, inserting $p = -1/2$ into Property 3 above gives

$$J_{-3/2}(x) = -\frac{1}{x} J_{-1/2}(x) - J_{1/2}(x) = -\frac{1}{x} \cdot \sqrt{\frac{2}{\pi x}} \cos x - \sqrt{\frac{2}{\pi x}} \sin x = -\sqrt{\frac{2}{\pi x}} \left(\frac{\cos x + x \sin x}{x} \right).$$

8. (a) By Property 1 of the Bessel functions,

$$\frac{d}{dx} [x^p J_p(x)] = x^p J_{p-1}(x)$$

so that

$$\int x^p J_{p-1}(x) dx = x^p J_p(x) + c.$$

(b) By Property 2 of the Bessel functions,

$$\frac{d}{dx} [x^{-p} J_p(x)] = -x^{-p} J_{p+1}(x)$$

so that

$$\int x^{-p} J_{p+1}(x) dx = -x^{-p} J_p(x) + c.$$

9. (a) $J_0(x)$ is a solution to the Bessel equation of order zero. Consequently,

$$x^2 J_0''(x) + x J_0'(x) + x^2 J_0(x) = 0,$$

or equivalently,

$$J_0''(x) = -J_0(x) - x^{-1} J_0'(x).$$

(b) Differentiating the result from part (a) yields

$$J_0'''(x) = -J_0'(x) - \frac{1}{x} J_0''(x) + \frac{1}{x^2} J_0'(x).$$

Substituting for $J_0''(x)$ from (a) and simplifying we obtain the desired result.

10. Inserting $p = 3$ into Property 3 of the Bessel functions gives

$$J_4(x) = 6x^{-1} J_3(x) - J_2(x).$$

Substituting the expressions obtained in Example 9.6.3 for $J_3(x)$ and $J_2(x)$ yields

$$\begin{aligned} J_4(x) &= 6x^{-1} [x^{-2}(8 - x^2)J_1(x) - 4x^{-1}J_0(x)] - 2x^{-1}J_1(x) + J_0(x) \\ &= [6x^{-3}(8 - x^2) - 2x^{-1}] J_1(x) - (24x^{-2} - 1)J_0(x) \\ &= 8x^{-3}(6 - x^2)J_1(x) - x^{-2}(24 - x^2)J_0(x). \end{aligned}$$

11. Inserting $p = 2$ into Property 4 of the Bessel functions yields

$$J_2'(x) = \frac{1}{2} [J_1(x) - J_3(x)]. \quad (0.0.97)$$

But, from Example 9.6.3,

$$J_3(x) = x^{-2}(8 - x^2)J_1(x) - 4x^{-1}J_0(x).$$

Substituting this expression for $J_3(x)$ into Equation (0.0.97) gives

$$\begin{aligned} J_2'(x) &= \frac{1}{2} [J_1(x) - x^{-2}(8 - x^2)J_1(x) + 4x^{-1}J_0(x)] = 2x^{-1}J_0(x) + x^{-2}(x^2 - 4)J_1(x) \\ &= 2x^{-1}J_0(x) + x^{-2}(4 - x^2)J_1'(x), \end{aligned}$$

where we have used (see (9.6.26) with $p = 0$ in the text) $J_1(x) = -J_0'(x)$.

12. (a) Inserting $p = 1$ into Property 4 of the Bessel functions yields

$$J_1'(x) = \frac{1}{2} [J_0(x) - J_2(x)].$$

Hence,

$$J_2(x) = J_0(x) - 2J_1'(x) = J_0(x) + 2J_0''(x),$$

where we have used (see (9.6.26) with $p = 0$ in the text) $J_1(x) = -J_0'(x)$.

(b) Differentiating the result from (a) gives

$$J_2'(x) = J_0'(x) + 2J_0'''(x)$$

which can be written as

$$J_2'(x) = -J_1(x) - 2J_1''(x). \quad (0.0.98)$$

But, inserting $p = 2$ into Property 4 of the Bessel functions yields

$$J_2'(x) = \frac{1}{2} [J_1(x) - J_3(x)].$$

Substituting this expression for $J_2'(x)$ into Equation (0.0.98) and rearranging gives

$$J_3(x) = 3J_1(x) + 4J_1''(x).$$

13. By Theorem 9.6.4, on the interval $(0, 1)$,

$$x^p = \sum_{n=1}^{\infty} a_n J_p(\lambda_n x) \quad (0.0.99)$$

where

$$a_n = \frac{2}{[J_{p+1}(\lambda_n)]^2} \int_0^1 x^{p+1} J_p(\lambda_n x) dx. \quad (0.0.100)$$

Making the change of variables $u = \lambda_n x$ in Equation (0.0.100) yields

$$a_n = \frac{2}{\lambda_n^{p+2} [J_{p+1}(\lambda_n)]^2} \int_0^{\lambda_n} u^{p+1} J_p(u) du. \quad (0.0.101)$$

But, from part (a) of Problem 8,

$$\int u^{p+1} J_p(u) du = u^{p+1} J_{p+1}(u) + c.$$

Inserting this result into Equation (0.0.101) yields

$$a_n = \frac{2}{\lambda_n J_{p+1}(\lambda_n)}$$

so that, from Equation (0.0.99),

$$x^p = \sum_{n=1}^{\infty} \frac{2}{\lambda_n J_{p+1}(\lambda_n)} J_p(\lambda_n x).$$

14. The coefficients in the Fourier-Bessel expansion are

$$a_n = \frac{2}{[J_1(\lambda_n)]^2} \int_0^1 x^3 J_0(\lambda_n x) dx = \frac{2}{\lambda_n^4 [J_1(\lambda_n)]^2} \int_0^{\lambda_n} u^3 J_0(u) du,$$

where $u = \lambda_n x$. Applying integration by parts on the last integral yields

$$\begin{aligned} a_n &= \frac{2}{\lambda_n^4 [J_1(\lambda_n)]^2} \left\{ [u^3 J_1(u)]_0^{\lambda_n} - 2 \int_0^{\lambda_n} u^2 J_1(u) du \right\} \\ &= \frac{2}{\lambda_n^4 [J_1(\lambda_n)]^2} \left\{ \lambda_n^3 J_1(\lambda_n) - 2 [-u^2 J_0(u)]_0^{\lambda_n} - 4 \int_0^{\lambda_n} u J_0(u) du \right\} \\ &= \frac{2}{\lambda_n^4 [J_1(\lambda_n)]^2} \left\{ \lambda_n^3 J_1(\lambda_n) + 2 \lambda_n^2 J_0(\lambda_n) - 4 [u J_1(u)]_0^{\lambda_n} \right\} \\ &= \frac{2}{\lambda_n^3 J_1(\lambda_n)} (\lambda_n^2 - 4), \end{aligned}$$

where we have used the fact that $J_0(\lambda_n) = 0$ to obtain the last step. Consequently,

$$x^2 = 2 \sum_{n=1}^{\infty} \frac{1}{\lambda_n^3 J_1(\lambda_n)} (\lambda_n^2 - 4) J_0(\lambda_n x).$$

15. If $u(x) = J_p(\lambda x)$, then setting $w = \lambda x$ and using the chain rule, we have

$$\frac{du}{dx} = \lambda J'_p(w) \quad \text{and} \quad \frac{d^2 u}{dx^2} = \lambda^2 J''_p(w),$$

where the prime symbol denotes differentiation with respect to w . Consequently,

$$\begin{aligned} \frac{d^2 u}{dx^2} + \frac{1}{x} \frac{du}{dx} + \left(\lambda^2 - \frac{p^2}{x^2} \right) u &= \lambda^2 J''_p(\lambda x) + \frac{\lambda}{x} J'_p(\lambda x) + \left(\lambda^2 - \frac{p^2}{x^2} \right) J_p(\lambda x) \\ &= \frac{1}{x^2} [\lambda^2 x^2 J''_p(\lambda x) + \lambda x J'_p(\lambda x) + (\lambda^2 x^2 - p^2) J_p(\lambda x)] \\ &= \frac{\lambda^2}{w^2} [w^2 J''_p(w) + w J'_p(w) + (w^2 - p^2) J_p(w)] \\ &= \frac{\lambda^2}{w^2} (0) = 0. \end{aligned}$$

16. (a) Multiply (9.6.36) by $J_p(\mu x)$, and (9.6.37) by $J_p(\lambda x)$, subtract the resulting equations, and integrate over $(0, 1)$ to find

$$\begin{aligned} \left[x \frac{d}{dx} [J_p(\lambda x)] J_p(\mu x) \right]_0^1 - \int_0^1 \mu x \frac{d}{dx} [J_p(\lambda x)] J_p'(\mu x) dx - \left[x \frac{d}{dx} [J_p(\mu x)] J_p(\lambda x) \right]_0^1 \\ + \int_0^1 \lambda x \frac{d}{dx} [J_p(\mu x)] J_p'(\lambda x) dx + (\lambda^2 - \mu^2) \int_0^1 x J_p(\lambda x) J_p(\mu x) dx = 0. \end{aligned}$$

That is,

$$\begin{aligned} \left[\lambda x J_p'(\lambda x) J_p(\mu x) \right]_0^1 - \int_0^1 \lambda \mu x J_p'(\lambda x) J_p'(\mu x) dx - \left[\mu x J_p'(\mu x) J_p(\lambda x) \right]_0^1 \\ + \int_0^1 \lambda \mu x J_p'(\mu x) J_p'(\lambda x) dx + (\lambda^2 - \mu^2) \int_0^1 x J_p(\lambda x) J_p(\mu x) dx = 0. \end{aligned}$$

Evaluation and simplification of this expression yields

$$\lambda J_p'(\lambda) J_p(\mu) - \mu J_p'(\mu) J_p(\lambda) + (\lambda^2 - \mu^2) \int_0^1 x J_p(\lambda x) J_p(\mu x) dx = 0.$$

Hence, for $\lambda \neq \mu$,

$$\int_0^1 x J_p(\lambda x) J_p(\mu x) dx = \frac{\mu J_p(\lambda) J_p'(\mu) - \lambda J_p(\mu) J_p'(\lambda)}{\lambda^2 - \mu^2}. \quad (0.0.102)$$

If λ and μ are distinct zeros of $J_p(x)$, then the preceding formula reduces to

$$\int_0^1 x J_p(\lambda_m x) J_p(\lambda_n x) dx = 0$$

which establishes the orthogonality of the set of functions $\{J_p(\lambda_n x)\}_{n=1}^\infty$ on the interval $(0, 1)$ relative to weight function $w(x) = x$.

(b) Taking the limit as $\lambda \rightarrow \mu$ in Equation (0.0.102) and using L'Hôpital's rule on the right-hand side (with μ viewed as a constant and λ as the limiting variable) gives

$$\int_0^1 x J_p^2(\mu x) dx = \lim_{\lambda \rightarrow \mu} \frac{\mu J_p'(\lambda) J_p'(\mu) - J_p(\mu) J_p'(\lambda) - \lambda J_p(\mu) J_p''(\lambda)}{2\lambda}.$$

That is,

$$\int_0^1 x J_p^2(\mu x) dx = \frac{\mu [J_p'(\mu)]^2 - J_p(\mu) J_p'(\mu) - \mu J_p(\mu) J_p''(\mu)}{2\mu}. \quad (0.0.103)$$

From Bessel's equation,

$$J_p''(\mu) = -\frac{1}{\mu} J_p'(\mu) - \frac{(\mu^2 - p^2)}{\mu^2} J_p(\mu).$$

Inserting this into Equation (0.0.103) yields

$$\int_0^1 x J_p^2(\mu x) dx = \frac{\mu [J_p'(\mu)]^2 + [J_p(\mu)]^2 (\mu^2 - p^2) / \mu}{2\mu} = \frac{1}{2} \left\{ [J_p'(\mu)]^2 + \left(1 - \frac{p^2}{\mu^2}\right) [J_p(\mu)]^2 \right\}.$$

(c) If μ is zero of $J_p(x)$, then $J_p(\mu) = 0$, and by using (9.6.26), we have $J'_p(\mu) = -J_{p+1}(\mu)$. Thus, the formula in part (b) reduces to

$$\int_0^1 x J_p^2(\mu x) dx = \frac{1}{2} [J_{p+1}(\mu)]^2.$$

Solutions to Section 9.7

Problems:

1. From the form (9.7.4), we see that $P(x) = 0$ and $Q(x) = x$. Both of these functions are analytic at $x = 0$ with infinite radius of convergence. Therefore, $x = 0$ is an **ordinary point** of the differential equation, and according to Theorem 9.2.4, the general solution can be represented as a power series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

with infinite radius of convergence. Differentiating, we obtain

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting into the differential equation $y'' + xy = 0$, we obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} a_n x^n = 0.$$

Changing the indices of the two summations appearing here, we find

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=1}^{\infty} a_{k-1} x^k = 0,$$

which can be written in the equivalent form

$$2a_2 + \sum_{k=1}^{\infty} [(k+2)(k+1) a_{k+2} + a_{k-1}] x^k = 0.$$

Consequently, $a_2 = 0$ and

$$(k+2)(k+1) a_{k+2} + a_{k-1} = 0, \quad k = 1, 2, \dots$$

Rearranging this relation gives

$$a_{k+2} = -\frac{1}{(k+1)(k+2)} a_{k-1}, \quad k = 1, 2, \dots$$

This recurrence relation leads to the following expressions for the coefficients of the power series for all $n \geq 1$:

$$a_{3n} = \frac{4 \cdot 7 \cdots (3n-2)}{(3n)!} (-1)^n a_0, \quad a_{3n+1} = \frac{2 \cdot 5 \cdots (3n-1)}{(3n+1)!} (-1)^n a_1, \quad a_{3n+2} = a_2 = 0.$$

Consequently,

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_{3n} x^{3n} + \sum_{n=0}^{\infty} a_{3n+1} x^{3n+1} + \sum_{n=0}^{\infty} a_{3n+2} x^{3n+2} \\ &= a_0 \left[1 + \sum_{n=1}^{\infty} \frac{4 \cdot 7 \cdots (3n-2)}{(3n)!} (-1)^n x^{3n} \right] + a_1 \left[x + \sum_{n=1}^{\infty} \frac{2 \cdot 5 \cdots (3n-1)}{(3n+1)!} (-1)^n x^{3n+1} \right]. \end{aligned}$$

Thus, we obtain the two linearly independent solutions

$$y_1(x) = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{4 \cdot 7 \cdots (3n-2)}{(3n)!} x^{3n} \quad \text{and} \quad y_2(x) = x + \sum_{n=1}^{\infty} (-1)^n \frac{2 \cdot 5 \cdots (3n-1)}{(3n+1)!} x^{3n+1}.$$

These solutions are valid on the interval $(-\infty, \infty)$.

2. From the form (9.7.4), we see that $P(x) = 0$ and $Q(x) = -x^2$. Both of these functions are analytic at $x = 0$ with infinite radius of convergence. Therefore, $x = 0$ is an **ordinary point** of the differential equation, and according to Theorem 9.2.4, the general solution can be represented as a power series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

with infinite radius of convergence. Differentiating, we obtain

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting into the differential equation $y'' - x^2 y = 0$, we obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x^2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

Changing the indices of the two summations appearing here, we find

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=2}^{\infty} a_{k-2} x^k = 0.$$

If we explicitly evaluate the first two terms of the first summation and combine the remaining terms in both summations, we obtain

$$2a_2 + 6a_3 x + \sum_{k=2}^{\infty} [(k+2)(k+1) a_{k+2} - a_{k-2}] x^k = 0.$$

Thus, we find that $a_2 = a_3 = 0$ and that

$$(k+2)(k+1) a_{k+2} - a_{k-2} = 0, \quad k = 2, 3, \dots$$

Hence,

$$a_{k+2} = \frac{a_{k-2}}{(k+2)(k+1)}, \quad k = 2, 3, \dots$$

This recurrence relation leads to the following expressions for the coefficients of the power series for all $n \geq 1$:

$$a_{4n} = \frac{1 \cdot 2 \cdot 5 \cdot 6 \cdots (4n-3)(4n-2)}{(4n)!} a_0, \quad a_{4n+1} = \frac{2 \cdot 3 \cdot 6 \cdot 7 \cdots (4n-2)(4n-1)}{(4n+1)!} a_1, \quad a_{4n+2} = a_{4n+3} = 0.$$

Consequently,

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_{4n} x^{4n} + \sum_{n=0}^{\infty} a_{4n+1} x^{4n+1} + \sum_{n=0}^{\infty} a_{4n+2} x^{4n+2} + \sum_{n=0}^{\infty} a_{4n+3} x^{4n+3} \\ &= a_0 \left[1 + \sum_{n=1}^{\infty} \frac{1 \cdot 2 \cdot 5 \cdot 6 \cdots (4n-3)(4n-2)}{(4n)!} x^{4n} \right] + a_1 \left[x + \sum_{n=1}^{\infty} \frac{2 \cdot 3 \cdot 6 \cdot 7 \cdots (4n-2)(4n-1)}{(4n+1)!} x^{4n+1} \right]. \end{aligned}$$

Thus, we obtain the two linearly independent solutions

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 2 \cdot 5 \cdot 6 \cdots (4n-3)(4n-2)}{(4n)!} x^{4n} \quad \text{and} \quad y_2(x) = x + \sum_{n=1}^{\infty} \frac{2 \cdot 3 \cdot 6 \cdot 7 \cdots (4n-2)(4n-1)}{(4n+1)!} x^{4n+1}.$$

These solutions are valid on the interval $(-\infty, \infty)$.

3. We begin by rewriting the given differential equation as

$$y'' - \frac{6x}{1-x^2} y' - \frac{4}{1-x^2} y = 0.$$

We see that both

$$P(x) = -\frac{6x}{1-x^2} \quad \text{and} \quad Q(x) = -\frac{4}{1-x^2}$$

are analytic at $x = 0$. Therefore $x = 0$ is an **ordinary point** of the differential equation, and according to Theorem 9.2.4, the general solution can be represented as a power series of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

with radius of convergence at least 1. Differentiating, we obtain

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting into the differential equation $(1-x^2)y'' - 6xy' - 4y = 0$, we obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 6 \sum_{n=1}^{\infty} n a_n x^n - 4 \sum_{n=0}^{\infty} a_n x^n = 0.$$

Changing the indices of the summations appearing here, we find

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{k=2}^{\infty} k(k-1) a_k x^k - 6 \sum_{k=1}^{\infty} k a_k x^k - 4 \sum_{k=0}^{\infty} a_k x^k = 0,$$

which can be simplified to

$$(2a_2 - 4a_0) + (6a_3 - 10a_1)x + \sum_{k=2}^{\infty} [(k+2)(k+1)a_{k+2} - (k+4)(k+1)a_k] x^k = 0.$$

Therefore,

$$(k+2)(k+1)a_{k+2} - (k+4)(k+1)a_k = 0, \quad k = 2, 3, \dots,$$

so that,

$$a_{k+2} = \frac{k+4}{k+2}a_k, \quad k = 0, 1, \dots$$

Hence,

$$a_{2n} = (n+1)a_0, \quad n = 1, 2, \dots, \quad \text{and} \quad a_{2n+1} = \frac{2n+3}{3}a_1, \quad n = 0, 1, \dots$$

Consequently,

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_{2n}x^{2n} + \sum_{n=0}^{\infty} a_{2n+1}x^{2n+1} \\ &= a_0 \sum_{n=1}^{\infty} (n+1)x^{2n} + \frac{1}{3}a_1 \sum_{n=1}^{\infty} (2n+3)x^{2n+1}. \end{aligned}$$

Therefore, we obtain two linearly independent solutions

$$y_1(x) = \sum_{n=0}^{\infty} (n+1)x^{2n} \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} (2n+3)x^{2n+1}.$$

These solutions are valid (at least) on the interval $(-1, 1)$.

4. We begin by rewriting the given differential equation as

$$y'' + \frac{1}{x}y' + \frac{2}{x}y = 0.$$

We see that $P(x) = \frac{1}{x}$ and $Q(x) = \frac{2}{x}$ are not analytic at $x = 0$. Therefore, $x = 0$ is not an ordinary point. Writing the differential equation in the form of (9.7.5), we have

$$x^2y'' + xy' + 2xy = 0.$$

Hence, $p(x) = 1$ and $q(x) = 2x$. Therefore, $x = 0$ is a **regular singular point** of the differential equation. The indicial equation (9.7.6) becomes $r(r-1) + r = 0$, or $r^2 = 0$. By Theorem 9.5.1, there exist two linearly independent solutions to the differential equation of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0, \quad \text{and} \quad y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} b_n x^n.$$

These solutions are valid on $(0, \infty)$. Beginning with $y_1(x)$, we differentiate to obtain

$$y_1'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y_1''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting into the differential equation

$$xy'' + y' + 2y = 0,$$

we obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n = 0,$$

which can be written in the equivalent form

$$\sum_{k=1}^{\infty} (k+1)ka_{k+1}x^k + \sum_{k=0}^{\infty} (k+1)a_{k+1}x^k + 2\sum_{k=0}^{\infty} a_kx^k = 0.$$

Extracting the $k = 0$ term and combining the remaining terms, this becomes

$$(a_1 + 2a_0) + \sum_{k=1}^{\infty} [(k+1)^2a_{k+1} + 2a_k]x^k = 0.$$

Hence, we have

$$(k+1)^2a_{k+1} + 2a_k = 0, \quad k = 1, 2, \dots,$$

so that

$$a_{k+1} = -\frac{2}{(k+1)^2}a_k, \quad k = 1, 2, \dots$$

Since $a_1 = -2a_0$, this recurrence relation holds for $k = 0$ as well. Iterative substitution into this formula gives

$$a_k = \frac{(-2)^k}{(k!)^2}a_0, \quad k = 1, 2, \dots$$

Hence,

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-2)^n}{(n!)^2}x^n. \quad (0.0.104)$$

We now determine a second solution of the form

$$y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} b_nx^n.$$

Differentiating this expression for $y_2(x)$ yields

$$\begin{aligned} y_2'(x) &= y_1'(x) \ln x + \frac{1}{x}y_1(x) + \sum_{n=1}^{\infty} nb_nx^{n-1}, \\ y_2''(x) &= y_1''(x) \ln x + \frac{2}{x}y_1'(x) - \frac{1}{x^2}y_1(x) + \sum_{n=1}^{\infty} n(n-1)b_nx^{n-2}. \end{aligned}$$

Inserting these results into the given differential equation and simplifying we find

$$2y_1'(x) + \sum_{n=1}^{\infty} n^2b_nx^{n-1} + 2\sum_{n=2}^{\infty} b_{n-1}x^{n-1} = 0.$$

That is, using (0.0.104),

$$2\sum_{n=0}^{\infty} \frac{(-2)^nn}{(n!)^2}x^{n-1} + \sum_{n=1}^{\infty} n^2b_nx^{n-1} + 2\sum_{n=2}^{\infty} b_{n-1}x^{n-1} = 0,$$

or equivalently,

$$(-4 + b_1) + \sum_{n=2}^{\infty} \left[\frac{2(-2)^nn}{(n!)^2} + n^2b_n + 2b_{n-1} \right] x^n = 0.$$

Hence,

$$b_1 = 4 \quad \text{and} \quad b_n = \frac{2}{n^2} \left[-\frac{n(-2)^n}{(n!)^2} - b_{n-1} \right], \quad n = 2, 3, \dots,$$

so that

$$b_1 = 4, \quad b_2 = -3, \quad b_3 = \frac{22}{27}, \quad b_4 = -\frac{25}{216} \dots$$

Consequently,

$$y_2(x) = y_1(x) \ln x + x \left(4 - 3x + \frac{22}{27}x^2 - \frac{25}{216}x^3 + \dots \right).$$

5. We begin by rewriting the given differential equation as

$$y'' + \frac{2}{x}y' + y = 0.$$

We see that $P(x) = \frac{2}{x}$ is not analytic at $x = 0$. Therefore, $x = 0$ is not an ordinary point. Writing the differential equation in the form of (9.7.5), we have

$$x^2 y'' + 2xy' + x^2 y = 0.$$

Hence, $p(x) = 2$ and $q(x) = x^2$. Therefore, $x = 0$ is a **regular singular point** of the differential equation, and the solutions will be valid on $(0, \infty)$. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} (r+n)(r+n+1)a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

When $n = 0$, we obtain the indicial equation

$$r(r+1) = 0,$$

with roots $r = 0, -1$. The remaining coefficients must satisfy

$$(r+1)(r+2)a_1 = 0, \tag{0.0.105}$$

and

$$(r+n)(r+n+1)a_n = -a_{n-2}, \quad n = 2, 3, \dots \tag{0.0.106}$$

When $r = 0$, Equation (0.0.105) implies that $a_1 = 0$, and Equation (0.0.106) reduces to

$$n(n+1)a_n = -a_{n-2}, \quad n = 2, 3, \dots$$

so that

$$a_n = -\frac{1}{n(n+1)}a_{n-2}, \quad n = 2, 3, \dots$$

Therefore,

$$a_{2k+1} = 0, \quad k = 0, 1, \dots,$$

and

$$a_{2k} = -\frac{1}{2k(2k+1)}a_{2k-2}, \quad k = 1, 2, \dots$$

The preceding recurrence relation implies that

$$a_2 = -\frac{1}{2 \cdot 1 \cdot 3}a_0, \quad a_4 = \frac{1}{2^2 \cdot (1 \cdot 2)(3 \cdot 5)}a_0, \quad \dots$$

and in general,

$$a_{2n} = \frac{(-1)^n}{2^n \cdot n! \cdot [1 \cdot 3 \cdots (2n+1)]}a_0, \quad n = 1, 2, \dots$$

Consequently,

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \cdot n! \cdot [1 \cdot 3 \cdots (2n+1)]}x^{2n}.$$

By combining factors in the denominator, this can be re-expressed as

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n}.$$

When $r = -1$, Equation (0.0.105) reduces to

$$0 \cdot a_1 = 0$$

which is satisfied for any value of a_1 and, from Equation (0.0.106), the remaining coefficients are obtained from

$$(n-1)na_n = -a_{n-2}, \quad n = 2, 3, \dots$$

Choosing $a_1 = 0$ it follows from the preceding equation that all odd coefficients are then zero, whereas the even coefficients are determined from the recurrence relation

$$a_{2k} = -\frac{1}{2k(2k-1)}a_{2k-2}, \quad k = 1, 2, \dots$$

Therefore,

$$a_2 = -\frac{1}{2 \cdot 1 \cdot 1}a_0, \quad a_4 = \frac{1}{2^2 \cdot (1 \cdot 3)(1 \cdot 2)}a_0, \quad \dots,$$

and in general,

$$a_{2n} = \frac{(-1)^n}{2^n [1 \cdot 3 \cdots (2n-1)]n!}a_0, \quad n = 1, 2, \dots$$

Consequently,

$$y_2(x) = x^{-1} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n [1 \cdot 3 \cdots (2n-1)]n!}x^{2n} \right\}.$$

By combining factors in the denominator, this can be re-expressed as

$$y_2(x) = x^{-1} \left\{ 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} \right\}.$$

6. We begin by rewriting the given differential equation as

$$y'' + \frac{5(1-2x)}{2x}y' - \frac{5}{2x}y = 0.$$

We see that

$$P(x) = \frac{5(1-2x)}{2x} \quad \text{and} \quad Q(x) = -\frac{5}{2x}$$

are not analytic at $x = 0$. Therefore, $x = 0$ is not an ordinary point. Writing the differential equation in the form of (9.7.5), we have

$$x^2y'' + \frac{5}{2}x(1-2x)y' - \frac{5}{2}xy = 0.$$

Hence,

$$p(x) = \frac{5}{2}(1-2x) \quad \text{and} \quad q(x) = -\frac{5}{2}x.$$

Therefore, $x = 0$ is a **regular singular point** of the differential equation, and the solutions will be valid on $(0, \infty)$. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} (r+n)(2r+2n+3)a_n x^n - 5 \sum_{n=1}^{\infty} (2r+2n-1)a_{n-1} x^n = 0.$$

When $n = 0$, we obtain the indicial equation

$$r(2r+3) = 0,$$

with roots $r = 0, -3/2$. The remaining coefficients are obtained from the recurrence relation

$$a_n = \frac{5(2r+2n-1)}{(r+n)(2r+2n+3)}a_{n-1}, \quad n = 1, 2, \dots \quad (0.0.107)$$

When $r = 0$, the preceding recurrence relation reduces to

$$a_n = \frac{5(2n-1)}{n(2n+3)}a_{n-1}, \quad n = 1, 2, \dots,$$

so that

$$a_1 = \frac{5 \cdot 1}{1 \cdot 5}a_0, \quad a_2 = \frac{5^2 \cdot 1 \cdot 3}{(2 \cdot 1)(5 \cdot 7)}a_0, \dots,$$

and in general,

$$a_n = \frac{5^n \cdot 1 \cdot 3 \cdots (2n-1)}{n! \cdot [5 \cdot 7 \cdots (2n+3)]}a_0, \quad n = 1, 2, \dots$$

The choice $a_0 = \frac{1}{3}$ allows us to simplify the preceding formula to

$$a_n = \frac{5^n}{n!(2n+3)(2n+1)}, \quad n = 1, 2, \dots,$$

so that

$$y_1(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{5^n}{n!(2n+3)(2n+1)} x^n = \sum_{n=0}^{\infty} \frac{5^n}{n!(2n+3)(2n+1)} x^n.$$

When $r = -\frac{3}{2}$, the recurrence relation (0.0.107) reduces to

$$a_n = \frac{10(n-2)}{n(2n-3)} a_{n-1}, \quad n = 1, 2, \dots,$$

so that

$$a_1 = 10a_0, \quad a_n = 0, \quad n = 2, 3, \dots$$

Consequently,

$$y_2(x) = x^{-3/2}(1 + 10x).$$

7. We begin by rewriting the given differential equation as

$$y'' + \frac{1}{x}y' + y = 0.$$

We see that

$$P(x) = \frac{1}{x}$$

is not analytic at $x = 0$. Therefore, $x = 0$ is not an ordinary point. Writing the differential equation in the form of (9.7.5), we have

$$x^2 y'' + xy' + x^2 y = 0.$$

Hence,

$$p(x) = 1 \quad \text{and} \quad q(x) = x^2.$$

Therefore, $x = 0$ is a **regular singular point** of the differential equation, and the solutions will be valid on $(0, \infty)$. The indicial equation for the given differential equation is

$$r(r-1) + p(0)r + q(0) = 0$$

which in this case gives

$$r(r-1) + r = 0,$$

or equivalently, $r^2 = 0$. Consequently, there is the single repeated root $r = 0$. The corresponding Frobenius series solution to the given differential equation is of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0.$$

Substituting the preceding expression for $y(x)$ into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} n^2 a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0,$$

from which we can conclude that

$$a_1 = 0, \quad a_n = -\frac{1}{n^2} a_{n-2}, \quad n = 2, 3, \dots$$

Consequently,

$$a_{2k+1} = 0, \quad k = 0, 1, \dots,$$

and

$$a_{2k} = -\frac{1}{(2k)^2} a_{2k-2}, \quad k = 1, 2, \dots$$

Therefore,

$$a_2 = -\frac{1}{2^2} a_0, \quad a_4 = \frac{1}{2^2 \cdot 4^2} a_0, \dots,$$

and in general,

$$a_{2n} = \frac{(-1)^n}{2^2 \cdot 4^2 \dots (2n)^2} a_0 = \frac{(-1)^n}{2^{2n} \cdot (n!)^2} a_0, \quad n = 1, 2, \dots$$

Hence,

$$y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} \cdot (n!)^2} x^{2n}. \quad (0.0.108)$$

Since the indicial equation has a repeated root, there is a second linearly independent solution to the given differential equation of the form

$$y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} b_n x^n.$$

Differentiating this expression for $y_2(x)$ yields

$$\begin{aligned} y_2'(x) &= y_1'(x) \ln x + \frac{1}{x} y_1(x) + \sum_{n=1}^{\infty} n b_n x^{n-1}, \\ y_2''(x) &= y_1''(x) \ln x + \frac{2}{x} y_1'(x) - \frac{1}{x^2} y_1(x) + \sum_{n=1}^{\infty} n(n-1) b_n x^{n-2}. \end{aligned}$$

Inserting these results into the given differential equation and simplifying we find

$$2y_1'(x) + \sum_{n=1}^{\infty} n^2 b_n x^{n-1} + \sum_{n=3}^{\infty} b_{n-2} x^{n-1} = 0.$$

That is, using (0.0.108),

$$4 \sum_{n=0}^{\infty} \frac{(-1)^n n}{2^{2n} \cdot (n!)^2} x^{2n} + \sum_{n=1}^{\infty} n^2 b_n x^n + \sum_{n=3}^{\infty} b_{n-2} x^n = 0,$$

or equivalently,

$$\begin{aligned} & b_1 x + (4b_2 - 1)x^2 + (9b_3 + b_1)x^3 + \left(\frac{1}{8} + 16b_4 + b_2\right)x^4 + (25b_5 + b_3)x^5 \\ & + \left(b_4 + 36b_6 - \frac{1}{192}\right)x^6 + 4 \sum_{n=4}^{\infty} \frac{(-1)^n n}{2^{2n} \cdot (n!)^2} x^{2n} + \sum_{n=7}^{\infty} (n^2 b_n + b_{n-2})x^n = 0. \end{aligned}$$

Therefore,

$$b_1 = b_3 = b_5 = \dots = 0, \quad b_2 = \frac{1}{4}, \quad b_4 = -\frac{1}{128}, \quad b_6 = \frac{11}{13824}, \quad \dots,$$

so that

$$y_2(x) = y_1(x) \ln x + \frac{1}{4}x^2 - \frac{1}{128}x^4 + \frac{11}{13824}x^6 + \dots$$

8. We begin by rewriting the given differential equation as

$$y'' - \frac{8}{1+4x^2}y = 0.$$

We see that

$$P(x) = 0, \quad \text{and} \quad Q(x) = -\frac{8}{1+4x^2}$$

are both analytic at $x = 0$. Therefore, $x = 0$ is an ordinary point of the given differential equation. Hence there exist two linearly independent solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

which will be valid (at least) on the interval $(-\frac{1}{2}, \frac{1}{2})$. Substituting the preceding expression for $y(x)$ into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} (n+1) [(n+2)a_{n+2} + 4(n-2)a_n] x^n = 0,$$

from which we obtain the recurrence relation

$$a_{n+2} = -4 \frac{(n-2)}{n+2} a_n, \quad n = 0, 1, \dots \quad (0.0.109)$$

For even n , the preceding recurrence relation gives

$$a_2 = 4a_0, \quad a_{2n} = 0, \quad n = 2, 3, \dots$$

Hence,

$$y_1(x) = 1 + 4x^2.$$

For odd n , the recurrence relation (0.0.109) gives

$$a_3 = \frac{4}{3}a_1, \quad a_5 = -\frac{4^2}{3 \cdot 5}a_1, \quad a_7 = \frac{4^3}{5 \cdot 7}a_1, \quad a_9 = -\frac{4^4}{7 \cdot 9}a_1,$$

and in general,

$$a_{2n+1} = (-1)^{n+1} \frac{4^n}{(2n-1)(2n+1)} a_1, \quad n = 1, 2, \dots$$

Consequently,

$$y_2(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{4^n}{(2n-1)(2n+1)} x^{2n+1}.$$

9. The given differential equation is Bessel's equation with $p = \frac{1}{2}$. Consequently,

$$y_1(x) = J_{1/2}(x), \quad y_2(x) = J_{-1/2}(x)$$

which can be written in the equivalent form (see Problems 6 and 7 in Section 9.6)

$$y_1(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad y_2(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

10. We begin by rewriting the given differential equation as

$$y'' + \frac{3}{4x}y' + \frac{3}{4x}y = 0.$$

We see that

$$P(x) = Q(x) = \frac{3}{4x}$$

is not analytic at $x = 0$. Therefore, $x = 0$ is not an ordinary point. Writing the differential equation in the form of (9.7.5), we have

$$x^2 y'' + \frac{3}{4} x y' + \frac{3}{4} x y = 0.$$

Hence,

$$p(x) = \frac{3}{4} \quad \text{and} \quad q(x) = \frac{3}{4}x.$$

Therefore, $x = 0$ is a **regular singular point** of the differential equation, and the solutions will be valid on $(0, \infty)$. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying gives

$$\sum_{n=0}^{\infty} (r+n)(4r+4n-1)a_n x^n + 3 \sum_{n=1}^{\infty} a_{n-1} x^n = 0.$$

When $n = 0$, we obtain the indicial equation

$$r(4r-1) = 0,$$

with roots $r = 0, \frac{1}{4}$. The remaining coefficients are determined from the recurrence relation

$$a_n = -\frac{3}{(r+n)(4r+4n-1)} a_{n-1}, \quad n = 1, 2, \dots \quad (0.0.110)$$

When $r = 0$, the preceding recurrence relation reduces to

$$a_n = -\frac{3}{n(4n-1)} a_{n-1}, \quad n = 1, 2, \dots,$$

so that

$$a_1 = -\frac{3}{1 \cdot 3} a_0, \quad a_2 = -\frac{3^2}{(1 \cdot 2)(3 \cdot 7)} a_0, \quad a_3 = -\frac{3^3}{(1 \cdot 2 \cdot 3)(3 \cdot 7 \cdot 11)} a_0,$$

and in general,

$$a_n = \frac{(-3)^n}{n![3 \cdot 7 \cdots (4n-1)]} a_0, \quad n = 1, 2, \dots$$

Consequently,

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-3)^n}{n![3 \cdot 7 \cdots (4n-1)]} x^n.$$

When $r = \frac{1}{4}$, the recurrence relation (0.0.110) reduces to

$$a_n = -\frac{3}{n(4n+1)} a_{n-1}, \quad n = 1, 2, \dots,$$

so that

$$a_1 = -\frac{3}{1 \cdot 5} a_0, \quad a_2 = \frac{3^2}{(1 \cdot 2)(5 \cdot 9)} a_0, \quad a_3 = -\frac{3^3}{(1 \cdot 2 \cdot 3)(5 \cdot 9 \cdot 13)} a_0,$$

and in general,

$$a_n = \frac{(-3)^n}{n![1 \cdot 5 \cdots (4n+1)]} a_0, \quad n = 1, 2, \dots$$

Consequently,

$$y_2(x) = x^{1/4} \sum_{n=0}^{\infty} \frac{(-3)^n}{n![1 \cdot 5 \cdots (4n+1)]} x^n.$$

11. We begin by rewriting the given differential equation as

$$y'' + \frac{3}{2x} y' - \frac{1}{2x^2} (1+x)y = 0.$$

We see that both

$$P(x) = \frac{3}{2x} \quad \text{and} \quad Q(x) = -\frac{1}{2x^2} (1+x)$$

are not analytic at $x = 0$. Therefore, $x = 0$ is not an ordinary point. Writing the differential equation in the form of (9.7.5), we have

$$x^2 y'' + \frac{3}{2} x y' - \frac{1}{2} (1+x)y = 0.$$

Hence,

$$p(x) = \frac{3}{2} \quad \text{and} \quad q(x) = -\frac{1}{2} (1+x).$$

Therefore, $x = 0$ is a **regular singular point** of the differential equation, and the solutions will be valid on $(0, \infty)$. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying gives

$$\sum_{n=0}^{\infty} [(r+n)(2r+2n+1) - 1] a_n x^{r+n} - \sum_{n=1}^{\infty} a_{n-1} x^{r+n} = 0.$$

When $n = 0$ we obtain the indicial equation

$$r(2r+1) - 1 = 0.$$

That is,

$$2r^2 + r - 1 = (r+1)(2r-1) = 0,$$

with roots $r = \frac{1}{2}, -1$. The remaining coefficients are determined from the recurrence relation

$$a_n = \frac{1}{[(r+n)(2r+2n+1)-1]}a_{n-1}, \quad n = 1, 2, \dots \quad (0.0.111)$$

When $r = \frac{1}{2}$, the preceding recurrence relation reduces to

$$a_n = \frac{1}{n(2n+3)}a_{n-1}, \quad n = 1, 2, \dots,$$

so that

$$a_1 = \frac{1}{1 \cdot 5}a_0, \quad a_2 = \frac{1}{(1 \cdot 2)(5 \cdot 7)}a_0,$$

and in general,

$$a_n = \frac{1}{n![5 \cdot 7 \cdots (2n+3)]}a_0, \quad n = 1, 2, \dots$$

Consequently,

$$y_1(x) = a_0 x^{1/2} \left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{n![5 \cdot 7 \cdots (2n+3)]} x^n \right\}.$$

The choice $a_0 = \frac{1}{3}$ gives

$$y_1(x) = x^{1/2} \sum_{n=0}^{\infty} \frac{1}{n![3 \cdot 5 \cdots (2n+3)]} x^n.$$

When $r = -1$, the recurrence relation (0.0.111) reduces to

$$a_n = \frac{1}{n(2n-3)}a_{n-1}, \quad n = 1, 2, \dots,$$

so that

$$a_1 = -a_0, \quad a_2 = -\frac{1}{2!}a_0, \quad a_3 = -\frac{1}{3! \cdot (1 \cdot 3)}a_0, \quad a_4 = -\frac{1}{4! \cdot (1 \cdot 3 \cdot 5)}a_0,$$

and in general,

$$a_n = -\frac{1}{n![1 \cdot 3 \cdots (2n-3)]}a_0, \quad n = 2, 3, \dots$$

Consequently,

$$y_2(x) = x^{-1} \left\{ 1 - x - \sum_{n=2}^{\infty} \frac{1}{n![1 \cdot 3 \cdots (2n-3)]} x^n \right\}.$$

12. We begin by rewriting the given differential equation as

$$y'' - \frac{1}{x}(2-x)y' + \frac{1}{x^2}(2+x^2)y = 0.$$

We see that both

$$P(x) = -\frac{1}{x}(2-x) \quad \text{and} \quad Q(x) = \frac{1}{x^2}(2+x^2)$$

are not analytic at $x = 0$. Therefore, $x = 0$ is not an ordinary point. Writing the differential equation in the form of (9.7.5), we have

$$x^2 y'' - x(2-x)y' + (2+x^2)y = 0.$$

Hence,

$$p(x) = -(2-x), \quad \text{and} \quad q(x) = 2+x^2.$$

Therefore, $x = 0$ is a **regular singular point** of the differential equation, and the solutions will be valid on $(0, \infty)$. Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying gives

$$\sum_{n=0}^{\infty} [(r+n)(r+n-3) + 2] a_n x^n + \sum_{n=1}^{\infty} (n+r-1) a_{n-1} x^n + \sum_{n=2}^{\infty} a_{n-2} x^n = 0. \quad (0.0.112)$$

When $n = 0$, we obtain the indicial equation

$$r(r-3) + 2 = r^2 - 3r + 2 = (r-2)(r-1) = 0,$$

with roots $r = 2, 1$. Equating the $n = 1$ terms in Equation (0.0.112) to zero gives

$$r(r-1)a_1 + ra_0 = 0, \quad (0.0.113)$$

and the remaining terms in Equation (0.0.112) yield the recurrence relation

$$[(r+n)(r+n-3) + 2]a_n + (n+r-1)a_{n-1} + a_{n-2} = 0, \quad n = 2, 3, \dots \quad (0.0.114)$$

When $r = 2$, Equation (0.0.113) gives $a_1 = -a_0$, and from Equation (0.0.114), the remaining coefficients must satisfy

$$a_n = -\frac{1}{n(n+1)} [(n+1)a_{n-1} + a_{n-2}], \quad n = 2, 3, \dots$$

Therefore,

$$\begin{aligned} a_2 &= -\frac{1}{6}(3a_1 + a_0) = \frac{1}{3}a_0, & a_3 &= -\frac{1}{12}(4a_2 + a_1) = -\frac{1}{36}a_0, \\ a_4 &= -\frac{1}{20}(5a_3 + a_2) = -\frac{7}{720}a_0, & \dots \end{aligned}$$

so that

$$y_1(x) = x^2 \left(1 - x + \frac{1}{3}x^2 - \frac{1}{36}x^3 - \frac{7}{720}x^4 + \dots \right). \quad (0.0.115)$$

When $r = 1$, Equation (0.0.113) gives the contradiction

$$0 \cdot a_1 + a_0 = 0.$$

Therefore we seek a second linearly independent solution to the given differential equation of the form

$$y_2(x) = Ay_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+1}.$$

Differentiating this expression for $y_2(x)$ yields

$$\begin{aligned} y_2'(x) &= Ay_1'(x) \ln x + \frac{A}{x} y_1(x) + \sum_{n=0}^{\infty} (n+1) b_n x^n, \\ y_2''(x) &= Ay_1''(x) \ln x + \frac{2A}{x} y_1'(x) - \frac{A}{x^2} y_1(x) + \sum_{n=0}^{\infty} n(n+1) b_n x^{n-1}. \end{aligned}$$

Inserting these results into the given differential equation and simplifying we find

$$A(2xy_1' - 3y_1 + xy_1) + b_0x^2 + \sum_{n=2}^{\infty} [n(n-1)b_n + nb_{n-1} + b_{n-2}]x^{n+1} = 0.$$

Substitution for $y_1(x)$ from Equation (0.0.115) gives

$$\begin{aligned} A \left[2 \left(2x^2 - 3x^3 + \frac{4}{3}x^4 - \frac{5}{36}x^5 - \frac{7}{120}x^6 + \cdots \right) - 3 \left(x^2 - x^3 + \frac{1}{3}x^4 - \frac{1}{36}x^5 - \frac{7}{720}x^6 + \cdots \right) \right. \\ \left. + \left(x^3 - x^4 + \frac{1}{3}x^5 - \frac{1}{36}x^6 + \cdots \right) \right] + b_0x^2 + \sum_{n=2}^{\infty} [n(n-1)b_n + nb_{n-1} + b_{n-2}]x^{n+1} = 0. \end{aligned}$$

Collecting terms we find

$$(A + b_0)x^2 + (-2A + 2b_2 + 2b_1 + b_0)x^3 + \left(\frac{2}{3}A + 6b_3 + 3b_2 + b_1 \right)x^4 + \left(\frac{5}{36}A + 12b_4 + 4b_3 + b_2 \right)x^5 + \cdots = 0,$$

from which we conclude

$$\begin{aligned} A + b_0 &= 0, \\ -2A + 2b_2 + 2b_1 + b_0 &= 0, \\ \frac{2}{3}A + 6b_3 + 3b_2 + b_1 &= 0, \\ \frac{5}{36}A + 12b_4 + 4b_3 + b_2 &= 0, \\ &\vdots \end{aligned}$$

We see that b_1 can be chosen arbitrarily, and for simplicity set $b_1 = 0$. Then,

$$A = -b_0, \quad b_2 = -\frac{3}{2}b_0, \quad b_3 = \frac{31}{36}b_0, \quad b_4 = -\frac{65}{432}b_0, \quad \dots,$$

so that

$$y_2(x) = -y_1(x) \ln x + x \left(1 - \frac{3}{2}x^2 + \frac{31}{36}x^3 - \frac{65}{432}x^4 + \cdots \right).$$

13. We begin by rewriting the given differential equation as

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}(x+1)y = 0.$$

We see that both

$$P(x) = -\frac{3}{x} \quad \text{and} \quad Q(x) = \frac{4}{x^2}(x+1)$$

are not analytic at $x = 0$. Therefore, $x = 0$ is not an ordinary point. Writing the differential equation in the form of (9.7.5), we have

$$x^2y'' - 3xy' + 4(x+1)y = 0.$$

Hence,

$$p(x) = -3 \quad \text{and} \quad q(x) = 4(x+1).$$

Therefore, $x = 0$ is a **regular singular point** of the differential equation, and the solutions will be valid on $(0, \infty)$. The indicial equation for the given differential equation is

$$r(r-1) + p(0)r + q(0) = 0$$

which in this case gives

$$r(r-1) - 3r + 4 = 0,$$

or equivalently,

$$r^2 - 4r + 4 = (r-2)^2 = 0.$$

Consequently, there is the single repeated root $r = 2$. The corresponding Frobenius series solution to the given differential equation is of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+2}, \quad a_0 \neq 0.$$

Substituting the preceding expression for $y(x)$ into the given differential equation and simplifying yields

$$\sum_{n=1}^{\infty} (n^2 a_n + 4a_{n-1}) x^n = 0,$$

from which we obtain the recurrence relation

$$a_n = -\frac{4}{n^2} a_{n-1}, \quad n = 1, 2, \dots$$

Consequently,

$$a_1 = -4a_0, \quad a_2 = \frac{4^2}{2^2} a_0, \quad a_3 = -\frac{4^3}{2^2 \cdot 3^2} a_0,$$

and in general,

$$a_n = \frac{(-4)^n}{(n!)^2} a_0, \quad n = 1, 2, \dots$$

Therefore,

$$y_1(x) = x^2 \sum_{n=0}^{\infty} \frac{(-4)^n}{(n!)^2} x^n. \quad (0.0.116)$$

Since the indicial equation has the repeated root $r = 2$, there is a second linearly independent solution to the given differential equation of the form

$$y_2(x) = y_1(x) \ln x + \sum_{n=1}^{\infty} b_n x^{n+2}.$$

Differentiating this expression for $y_2(x)$ yields

$$\begin{aligned} y_2'(x) &= y_1'(x) \ln x + \frac{1}{x} y_1(x) + \sum_{n=1}^{\infty} (n+2) b_n x^{n+1}, \\ y_2''(x) &= y_1''(x) \ln x + \frac{2}{x} y_1'(x) - \frac{1}{x^2} y_1(x) + \sum_{n=1}^{\infty} (n+2)(n+1) b_n x^n. \end{aligned}$$

Inserting these results into the given differential equation and simplifying we find

$$2xy_1'(x) - 4y_1(x) + \sum_{n=1}^{\infty} n^2 b_n x^{n+2} + 4 \sum_{n=2}^{\infty} b_{n-1} x^{n+2} = 0,$$

that is, using (0.0.116),

$$2 \sum_{n=0}^{\infty} \frac{(-4)^n (n+2)}{(n!)^2} x^{n+2} - 4 \sum_{n=0}^{\infty} \frac{(-4)^n}{(n!)^2} x^{n+2} + \sum_{n=1}^{\infty} n^2 b_n x^{n+2} + 4 \sum_{n=2}^{\infty} b_{n-1} x^{n+2} = 0,$$

or equivalently

$$(-8 + b_1) + \sum_{n=2}^{\infty} \left[\frac{2n(-4)^n}{(n!)^2} + n^2 b_n + 4b_{n-1} \right] x^n = 0.$$

Hence,

$$b_1 = 8 \quad \text{and} \quad b_n = \frac{4}{n^2} \left[\frac{2n(-4)^{n-1}}{(n!)^2} - b_{n-1} \right], \quad n = 2, 3, \dots,$$

so that

$$b_1 = 8, \quad b_2 = -12, \quad b_3 = \frac{176}{27}, \quad b_4 = -\frac{50}{27}, \quad b_5 = \frac{1096}{3375} \dots$$

Consequently,

$$y_2(x) = y_1(x) \ln x + x^3 \left(8 - 12x + \frac{176}{27}x^2 - \frac{50}{27}x^3 + \frac{1096}{3375}x^4 + \dots \right).$$

14. (a) We begin by rewriting the given differential equation as

$$y'' + \frac{[c - (a + b + 1)x]}{x(1-x)} y' - \frac{ab}{x(1-x)} y = 0.$$

We see that for $c \neq 0$,

$$P(x) = \frac{[c - (a + b + 1)x]}{x(1-x)}$$

is not analytic at $x = 0$. Therefore, $x = 0$ is not an ordinary point. Writing the differential equation in the form of (9.7.5), we have

$$x^2 y'' + \frac{x[c - (a + b + 1)x]}{(1-x)} y' - \frac{abx}{(1-x)} y = 0.$$

Hence,

$$p(x) = \frac{[c - (a + b + 1)x]}{(1-x)} \quad \text{and} \quad q(x) = -\frac{abx}{(1-x)}.$$

Therefore, $x = 0$ is a **regular singular point** of the differential equation.

(b) The indicial equation is

$$r(r-1) + p(0)r + q(0) = 0,$$

that is

$$r(r-1) + cr = 0,$$

or equivalently,

$$r[r - (1-c)] = 0.$$

Hence the roots are $r = 0, 1 - c$.

(c) Inserting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} (n+r)(n+r-1+c) a_n x^{n-1} - \sum_{n=0}^{\infty} [(n+r)(n+r+a+b) + ab] a_n x^n = 0,$$

that is,

$$r(r-1+c)x^{-1} + \sum_{n=0}^{\infty} (n+r+1)(n+r+c) a_{n+1} x^n - \sum_{n=0}^{\infty} [(n+r)(n+r+a+b) + ab] a_n x^n = 0$$

or equivalently,

$$r(r-1+c)x^{-1} + \sum_{n=0}^{\infty} \{ (n+r+1)(n+r+c) a_{n+1} - [(n+r)^2 + (n+r)(a+b) + ab] a_n \} x^n = 0.$$

The coefficient of a_n can be factored to obtain

$$r(r-1+c)x^{-1} + \sum_{n=0}^{\infty} [(n+r+1)(n+r+c) a_{n+1} - (n+r+a)(n+r+b) a_n] x^n = 0.$$

Therefore, we obtain the indicial equation

$$r(r-1+c) = 0$$

and the recurrence relation

$$a_{n+1} = \frac{(n+r+a)(n+r+b)}{(n+r+1)(n+r+c)} a_n, \quad n = 0, 1, \dots \quad (0.0.117)$$

(d) From (0.0.117) we have

$$\begin{aligned} a_1 &= \frac{(r+a)(r+b)}{(r+1)(r+c)} a_0, \\ a_2 &= \frac{(r+a+1)(r+b+1)}{(r+c+1)(r+2)} a_1 = \frac{(r+a)(r+a+1)(r+b)(r+b+1)}{(r+c)(r+c+1)(r+1)(r+2)} a_0, \\ a_3 &= \frac{(r+a+2)(r+b+2)}{(r+c+2)(r+3)} a_2 = \frac{(r+a)(r+a+1)(r+a+2)(r+b)(r+b+1)(r+b+2)}{(r+c)(r+c+1)(r+c+2)(r+1)(r+2)(r+3)} a_0, \\ &\vdots \\ a_n &= \frac{(r+a)(r+a+1) \cdots (r+a+n-1)(r+b)(r+b+1) \cdots (r+b+n-1)}{(r+c)(r+c+1) \cdots (r+c+n-1)(r+1)(r+2) \cdots (r+n)} a_0, \quad n = 1, 2, \dots \end{aligned}$$

We now let

$$F(a+r, b+r, 1+r; x) = \sum_{n=0}^{\infty} \frac{(r+a)(r+a+1) \cdots (r+a+n-1)(r+b)(r+b+1) \cdots (r+b+n-1)}{(r+c)(r+c+1) \cdots (r+c+n-1)(r+1)(r+2) \cdots (r+n)} x^n.$$

Then, the first Frobenius series solution is (corresponding to $r = 0$) is

$$\begin{aligned} y_1(x) &= F(a, b, 1; x) = \sum_{n=0}^{\infty} \frac{a(a+1) \cdots (a+n-1)b(b+1) \cdots (b+n-1)}{(c)(c+1) \cdots (c+n-1)(1)(2) \cdots (n)} x^n \\ &= \sum_{n=0}^{\infty} \frac{a(a+1) \cdots (a+n-1)b(b+1) \cdots (b+n-1)}{(c)(c+1) \cdots (c+n-1)n!} x^n, \end{aligned}$$

and the second Frobenius solution (corresponding to $r = 1 - c$) is

$$y_2(x) = x^{1-c} F(a+1-c, b+1-c, 2-c; x).$$

15. (a) The point $x = 0$ is a regular point of the given differential equation. Therefore the differential equation has two linearly independent solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Differentiating twice, we obtain

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting into (9.7.9), this gives

$$(x^2 - 1) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + [1 - (a+b)]x \sum_{n=1}^{\infty} n a_n x^{n-1} + ab \sum_{n=0}^{\infty} a_n x^n = 0,$$

which simplifies to

$$\sum_{n=0}^{\infty} [(n-a)(n-b)a_n - (n+1)(n+2)a_{n+2}] x^n = 0.$$

Hence,

$$(n-a)(n-b)a_n - (n+1)(n+2)a_{n+2} = 0, \quad n = 0, 1, \dots,$$

so that

$$a_{n+2} = \frac{(n-a)(n-b)}{(n+1)(n+2)} a_n, \quad n = 0, 1, \dots \quad (0.0.118)$$

When n is even, (0.0.118) implies that

$$a_2 = \frac{ab}{2!} a_0, \quad a_4 = \frac{ab(2-a)(2-b)}{4!} a_0, \quad a_6 = \frac{ab(2-a)(2-b)(4-a)(4-b)}{6!} a_0,$$

and in general,

$$a_{2n} = \frac{ab(2-a)(2-b)(4-a)(4-b) \cdots (2n-2-a)(2n-2-b)}{(2n)!} a_0, \quad n = 1, 2, \dots$$

When n is odd, (0.0.118) implies that

$$\begin{aligned} a_3 &= \frac{(1-a)(1-b)}{2 \cdot 3} a_1, \quad a_5 = \frac{(1-a)(1-b)(3-a)(3-b)}{5!} a_1, \\ a_7 &= \frac{(1-a)(1-b)(3-a)(3-b)(5-a)(5-b)}{7!} a_1, \end{aligned}$$

and in general,

$$a_{2n+1} = \frac{(1-a)(1-b)(3-a)(3-b)\cdots(2n-1-a)(2n-1-b)}{(2n+1)!}a_1, \quad n = 1, 2, \dots$$

Therefore, we obtain the two linearly independent series solutions

$$y_1(x) = 1 + \frac{ab}{2!}x^2 + \frac{ab(2-a)(2-b)}{4!}x^4 + \frac{ab(2-a)(2-b)(4-a)(4-b)}{6!}x^6 + \dots$$

and

$$y_2(x) = x + \frac{(1-a)(1-b)}{3!}x^3 + \frac{(1-a)(1-b)(3-a)(3-b)}{5!}x^5 + \frac{(1-a)(1-b)(3-a)(3-b)(5-a)(5-b)}{7!}x^7 + \dots$$

(b) If a or b is an even non-negative integer, say $2k$ (where k is an integer), then the coefficients of x^{2k+2} , x^{2k+4} , $\dots$ in the solution $y_1(x)$ become zero, and therefore, $y_1(x)$ has only finitely many nonzero terms. That is, $y_1(x)$ is a polynomial. On the other hand, if either a or b is an odd non-negative integer, say $2k+1$ (where k is an integer), then the coefficients of x^{2k+3} , x^{2k+5} , $\dots$ in the solution $y_2(x)$ become zero, and therefore, $y_2(x)$ has only finite many nonzero terms. That is, $y_2(x)$ is a polynomial.

(c) If a is an odd positive integer, then from part (b), we conclude that $y_2(x)$ is a polynomial. If, in addition, b is an even positive integer, then from part (b), we conclude that $y_1(x)$ is also a polynomial. In this case, both $y_1(x)$ and $y_2(x)$ are polynomials.

(d) We can simply substitute the values $a = 5$ and $b = 4$ into the expressions for $y_1(x)$ and $y_2(x)$ that were derived in part (a). We get

$$y_1(x) = 1 + 10x^2 + 5x^4 \quad \text{and} \quad y_2(x) = x + 2x^3 + \frac{1}{5}x^5.$$

16. (a) We are considering the Chebyshev equation

$$(1-x^2)y'' - xy' + N^2y = 0,$$

where N is a nonnegative integer. Since $x = 0$ is a regular point of this differential equation, there exist two linearly independent solutions of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Differentiating twice, we obtain

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

Substituting into (9.7.10), this gives

$$(1-x^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n + N^2 \sum_{n=0}^{\infty} a_n x^n = 0,$$

which simplifies to

$$\sum_{k=0}^{\infty} [(k+2)(k+1)a_{k+2} - (k^2 - N^2)a_k] x^k = 0.$$

Hence,

$$(k+2)(k+1)a_{k+2} - (k^2 - N^2)a_k = 0, \quad k = 0, 1, \dots,$$

so that

$$a_{k+2} = \frac{(k-N)(k+N)}{(k+2)(k+1)}a_k, \quad k = 0, 1, \dots$$

Thus, $a_{N+2} = 0$, and hence $a_{N+4} = a_{N+6} = \dots = 0$. If N is even, then by choosing $a_0 = 1$ and $a_1 = 0$, we have $a_1 = a_3 = a_5 = a_7 = \dots = 0$, but $a_0, a_2, a_4, \dots, a_N \neq 0$. Hence, the solution $y(x)$ is a polynomial of degree N . On the other hand, if N is odd, then by choosing $a_0 = 0$ and $a_1 = 1$, we have $a_0 = a_2 = a_4 = \dots = 0$, but $a_1, a_3, a_5, \dots, a_N \neq 0$, and once more we have a polynomial solution of degree N for $y(x)$.

(b) By part (a), the polynomial $T_N(x)$ satisfies the differential equation (9.7.10). Thus,

$$(1-x^2)T_N'' - xT_N' + N^2T_N = 0.$$

Dividing through by $\sqrt{1-x^2}$, this becomes

$$\sqrt{1-x^2}T_N'' - \frac{x}{\sqrt{1-x^2}}T_N' + \frac{N^2}{\sqrt{1-x^2}}T_N = 0.$$

However, the first two terms on the left side can be combined as the derivative of a product:

$$\left[\sqrt{1-x^2} \cdot T_N' \right]' + \frac{N^2}{\sqrt{1-x^2}}T_N = 0.$$

(c) From part (b), we have that

$$\left[\sqrt{1-x^2}T_N' \right]' + \frac{N^2}{\sqrt{1-x^2}}T_N = 0 \quad \text{and} \quad \left[\sqrt{1-x^2}T_M' \right]' + \frac{M^2}{\sqrt{1-x^2}}T_M = 0.$$

Multiplying the first equation by T_M and the second equation by T_N gives

$$T_M \left[\sqrt{1-x^2}T_N' \right]' + \frac{T_M N^2}{\sqrt{1-x^2}}T_N = 0 \quad \text{and} \quad T_N \left[\sqrt{1-x^2}T_M' \right]' + \frac{T_N M^2}{\sqrt{1-x^2}}T_M = 0.$$

Subtracting the second of these equations from the first yields

$$T_M \left[\sqrt{1-x^2}T_N' \right]' - T_N \left[\sqrt{1-x^2}T_M' \right]' + \frac{1}{\sqrt{1-x^2}}(N^2 - M^2)T_N T_M = 0.$$

Using integration by parts to integrate over the interval $[-1, 1]$ gives

$$\begin{aligned} \left[T_M T_N' \sqrt{1-x^2} \right]_{-1}^1 - \int_{-1}^1 \sqrt{1-x^2} T_M' T_N' dx - \left[T_N T_M' \sqrt{1-x^2} \right]_{-1}^1 \\ + \int_{-1}^1 \sqrt{1-x^2} T_M' T_N' dx + (N^2 - M^2) \int_{-1}^1 \frac{T_N(x) T_M(x)}{\sqrt{1-x^2}} dx = 0, \end{aligned}$$

which simplifies to

$$(N^2 - M^2) \int_{-1}^1 \frac{T_N(x)T_M(x)}{\sqrt{1-x^2}} dx = 0.$$

Consequently, for $N \neq M$, we have

$$\int_{-1}^1 \frac{T_N(x)T_M(x)}{\sqrt{1-x^2}} dx = 0.$$

17.

(a) This differential equation is already in the form (9.7.5), with $p(x) = 1 + 2N - x$ and $q(x) = N^2$. Hence, $p(0) = 1 + 2N$ and $q(0) = N^2$. The indicial equation (9.7.6) is therefore

$$r(r-1) + (1+2N)r + N^2 = 0.$$

That is, $r^2 + 2Nr + N^2 = 0$, or $(r+N)^2 = 0$. We conclude that $r = -N$ is the only root of the indicial equation (of multiplicity two).

(b) We have a Frobenius series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n-N}, \quad a_0 \neq 0.$$

Differentiating twice, we have that

$$y'(x) = \sum_{n=0}^{\infty} a_n (n-N) x^{n-N-1} \quad \text{and} \quad y''(x) = \sum_{n=0}^{\infty} a_n (n-N)(n-N-1) x^{n-N-2}.$$

Substituting into the differential equation (9.7.11) and simplifying yields

$$\sum_{n=0}^{\infty} a_n (n-N)(n-N-1) x^n + (1+2N) \sum_{n=0}^{\infty} a_n (n-N) x^n - \sum_{n=0}^{\infty} a_n (n-N) x^{n+1} + N^2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

That is,

$$\sum_{n=0}^{\infty} [(n-N)(n-N-1) + (1+2N)(n-N) + N^2] a_n x^n - \sum_{n=1}^{\infty} (n-1-N) a_{n-1} x^n = 0,$$

or equivalently,

$$\sum_{n=1}^{\infty} [n^2 a_n - (n-1-N) a_{n-1}] x^n = 0.$$

Hence,

$$n^2 a_n - (n-1-N) a_{n-1} = 0, \quad n = 1, 2, \dots,$$

so that

$$a_n = \frac{(n-1-N)}{n^2} a_{n-1}, \quad n = 1, 2, \dots \quad (0.0.119)$$

Note that $a_{N+1} = 0$, and therefore $a_{N+2} = a_{N+3} = a_{N+4} = \dots = 0$. Hence, this Frobenius series solution terminates after $N+1$ terms.

(c) When $N = 0$, we see from (0.0.119) that all coefficients except a_0 are zero. Hence, setting $a_0 = 1$, the corresponding solution is

$$y(x) = 1.$$

When $N = 1$, we see from (0.0.119) that

$$a_1 = -a_0, \quad a_n = 0, \quad n = 2, 3, \dots$$

Hence, setting $a_0 = 1$, the corresponding solution is

$$y(x) = x^{-1}(1 - x).$$

When $N = 2$, we see from (0.0.119) that

$$a_1 = -2a_0, \quad a_2 = \frac{1}{2}a_0, \quad a_n = 0, \quad n = 3, 4, \dots$$

Hence, setting $a_0 = 1$, the corresponding solution is

$$y(x) = x^{-2} \left(1 - 2x + \frac{1}{2}x^2 \right).$$

When $N = 3$, we see from (0.0.119) that

$$a_1 = -3a_0, \quad a_2 = \frac{3}{2}a_0, \quad a_3 = -\frac{1}{6}a_0, \quad a_n = 0, \quad n = 4, 5, \dots$$

Hence, setting $a_0 = 1$, the corresponding solution is

$$y(x) = x^{-3} \left(1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3 \right).$$

(d) Iterative substitution into (0.0.119) yields

$$a_k = \frac{(-N)(1-N) \cdots (k-1-N)}{1^2 \cdot 2^2 \cdot 3^2 \cdots k^2} a_0.$$

Setting $a_0 = 1$ and factoring -1 out of each term in the numerator yields

$$a_k = (-1)^k \frac{N(N-1) \cdots (N+1-k)}{1^2 \cdot 2^2 \cdots (k-1)^2 \cdot k^2}.$$

Hence, the Frobenius series solution becomes

$$\begin{aligned} y(x) &= x^{-N} \left[1 + \sum_{k=1}^N (-1)^k \frac{N(N-1) \cdots (N+1-k)}{1^2 \cdot 2^2 \cdots k^2} x^k \right] \\ &= x^{-N} \left[1 + \sum_{k=1}^N (-1)^k \prod_{i=1}^k \frac{(N+1-i)}{i^2} x^k \right]. \end{aligned}$$

18. The given differential equation has the form $x^2 y'' + xp(x)y' + q(x)y = 0$, where

$$p(x) = 1 - (a+b) + cx \quad \text{and} \quad q(x) = ab + dx.$$

Hence,

$$p(0) = 1 - (a + b) \quad \text{and} \quad q(0) = ab.$$

Therefore, the indicial equation is

$$r(r - 1) + (1 - (a + b))r + ab = 0.$$

That is,

$$r^2 - (a + b)r + ab = 0,$$

from which the roots $r = a$ and $r = b$ are quickly derived. Since a and b are distinct and do not differ by an integer, there exist two linearly independent Frobenius series solutions of the form

$$y_1(x) = x^a \sum_{n=0}^{\infty} a_n x^n, \quad a_0 \neq 0, \quad \text{and} \quad y_2(x) = x^b \sum_{n=0}^{\infty} b_n x^n, \quad b_0 \neq 0.$$

Differentiating the expression for $y_1(x)$ yields

$$y_1'(x) = \sum_{n=0}^{\infty} a_n(a + n)x^{a+n-1}, \quad y_1''(x) = \sum_{n=0}^{\infty} a_n(a + n)(a + n - 1)x^{a+n-2}.$$

Substituting into the given differential equation gives

$$\sum_{n=0}^{\infty} a_n(a + n)(a + n - 1)x^{a+n} + [1 - (a + b) + cx] \sum_{n=0}^{\infty} a_n(a + n)x^{a+n} + (ab + dx) \sum_{n=0}^{\infty} a_n x^{a+n} = 0,$$

or equivalently,

$$\sum_{n=0}^{\infty} [(a + n)(a + n - 1) + (a + n)[1 - (a + b)] + ab] a_n x^n + \sum_{n=0}^{\infty} [(a + n)c + d] a_n x^{n+1} = 0,$$

which simplifies to

$$\sum_{n=1}^{\infty} \{n(n + a - b)a_n + [(a + n - 1)c + d]a_{n-1}\} x^n = 0.$$

Hence,

$$n(n + a - b)a_n + [(a + n - 1)c + d]a_{n-1} = 0, \quad n = 1, 2, \dots,$$

so that

$$a_n = \frac{[(1 - a - n)c - d]}{n(n + a - b)} a_{n-1}, \quad n = 1, 2, \dots$$

Iterative substitution into this expression gives

$$\begin{aligned} a_n &= \left(\frac{[(1 - a - n)c - d]}{n(n + a - b)} \right) \left(\frac{[(2 - a - n)c - d]}{(n - 1)(n - 1 + a - b)} \right) \left(\frac{[(3 - a - n)c - d]}{(n - 2)(n - 2 + a - b)} \right) \cdots \left(\frac{-ac - d}{1(1 + a - b)} \right) a_0 \\ &= \frac{1}{n!} \left(\frac{[(1 - a - n)c - d]}{n + a - b} \right) \left(\frac{[(2 - a - n)c - d]}{n - 1 + a - b} \right) \left(\frac{[(3 - a - n)c - d]}{n - 2 + a - b} \right) \cdots \left(\frac{-ac - d}{1 + a - b} \right) a_0. \end{aligned}$$

Thus, setting $a_0 = 1$, we obtain

$$y_1(x) = x^a \left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{[(1 - a - n)c - d]}{n + a - b} \right) \left(\frac{[(2 - a - n)c - d]}{n - 1 + a - b} \right) \left(\frac{[(3 - a - n)c - d]}{n - 2 + a - b} \right) \cdots \left(\frac{-ac - d}{1 + a - b} \right) x^n \right\}.$$

To obtain the second linearly independent solution to the differential equation, we use the root $r = b$. At this point, all of the ensuing calculations will follow the same lines as the case $r = a$ above with the roles of a and b reversed, since a and b are interchangeable in the differential equation. Therefore, by the symmetry of a and b , we can immediately write down the second solution by interchanging the roles of a and b in the solution $y_1(x)$ above:

$$y_2(x) = x^b \left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{[(1-b-n)c-d]}{n+b-a} \right) \left(\frac{[(2-b-n)c-d]}{n-1+b-a} \right) \left(\frac{[(3-b-n)c-d]}{n-2+b-a} \right) \cdots \left(\frac{-bc-d}{1+b-a} \right) x^n \right\}.$$

19. (a) The given differential equation has the form $x^2 y'' + xp(x)y' + q(x)y = 0$, where

$$p(x) = 1 + bx \quad \text{and} \quad q(x) = b(1 - N)x - N^2.$$

Hence, $p(0) = 1$ and $q(0) = -N^2$. Therefore, the indicial equation is

$$r(r-1) + r - N^2 = 0.$$

That is,

$$r^2 - N^2 = 0,$$

with roots $r = \pm N$.

(b) Substituting

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad a_0 \neq 0,$$

into the given differential equation and simplifying yields

$$\sum_{n=0}^{\infty} [(n+r)(n+r-1) + (n+r) - N^2] a_n x^n + b \sum_{n=0}^{\infty} (n+r+1-N) a_n x^{n+1} = 0,$$

which can be written in the equivalent form

$$(r^2 - N^2)a_0 + \sum_{n=1}^{\infty} \{ [(n+r)^2 - N^2] a_n + b(n+r-N) a_{n-1} \} x^n = 0.$$

Therefore, we obtain the indicial equation

$$r^2 - N^2 = 0,$$

and the general recurrence relation

$$[(n+r)^2 - N^2] a_n + b(n+r-N) a_{n-1} = 0, \quad n = 1, 2, \dots$$

That is,

$$[(n+r)^2 - N^2] a_n = -b(n+r-N) a_{n-1}, \quad n = 1, 2, \dots \quad (0.0.120)$$

When $r = N$, (0.0.120) reduces to

$$a_n = -\frac{b}{n+2N} a_{n-1}, \quad n = 1, 2, \dots$$

Iterative substitution into this expression yields

$$a_n = \frac{(-b)^n (2N)!}{(n+2N)!} a_0, \quad n = 1, 2, \dots$$

Hence,

$$y_1(x) = a_0 x^N \sum_{n=0}^{\infty} \frac{(-b)^n (2N)!}{(n+2N)!} x^n.$$

Using the fact that

$$e^{-bx} = \sum_{n=0}^{\infty} \frac{(-bx)^n}{n!},$$

it follows that we can write $y_1(x)$ as

$$\begin{aligned} y_1(x) &= a_0 b^{-2N} x^{-N} \sum_{n=0}^{\infty} \frac{(-b)^{n+2N} x^{n+2N}}{(n+2N)!} \\ &= a_0 b^{-2N} x^{-N} \sum_{n=2N}^{\infty} \frac{(-bx)^n}{n!} \\ &= a_0 b^{-2N} x^{-N} \left[e^{-bx} - \sum_{n=0}^{2N-1} \frac{(-bx)^n}{n!} \right]. \end{aligned}$$

Thus, letting $a_0 = b^{2N}$, we conclude that

$$y_1(x) = x^{-N} \left[e^{-bx} - \sum_{n=0}^{2N-1} \frac{(-bx)^n}{n!} \right].$$

(c) When $r = -N$ the recurrence relation (0.0.120) reduces to

$$n(n-2N)a_n = -b(n-2N)a_{n-1}, \quad n = 1, 2, \dots \quad (0.0.121)$$

When $n = 2N$ we obtain the following consistency condition for the existence of a second Frobenius solution:

$$n \cdot 0 \cdot a_{2N} = -b \cdot 0 \cdot a_{2N-1},$$

which is satisfied for any choice of a_{2N} . We choose $a_{2N} = 0$ in which case, from (0.0.121), all subsequent coefficients are also zero, and

$$a_n = -\frac{b}{n} a_{n-1}, \quad n = 1, 2, \dots, 2N-1.$$

Iterative substitution into this expression yields

$$a_n = \frac{(-b)^n}{n!} a_0, \quad n = 1, 2, \dots, 2N-1.$$

Therefore,

$$y_2(x) = x^{-N} \sum_{n=0}^{2N-1} \frac{(-bx)^n}{n!}.$$

20. Let $y = x^{1/2}u$. Then

$$y' = x^{1/2}u' + \frac{1}{2}x^{-1/2}u \quad \text{and} \quad y'' = x^{1/2}u'' + x^{-1/2}u' - \frac{1}{4}x^{-3/2}u.$$

Therefore, the given differential equation becomes

$$x^{1/2}u'' + x^{-1/2}u' - \frac{1}{4}x^{-3/2}u + \left(1 - \frac{3}{4x^2}\right)x^{1/2}u = 0.$$

Multiplying this through by $x^{3/2}$ and simplifying yields the Bessel equation of order 1:

$$x^2u'' + xu' + (x^2 - 1)u = 0.$$

Hence, the general solution to this differential equation is

$$u(x) = c_1J_1(x) + c_2Y_1(x),$$

and so

$$y(x) = x^{1/2} [c_1J_1(x) + c_2Y_1(x)].$$

Solutions to Appendix A

1. We have $\bar{z} = \overline{2+5i} = 2-5i$ and $|z| = |2+5i| = \sqrt{4+25} = \sqrt{29}$.

2. We have $\bar{z} = \overline{3-4i} = 3+4i$ and $|z| = |3-4i| = \sqrt{9+16} = \sqrt{25} = 5$.

3. We have $\bar{z} = \overline{5-2i} = 5+2i$ and $|z| = |5-2i| = \sqrt{25+4} = \sqrt{29}$.

4. We have $\bar{z} = \overline{7+i} = 7-i$ and $|z| = |7+i| = \sqrt{49+1} = \sqrt{50} = 5\sqrt{2}$.

5. We have $\bar{z} = \overline{1+2i} = 1-2i$ and $|z| = |1+2i| = \sqrt{1+4} = \sqrt{5}$.

6. We have

$$z_1 z_2 = (1+i)(3+2i) = 3+5i+2i^2 = 3+5i-2 = 1+5i$$

and

$$\frac{z_1}{z_2} = \frac{1+i}{3+2i} = \frac{1+i}{3+2i} \cdot \frac{3-2i}{3-2i} = \frac{3+i-2i^2}{9+4} = \frac{1}{13}(5+i).$$

7. We have

$$z_1 z_2 = (-1+3i)(2-i) = -2+7i-3i^2 = -2+7i+3 = 1+7i$$

and

$$\frac{z_1}{z_2} = \frac{-1+3i}{2-i} = \frac{-1+3i}{2-i} \cdot \frac{2+i}{2+i} = \frac{-2+5i+3i^2}{4+1} = \frac{-2+5i-3}{5} = -1+i.$$

8. We have

$$z_1 z_2 = (2+3i)(1-i) = 2+i-3i^2 = 2+i+3 = 5+i$$

and

$$\frac{z_1}{z_2} = \frac{2+3i}{1-i} = \frac{2+3i}{1-i} \cdot \frac{1+i}{1+i} = \frac{2+5i+3i^2}{1+1} = \frac{2+5i-3}{2} = \frac{1}{2}(-1+5i).$$

9. We have

$$z_1 z_2 = (4-i)(1+3i) = 4+11i-3i^2 = 4+11i+3 = 7+11i$$

and

$$\frac{z_1}{z_2} = \frac{4-i}{1+3i} = \frac{4-i}{1+3i} \cdot \frac{1-3i}{1-3i} = \frac{4-13i+3i^2}{1+9} = \frac{4-13i-3}{10} = \frac{1}{10}(1-13i).$$

10. We have

$$z_1 z_2 = (1-2i)(3+4i) = 3-2i-8i^2 = 3-2i+8 = 11-2i$$

and

$$\frac{z_1}{z_2} = \frac{1-2i}{3+4i} = \frac{1-2i}{3+4i} \cdot \frac{3-4i}{3-4i} = \frac{3-10i+8i^2}{9+16} = \frac{3-10i-8}{25} = -\frac{1}{5}(1+2i).$$

11. Let $z_1 = a+bi$ and $z_2 = c+di$, where a, b, c and $d \in \mathbb{R}$. Then

$$\overline{z_1 + z_2} = \overline{(a+bi) + (c+di)} = \overline{(a+c) + (b+d)i} = (a+c) - (b+d)i = (a-bi) + (c-di) = \bar{z}_1 + \bar{z}_2.$$

12. If $z_k = a_k + b_k i$, where $a_k, b_k \in \mathbb{R}$ and $k \in \{1, 2, 3, \dots, n\}$, then $\overline{\sum_{k=1}^n z_k} = \sum_{k=1}^n \bar{z}_k$ for all positive integers n .

Proof: The proposition is true for $n = 2$ by Problem 11. Suppose that the statement is true for some

positive integer $k \geq 2$, and consider the conjugate of the addition of $k + 1$ terms:

$$\overline{\sum_{j=1}^{k+1} z_j} = \overline{\sum_{j=1}^k z_j + z_{k+1}} = \overline{\sum_{j=1}^k z_j + \overline{\overline{z_{k+1}}}} = \sum_{k=1}^n \overline{\overline{z_k}} + \overline{\overline{z_{k+1}}} = \sum_{j=1}^{k+1} \overline{\overline{z_j}}.$$

Consequently, the proposition is true for $k + 1$ when it is true for k ; hence, by mathematical induction, the statement is true for all positive integers $n \geq 2$.

13. Let $z_1 = a + bi$ and $z_2 = c + di$, where $a, b, c, d \in \mathbb{R}$. Then

$$\begin{aligned} \overline{z_1 z_2} &= \overline{(a + bi)(c + di)} \\ &= \overline{ac + (ad + bc)i + bdi^2} \\ &= \overline{ac + (ad + bc)i - bd} \\ &= \overline{(ac - bd) + (ad + bc)i} \\ &= (ac - bd) - (ad + bc)i \\ &= ac - bd - adi - bci \\ &= ac - adi - bci + bdi^2 \\ &= (a - bi)(c - di) \\ &= \overline{a + bi} \overline{c + di} \\ &= \overline{z_1} \overline{z_2}. \end{aligned}$$

14. Let $z_1 = a + bi$ and $z_2 = c + di$, where $z_2 \neq 0$, and let $a, b, c, d \in \mathbb{R}$. Then

$$\begin{aligned} \overline{\left[\frac{z_1}{z_2}\right]} &= \overline{\left[\frac{a + bi}{c + di}\right]} \\ &= \overline{\left[\frac{a + bi}{c + di} \cdot \frac{c - di}{c - di}\right]} \\ &= \overline{\frac{ac - adi + bci - bdi^2}{c^2 + d^2}} \\ &= \overline{\frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}} \\ &= \overline{\frac{(ac + bd) - (bc - ad)i}{c^2 + d^2}} \\ &= \frac{ac + adi - bci - bdi^2}{c^2 + d^2} \\ &= \frac{a - bi}{c - di} \cdot \frac{c + di}{c + di} \\ &= \frac{a - bi}{c - di} \\ &= \frac{\overline{z_1}}{\overline{z_2}}. \end{aligned}$$

15. We have $e^{2ix} = \cos(2x) + i \sin(2x)$.

16. We have $e^{(3+4i)x} = e^{3x} \cos(4x) + ie^{3x} \sin(4x)$.

17. We have $e^{-5ix} = \cos(5x) - i \sin(5x)$.

18. We have $e^{-(2+i)x} = e^{-2x} \cos x - ie^{-2x} \sin x$.

19. We have $x^{2-i} = e^{\ln x^{2-i}} = e^{(2-i) \ln x} = e^{2 \ln x} \cos(\ln x) - ie^{2 \ln x} \sin(\ln x)$.

20. We have $x^{3i} = e^{\ln(x^{3i})} = e^{3i \ln x} = \cos(3 \ln x) + i \sin(3 \ln x)$.

21. We have

$$x^{-1+2i} = \frac{1}{x} \cdot x^{2i} = \frac{1}{x} \cdot e^{\ln(x^{2i})} = \frac{1}{x} \cdot e^{2i \ln x} = \frac{1}{x} [\cos(2 \ln x) + i \sin(2 \ln x)] = \frac{\cos(2 \ln x)}{x} + i \frac{\sin(2 \ln x)}{x}.$$

22. We have

$$\begin{aligned} x^{2i} e^{(3+4i)x} &= e^{3x} (\cos(2 \ln x) + i \sin(2 \ln x)) (\cos 4x + i \sin 4x) \\ &= e^{3x} [\cos(2 \ln x) \cos(4x) - \sin(2 \ln x) \sin(4x)] + ie^{3x} [\cos(2 \ln x) \sin(4x) + \sin(2 \ln x) \cos(4x)]. \end{aligned}$$

23. Using Euler's identity, $e^{ix} = \cos x + i \sin x$, we simply substitute $x = \pi$ and rearrange the resulting equality: $e^{i\pi} = \cos \pi + i \sin \pi = -1$.

24. We have

$$\begin{aligned} e^{ibx} + e^{-ibx} &= (\cos bx + i \sin bx + \cos(-bx) + i \sin(-bx)) \\ &= 2 \cos bx, \end{aligned}$$

from which the desired formula follows at once. The expression for $\sin bx$ is derived similarly.

25. We have

$$\sin 4x = \frac{1}{2i} (e^{4ix} - e^{-4ix}).$$

26. We have

$$\cos 8x = \frac{1}{2} (e^{8ix} + e^{-8ix}).$$

27. We have

$$\tan x = \frac{\sin x}{\cos x} = \frac{1}{i} \cdot \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}.$$

28. From Problem 24, we have

$$\cos^2 x = \frac{1}{4} (e^{ix} + e^{-ix})^2 = \frac{1}{4} (e^{2ix} + 2 + e^{-2ix})$$

and

$$\sin^2 x = \frac{1}{4} = -\frac{1}{4} (e^{ix} - e^{-ix})^2 = -\frac{1}{4} (e^{2ix} - 2 + e^{-2ix}).$$

Summing the two expressions gives 1, as desired.

Solutions to Appendix B

1. From $\frac{2x-1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$, we find that $2x-1 = A(x+2) + B(x+1)$. If $x = -2$, this implies that $-5 = (-1)B$, so that $B = 5$. If $x = -1$, on the other hand, we have $-3 = 1 \cdot A$, so that $A = -3$. Thus,

$$\frac{2x-1}{(x+1)(x+2)} = \frac{-3}{x+1} + \frac{5}{x+2}.$$

2. From $\frac{x-2}{(x-1)(x+4)} = \frac{A}{x-1} + \frac{B}{x+4}$, we find that $x-2 = A(x+4) + B(x-1)$. If $x = -4$, this implies that $-6 = -5B$, so that $B = 6/5$. If $x = 1$ on the other hand, we have $-1 = 5A$, so that $A = -1/5$. Thus,

$$\frac{x-2}{(x-1)(x+4)} = \frac{-1}{5(x-1)} - \frac{6}{5(x+4)}.$$

3. From $\frac{x+1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$, we find that $x+1 = A(x+2) + B(x-3)$. If $x = -2$, this implies that $-1 = -5B$, so that $B = 1/5$. If $x = 3$ on the other hand, we have $4 = 5A$, so that $A = 4/5$. Thus,

$$\frac{x+1}{(x-3)(x+2)} = \frac{4}{5(x-3)} + \frac{1}{5(x+2)}.$$

4. From $\frac{x^2-x+4}{(x+3)(x-1)(x+2)} = \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{x+2}$, we find that

$$x^2-x+4 = A(x-1)(x+2) + B(x+3)(x+2) + C(x+3)(x-1).$$

If $x = 1$, then $4 = 12B$, so that $B = 1/3$. If $x = -2$, then we obtain $10 = C(-3)$, so that $C = -10/3$. Finally, if $x = -3$, then we obtain $16 = 4A$, so that $A = 4$. Thus,

$$\frac{x^2-x+4}{(x+3)(x-1)(x+2)} = \frac{4}{x+3} + \frac{1}{3(x-1)} - \frac{10}{3(x+2)}.$$

5. From $\frac{2x-1}{(x+4)(x-2)(x+1)} = \frac{A}{x+4} + \frac{B}{x-2} + \frac{C}{x+1}$, we find that

$$2x-1 = A(x-2)(x+1) + B(x+4)(x+1) + C(x+4)(x-2).$$

If $x = -4$, this implies that $-9 = 18A$, so that $A = -1/2$. If $x = 2$, on the other hand, then $3 = 18B$, so that $B = 1/6$. Finally, if $x = -1$, then $-3 = C(-9)$, so that $C = 1/3$. Thus,

$$\frac{2x-1}{(x+4)(x-2)(x+1)} = -\frac{1}{2(x+4)} + \frac{1}{6(x-2)} + \frac{1}{3(x+1)}.$$

6. From $\frac{3x^2-2x+14}{(2x-1)(x+5)(x+2)} = \frac{A}{2x-1} + \frac{B}{x+5} + \frac{C}{x+2}$, we find that

$$3x^2-2x+14 = A(x+5)(x+2) + B(2x-1)(x+2) + C(2x-1)(x+5).$$

If $x = -2$, this implies that $30 = C(-15)$, so we have $C = -2$. If $x = -5$, on the other hand, we obtain $99 = 33B$, so that $B = 3$. Finally, if $x = 1/2$, we have $55/4 = A(55/4)$, which implies that $A = 1$. Thus,

$$\frac{3x^2-2x+14}{(2x-1)(x+5)(x+2)} = \frac{1}{2x-1} + \frac{3}{x+5} - \frac{2}{x+2}.$$

7. From $\frac{2x+1}{(x+2)(x+1)^2} = \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$, we find that

$$2x+1 = A(x+1)^2 + B(x+2)(x+1) + C(x+2).$$

Therefore, $2x+1 = (A+B)x^2 + (2A+3B+C)x + (A+2B+2C)$, which leads to the system

$$\begin{cases} A+B=0, \\ 2A+3B+C=2, \\ A+2B+2C=1 \end{cases}$$

Solving this system, we have $A=-3$, $B=3$, and $C=-1$. Thus,

$$\frac{2x+1}{(x+2)(x+1)^2} = -\frac{3}{x+2} + \frac{3}{x+1} - \frac{1}{(x+1)^2}.$$

8. From $\frac{5x^2+3}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$, we find that

$$5x^2+3 = A(x-1)^2 + B(x+1)(x-1) + C(x+1).$$

Therefore, $5x^2+3 = (A+B)x^2 + (-2A+C)x + (A-B+C)$, which leads to the system

$$\begin{cases} A+B=5, \\ -2A+C=0, \\ A-B+C=3 \end{cases}$$

Solving this system, we have $A=2$, $B=3$, and $C=4$. Thus,

$$\frac{5x^2+3}{(x+1)(x-1)^2} = \frac{2}{x+1} + \frac{3}{x-1} + \frac{4}{(x-1)^2}.$$

9. From $\frac{3x+4}{x^2(x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4}$, we find that

$$3x+4 = Ax(x^2+4) + B(x^2+4) + (Cx+D)x^2.$$

Therefore, $3x+4 = (A+C)x^3 + (B+D)x^2 + Ax + 4B$, which leads to the system

$$\begin{cases} A+C=0, \\ B+D=0, \\ 4A=3, \\ 4B=4 \end{cases}$$

Solving this system, we have $A=3/4$, $B=1$, $C=-3/4$, and $D=-1$. Thus,

$$\frac{3x+4}{x^2(x^2+4)} = \frac{3}{4x} + \frac{1}{x^2} - \frac{3x+4}{4(x^2+4)}.$$

10. From $\frac{3x-2}{(x-5)(x^2+1)} = \frac{A}{x-5} + \frac{Bx+C}{x^2+1}$, we find that

$$3x-2 = A(x^2+1) + (Bx+C)(x-5).$$

Therefore, $3x - 2 = (A + B)x^2 + (-5B + C)x + (A - 5C)$, which leads to the system

$$\begin{cases} A + B = 0, \\ -5B + C = 3, \\ A - 5C = -2 \end{cases}$$

Solving this system, we have $A = 1/2$, $B = -1/2$, and $C = 1/2$. Thus,

$$\frac{3x - 2}{(x - 5)(x^2 + 1)} = \frac{1}{2(x - 5)} - \frac{x - 1}{2(x^2 + 1)}.$$

11. From $\frac{x^2 + 6}{(x - 2)(x^2 + 16)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 16}$, we find that

$$x^2 + 6 = A(x^2 + 16) + (Bx + C)(x - 2).$$

Therefore, $x^2 + 6 = (A + B)x^2 + (-2B + C)x + (16A - 2C)$, which leads to the system

$$\begin{cases} A + B = 1, \\ -2B + C = 0, \\ 16A - 2C = 6 \end{cases}$$

Solving this system, we have $A = 1/2$, $B = 1/2$, and $C = 1$. Thus,

$$\frac{x^2 + 6}{(x - 2)(x^2 + 16)} = \frac{1}{2(x - 2)} + \frac{x + 2}{2(x^2 + 16)}.$$

12. From $\frac{10}{(x - 1)(x^2 + 9)} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 9}$, we find that

$$10 = A(x^2 + 9) + (Bx + C)(x - 1).$$

Therefore, $10 = (A + B)x^2 + (-B + C)x + (9A - C)$, which leads to the system

$$\begin{cases} A + B = 0, \\ -B + C = 0, \\ 9A - C = 10 \end{cases}$$

Solving this system, we have $A = 1$, $B = -1$, and $C = -1$. Thus,

$$\frac{10}{(x - 1)(x^2 + 9)} = \frac{1}{x - 1} - \frac{x + 1}{x^2 + 9}.$$

13. From $\frac{7x + 2}{(x - 2)(x + 2)^2} = \frac{A}{x - 2} + \frac{B}{x + 2} + \frac{C}{(x + 2)^2}$, we find that

$$7x + 2 = A(x + 2)^2 + B(x - 2)(x + 2) + C(x - 2).$$

Therefore, $7x - 2 = (A + B)x^2 + (4A + C)x + (4A - 4B - 2C)$, which leads to the system

$$\begin{cases} A + B = 0, \\ 4A + C = 7, \\ 4A - 4B - 2C = 2 \end{cases}$$

Solving this system, we have $A = 1$, $B = -1$, and $C = 3$. Thus,

$$\frac{7x+2}{(x-2)(x+2)^2} = \frac{1}{x-2} - \frac{1}{x+2} + \frac{3}{(x+2)^2}.$$

14. From $\frac{7x^2-20}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$, we find that

$$7x^2 - 20 = A(x^2 + 4) + (Bx + C)(x - 2).$$

Therefore, $7x^2 - 20 = (A + B)x^2 + (-2B + C)x + (4A - 2C)$, which leads to the system

$$\begin{cases} A + B = 7, \\ -2B + C = 0, \\ 4A - 2C = -20 \end{cases}$$

Solving this system, we have $A = 1$, $B = 6$, and $C = 12$. Thus,

$$\frac{7x^2-20}{(x-2)(x^2+4)} = \frac{1}{x-2} + \frac{6(x+2)}{x^2+4}.$$

15. From $\frac{7x+4}{(x+1)^3(x-2)} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} + \frac{D}{(x+1)^3}$, we find that

$$\begin{aligned} 7x + 4 &= A(x+1)^3 + B(x-2)(x+1)^2 + C(x-2)(x+1) + D(x-2) \\ &= (A+B)x^3 + (3A+C)x^2 + (3A-3B-C+D)x + (A-2B-2C-2D). \end{aligned}$$

This leads to the system

$$\begin{cases} A + B = 0, \\ 3A + C = 0, \\ 3A - 3B - C + D = 7, \\ A - 2B - 2C - 2D = 4 \end{cases}$$

Solving this system, we have $A = 2/3$, $B = -2/3$, $C = -2$, and $D = 1$. Thus,

$$\frac{7x+4}{(x+1)^3(x-2)} = \frac{2}{3(x-2)} - \frac{2}{3(x+1)} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3}.$$

16. From $\frac{2x^2+3x}{(x+1)(x^2+2x+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+2}$, we find that

$$2x^2 + 3x = A(x^2 + 2x + 2) + (Bx + C)(x + 1).$$

Therefore, $2x^2 + 3x = (A + B)x^2 + (2A + B + C)x + (2A + C)$, which leads to the system

$$\begin{cases} A + B = 2, \\ 2A + B + C = 3, \\ 2A + C = 0 \end{cases}$$

Solving this system, we have $A = -1$, $B = 3$, and $C = 2$. Thus,

$$\frac{2x^2+3x}{(x+1)(x^2+2x+2)} = -\frac{1}{x+1} + \frac{3x+2}{x^2+2x+2}.$$

17. From $\frac{3x+4}{(x-3)(x^2+4x+5)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+4x+5}$, we find that

$$3x+4 = A(x^2+4x+5) + (Bx+C)(x-3).$$

Therefore, $3x+4 = (A+B)x^2 + (4A-3B+C)x + (5A-3C)$, which leads to the system

$$\begin{cases} A+B=0, \\ 4A-3B+C=3, \\ 5A-3C=4 \end{cases}$$

Solving this system, we have $A=1/2$, $B=-1/2$, and $C=-1/2$. Thus,

$$\frac{3x+4}{(x-3)(x^2+4x+5)} = \frac{1}{2(x-3)} - \frac{x+1}{2(x^2+4x+5)}.$$

18. From $\frac{7-2x^2}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$, we find that

$$7-2x^2 = A(x^2+4) + (Bx+C)(x-1).$$

Therefore, $7-2x^2 = (A+B)x^2 + (-B+C)x + (4A-C)$, which leads to the system

$$\begin{cases} A+B=-2, \\ -B+C=0, \\ 4A-C=7 \end{cases}$$

Solving this system, we have $A=1$, $B=-3$, and $C=-3$. Thus,

$$\frac{7-2x^2}{(x-1)(x^2+4)} = \frac{1}{x-1} - \frac{3(x+1)}{x^2+4}.$$

Solutions to Appendix C

1. Let $u = x$ and $dv = \cos x$. Then $du = dx$ and $v = \sin x$. Using integration by parts,

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + C.$$

2. Use integration by parts twice. Let $u_1 = x^2$ and $dv_1 = e^{-x} dx$. Then $du_1 = 2x dx$ and $v_1 = -e^{-x}$ so that

$$\int x^2 e^{-x} dx = -x^2 e^{-x} + 2 \int x e^{-x} dx.$$

Now let $u_2 = x$ and $dv_2 = e^{-x} dx$. Then $du_2 = dx$ and $v_2 = -e^{-x}$ so that

$$\int x e^{-x} dx = -x e^{-x} + \int e^{-x} dx = -x e^{-x} - e^{-x}.$$

Substituting this expression into the integration above, we obtain

$$\int x^2 e^{-x} dx = -x^2 e^{-x} + 2(-x e^{-x} - e^{-x}) = -e^{-x}(x^2 + 2x + 2) + C.$$

3. The integrand is only defined for $x > 0$. Let $u = \ln x$ and $dv = dx$. Then $du = \frac{dx}{x}$ and $v = x$ so that

$$\int \ln x dx = x \ln x - \int dx = x \ln x - x + C.$$

4. Let $u = \tan^{-1} x$ and $dv = dx$. Then $du = \frac{dx}{1+x^2}$ and $v = x$ so that

$$\int \tan^{-1} x dx = x \tan^{-1} x - \int \frac{x}{1+x^2} dx = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C.$$

5. Let $w = x^2$ so that $dw = 2x dx$. Then

$$\int x^3 e^{x^2} dx = \frac{1}{2} \int w e^w dw.$$

Now let $u = w$ and $dv = e^w dw$. Then $du = dw$ and $v = e^w$, so that

$$\int w e^w dw = w e^w - \int e^w dw = w e^w - e^w.$$

Substituting this expression into the above integral yields

$$\int x^3 e^{x^2} dx = \frac{1}{2} [w e^w - e^w] = \frac{1}{2} e^{x^2} (x^2 - 1).$$

6. $\int \frac{x}{x^2+1} dx = \frac{1}{2} \int \frac{2x}{x^2+1} dx = \frac{1}{2} \ln(x^2+1) + C.$

7. $\int \frac{(x-1)}{x+2} dx = \int \left(1 - \frac{3}{x+2}\right) dx = \int dx - 3 \int \frac{1}{x+2} dx = x - 3 \ln|x+2| + C.$

8. From $\frac{x+2}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$, we obtain $x+2 = A(x+3) + B(x-1)$. Therefore, $A = \frac{3}{4}$ and $B = \frac{1}{4}$. Thus,

$$\int \frac{(x+2)}{(x-1)(x+3)} dx = \frac{3}{4} \int \frac{1}{x-1} dx + \frac{1}{4} \int \frac{1}{x+3} dx = \frac{3}{4} \ln|x-1| + \frac{1}{4} \ln|x+3| + C.$$

9. From $\frac{2x+1}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$, we find that

$$2x+1 = A(x^2+4) + x(Bx+C).$$

Therefore, $2x+1 = (A+B)x^2 + Cx + 4A$, which leads to the system

$$\begin{cases} A+B=0, \\ C=2, \\ 4A=1 \end{cases}$$

Solving the system, we have $A = \frac{1}{4}$, $B = -\frac{1}{4}$, and $C = 2$. Thus,

$$\int \frac{2x+1}{x(x^2+4)} dx = \frac{1}{4} \int \frac{dx}{x} - \frac{1}{4} \int \frac{(x-2)}{x^2+4} dx = \frac{1}{2} \ln|x| - \frac{1}{8} \int \frac{2x}{x^2+4} dx + \frac{1}{2} \int \frac{dx}{(x^2+4)}$$

$$= \frac{1}{4} \ln |x| - \frac{1}{8} \ln(x^2 + 4) + \tan^{-1}(x/2) + C.$$

10. We have $\frac{x^2 + 5}{(x-1)(x+4)} = 1 - \frac{3x-9}{(x-1)(x+4)}$, and we write $\frac{3x-9}{(x-1)(x+4)} = \frac{A}{x-1} + \frac{B}{x+4}$. Therefore, $3x-9 = A(x+4) + B(x-1)$. If $x = 1$, we have $-6 = 5A$, so that $A = -6/5$. If $x = -4$, on the other hand, then $-21 = -5B$, so that $B = 21/5$. Thus,

$$\begin{aligned} \int \frac{(x^2 + 5)}{(x-1)(x+4)} dx &= \int \left[1 + \frac{6}{5(x-1)} - \frac{21}{5(x+4)} \right] dx \\ &= \int dx + \frac{6}{5} \int \frac{1}{x-1} dx - \frac{21}{5} \int \frac{1}{x+4} dx \\ &= x + \frac{6}{5} \ln |x-1| - \frac{21}{5} \ln |x+4| + C. \end{aligned}$$

11. We have

$$\int \frac{(x+3)}{2x-1} dx = \int \left[\frac{1}{2} + \frac{7}{2(2x-1)} \right] dx = \frac{1}{2} \int dx + \frac{7}{4} \int \frac{2dx}{2x-1} = \frac{x}{2} + \frac{7}{4} \ln |2x-1| + C.$$

12. Let $w = x^2 + 3x + 4$ so that $dw = (2x+3)dx$. Then

$$\int \frac{(2x-3)}{x^2 + 3x + 4} dx = \int \frac{dw}{w} = \ln |w| + C = \ln |x^2 + 3x + 4| + C.$$

13. From $\frac{3x+2}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$, we find that

$$3x+2 = A(x+1)^2 + Bx(x+1) + Cx.$$

Therefore, $3x+2 = (A+B)x^2 + (2A+B+C)x + A$, which leads to the system

$$\begin{cases} A+B=0, \\ 2A+B+C=3, \\ A=2 \end{cases}$$

Solving the system, we have $A = 2$, $B = -2$, and $C = 1$. Thus,

$$\int \frac{(3x+2)}{x(x+1)^2} dx = 2 \int \frac{dx}{x} - 2 \int \frac{dx}{x+1} + \int \frac{dx}{(x+1)^2} = 2 \ln |x| - 2 \ln |x+1| - \frac{1}{x-1} + C.$$

14. Let $u = \frac{x}{2}$ so that $du = \frac{dx}{2}$. Thus,

$$\int \frac{1}{\sqrt{4-x^2}} dx = \int \frac{1}{\sqrt{1-u^2}} du = \sin^{-1} u + C = \sin^{-1} \left(\frac{x}{2} \right) + C.$$

15. Let $u = x+1$ so that $du = dx$. Then

$$\int \frac{1}{x^2 + 2x + 2} dx = \int \frac{1}{(x+1)^2 + 1} dx = \int \frac{1}{u^2 + 1} du = \tan^{-1}(u) + C = \tan^{-1}(x+1) + C.$$

16. The integrand is only defined for $x > 0$. Let $u = \ln x$ so that $du = \frac{dx}{x}$. Then

$$\int \frac{1}{x \ln x} dx = \int \frac{1}{u} du = \ln |u| + C = \ln |\ln x| + C.$$

17. Let $u = \cos x$ so that $du = -\sin x dx$. Then

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = - \int \frac{1}{u} du = -\ln |u| + C = -\ln |\cos x| + C.$$

18. Writing $\frac{x+1}{x^2-x-6} = \frac{A}{x+2} + \frac{B}{x-3}$, we find that

$$x+1 = A(x-3) + B(x+2).$$

If $x = -2$, this gives $-1 = -5A$, so that $A = 1/5$. If $x = 3$, on the other hand, then $4 = 5B$, so that $B = 4/5$. Hence,

$$\int \frac{(x+1)}{x^2-x-6} dx = \frac{1}{5} \int \frac{1}{x+2} dx + \frac{4}{5} \int \frac{1}{x-3} dx = \frac{1}{5} \ln |x+2| + \frac{4}{5} \ln |x-3| + C.$$

19. $\int \cos^2 x dx = \int \frac{(1+\cos 2x)}{2} dx = \frac{x}{2} + \frac{1}{4} \sin 2x = \frac{1}{4}(2x + \sin 2x) + C.$

20. Let $x = \sin \theta$ where $-\pi/2 \leq \theta \leq \pi/2$ so that $dx = \cos \theta d\theta$. Then

$\sqrt{1-x^2} = \sqrt{1-\sin^2 \theta} = \sqrt{\cos^2 \theta} = |\cos \theta| = \cos \theta$. Thus,

$$\begin{aligned} \int \sqrt{1-x^2} dx &= \int \cos^2 \theta d\theta = \int \frac{(1+\cos 2\theta)}{2} d\theta = \frac{1}{2} \int d\theta + \frac{1}{4} \int \cos 2\theta d\theta = \frac{\theta}{2} + \frac{1}{4} \sin 2\theta + C \\ &= \frac{1}{2} \sin^{-1} x + \frac{1}{2} \sin \theta \cos \theta + C = \frac{1}{2} (\sin^{-1} x + x \sqrt{1-x^2}) + C. \end{aligned}$$

21. Integration by parts will be applied twice. Let $u_1 = e^{3x}$ and $dv_1 = \sin 2x dx$.

Then $du_1 = 3e^{3x} dx$ and $v_1 = -\frac{1}{2} \cos 2x$ so that

$$\int e^{3x} \sin 2x dx = -\frac{e^{3x}}{2} \cos 2x + \frac{3}{2} \int e^{3x} \cos 2x dx. \quad (21.1)$$

Now let $u_2 = e^{3x}$ and $dv_2 = \cos 2x dx$. Then $du_2 = 3e^{3x} dx$ and $v_2 = \frac{1}{2} \sin 2x$ so that

$$\int e^{3x} \cos 2x dx = \frac{e^{3x}}{2} \sin 2x - \frac{3}{2} \int e^{3x} \sin 2x dx. \quad (21.2)$$

Substituting Equation (21.2) into (21.1) yields

$$\begin{aligned} \int e^{3x} \sin 2x dx &= -\frac{e^{3x}}{2} \cos 2x + \frac{3}{2} \left(\frac{e^{3x}}{2} \sin 2x - \frac{3}{2} \int e^{3x} \sin 2x dx \right) \\ \Rightarrow \int e^{3x} \sin 2x dx &= -\frac{e^{3x}}{2} \cos 2x + \frac{3e^{3x}}{4} \sin 2x - \frac{9}{4} \int e^{3x} \sin 2x dx \\ \Rightarrow \frac{13}{4} \int e^{3x} \sin 2x dx &= \frac{1}{4} (3e^{3x} \sin 2x - 2e^{3x} \cos 2x) \\ \Rightarrow \int e^{3x} \sin 2x dx &= \frac{e^{3x}}{13} (3 \sin 2x - 2 \cos 2x) + C. \end{aligned}$$

22. Since $\int e^x \sin^2 x dx = \int \frac{e^x(1 - \cos 2x)}{2} dx = \frac{1}{2} \int e^x dx - \frac{1}{2} \int e^x \cos 2x dx$, we have $\int e^x \sin^2 x dx = \frac{e^x}{2} - \frac{1}{2} \int e^x \cos 2x dx$. (22.1)

Integration by parts will be performed twice to evaluate the right integral in Equation (22.1).

Let $u_1 = e^x$ and $dv_1 = \cos 2x dx$. Then $du_1 = e^x dx$ and $v_1 = \frac{1}{2} \sin 2x$ so that

$$\int e^x \cos 2x dx = \frac{e^x}{2} \sin 2x - \frac{1}{2} \int e^x \sin 2x dx. \quad (22.2)$$

Now let $u_2 = e^x$ and $dv_2 = \sin 2x dx$. Then $du_2 = e^x dx$ and $v_2 = -\frac{1}{2} \cos 2x$ so that

$$\int e^x \sin 2x dx = -\frac{e^x}{2} \cos 2x + \frac{1}{2} \int e^x \cos 2x dx. \quad (22.3)$$

Substituting from (22.3) into (22.2) yields

$$\begin{aligned} \int e^x \cos 2x dx &= \frac{e^x}{2} \sin 2x - \frac{1}{2} \left[-\frac{e^x}{2} \cos 2x + \frac{1}{2} \int e^x \cos 2x dx \right] \\ \Rightarrow \int e^x \cos 2x dx &= \frac{e^x}{2} \sin 2x + \frac{e^x}{4} \cos 2x - \frac{1}{4} \int e^x \cos 2x dx \\ \Rightarrow \frac{5}{4} \int e^x \cos 2x dx &= \frac{e^x}{4} (\cos 2x + 2 \sin 2x) \\ \Rightarrow \int e^x \cos 2x dx &= \frac{e^x}{5} (\cos 2x + 2 \sin 2x). \end{aligned} \quad (22.4)$$

Substituting from (22.4) into (22.1) yields

$$\begin{aligned} \int e^x \sin^2 x dx &= \frac{e^x}{2} - \frac{1}{2} \frac{e^x}{5} (\cos 2x + 2 \sin 2x) \\ \Rightarrow \int e^x \sin^2 x dx &= \frac{e^x}{10} [5 - (\cos 2x + 2 \sin 2x)] + C. \end{aligned}$$